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Asymptotic Properties of Stieltjes Polynomials and Gauss–Kronrod Quadrature Formulae

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Stieltjes polynomials are orthogonal polynomials with respect to the sign changing weight function $wP_n(\cdot, w)$, where $P_n(\cdot, w)$ is the nth orthogonal polynomial with respect to w. Zeros of Stieltjes polynomials are nodes of Gauss–Kronrod quadrature formulae, which are basic for the most frequently used quadrature routines with combined practical error estimate. For the ultraspherical weight function $w_\lambda(x) = (1-x^2)^{\lambda - 1/2}$, $0 \leqslant \lambda \leqslant 1$, we prove asymptotic representations of the Stieltjes polynomials and of their first derivative, which hold uniformly for $x = \cos\theta$, $\varepsilon \leqslant \theta \leqslant \pi - \varepsilon$, where $\varepsilon \in (0, \pi/2)$ is fixed. Some conclusions are made with respect to the distribution of the zeros of Stieltjes polynomials, proving an open problem of Monegato [15, p. 235] and Peherstorfer [23, p. 186]. As a further application, we prove an asymptotic representation of the weights of Gauss–Kronrod quadrature formulae with respect to w_λ, $0 \leqslant \lambda \leqslant 1$, and we prove the precise asymptotical value for the variance of Gauss–Kronrod quadrature formulae in these cases.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

Let $\mathscr{P}_n$ be the space of polynomials of degree less than or equal to n. Let the weight function w on $[-1, 1]$ be such that there exists a sequence of orthogonal polynomials $P_n(\cdot, w)$, $n = 0, 1, 2, \ldots$, $P_n(\cdot, w) \in \mathscr{P}_n$, i.e.

$$\int_{-1}^{1} w(x)\, P_n(x, w)\, x^m\, dx \begin{cases} = 0 & 0 \leqslant m < n, \\ \neq 0 & m = n. \end{cases} \tag{1}$$

Regarding $wP_n(\cdot, w)$ as a sign-changing weight function, $E_{n+1}(\cdot, w) \in \mathscr{P}_{n+1}$ is called a Stieltjes polynomial if it satisfies

$$\int_{-1}^{1} w(x)\, P_n(x, w)\, E_{n+1}(x, w)\, x^m\, dx \begin{cases} = 0 & 0 \leqslant m < n+1, \\ \neq 0 & m = n+1. \end{cases} \tag{2}$$

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Depending on w these equations may not be sufficient for the zeros of $E_{n+1}(\cdot, w)$ neither to lie in $[-1, 1]$ nor to be real. However, for the ultraspherical weight function w_λ, $w_\lambda(x) = (1-x^2)^{\lambda - 1/2}$ and $\lambda \in [0, 2]$, Szegö [25] proved that these properties hold for all $n \in \mathbb{N}$. Moreover, Szegö proved that the zeros of $E_{n+1}(\cdot, w_\lambda)$ and the zeros of $P_n(\cdot, w_\lambda)$ interlace, and he gave explicit expressions for $E_{n+1}(\cdot, w_\lambda)$ in each of the cases $\lambda = 0$, $\lambda = 1$ respectively $\lambda = 2$. Since Szegös paper, many results and new questions with respect to the location of the zeros of Stieltjes polynomials appeared in the literature. For the Legendre weight function $w_{1/2}$, Monegato [15] conjectured the interlacing property for the zeros of $E_{n+1}(\cdot, w_{1/2})$ and $E_n(\cdot, w_{1/2})$, which is the Stieltjes polynomial with respect to $P_{n-1}(\cdot, w_{1/2})$. Furthermore, Monegato [15] conjectured from numerical results that for the zeros $\xi_{\mu, n+1}$ of $E_{n+1}(\cdot, w_{1/2})$ there holds

$$\xi_{n+2-\mu, n+1} \approx \cos \frac{\mu - 3/4}{n + 1/2} \pi, \qquad \mu = 1, ..., n+1. \tag{3}$$

In a recent paper, Peherstorfer [23] proved the important and very general result that there hold [23, Theorem 4.1 and Corollary 4.1]

(a) $k_n E_{n+1}(x, (1-x^2) w) = P_{n+1}(x, w) + \delta_n(x)$, where

$$|\delta_n(x)| \leqslant \text{const} \frac{\log n}{n}, \qquad x \in [-1, 1],$$

whenever there exists a $m \in \mathbb{R}$ such that $0 < m \leqslant \sqrt{1-x^2}\, w(x)$, $x \in [-1, 1]$, and $\sqrt{1-x^2}\, w(x) \in C^2[-1, 1]$;

(b) $k_n E_{n+1}(x, (1-x^2) w) + 2^{-n-1} k_n d_{n+1, n} = P_{n+1}(x, w) + \tilde{\delta}_n(x)$, where

$$\lim_{n \to \infty} \tilde{\delta}_n(x) = 0$$

uniformly for $x \in [\eta_1 + \delta, \eta_2 - \delta]$, $\delta > 0$, $-1 \leqslant \eta_1 < \eta_2 \leqslant 1$ and $d_{n+1, n}$ is defined in [23, (4.1)], whenever there exists a $m \in \mathbb{R}$ such that $w(x)/\sqrt{1-x^2} \in L^1[-1, 1]$, $\sqrt{1-x^2}\, w(x) \geqslant m > 0$ for $x \in [\eta_1, \eta_2] \subset [-1, 1]$ and $\sqrt{1-x^2}\, w(x) \in C^2[\eta_1, \eta_2]$.

In both cases, k_n is defined by

$$P_n(x, (1-x^2) w) = k_n x^n + p(x), \qquad p \in \mathscr{P}_{n-1}. \tag{4}$$

Under these general assumptions, Peherstorfer proved several interlacing properties (cf. [23, Corollary 4.3]) to hold for sufficiently large n.

In the case of the ultraspherical weight function w_λ, the conditions in part (a) are satisfied for $\lambda = 0$, while the conditions of part (b) are satisfied for w_λ whenever $\lambda > 0$. Hence, an asymptotic representation of $E_{n+1}(\cdot, w_\lambda)$ is given in part (a) for $\lambda = 1$, and could be derived from part (b) for $\lambda > 1$ by proving that

$$2^{-n-1} d_{n+1,n} = o(E_{n+1}(x, w_\lambda)) \tag{5}$$

holds uniformly for $x \in [\eta_1 + \delta, \eta_2 - \delta] \subset [-1, 1]$. However, the question of an asymptotic representation of Stieltjes polynomials $E_{n+1}(\cdot, 1)$ for the Legendre weight function $w_{1/2}$ as well as Monegato's conjectures still remain open (cf. Peherstorfer [23, p. 186]).

In this paper, we investigate these problems for w_λ and $0 \leqslant \lambda \leqslant 1$. As our first result, we state an asymptotic representation for $E_{n+1}(\cdot, w_\lambda)$, as well as for the first derivative $E'_{n+1}(\cdot, w_\lambda)$, $0 \leqslant \lambda \leqslant 1$.

THEOREM. *Let $0 \leqslant \lambda \leqslant 1$, $w_\lambda(x) = (1-x^2)^{\lambda - 1/2}$ and let $E_{n+1}(\cdot, w_\lambda)$ be the Stieltjes polynomial with respect to w_λ. For $\varepsilon \leqslant \theta \leqslant \pi - \varepsilon$, with fixed $\varepsilon \in (0, \pi/2)$, we have uniformly*

(i) $E_{n+1}(\cos\theta, w_\lambda) = n^{1-\lambda} \pi^{-1/2} 2^{2-\lambda} \sin^{1-\lambda}\theta \cos\{(n+\lambda)\theta - (\lambda - 1)\pi/2\} + o(n^{1-\lambda})$,

(ii) $E'_{n+1}(\cos\theta, w_\lambda) = n^{2-\lambda} \pi^{-1/2} 2^{2-\lambda} \sin^{-\lambda}\theta \sin\{(n+\lambda)\theta - (\lambda - 1)\pi/2\} + O(n^{1-\lambda})$.

With respect to Monegato's conjecture (3), the following corollary is a direct consequence of the Theorem.

COROLLARY 1. *Let $0 \leqslant \lambda \leqslant 1$, let $\varepsilon \in (0, \pi/2)$ be fixed, and let $\pi \geqslant \theta_{1,n+1} > \theta_{2,n+1} > \cdots > \theta_{n+1,n+1} \geqslant 0$ such that $E_{n+1}(\cos\theta_{\mu,n+1}, w_\lambda) = 0$, $\mu = 1, 2, ..., n+1$. Then there holds uniformly for all $\varepsilon \leqslant \theta_{n+2-\mu,n+1} \leqslant \pi - \varepsilon$ that*

$$\theta_{n+2-\mu,n+1} = \frac{\mu + (\lambda - 2)/2 + o(1)}{n + \lambda}\pi. \tag{6}$$

As a second corollary, the following interlacing property can be shown.

COROLLARY 2. *Let $0 \leqslant \lambda \leqslant 1$, $0 < C \leqslant \frac{1}{2}$ and let $\varepsilon \in (0, \pi/2)$ be fixed. Let $\pi \geqslant \theta_{1,n+1} > \theta_{2,n+1} > \cdots > \theta_{n+1,n+1} \geqslant 0$ such that $E_{n+1}(\cos\theta_{\mu,n+1}, w_\lambda) = 0$, $\mu = 1, 2, ..., n+1$, and let $\pi \geqslant \theta_{1,n} > \theta_{2,n} > \cdots > \theta_{n,n} \geqslant 0$ such that $E_n(\cos\theta_{\mu,n}, w_\lambda) = 0$, $\mu = 1, 2, ..., n$. There exists a $N \in \mathbb{N}$ such that for $n \geqslant N$ and $Cn \leqslant \mu \leqslant (1-C)n$, $\varepsilon \leqslant \theta_{\mu+1,n+1} < \theta_{\mu,n+1} \leqslant \pi - \varepsilon$ and $\varepsilon \leqslant \theta_{\mu+1,n} < \theta_{\mu,n} \leqslant \pi - \varepsilon$ there hold*

(i) $\theta_{\mu+1,n+1} < \theta_{\mu,n} < \theta_{\mu,n+1}$,

(ii) $\theta_{\mu+1,n} < \theta_{\mu+1,n+1} < \theta_{\mu,n}$.

2. Application to Gauss–Kronrod Quadrature

In addition to the interesting theoretic aspects which Stieltjes polynomials offer per se as a remarkable special case of orthogonal polynomials, the study of Stieltjes polynomials is motivated by their importance for the practically used Gauss–Kronrod quadrature formulae. A minimum of notation is necessary for a further study.

Let $p_\mu(x) = x^\mu$. A quadrature formula Q_n with remainder R_n of polynomial degree of exactness $\deg(R_n) = s \geq 0$ is a real linear functional of the type (cf. Brass [1])

$$Q_n[f] = \sum_{\nu=1}^{n} a_\nu f(x_\nu), \qquad -\infty < x_1 < \cdots < x_n < \infty, \tag{7}$$

$$\int_{-1}^{1} w(x)\, f(x)\, dx = Q_n[f] + R_n[f], \qquad R_n[p_\mu] \begin{cases} = 0 & \mu = 0, ..., s, \\ \neq 0 & \mu = s+1. \end{cases}$$

Q_n is called interpolatory if $\deg(R_n) \geq n-1$. For suitable weight functions w, the Gaussian quadrature formula $Q_n^G[f] = \sum_{\nu=1}^n a_{\nu,n}^G f(x_{\nu,n}^G)$ can be defined by $\deg(R_n^G) = 2n-1$, and it is well known that $P_n(x_{\nu,n}^G, w) = 0$, $\nu = 1, ..., n$. If a quadrature formula

$$Q_{2n+1}^{GK}[f] = \sum_{\nu=1}^{n} A_{\nu,n}^{GK} f(x_{\nu,n}^G) + \sum_{\mu=1}^{n+1} B_{\mu,n+1}^{GK} f(\xi_{\mu,n+1}^K) \tag{8}$$

exists such that $\deg(R_{2n+1}^{GK}) \geq 3n+1$, then Q_{2n+1}^{GK} is called a Gauss–Kronrod quadrature formula.

The Gauss–Kronrod quadrature formula is used to compute a second approximation that is considered to improve upon Q_n^G, but which involves only $n+1$ new functional values in addition to the ones used by Q_n^G. This economic advantage makes Gauss–Kronrod quadrature formulas a basis for the most frequently used quadrature routines with practical error estimate (cf. Piessens *et al.* [24]).

Due to a well known characterization of Gauss–Kronrod quadrature formulae, the nodes $\xi_{\mu,n+1}^K$, $\mu = 1, ..., n+1$, in (8) have to be the zeros of the Stieltjes polynomial $E_{n+1}(\cdot, w)$ satisfying the orthogonality property (2). Hence, a Gauss–Kronrod formula is said to exist if all zeros of $E_{n+1}(\cdot, w)$ are real and contained in the interval of integration.

Surveys on Stieltjes polynomials and Gauss–Kronrod quadrature formulae are given by Monegato [15, 16] and by Gautschi [8]. More recent results have been obtained by Gautschi and Notaris [9, 10, 11], Notaris [17, 18, 19, 20], Peherstorfer [21, 22, 23] and in [3].

Monegato [13, 14] proved that for the weight function w_λ, $0 \leqslant \lambda \leqslant 1$, the weights $A^{GK}_{\nu,n}$, $\nu = 1, ..., n$, $B^{GK}_{\mu,n+1}$, $\mu = 1, ..., n+1$ are positive for all $n \in \mathbb{N}$. Using the Theorem from Section 1, we prove an asymptotic representation of the weights in (8).

COROLLARY 3. *Let* $0 \leqslant \lambda \leqslant 1$, *let* $\varepsilon \in (0, \pi/2)$ *be fixed, and for the Gauss–Kronrod quadrature formula* (8) *let* $x^G_{\nu,n} = \cos \phi^G_{\nu,n}$ *and* $\xi^K_{\mu,n+1} = \cos \theta^K_{\mu,n+1}$. *Then there holds uniformly for all* $\varepsilon \leqslant \phi^G_{\nu,n} \leqslant \pi - \varepsilon$ *that*

$$A^{GK}_{\nu,n} = \frac{\pi}{2n+1+\lambda} \sin^{2\lambda} \phi^G_{\nu,n}(1 + o(1)). \tag{9}$$

For all $\varepsilon \leqslant \theta^K_{\mu,n+1} \leqslant \pi - \varepsilon$ *there holds uniformly that*

$$B^{GK}_{\mu,n+1} = \frac{\pi}{2n+1+\lambda} \sin^{2\lambda} \theta^K_{\mu,n+1}(1 + o(1)). \tag{10}$$

Our last result is concerned with the socalled variance of quadrature formulae. For $Q_n[f] = \sum_{\nu=1}^n a_\nu f(x_\nu)$, the variance

$$\text{Var}(Q_n) = \sum_{\nu=1}^n a_\nu^2 \tag{11}$$

plays an important rôle in the numerical stability of the quadrature formula Q_n (for a recent survey, cf. Förster [5]). In [5], precise values of $\lim_{n\to\infty} n \,\text{Var}(Q^G_n)$ for the Gaussian quadrature formulae Q^G_n with respect to many different weight functions, in particular to ultraspherical weight functions are given. For Gauss–Kronrod formulae, Notaris [19] proved that there do not exist Gauss–Kronrod formulae such that all weights are equal for each $n \in \mathbb{N}$, which would minimize (11). Furthermore, we conclude from [5, Eq. (4.9) and Eq. (4.16)] that for the Gauss–Kronrod formula with respect to w_λ, $0 \leqslant \lambda \leqslant 1$, we have

$$\liminf_{n\to\infty} (2n+1) \,\text{Var}(Q^{GK}_{2n+1}) \quad > \quad \pi \frac{\Gamma^2(\lambda + 1/2)}{\Gamma^2(\lambda+1)} \tag{12}$$

as well as

$$\limsup_{n\to\infty} (2n+1) \,\text{Var}(Q^{GK}_{2n+1}) \quad < \quad \frac{4}{3} \pi^{3/2} \frac{\Gamma(2\lambda + 1/2)}{\Gamma(2\lambda+1)}. \tag{13}$$

However, the precise value of $\lim_{n\to\infty}(2n+1) \,\text{Var}(Q^{GK}_{2n+1})$ is unknown until now. The following result can be shown with the help of Corollary 3.

COROLLARY 4. *Let $0 \leqslant \lambda \leqslant 1$, and let Q_{2n+1}^{GK} be the Gauss–Kronrod quadrature formula with respect to w_λ. Then*

$$\lim_{n\to\infty} (2n+1)\operatorname{Var}(Q_{2n+1}^{GK}) = \pi^{3/2}\frac{\Gamma(2\lambda+1/2)}{\Gamma(2\lambda+1)}. \tag{14}$$

3. PROOFS

Let $0 \leqslant \lambda \leqslant 1$. In the sequel, Stieltjes polynomials will be normalized by

$$E_{n+1}(x, w_\lambda) = \frac{2^{n+1}}{\gamma_n} x^{n+1} + p(x), \qquad p \in \mathscr{P}_n, \tag{15}$$

where

$$\gamma_n = \sqrt{\pi}\,\frac{\Gamma(n+2\lambda)}{\Gamma(n+\lambda+1)}. \tag{16}$$

The orthogonal polynomials with respect to w_λ are the ultraspherical polynomials $P_n^{(\lambda)}$ (cf. Szegö [26, §4.7]).

Proof of the Theorem. (i) Note that for $0 \leqslant \theta \leqslant \pi$ there hold (cf. Szegö [25])

$$E_{n+1}(\cos\theta, w_0) = \frac{2n}{\sqrt{\pi}}\,[\cos(n+1)\,\theta - \cos(n-1)\,\theta], \tag{17}$$

$$E_{n+1}(\cos\theta, w_1) = \frac{2}{\sqrt{\pi}}\cos(n+1)\,\theta. \tag{18}$$

Hence we only have to consider $0 < \lambda < 1$.

Let $Q_n^{(\lambda)}$ be the ultraspherical function of the second kind, defined by

$$(1-y^2)^{\lambda-1/2}\,Q_n^{(\lambda)}(y) = \frac{1}{2}\frac{\Gamma(2\lambda)}{\Gamma(\lambda+1/2)}\int_{-1}^{1}(1-t^2)^{\lambda-1/2}\frac{P_n^{(\lambda)}(t)}{y-t}\,dt \tag{19}$$

for $y \notin [-1, 1]$, $\lambda > -1/2$. For $-1 < x < 1$, $Q_n^{(\lambda)}$ is defined by [26, (4.62.9)], or, equivalently, by a Cauchy principal value integral,

$$(1-x^2)^{\lambda-1/2}\,Q_n^{(\lambda)}(x) = \frac{1}{2}\frac{\Gamma(2\lambda)}{\Gamma(\lambda+1/2)}\,⨍_{-1}^{1}(1-t^2)^{\lambda-1/2}\frac{P_n^{(\lambda)}(t)}{x-t}\,dt. \tag{20}$$

Using the method described by Szegö [26, §8.71(5)], it can be proved that

$$(1-x^2)^{\lambda-1/2}\,Q_n^{(\lambda)}(x)=n^{\lambda-1}\pi^{1/2}2^{\lambda-1}\sin^{\lambda-1}\theta\cos\{(n+\lambda)\,\theta-(\lambda-1)\,\pi/2\}+O(n^{\lambda-2}) \tag{21}$$

as well as

$$\frac{d}{dx}\{(1-x^2)^{\lambda-1/2}\,Q_n^{(\lambda)}(x)\}=-n^{\lambda}\pi^{1/2}2^{\lambda-1}\sin^{\lambda-2}\theta\sin\{(n+\lambda)\,\theta-(\lambda-1)\,\pi/2\}+O(n^{\lambda-1}) \tag{22}$$

uniformly for $x=\cos\theta$, $\varepsilon\leqslant\theta\leqslant\pi-\varepsilon$, ε fixed.

Szegö [25] proved that the coefficients of the Chebyshev polynomial representation of $E_{n+1}(\cdot, w_\lambda)$,

$$E_{n+1}(\cdot, w_\lambda)=\frac{2}{\gamma_n}\sum_{\nu=0}^{\lfloor (n+1)/2\rfloor}{}' \alpha_\nu T_{n+1-2\nu} \tag{23}$$

(the prime indicates that the last term should be halved if n is odd), can be obtained from the recurrence formula

$$\alpha_0=1, \qquad \sum_{\mu=0}^{\nu}\alpha_\mu f_{\nu-\mu}=0, \qquad \nu\geqslant 1, \tag{24}$$

where $\alpha_\nu=\alpha_\nu^{(n,\lambda)}$ depends on n and λ also, since

$$f_0=f_0^{(n,\lambda)}=1, \qquad f_\nu=f_\nu^{(n,\lambda)}=\left(1-\frac{\lambda}{\nu}\right)\left(1-\frac{\lambda}{n+\lambda+\nu}\right)f_{\nu-1}, \quad \nu\geqslant 1, \tag{25}$$

are the coefficients in the expansion

$$\sin^{2\lambda-1}\theta\left(Q_n^{(\lambda)}(\cos\theta)+\frac{i\pi}{2}\frac{\Gamma(2\lambda)}{\Gamma(\lambda+1/2)}P_n^{(\lambda)}(\cos\theta)\right)=\gamma_n\sum_{\nu=0}^{\infty}f_\nu e^{i(n+1+2\nu)\theta} \tag{26}$$

(cf Szegö [25, p. 533]) of the ultraspherical polynomials and functions of the second kind. The latter series converges uniformly for $\varepsilon\leqslant\theta\leqslant\pi-\varepsilon$, ε fixed.

Let $m=\lfloor (n+1)/2\rfloor$. Starting as in the proof of Laplaces formula in [26, p. 205] we write

$$E_{n+1}(\cos\theta, w_\lambda)=\frac{2}{\gamma_n}\,\Re\left\{e^{i(n+1)\theta}\sum_{\nu=0}^{m}{}'\alpha_\nu e^{-2i\nu\theta}\right\}, \qquad 0\leqslant\theta\leqslant\pi. \tag{27}$$

Szegö [25, p. 509] proved

$$\alpha_1 < \alpha_2 < \alpha_3 < \cdots < 0, \qquad 0 \leqslant \sum_{\nu=0}^{\infty} \alpha_\nu < 1. \tag{28}$$

Hence, we have $|\sum_{\nu=0}^{\infty} \alpha_\nu e^{-2i\nu\theta}| \leqslant \sum_{\nu=0}^{\infty} |\alpha_\nu| \leqslant 2$, and we can use

$$\sum_{\nu=0}^{m}{}' \alpha_\nu e^{-2i\nu\theta} = \sum_{\nu=0}^{\infty} \alpha_\nu e^{-2i\nu\theta} - \sum_{\nu=m+1}^{\infty}{}^{*} \alpha_\nu e^{-2i\nu\theta}, \tag{29}$$

where the asterisk indicates that $\frac{1}{2}\alpha_m e^{-2im\theta}$ should be added if n is odd. Regarding (24) as the coefficients of the Cauchy product of two power series, and using (26) we obtain

$$\begin{aligned} e^{i(n+1)\theta} &\sum_{\nu=0}^{\infty} \alpha_\nu e^{-2i\nu\theta} \\ &= \left(\sum_{\nu=0}^{\infty} f_\nu e^{-i(n+1+2\nu)\theta} \right)^{-1} \\ &= \gamma_n \sin^{1-2\lambda}\theta \left(Q_n^{(\lambda)}(\cos\theta) + \frac{i\pi}{2} \frac{\Gamma(2\lambda)}{\Gamma(\lambda+1/2)} P_n^{(\lambda)}(\cos\theta) \right) \\ &\quad \times \left([Q_n^{(\lambda)}(\cos\theta)]^2 + \left[\frac{\pi}{2} \frac{\Gamma(2\lambda)}{\Gamma(\lambda+1/2)} P_n^{(\lambda)}(\cos\theta) \right]^2 \right)^{-1}. \end{aligned} \tag{30}$$

Since $Q_n^{(\lambda)}$ and $P_n^{(\lambda)}$ are linearly independent solutions of the same second order differential equation (cf. [26, p. 78]), their zeros interlace and the denominator in (30) cannot vanish. Using (21) as well as

$$\begin{aligned} &\frac{\pi}{2} \frac{\Gamma(2\lambda)}{\Gamma(\lambda+1/2)} P_n^{(\lambda)}(\cos\theta) \\ &\quad = n^{\lambda-1}\pi^{1/2}2^{\lambda-1} \sin^{-\lambda}\theta \cos\{(n+\lambda)\theta - \lambda\pi/2\} + O(n^{\lambda-2}) \end{aligned} \tag{31}$$

(cf. Szegö [26, (8.21.10)]) for $\varepsilon \leqslant \theta \leqslant \pi - \varepsilon$ we obtain that

$$\begin{aligned} &[Q_n^{(\lambda)}(\cos\theta)]^2 + \left[\frac{\pi}{2} \frac{\Gamma(2\lambda)}{\Gamma(\lambda+1/2)} P_n^{(\lambda)}(\cos\theta) \right]^2 \\ &\quad = n^{2\lambda-2}\pi 2^{2\lambda-2} \sin^{-2\lambda}\theta + O(n^{2\lambda-3}) \end{aligned} \tag{32}$$

converges uniformly for $\varepsilon \leqslant \theta \leqslant \pi - \varepsilon$. Therefore part (i) of the Theorem will follow with the help of (21) and (27) if

$$e^{i(n+1)\theta} \sum_{\nu=m+1}^{\infty}{}^{*} \alpha_\nu e^{-2i\nu\theta} = o(1). \tag{33}$$

In view of (28), we can estimate

$$\left| \sum_{\nu=m+1}^{\infty}{}^{*} \alpha_\nu e^{-2i\nu\theta} \right| < - \sum_{\nu=m+1}^{\infty} \alpha_m \leqslant \sum_{\nu=0}^{m} \alpha_\nu. \tag{34}$$

Using (24), we obtain that for $k > 0$

$$(\alpha_0 + \alpha_1 + \cdots + \alpha_k)(f_0 + f_1 + \cdots + f_k) = 1 + R_k, \tag{35}$$

where $R_k < 0$, hence

$$\alpha_0 + \alpha_1 + \cdots + \alpha_m < (f_0 + f_1 + \cdots + f_m)^{-1}. \tag{36}$$

Recalling the definition of m, we now show that $f_0 + f_1 + \cdots + f_m$ is unbounded as n increases. An explicit representation for f_ν and $0 < \lambda < 1$ can easily be calculated from (25),

$$f_\nu = \frac{1}{\Gamma(1-\lambda)} \frac{\Gamma(\nu+1-\lambda)}{\Gamma(\nu+1)} \frac{\Gamma(n+\lambda+1)}{\Gamma(n+1)} \frac{\Gamma(n+\nu+1)}{\Gamma(n+\nu+\lambda+1)}. \tag{37}$$

LEMMA (Laforgia [12]). *Let $x, \mu \in \mathbb{R}$, $x \geqslant 1$. Then*

(i) $\left(x + \frac{2}{3}\mu\right)^{\mu-1} < \frac{\Gamma(x+\mu)}{\Gamma(x+1)} < \left(x + \frac{\mu}{2}\right)^{\mu-1}, \qquad 0 < \mu < 1;$

(ii) $\left(x + \frac{\mu}{2}\right)^{\mu-1} < \frac{\Gamma(x+\mu)}{\Gamma(x+1)} < \left(x + \frac{\mu}{2} + \frac{1}{10}\right)^{\mu-1}, \qquad 1 < \mu < 2;$

Application of the Lemma with $x = n$, $\mu = 1 + \lambda$ yields

$$\frac{\Gamma(n+1+\lambda)}{\Gamma(n+1)} > \left(n + \frac{1+\lambda}{2}\right)^{\lambda}. \tag{38}$$

Application of the Lemma with $x = n + \nu$, $\mu = 1 + \lambda$ yields

$$\frac{\Gamma(n+\nu+\lambda+1)}{\Gamma(n+\nu+1)} < \left(n + \nu + \frac{1+\lambda}{2} + \frac{1}{10}\right)^{\lambda}. \tag{39}$$

Hence, for $0<\lambda<1$ and $\nu \leqslant m$ we obtain

$$f_\nu^{(\lambda)} > \frac{1}{\Gamma(1-\lambda)} \frac{\Gamma(\nu+1-\lambda)}{\Gamma(\nu+1)} \left(\frac{n+\dfrac{1+\lambda}{2}}{n+\nu+\dfrac{1+\lambda}{2}+\dfrac{1}{10}} \right)^{\lambda}$$

$$\geqslant \frac{1}{\Gamma(1-\lambda)} \frac{\Gamma(\nu+1-\lambda)}{\Gamma(\nu+1)} \left(\frac{2\nu+\dfrac{1+\lambda}{2}}{3\nu+\dfrac{1+\lambda}{2}+\dfrac{1}{10}} \right) =: g_\nu^{(\lambda)}. \qquad (40)$$

Now $g_\nu^{(\lambda)}$ is independent of n, and

$$g_\nu^{(\lambda)} = O(\nu^{-\lambda}) \qquad (41)$$

leads to the conclusion.

Proof of Corollary 1 *and Corollary* 2. Setting

$$\theta^{(\pm\delta)}_{n+2-\mu, n+1} := \frac{\mu+(\lambda-2)/2 \pm \delta}{n+\lambda} \pi, \qquad (42)$$

by part (i) of the Theorem it follows that for every $\delta>0$ and sufficiently large n there is a zero of $E_{n+1}(\cdot, w_\lambda)$ in $(\cos \theta^{(+\delta)}_{n+2-\mu,n+1}, \cos \theta^{(-\delta)}_{n+2-\mu,n+1})$, which proves Corollary 1. We now set

$$\theta_{n+2-\mu,n+1} = \frac{\mu+(\lambda-2)/2+\delta_{\mu,n+1}}{n+\lambda}\pi = \bar{\theta}_{n+2-\mu,n+1} + \frac{\delta_{\mu,n+1}}{n+\lambda}\pi. \qquad (43)$$

For the inequalities (i) and (ii) of Corollary 2 we shall prove that the δ-terms in (43) are less than half the differences of the $\bar{\theta}$-terms for sufficiently large n. After some elementary calculations, we obtain the sufficient condition

$$\max\{|\delta_{\mu+1,n}|, |\delta_{\mu+1,n+1}|, |\delta_{\mu,n}|, |\delta_{\mu,n+1}|\} < \min\left\{\frac{\mu-1+\lambda/2}{2n+2\lambda}, \frac{n-\mu+\lambda/2}{2n+2\lambda}\right\}. \qquad (44)$$

For $Cn \leqslant \mu \leqslant \lfloor (n+1)/2 \rfloor$, we have

$$\frac{\mu+(\lambda-2)/2}{2n+2\lambda} > \frac{C}{2} + O(n^{-1}), \qquad (45)$$

while for $\lfloor (n+1)/2 \rfloor < \mu \leqslant (1-C)\,n$ we have

$$\frac{n-\mu+\lambda/2}{2n+2\lambda} > \frac{C}{2} + O(n^{-1}). \tag{46}$$

We conclude from Corollary 1 that

$$\max\{|\delta_{\mu+1,n}|, |\delta_{\mu+1,n+1}|, |\delta_{\mu,n}|, |\delta_{\mu,n+1}|\} = o(1), \tag{47}$$

which leads to the conclusion.

Proof of the Theorem. (ii) Let $m = \lfloor (n+1)/2 \rfloor$. Setting again $x = \cos\theta$, $0 < \theta < \pi$, we obtain from (27), (29) and (30) that

$$\begin{aligned} &E'_{n+1}(x, w_\lambda) \\ &= 2\frac{d}{dx}\left\{\frac{(1-x^2)^{\lambda-1/2}\,Q_n^{(\lambda)}(x)}{[(1-x^2)^{\lambda-1/2}\,Q_n^{(\lambda)}(x)]^2 + \left[(1-x^2)^{\lambda-1/2}\frac{\pi}{2}\frac{\Gamma(2\lambda)}{\Gamma(\lambda+1/2)}\,[P_n^{(\lambda)}(x)]\right]^2}\right\} \\ &\quad - \frac{2}{\gamma_n \sin\theta}\frac{d}{d\theta}\Re\left\{e^{i(n+1)\theta}\sum_{\nu=m+1}^{\infty}{}^{*}\,\alpha_\nu e^{-2i\nu\theta}\right\}. \end{aligned} \tag{48}$$

It can easily be shown with the help of (31) and

$$\frac{d}{dx}P_n^{(\lambda)}(x) = 2\lambda P_{n-1}^{(\lambda+1)}(x) \tag{49}$$

(cf. [26, (4.7.17)]), that

$$\begin{aligned} &\frac{d}{dx}\left\{(1-x^2)^{\lambda-1/2}\frac{\pi}{2}\frac{\Gamma(2\lambda)}{\Gamma(\lambda+1/2)}\,P_n^{(\lambda)}(x)\right\} \\ &\qquad = n^\lambda \pi^{1/2} 2^{\lambda-1}\sin^{\lambda-2}\theta\,\sin\{(n+\lambda)\,\theta - \lambda\pi/2\} + O(n^{\lambda-1}). \end{aligned} \tag{50}$$

Using (21), (22), (31) and (50), it follows that

$$\begin{aligned} &2\frac{d}{dx}\left\{\frac{(1-x^2)^{\lambda-1/2}\,Q_n^{(\lambda)}(x)}{[(1-x^2)^{\lambda-1/2}\,Q_n^{(\lambda)}(x)]^2 + \left[(1-x^2)^{\lambda-1/2}\frac{\pi}{2}\frac{\Gamma(2\lambda)}{\Gamma(\lambda+1/2)}\,[P_n^{(\lambda)}(x)]\right]^2}\right\} \\ &\quad = n^{2-\lambda}\pi^{-1/2}2^{2-\lambda}\sin^{-\lambda}\theta\,\sin\{(n+\lambda)\,\theta - (\lambda-1)\,\pi/2\} + O(n^{1-\lambda}). \end{aligned} \tag{51}$$

Therefore, part (ii) of the Theorem follows if we prove that there holds uniformly for $\varepsilon \leqslant \theta \leqslant \pi - \varepsilon$, ε fixed, that

$$\frac{d}{d\theta} \Re \left\{ e^{i(n+1)\theta} \sum_{\nu=m+1}^{\infty}{}^{*} \alpha_\nu e^{-2i\nu\theta} \right\} = O(1). \tag{52}$$

Let n be odd; an analogous proof holds for even n. We have

$$\frac{d}{d\theta} \Re \left\{ e^{i(n+1)\theta} \sum_{\nu=m+1}^{\infty}{}^{*} \alpha_\nu e^{-2i\nu\theta} \right\} = -2 \sum_{\nu=1}^{\infty} \nu \alpha_{m+\nu} \sin 2\nu\theta. \tag{53}$$

Using partial summation we obtain

$$\begin{aligned} &\sum_{\nu=1}^{\infty} \nu \alpha_{m+\nu} \sin 2\nu\theta \\ &\quad = \lim_{K \to \infty} \left[\sum_{\nu=1}^{K-1} (\nu \alpha_{m+1} - (\nu+1)\, \alpha_{m+\nu+1}) \sum_{\mu=1}^{\nu} \sin 2\mu\theta \right. \\ &\qquad \left. + K \alpha_{m+K} \sum_{\mu=1}^{K} \sin 2\mu\theta \right]. \end{aligned} \tag{54}$$

Now

$$\left| \sum_{\mu=1}^{\nu} \sin 2\mu\theta \right| = \left| \frac{\cos \theta - \cos(2\nu+1)\,\theta}{2 \sin \theta} \right| < \frac{1}{\sin \varepsilon} \tag{55}$$

is bounded for $\varepsilon \leqslant \theta \leqslant \pi - \varepsilon$ and all $\nu \in \mathbb{N}$, and

$$\lim_{K \to \infty} |K \alpha_{m+K}| = 0 \tag{56}$$

holds since $\sum_{\nu=1}^{\infty} \alpha_{m+\nu}$ is convergent. Furthermore,

$$\begin{aligned} \sum_{\nu=1}^{K-1} |\nu \alpha_{m+\nu} - (\nu+1)\, \alpha_{m+\nu+1}| \leqslant{} & \sum_{\nu=1}^{K-1} \nu\, |\alpha_{m+\nu} - \alpha_{m+\nu+1}| \\ & + \sum_{\nu=1}^{K-1} |\alpha_{m+\nu+1}|, \end{aligned} \tag{57}$$

where

$$\lim_{K \to \infty} \sum_{\nu=1}^{K-1} |\alpha_{m+\nu+1}| = \sum_{\nu=1}^{\infty} |\alpha_{m+\nu+1}| < 1. \tag{58}$$

For the first term in the right side of (57), it follows from (28) that

$$\begin{aligned}\lim_{K\to\infty}\sum_{\nu=1}^{K-1}\nu\,|\alpha_{m+\nu}-\alpha_{m+\nu+1}| &= \lim_{K\to\infty}\left(-\sum_{\nu=1}^{K-1}\alpha_{m+\nu}+(K-1)\,\alpha_{m+K}\right)\\ &\leqslant -\lim_{K\to\infty}\sum_{\nu=1}^{K-1}\alpha_{m+\nu}\leqslant 1,\end{aligned} \tag{59}$$

and the proof is complete.

Proof of Corollary 3. For the proof of (9) and (10) note that

$$Q^{GK}_{2n+1}=Q^{G}_{n}-R^{G}_{n}[p_{2n}]\,\mathrm{dvd}(x^{G}_{1,n},...,x^{G}_{n,n},\xi^{K}_{1,n+1},...,\xi^{K}_{n+1,n+1}), \tag{60}$$

where $\mathrm{dvd}(y_1,...,y_k)[f]=\sum_{\nu=1}^{k}b_\nu f(y_\nu)$ is the divided difference defined by

$$\mathrm{dvd}(y_1,...,y_k)[p_\nu]=\begin{cases}0 & \nu=0,1,...,k-2,\\ 1 & \nu=k-1,\end{cases} \tag{61}$$

which leads to

$$b_\nu=\prod_{\substack{\mu=1\\ \mu\neq\nu}}^{k}(y_\nu-y_\mu)^{-1}. \tag{62}$$

Therefore, the weights of Gauss–Kronrod quadrature formulae can be written as

$$A^{GK}_{\nu,n}=a^{G}_{\nu,n}+\frac{2^{2-2\lambda}\sqrt{\pi}}{\Gamma(\lambda)\,P^{(\lambda)\prime}_{n}(x^{G}_{\nu,n})\,E_{n+1}(x^{G}_{\nu,n},w_\lambda)},\qquad \nu=1,...,n, \tag{63}$$

$$B^{GK}_{\mu,n+1}=\frac{2^{2-2\lambda}\sqrt{\pi}}{\Gamma(\lambda)\,P^{(\lambda)}_{n}(\xi^{K}_{\mu,n+1})\,E'_{n+1}(\xi^{K}_{\mu,n+1},w_\lambda)},\qquad \mu=1,...,n+1. \tag{64}$$

It is known (cf. e.g. Gatteschi [7] for a stronger result) that for $x^{G}_{\nu,n}=\cos\phi^{G}_{\nu,n}$ we have uniformly for $\varepsilon\leqslant\phi^{G}_{\nu,n}\leqslant\pi-\varepsilon$, ε fixed, that

$$\phi^{G}_{n+1-\nu,n}=\frac{\nu+(\lambda-1)/2+o(1)}{n+\lambda}\,\pi \tag{65}$$

and (cf. [26, §15.3])

$$a^G_{\nu,n} = \frac{\pi}{n+\lambda} \sin^{2\lambda} \phi^G_{\nu,n}(1+o(1)). \tag{66}$$

Furthermore, it follows from (31) and (49) that

$$(-1)^{n-\nu} P_n^{(\lambda)\prime}(x^G_{\nu,n}) = \frac{\Gamma(\lambda+1/2)}{\Gamma(2\lambda)} n^\lambda \pi^{-1/2} 2^\lambda \sin^{-\lambda-1} \phi^G_{\nu,n}(1+o(1)) \tag{67}$$

for $x^G_{\nu,n} = \cos \phi^G_{\nu,n}$, $\varepsilon \leqslant \phi^G_{\nu,n} \leqslant \pi - \varepsilon$. Using now part (i) of the Theorem and (65) for an asymptotic representation of $E_{n+1}(x^G_{\nu,n}, w_\lambda)$, (31) and Corollary 1 for an asymptotic representation of $P_n^{(\lambda)}(\xi^K_{\mu,n+1})$ as well as part (ii) of the Theorem and Corollary 1 for an asymptotic representation of $E'_{n+1}(\xi^K_{\mu,n+1}, w_\lambda)$, we obtain from (63) respectively (64) that

$$A^{GK}_{\nu,n} = \frac{\pi}{2n+1+\lambda} \sin^{2\lambda} \phi^G_{\nu,n}(1+o(1)), \tag{68}$$

$$B^{GK}_{\mu,n+1} = \frac{\pi}{2n+1+\lambda} \sin^{2\lambda} \theta^K_{\mu,n+1}(1+o(1)) \tag{69}$$

hold uniformly for $\varepsilon \leqslant \phi^G_{\nu,n+1} \leqslant \pi - \varepsilon$ and $\varepsilon \leqslant \theta^K_{\mu,n+1} \leqslant \pi - \varepsilon$, ε fixed.

Proof of Corollary 4. Let $\varepsilon \in (0,1)$ be fixed and let $I_\varepsilon = [-1, -1+\varepsilon] \cup [1-\varepsilon, 1]$; let $x^G_{\nu,n} = \cos \phi^G_{\nu,n}$ and $\xi^K_{\mu,n+1} = \cos \theta^K_{\mu,n+1}$. Then

$$\begin{aligned} \operatorname{Var}(Q^{GK}_{2n+1}) = &\sum_{x^G_{\nu,n} \notin I_\varepsilon} (A^{GK}_{\nu,n})^2 + \sum_{\xi^K_{\mu,n+1} \notin I_\varepsilon} (B^{GK}_{\mu,n+1})^2 \\ &+ \sum_{x^G_{\nu,n} \in I_\varepsilon} (A^{GK}_{\nu,n})^2 + \sum_{\xi^K_{\mu,n+1} \in I_\varepsilon} (B^{GK}_{\mu,n+1})^2. \end{aligned} \tag{70}$$

We deduce from Corollary 3 that there hold uniformly

$$\begin{aligned} &\sum_{x^G_{\nu,n} \notin I_\varepsilon} (A^{GK}_{\nu,n})^2 + \sum_{\xi^K_{\mu,n+1} \notin I_\varepsilon} (B^{GK}_{\mu,n+1})^2 \\ &= \frac{\pi}{2n+1+\lambda} \Bigg(\sum_{x^G_{\nu,n} \notin I_\varepsilon} A^{GK}_{\nu,n}(1-[x^G_{\nu,n}]^2)^\lambda \\ &\quad + \sum_{\xi^K_{\mu,n+1} \notin I_\varepsilon} B^{GK}_{\mu,n+1}(1-[\xi^K_{\mu,n+1}]^2)^\lambda \Bigg) (1+o(1)) \\ &= \frac{\pi}{2n+1+\lambda} Q^{GK}_{2n+1}[f](1+o(1)), \end{aligned} \tag{71}$$

where

$$f(x) := \begin{cases} 0 & x \in I_\varepsilon, \\ (1-x^2)^\lambda & x \notin I_\varepsilon. \end{cases} \tag{72}$$

Since f is bounded and Riemann integrable, it follows from the positivity of Q_{2n+1}^{GK} and from $\deg(Q_{2n+1}^{GK}) \geq 3n+1$ that (c.f. e.g. Davis and Rabinowitz [2, pp. 129/130])

$$\lim_{n\to\infty} Q_{2n+1}^{GK}[f] = \int_{-1+\varepsilon}^{1-\varepsilon} w_\lambda(x)(1-x^2)^\lambda \, dx$$

$$= \sqrt{\pi}\,\frac{\Gamma(2\lambda+1/2)}{\Gamma(2\lambda+1)} + \delta_\varepsilon^{(1)}, \tag{73}$$

where

$$|\delta_\varepsilon^{(1)}| \leq 2\int_{-1}^{-1+\varepsilon} (1-x^2)^{-1/2}\, dx = 2\pi - 2\arccos(-1+\varepsilon). \tag{74}$$

Let now $m = (\deg(Q_{2n+1}^{GK})+1)/2$, and let Q_m^G be the Gaussian formula with respect to w_λ. Let $N \in \mathbb{N}$ be defined by $-1+\varepsilon \in (x_{N-1,m}^G, x_{N,m}^G]$. Let, for notational convenience, $x_{2\nu-1,2n+1}^{GK} = \xi_{\nu,n+1}^K$, $a_{2\nu-1,2n+1}^{GK} = B_{\nu,n+1}^{GK}$, $\nu = 1, ..., n+1$, $x_{2\nu,2n+1}^{GK} = x_{\nu,n}^G$, $a_{2\nu,2n+1}^{GK} = A_{\nu,n}^{GK}$, $\nu = 1, ..., n$. Using a result of Förster [4, Theorem 2.1], it follows that

$$\sum_{x_{\nu,2n+1}^{GK} \in I_\varepsilon} (a_{\nu,2n+1}^{GK})^2 \leq 2\sum_{\nu=0}^{N} \left(\sum_{x_{\nu,m}^G \leq x_{\nu,2n+1}^{GK} \leq x_{\nu+1,m}^G} a_{\nu,2n+1}^{GK} \right)^2$$

$$\leq 2\sum_{\nu=0}^{N} (a_{\nu,m}^G + a_{\nu+1,m}^G)^2. \tag{75}$$

From a result of Förster and Petras [6, Theorem 1] we obtain that this is bounded by

$$8\sum_{\nu=1}^{N+1} (a_{\nu,m}^G)^2. \tag{76}$$

Using [6, Corollary 1] we obtain

$$8\sum_{\nu=1}^{N+1} (a_{\nu,m}^G)^2 \leq \frac{8\pi}{m+\lambda} \sum_{\nu=1}^{N+1} a_{\nu,m}^G \sin^{2\lambda} \theta_{\nu,m}^G, \tag{77}$$

where $x^G_{\nu,m} = \cos \theta^G_{\nu,m}$. Using the same argument as above, we obtain that

$$\limsup_{n \to \infty} (2n+1) \sum_{x^{GK}_{\nu,2n+1} \in I_\varepsilon} (a^{GK}_{\nu,2n+1})^2 \leqslant \frac{32\pi}{3} (\pi - \arccos(-1+\varepsilon)). \quad (78)$$

Since the arccos function is continuous, it follows that the right hand sides of (74) and (78) can be made arbitrarily small by suitable choice of ε, which leads to the result.

References

1. H. Brass, "Quadraturverfahren," Vandenhoeck und Ruprecht, Göttingen, 1977.
2. P. J. Davis and P. Rabinowitz, "Methods of Numerical Integration," Academic Press, Orlando, 1984.
3. S. Ehrich, Error bounds for Gauss–Kronrod quadrature formulae, *Math. Comp.* **62** (1994), 295–304.
4. K.-J. Förster, A comparison theorem for linear functionals and its application to quadrature, *in* "Numerical Integration" (G. Hämmerlin, Ed.), Proc. Conf. Oberwolfach, ISNM 57, pp. 66–76, Birkhäuser, Basel, 1982.
5. K.-J. Förster, Variance in quadrature—A survey, *in* "Numerical Integration IV" (H. Brass and G. Hämmerlin, Eds.), Proc. Conf. Oberwolfach, ISNM, Vol. 112, pp. 91–110, Birkhäuser, Basel, 1993.
6. K.-J. Förster and K. Petras, On estimates for the weights in Gaussian quadrature in the ultraspherical case, *Math. Comp.* **55** (1990), 243–264.
7. L. Gatteschi, Una nova rappresentatione asintotica dei polinomi ultrasferici, *Calcolo* **16** (1979), 447–458.
8. W. Gautschi, Gauss–Kronrod quadrature—A survey, *in* "Numerical Methods and Approximation Theory III" (G. V. Milovanović, Ed.), pp. 39–66, Niš, 1988.
9. W. Gautschi and S. E. Notaris, An algebraic study of Gauss–Kronrod quadrature formulae for Jacobi weight functions, *Math. Comp.* **51** (1988), 231–248.
10. W. Gautschi and S. E. Notaris, Newton's method and Gauss–Kronrod quadrature, *in* "Numerical Integration III" (H. Brass and G. Hämmerlin, Eds.), Proc. Conf. Oberwolfach, ISNM, Vol. 85, pp. 60–71, Birkhäuser, Basel, 1988.
11. W. Gautschi and S. E. Notaris, Gauss–Kronrod quadrature formulae for weight functions of Bernstein–Szegö type, *J. Comp. Appl. Math.* **25** (1989), 199–224; errata, **27** (1989), 429.
12. A. Laforgia, Further inequalities for the gamma-function, *Math. Comp.* **42** (1984), 597–600.
13. G. Monegato, A note on extended Gaussian quadrature rules, *Math. Comp.* **30** (1976), 812–817.
14. G. Monegato, Positivity of weights of extended Gauss–Legendre quadrature rules, *Math. Comp.* **32** (1978), 243–245.
15. G. Monegato, An overview of results and questions related to Kronrod schemes, *in* "Numerische Integration" (G. Hämmerlin, Ed.), Proc. Conf. Oberwolfach, ISNM, Vol. 57, pp. 231–240, Birkhäuser, Basel, 1979.
16. G. Monegato, Stieltjes polynomials and related quadrature rules, *SIAM Rev.* **24**, 2 (1982), 137–158.
17. S. E. Notaris, Gauss–Kronrod quadrature formulae for weight functions of Bernstein–Szegö type, *J. Comp. Appl. Math.* **29** (1990), 161–169.

18. S. E. NOTARIS, Some new formulae for the Stieltjes polynomials relative to classical weight functions, *SIAM J. Numer. Anal.* **28** (1991), 1196–1206.
19. S. E. NOTARIS, On Gauss–Kronrod quadrature of Chebyshev type, *Math. Comp.* **58** (1992), 745–753.
20. S. E. NOTARIS, Error bounds for Gauss–Kronrod quadrature formulae of analytic functions, *Numer. Math.* **64** (1993), 371–380.
21. F. PEHERSTORFER, On Stieltjes polynomials and Gauss–Kronrod quadrature, *Math. Comp.* **55** (1990), 649–664.
22. F. PEHERSTORFER, Weight functions admitting repeated positive Kronrod quadrature, *BIT* **30** (1990), 145–151.
23. F. PEHERSTORFER, On the asymptotic behaviour of functions of the second kind and Stieltjes polynomials and on Gauss–Kronrod quadrature formulae, *J. Approx. Theory* **70** (1992), 156–190.
24. R. PIESSENS, E. DE DONCKER, C. ÜBERHUBER, AND D. K. KAHANER, "QUADPACK—A subroutine Package for Automatic Integration," Springer Series in Computational Mathematics, Vol. 1, Springer, Berlin, 1983.
25. G. SZEGÖ, Über gewisse orthogonale Polynome, die zu einer oszillierenden Belegungsfunktion gehören, *Math. Ann.* **110** (1934), 501–513.
26. G. SZEGÖ, "Orthogonal Polynomials," AMS Colloq. Publ., Vol. 23, Providence, RI, 1975.