Journal of Computational and Applied Mathematics 236 (2012) 2685–2695

Contents lists available at SciVerse ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A parametric approach for solving a class of generalized quadratic-transformable rank-two nonconvex programs

Riccardo Cambini *, Claudio Sodini

Department of Statistics and Applied Mathematics, Faculty of Economics, University of Pisa, Via Cosimo Ridolfi 10, 56124 Pisa, Italy

ARTICLE INFO

Article history:
Received 27 October 2009
Received in revised form 3 June 2011

MSC:
90C20
90C26
90C31

Keywords:
Generalized quadratic programming
Low rank structures
Optimal level solutions
Global optimization

ABSTRACT

The aim of this paper is to propose a solution algorithm for a particular class of rank-two nonconvex programs having a polyhedral feasible region. The algorithm is based on the so-called "optimal level solutions" method. Various global optimality conditions are discussed and implemented in order to improve the efficiency of the algorithm.

1. Introduction

The aim of this paper is to study, from a theoretical, an algorithmic, and a computational point of view, the following class of rank-two nonconvex programs:

$$P: \begin{cases} \inf f(x) = \phi\left(\frac{1}{2}x^TQx + q^Tx,\, d^Tx\right) \\ x \in X = \{x \in \Re^n : Ax \leq b\}, \end{cases} \tag{1}$$

where $A \in \Re^{m\times n}$, $b \in \Re^m$, $q, d \in \Re^n$, $Q \in \Re^{n\times n}$ is positive definite, and $X \neq \emptyset$. The scalar function $\phi(y_1, y_2)$ is assumed to be continuous and strictly increasing with respect to variable y_1, and is defined for all values in Ω, where

$$\Omega = \left\{(y_1, y_2) \in \Re^2 : y_1 = \frac{1}{2}x^TQx + q^Tx,\ y_2 = d^Tx,\ x \in X\right\}.$$

The considered class of objective functions $f(x)$ is extremely wide, and it covers multiplicative, fractional, and d.c. quadratic functions (as it is known, a d.c. function is a function expressed by the difference of two convex functions). Just as an example, given any strictly increasing real function g_1, any positive function g_2, and any real function g_3, then the following function $f(x)$ verifies the assumptions of problem P by using $\phi(y_1, y_2) = g_1(y_1)g_2(y_2) + g_3(y_2)$ (see also [1]):

$$f(x) = g_1\left(\frac{1}{2}x^TQx + q^Tx\right) g_2\left(d^Tx\right) + g_3\left(d^Tx\right). \tag{2}$$

* Corresponding author. Tel.: +39 0502216249; fax: +39 0502216375.
E-mail addresses: cambric@ec.unipi.it (R. Cambini), csodini@ec.unipi.it (C. Sodini).

0377-0427/$ – see front matter © 2012 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2012.01.004

Various particular problems belonging to this class have been studied in the literature of mathematical programming and global optimization, from both a theoretic and an applicative point of view [2–8]. For these particular problems the proposed solution algorithms are often based on branch and bound, branch and cut, or branch and reduce methods. It is worth noticing that this class covers several multiplicative, fractional, and d.c. quadratic problems (see [9–11,1,12–15]) which are used in applications such as location models, tax programming models, portfolio theory, risk theory, and data envelopment analysis (see [16,17,13,18,14,19]).

Unfortunately, the current literature does not provide any algorithm which can determine the global solution of all the problems P belonging to the class described in (1).

The aim of this paper is to propose a solution algorithm which is able to solve in an unifying approach all of the problems considered in (1) by means of the so-called "optimal level solutions" method (see [20,9,21,10,11,1,12,22,23,15,24]). It is known that this is a parametric method, which finds the optimum of the problem by determining the minima of particular subproblems. In particular, the optimal solutions of these subproblems are obtained by means of a sensitivity analysis aimed at maintaining the Karush–Kuhn–Tucker optimality conditions. Applying the optimal level solutions method to problem P, we obtain some strictly convex quadratic subproblems which are independent of function $\phi(y_1, y_2)$. In other words, different problems share the same set of optimal level solutions, and this allows us to propose a unifying method to solve all of them.

In Section 2, we describe how the optimal level solutions method can be applied to problem P; in Section 3, a solution algorithm is proposed and fully described; in Section 4 some results are proposed in order to improve the performance of the method; finally, in Section 5 the results of a deep computational test are provided and discussed, while in Section 6 some real applications are described.

2. A parametric approach

In this section, we show how problem P can be solved by means of the so-called *optimal level solutions approach* (see [10,11,1,23]). With this aim, let $\xi \in \Re$ be a real parameter, and let us define the corresponding parametrical subset of X:

$$X_\xi = \{x \in \Re^n : Ax \leqq b,\ d^T x = \xi\}.$$

In the same way, the following further subset of X can be defined:

$$X_{[\xi_1,\xi_2]} = \{x \in \Re^n : Ax \leqq b,\ \xi_1 \leq d^T x \leq \xi_2\}.$$

The following parametric subproblem can then be obtained just by adding to problem P the constraint $d^T x = \xi$:

$$P_\xi : \begin{cases} \min \phi\left(\frac{1}{2} x^T Q x + q^T x, \xi\right) \\ x \in X_\xi = \{x \in \Re^n : Ax \leqq b,\ d^T x = \xi\}. \end{cases}$$

The parameter ξ is said to be a *feasible level* if the set X_ξ is nonempty. An optimal solution of problem P_ξ is called an *optimal level solution*. Since $\phi(y_1, y_2)$ is strictly increasing with respect to variable y_1, for any feasible level ξ the optimal solution of problem P_ξ coincides with the optimal solution of the following strictly convex quadratic problem $\overline{P}_\xi$:

$$\overline{P}_\xi : \begin{cases} \min \frac{1}{2} x^T Q x + q^T x \\ x \in X_\xi = \{x \in \Re^n : Ax \leqq b,\ d^T x = \xi\}. \end{cases}$$

Obviously, an optimal solution of problem P is also an optimal level solution and, in particular, it is the optimal level solution with the smallest value; the idea of this approach is then to scan all the feasible levels, studying the corresponding optimal level solutions, until the minimizer of the problem is reached. Starting from an incumbent optimal level solution, this can be done by means of a sensitivity analysis on the parameter ξ, which allows us to move in the various steps through several optimal level solutions until the optimal solution is found (see [1]).

Remark 2.1. Notice that problem $\overline{P}_\xi$ admits one and only one minimum point, since its objective function is quadratic and positive definite and the feasible region X_ξ is closed. Since function $\phi(y_1, y_2)$ is strictly increasing with respect to variable y_1 and is defined for all the values in Ω, problem P_ξ admits one and only one minimum point too, the same as $\overline{P}_\xi$. As a consequence, the following logical implication holds:

$$\xi \in \Re \text{ is a feasible level} \quad \Rightarrow \quad \arg\min_{x \in X_\xi} f(x) \neq \emptyset.$$

2.1. Sensitivity analysis

Let x' be the optimal solution of problem $\overline{P}_{\xi'}$, where $d^T x' = \xi'$, and let us consider the following Karush–Kuhn–Tucker conditions for $\overline{P}_{\xi'}$:

$$\begin{cases} Qx' + q = A^T\mu + d\lambda & \\ d^Tx' = \xi' & \\ Ax' \leqq b & \textit{feasibility} \\ \mu \leqq 0 & \textit{optimality} \\ \mu^T(Ax' - b) = 0 & \textit{complementarity} \\ \mu \in \Re^m,\ \lambda \in \Re. & \end{cases} \tag{3}$$

Since $\overline{P}_{\xi'}$ is a quadratic strictly convex problem, the previous system has at least one solution (μ', λ'). By means of a sort of sensitivity analysis, we now aim to study the optimal level solutions of problems $\overline{P}_{\xi'+\theta}, \theta \in (0, \epsilon)$, with $\epsilon > 0$ small enough. This can be done by maintaining the consistence of the Karush–Kuhn–Tucker systems corresponding to these problems. Since the Karush–Kuhn–Tucker systems are linear whenever the complementarity conditions are implicitly handled, the solution of the optimality conditions regarding $\overline{P}_{\xi'+\theta}, \theta \in (0, \epsilon)$, with $\epsilon > 0$ small enough, is of the form

$$x'(\theta) = x' + \theta\Delta_x, \qquad \mu'(\theta) = \mu' + \theta\Delta_\mu, \qquad \lambda'(\theta) = \lambda' + \theta\Delta_\lambda. \tag{4}$$

It is worth pointing out that the strict convexity of problem $\overline{P}_{\xi'+\theta}$ guarantees for any $\theta \in [0, \epsilon)$ the uniqueness of the optimal level solution $x'(\theta) = x' + \theta\Delta_x$; this implies also the following important property:

vector Δ_x *is unique and different from* 0.

Clearly, the Karush–Kuhn–Tucker conditions are verified for values of $\theta \geq 0$ such that the following hold

Feasibility conditions: $Ax' + \theta A\Delta_x \leqq b$

Optimality conditions: $\mu' + \theta\Delta_\mu \leqq 0$.

Our aim is to determine the values of x', Δ_x, μ', Δ_μ, λ', and Δ_λ. By means of these parameters, the value $\theta_m = \min\{F, O\}$ can also be computed, where

$$F = \sup\{\theta \geq 0 : Ax' + \theta A\Delta_x \leqq b\}$$

$$O = \sup\{\theta \geq 0 : \mu' + \theta\Delta_\mu \leqq 0\}.$$

Observe that in [1] it has been proved that F and O are positive values whenever $\xi' < \xi_{\max}$, so θ_m results in being positive too.

Notice that for $\theta \in [0, \theta_m]$ both the optimality and the feasibility of $x'(\theta)$ are guaranteed, so $x'(\theta)$ represents a segment of optimal level solutions. Starting from $x'(\theta_m)$, we can iterate the process determining a new segment of optimal level solutions. As a consequence, this yields that the set of the optimal level solutions is nothing but a connected set given by the union of segments.

In [1], various results are given for determining the values of the feasibility and optimality parameters. In this paper, we aim to propose a simplified approach for determining them from a computational point of view, taking into account that the starting optimal level solution $x' = x'(0)$ is known.

Let x' be the optimal level solution corresponding to level ξ', and let $x'(\delta) = x' + \delta\Delta_x$ be the optimal level solution corresponding to level $\xi' + \delta$, with $\delta > 0$ small enough to guarantee that x' and $x'(\delta)$ belong to the same segment of optimal level solutions. Hence,

$$\Delta_x = \frac{x'(\delta) - x'}{\delta}.$$

Once Δ_x is computed, the set of binding constraints for $\theta \in [0, \delta]$ can be easily determined, so the complementarity conditions in the Karush–Kuhn–Tucker system can be implicitly handled.

With this aim, let A_B be the largest submatrix of A (made by rows of A) such that $A_B(x' + \theta\Delta_x) = b_B$ for all $\theta \in [0, \delta]$, where b_B is the subvector of b corresponding to A_B. Notice that the positivity of δ implies that such a condition is equivalent to the following one:

$$A_Bx' = b_B \quad \text{and} \quad A_B\Delta_x = 0.$$

By implicitly handling the complementarity conditions, the Karush–Kuhn–Tucker system for $\theta \in [0, \delta]$ becomes the following one:

$$\begin{cases} -A_B^T\mu_B & -d\lambda & +Qx & = -q \\ & & A_Bx & = b_B \\ & & d^Tx & = \xi' + \theta, \end{cases}$$

which can be expressed in matrix form as

$$S\begin{bmatrix} \mu_B \\ \lambda \\ x \end{bmatrix} = \begin{bmatrix} -q \\ b_B \\ \xi' + \theta \end{bmatrix}, \quad \text{where } S = \begin{bmatrix} -M^T & Q \\ 0 & M \end{bmatrix},\ M = \begin{bmatrix} A_B \\ d^T \end{bmatrix}. \tag{5}$$

***Procedure* Parameters**(*inputs*: x'; *outputs*: $\Delta_x, \mu', \Delta_\mu, \lambda', \Delta_\lambda, F, O, \theta_m$)
let $\delta > 0$ be the step parameter; set $\xi' := d^T x'$;
let $x'_\delta := \arg\min\{\overline{P}_{\xi'+\delta}\}$ and set $\Delta_x := \frac{x'_\delta - x'}{\delta}$;
let A_B be the submatrix of A such that $A_B x' = b_B$ and $A_B \Delta_x = 0$;
if rank $\begin{bmatrix} A_B \\ d^T \end{bmatrix}$ < *rows* $\begin{bmatrix} A_B \\ d^T \end{bmatrix}$ *then* delete the redundant rows of A_B;
set $M := \begin{bmatrix} A_B \\ d^T \end{bmatrix}, S := \begin{bmatrix} -M^T & Q \\ 0 & M \end{bmatrix}$ and compute S^{-1};
set $\begin{bmatrix} \mu'_B \\ \lambda' \\ x' \end{bmatrix} := S^{-1} \begin{bmatrix} -q \\ b_B \\ \xi' \end{bmatrix}$ and $\begin{bmatrix} \Delta_{\mu_B} \\ \Delta_\lambda \\ \Delta_x \end{bmatrix} := S^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$;
set $F := \sup\{\theta \geq 0 : Ax' + \theta A \Delta_x \leqq b\}$;
set $O := \sup\{\theta \geq 0 : \mu'_B + \theta \Delta_{\mu_B} \leqq 0\}$ and $\theta_m := \min\{F, O\}$;
end proc.

Assuming the rows of matrix M to be linearly independent (which can be obtained by eventually deleting some redundant rows of A_B), we have that matrix S is nonsingular (for the positive definiteness of Q). As a consequence, the solution of (5) is unique, and it is given by

$$\begin{bmatrix} \mu'_B(\theta) \\ \lambda'(\theta) \\ x'(\theta) \end{bmatrix} = \begin{bmatrix} \mu'_B \\ \lambda' \\ x' \end{bmatrix} + \theta \begin{bmatrix} \Delta_{\mu_B} \\ \Delta_\lambda \\ \Delta_x \end{bmatrix} = S^{-1} \begin{bmatrix} -q \\ b_B \\ \xi' \end{bmatrix} + \theta S^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Clearly, the parameters μ_i and Δ_{μ_i} corresponding to the nonbasic rows of A are equal to zero. Notice also that the value O can be computed by using the parameters μ'_B and Δ_{μ_B} only. The described approach is summarized in the following procedure "*Parameters*()".

In the solution algorithm, there will be the need to evaluate the objective function $f(x) = \phi(\frac{1}{2}x^T Qx + q^T x, d^T x)$ along the obtained segment of optimal level solutions $x'(\theta)$, $\theta \in [0, \theta_m]$. With this aim, it is worth defining the following restriction function:

$$\begin{aligned} z(\theta) &= f\left(x' + \theta \Delta_x\right) \\ &= \phi\left(\frac{1}{2}\theta^2 \Delta_\lambda + \theta\lambda' + \frac{1}{2}x'^T Qx' + q^T x', \xi' + \theta\right), \end{aligned}$$

where we have taken into account (see for example [1]) that from the Karush–Kuhn–Tucker conditions $d^T \Delta_x = 1$, $\Delta_x^T Q \Delta_x = \Delta_\lambda$, $\Delta_x^T A^T \Delta_\mu = 0$, and $\Delta_x^T (Qx' + q) = \lambda'$.

2.2. Underestimation function

A key role in the study of problem P will be played by the use of a proper underestimation function, that is, a function $\psi(\xi)$ which verifies the following property for all the feasible levels ξ:

$$\min_{x \in X_\xi} f(x) \geq \psi(\xi).$$

In order to determine such an underestimation function, the following notation can be introduced:

$$\gamma = 1/d^T Q^{-1} d, \qquad \xi_u = -d^T Q^{-1} q,$$

where γ is positive due to the positive definiteness of Q.

Lemma 2.1. *The following strictly convex quadratic parametric problem*

$$\begin{cases} \min \ \frac{1}{2}x^T Qx + q^T x \\ d^T x = \xi \end{cases}$$

attains the minimum at $\hat{x}(\xi) = \gamma(\xi - \xi_u)Q^{-1}d - Q^{-1}q$ *with minimum value* $\hat{g}(\xi) = \frac{1}{2}\gamma(\xi - \xi_u)^2 - \frac{1}{2}q^T Q^{-1} q$.

The previous lemma (which follows directly by applying the Lagrange conditions) shows that it is possible to explicitly determine the line of unconstrained minima corresponding to problem P, which from now on will be denoted as follows:

$$U_P = \left\{x \in \Re^n : x = \hat{x}(\xi), \xi \in \Re\right\}.$$

The positiveness of γ implies that function $\hat{g}(\xi)$ is a convex parabola with minimum value $\hat{g}(\xi_u) = -\frac{1}{2}q^T Q^{-1} q$. The following result suggests a first possible underestimation function for problem P.

Theorem 2.1. *Consider problem P. Then, for any feasible level* ξ,

$$\min_{x\in X_\xi} f(x) \geq \phi(\hat{g}(\xi), \xi).$$

Proof. Since the scalar function $\phi(y_1, y_2)$ is strictly increasing with respect to variable y_1, and by means of Lemma 2.1,

$$\begin{aligned}\min_{x\in X,\ d^Tx=\xi} f(x) &\geq \min_{x\in\Re^n,\ d^Tx=\xi} f(x)\\ &= \min_{x\in\Re^n,\ d^Tx=\xi} \phi\left(\frac{1}{2}x^TQx + q^Tx, \xi\right)\\ &= \phi\left(\min_{x\in\Re^n,\ d^Tx=\xi}\left\{\frac{1}{2}x^TQx + q^Tx\right\}, \xi\right)\\ &= \phi(\hat{g}(\xi), \xi). \quad \square\end{aligned}$$

When the line of unconstrained minima U_P does not intersect the feasible region X, the underestimation function can be further improved.

With this aim, assume that $U_P \cap X = \emptyset$, and let $x_s \in X$ and $v_s \in (Q^{-1}d)^\perp$ be such that $\{x \in \Re^n : v_s^Tx = v_s^Tx_s\}$ is a support hyperplane for X separating region X itself and the unconstrained minima line U_P, with $v_s^Tx \leq v_s^Tx_s$ for all $x \in X$. Notice that

$$(Q^{-1}d)^\perp = \left\{v \in \Re^n : v = Mw,\ w \in \Re^n\right\},$$

where

$$M = I - \frac{Q^{-1}dd^TQ^{-1}}{d^TQ^{-1}Q^{-1}d}$$

is a symmetric singular positive semidefinite matrix such that $M^2 = M$, with one eigenvalue equal to 0 (and corresponding eigenvector $Q^{-1}d$) and $n-1$ eigenvalues equal to 1 (and corresponding eigenvectors in $(Q^{-1}d)^\perp$).

Since $MQ^{-1}d = 0$, $M\hat{x}(\xi) = -MQ^{-1}q$ for all $\xi \in \Re$, so, given a point $x \in X$,

$$v = M(\hat{x}(\xi) - x) = M(-Q^{-1}q - x).$$

Such a vector v is nothing but the vector starting from point $x \in X$ and reaching the unconstrained minima line in an orthogonal way. To determine the separating hyperplane we are then left to determine the point $x_s \in X$ which is as close as possible to the unconstrained minima line, that is, the one providing the smallest vector $M(-Q^{-1}q - x)$. In other words, we have to minimize the quadratic form $(M(-Q^{-1}q - x))^T(M(-Q^{-1}q - x))$, and this can be done by solving the following equivalent convex quadratic problem (recall that $M^2 = M$):

$$x_s = \arg\min_{x\in X}\left\{\frac{1}{2}x^TMx + q^TQ^{-1}Mx\right\}.$$

From now on we can then assume that

$$v_s = M(-Q^{-1}q - x_s).$$

Notice that, since $M^2 = M$ and $M\hat{x}(\xi) = -MQ^{-1}q$,

$$v_s^T\hat{x}(\xi) = -v_s^TQ^{-1}q \quad \text{and} \quad v_s^T(\hat{x}(\xi) - x_s) = v_s^Tv_s.$$

Clearly, $v_s \neq 0$ if and only if the unconstrained minima line U_P does not intersect the feasible region X. To determine a tighter underestimation function let us define, when $X \cap U_P = \emptyset$, the following:

$$\nu = \frac{v_s^Tv_s}{v_s^TQ^{-1}v_s} > 0.$$

Lemma 2.2. *The following strictly convex quadratic parametric problem*

$$\begin{cases}\min\ \frac{1}{2}x^TQx + q^Tx\\ d^Tx = \xi\\ v_s^Tx \leq v_s^Tx_s\end{cases}$$

attains the minimum at $\ddot{x}(\xi) = \hat{x}(\xi) - \nu(Q^{-1}v_s)$ *with minimum value* $\ddot{g}(\xi) = \hat{g}(\xi) + \frac{1}{2}\nu(v_s^Tv_s)$.

Proof. The minimum point of the problem verifies the following necessary and sufficient optimality condition:

$$\begin{cases} Qx + q = \lambda d + \alpha v_s \\ d^T x = \xi,\ v_s^T x \le v_s^T x_s \\ \alpha(v_s^T x - v_s^T x_s) = 0 \\ \alpha \le 0,\ \lambda \in \Re. \end{cases}$$

Since Q is positive definite it is also nonsingular; hence, $\ddot{x}(\xi) = \lambda Q^{-1}d - Q^{-1}q + \alpha Q^{-1}v_s$. By means of simple calculations, since $\xi = d^T\ddot{x}(\xi)$ and $v_s \in (Q^{-1}d)^\perp$, we have $\lambda = \gamma(\xi - \xi_u)$, so $\ddot{x}(\xi) = \hat{x}(\xi) + \alpha Q^{-1}v_s$. From the feasibility conditions, we have also

$$v_s^T x_s \ge v_s^T \ddot{x}(\xi) = v_s^T \hat{x}(\xi) + \alpha \frac{v_s^T v_s}{\nu}$$

and hence, since $v_s^T(\hat{x}(\xi) - x_s) = v_s^T v_s$,

$$\alpha \le \frac{v_s^T(x_s - \hat{x}(\xi))}{v_s^T v_s}\nu = -\nu < 0.$$

With $\alpha < 0$ from the complementarity conditions, we get $v_s^T\ddot{x}(\xi) = v_s^T x_s$, which yields $\alpha = -\nu$ and $\ddot{x}(\xi) = \hat{x}(\xi) - \nu(Q^{-1}v_s)$. Finally,

$$\begin{aligned} \ddot{g}(\xi) &= \frac{1}{2}\ddot{x}(\xi)^T Q\ddot{x}(\xi) + q^T\ddot{x}(\xi) \\ &= \hat{g}(\xi) - \nu(v_s^T\hat{x}(\xi)) + \frac{1}{2}\nu(v_s^T v_s) - \nu(q^T Q^{-1} v_s) \\ &= \hat{g}(\xi) + \nu(v_s^T Q^{-1} q) + \frac{1}{2}\nu(v_s^T v_s) - \nu(q^T Q^{-1} v_s) = \hat{g}(\xi) + \frac{1}{2}\nu(v_s^T v_s). \quad \square \end{aligned}$$

The proof of the following result is analogous to that of Theorem 2.1.

Theorem 2.2. *Consider problem P and assume that $X \cap U_P = \emptyset$. Then, for any feasible level ξ,*

$$\min_{x \in X_\xi} f(x) \ge \phi\left(\hat{g}(\xi) + \frac{1}{2}\nu(v_s^T v_s), \xi\right).$$

As a conclusion, the following underestimation function can be defined:

$$\psi(\xi) = \phi\left(\hat{g}(\xi) + \hat{g}_0, \xi\right),$$

where

$$\hat{g}_0 = \begin{cases} 0 & \text{if } X \cap U_P \ne \emptyset \\ \frac{1}{2}\nu(v_s^T v_s) & \text{if } X \cap U_P = \emptyset. \end{cases}$$

Notice that, when $X \cap U_P = \emptyset$,

$$\phi\left(\hat{g}(\xi) + \hat{g}_0, \xi\right) > \phi\left(\hat{g}(\xi), \xi\right),$$

since function $\phi(y_1, y_2)$ is strictly increasing with respect to variable y_1, $v_s \ne 0$, and $\nu > 0$. Notice also that the continuity of $\phi(y_1, y_2)$ implies the continuity of $\psi(\xi)$. From a theoretical point of view, the previous underestimation function $\psi(\xi)$ allows us to prove the following result, which generalizes the one provided in Remark 2.1.

Corollary 2.1. *Consider problem P. Then, for any compact interval of feasible levels $[\xi_1, \xi_2]$,*

$$\arg\min_{x \in X_{[\xi_1,\xi_2]}} f(x) \ne \emptyset \quad \text{and} \quad \min_{x \in X_{[\xi_1,\xi_2]}} f(x) \ge \min_{\xi \in [\xi_1,\xi_2]} \psi(\xi).$$

Proof. Since $X_{[\xi_1,\xi_2]}$ is a closed set and $f(x)$ is a continuous function, the image set $f(X_{[\xi_1,\xi_2]})$ is closed too. Since $\psi(\xi)$ is continuous and the interval $[\xi_1, \xi_2]$ is compact, $\min_{\xi \in [\xi_1,\xi_2]} \psi(\xi)$ exists. Hence, for Theorem 2.1,

$$f(x) \ge \min_{\xi \in [\xi_1,\xi_2]} \psi(\xi) \quad \forall x \in X_{[\xi_1,\xi_2]}.$$

As a consequence, the set $f(X_{[\xi_1,\xi_2]})$ is closed and lower bounded, so the result is proved. $\square$

From the previous corollary we see that problem P can be unbounded only along extreme rays with feasible levels ξ going towards $+\infty$ or $-\infty$, while it admits a minimum in any compact set of feasible levels.

***Procedure* Main**(*inputs*: *P*; *outputs*: *Opt*, *OptVal*)

Compute the values $\xi_{\min} := \inf_{x\in X} d^T x$ and $\xi_{\max} := \sup_{x\in X} d^T x$;
Let $\xi_{big} >> 0$ and set $\xi_1 := \max\{-\xi_{big}, \xi_{\min}\}$; $\xi_2 := \min\{\xi_{big}, \xi_{\max}\}$;
Compute $x_1' := \arg\min\{\overline{P}_{\xi_1}\}$ and $x_2' := \arg\min\{\overline{P}_{\xi_2}\}$;
if $f(x_1') < f(x_2')$ *then* $\bar{x} := x_1'$ *else* $\bar{x} := x_2'$ *end if*;
Set $UB := f(\bar{x})$ and let $I_P = \left\{\xi \in \Re : A\hat{x}(\xi) \underline{\leq} b\right\}$;
if $I_P = \emptyset$ *then* $\xi_F := d^T x_s$; $x_F := \arg\min\{\overline{P}_{\xi_F}\}$;
else if $I_P \cap [\xi_1, \xi_2] = \emptyset$ *then* $\xi_F := \frac{\xi_1+\xi_2}{2}$; $x_F := \arg\min\{\overline{P}_{\xi_F}\}$;
else $\xi_F := \arg\min_{\xi\in I_P\cap[\xi_1,\xi_2]}\{\psi(\xi)\}$; $x_F := \hat{x}(\xi_F)$;
end if;
end if;
if $f(x_F) <$ *UB then* $\bar{x} := x_F$ and $UB := f(x_F)$ *end if*;
if $d^T\bar{x} \geq \xi_F$ *then*
$[\bar{x}, UB] := Visit(P, \xi_F, \xi_{\max}, \bar{x}, UB)$;
$[\bar{x}, UB] := Visit(\tilde{P}, -\xi_F, -\xi_{\min}, \bar{x}, UB)$;
else
$[\bar{x}, UB] := Visit(\tilde{P}, -\xi_F, -\xi_{\min}, \bar{x}, UB)$;
$[\bar{x}, UB] := Visit(P, \xi_F, \xi_{\max}, \bar{x}, UB)$;
end if ;
$Opt := \bar{x}$ and $OptVal := UB$;

end proc.

3. Solution algorithm

In order to find a global minimum (assuming that one exists) it would be necessary to solve problems $\overline{P}_\xi$ for all feasible levels. In this section, we will show that this can be done by means of a finite number of iterations, using the results of the previous section.

The method scans all the feasible levels, looking for the optimal solution starting from a certain feasible level ξ_F. With this aim, there will be the need to visit feasible levels lower than ξ_F in decreasing order. This can be done by reversing the problem itself, observing that problem P can be equivalently rewritten in the following form:

$$P \equiv \tilde{P} : \begin{cases} \inf f(x) = \tilde{\phi}\left(\frac{1}{2}x^T Qx + q^T x, \tilde{d}^T x\right) \\ x \in X, \end{cases}$$

where $\tilde{\phi}(y_1, y_2) = \phi(y_1, -y_2)$ and $\tilde{d} = -d$. In this light, if the feasible levels of P decrease then the feasible levels of $\tilde{P}$ increase.

The following procedures "*Main*()" and "*Visit*()" can then be proposed. Procedure "*Main*()" initializes the algorithm by determining the set of feasible levels and a "good" starting incumbent solution; then it uses procedure "*Visit*()" to obtain the global optimal solution (if it exists). As will be shown in detail in the next section, a "good" incumbent solution is useful in order to reduce the set of feasible levels to be explicitly scanned, thus improving the performance of the proposed method.

In particular, the optimal level solutions x_1' and x_2' are determined in order to have a good starting incumbent solution. The obtained starting incumbent solution results in being extremely effective when the objective function of problem P is unbounded along a feasible extremum ray. The starting feasible level ξ_F and its corresponding optimal level solution x_F are determined taking into account of the possibility of having $U_P \cap X = \emptyset$ or not.

Procedure "*Visit*()" scans the given set of feasible levels iteratively, obtaining the best solution. Notice that "*Visit*()" uses two subprocedures: the first one is procedure "*Parameters*()", which has been already described in Section 3, and the latter one is procedure "*MinRestriction*()", which determines the minimum of the continuous single-valued function $z(\theta)$ in the closed interval $[0, \theta_m]$. Observe that procedure "*MinRestriction*()" can be implemented numerically, and eventually improved for specific functions $f(x)$ (see [10,11,1,23]). Notice finally that in procedure "*Visit*()" there is also one more optional subprocedure, namely "*ImplicitVisit*()", which is aimed at improving the performance of the solution algorithm by implicitly visiting some of the feasible levels to be scanned. This optional procedure will be discussed in the next section.

The correctness of the proposed algorithm follows since all the feasible levels are scanned, and the optimal solution, if it exists, is also an optimal level solution.

It remains to verify the convergence (finiteness), that is to say that the procedure stops after a finite number of steps. First note that, at every iterative step of the proposed algorithm, the set of binding constraints changes; note also that the level is increased from ξ' to $\xi' + \theta_m > \xi'$, so it is not possible to obtain again an already used set of binding constraints; the convergence then follows, since we have a finite number of possible sets of binding constraints.

Procedure **Visit**(*inputs*: $P, \xi_F, \xi_{\max}, \bar{x}, UB$; *outputs*: *Opt*, *OptVal*)
  $\xi' := \xi_F$; $x' := x_F$;
  # $[\xi', x'] := ImplicitVisit(\xi', x', \xi_{\max}, false)$;
  while $\xi' < \xi_{\max}$
    set $[\Delta_x, \mu', \Delta_\mu, \lambda', \Delta_\lambda, F, O, \theta_m] := Parameters(x')$;
    let $z(\theta) = \phi\left(\frac{1}{2}\theta^2\Delta_\lambda + \theta\lambda' + \frac{1}{2}x'^TQx' + q^Tx', \xi' + \theta\right)$;
    set $[\bar{\theta}, z_{inf}] := MinRestriction(z(\theta), [0, \theta_m])$;
    if $z_{inf} = -\infty$ then $\bar{x} := []$; $UB := -\infty$; $\xi' := +\infty$ *else*
      if $z_{inf} < UB$ *then*
        $UB := z_{inf}$;
        if $\bar{\theta} = +\infty$ *then* $\bar{x} := []$ *else* $\bar{x} := x' + \bar{\theta}\Delta_x$ *end if*;
      end if;
      set $\xi' := \xi' + \theta_m$ and $x' := x' + \theta_m\Delta_x$;
    end if;
    # $[\xi', x'] := ImplicitVisit(\xi', x', \xi_{\max}, true)$;
  end while;
  $Opt := \bar{x}$; $OptVal := UB$;
end proc.

Remark 3.1. Let us point out that problems $\bar{P}_\xi$ are independent of the function ϕ. This means that problems having the same feasible region, the same Q, q, and d, but different function ϕ (either multiplicative or fractional or d.c.), share the same set of optimal level solutions. As a consequence, when procedure "*Main*()" explicitly visits all the feasible levels, these different problems are solved by means of the same iterations of the while cycle in procedure "*Visit*()".

4. Algorithm improvements

In this section, we aim to discuss how the proposed algorithm can be improved.

First of all, let us notice that in the various iterations of procedure "*Visit*()" some feasible levels could be implicitly visited when $O > F$. With this aim, first note that, for all $\theta \in [0, O]$, the value $z(\theta)$ is a lower bound for the parametric problem $\bar{P}_{\xi'+\theta}$; in fact, if $\theta \in [0, \theta_m]$, then $x'(\theta)$ is an optimal level solution, while, if $\theta \in (F, O]$, then $x'(\theta)$ is unfeasible for $\bar{P}_{\xi'+\theta}$, but is an optimal solution of a problem with the same objective function as $\bar{P}_{\xi'+\theta}$ and a feasible region containing $X_{\xi'+\theta}$. As a consequence, if the minimum value of $z(\theta)$ in the interval $(F, O]$ is greater than or equal to UB, then the feasible levels $(F, O]$ can be skipped.

Analogously, some more feasible levels can be implicitly visited by using the underestimation function $\psi(\xi)$. In fact, given $\xi_a \in [\xi', \xi_{\max}]$ it can be easily proved that

$$\psi(\xi) \geq UB \quad \forall \xi \in [\xi', \xi_a] \Rightarrow \min_{x \in X_{[\xi', \xi_{\max}]}} f(x) = \min_{x \in X_{[\xi_a, \xi_{\max}]}} f(x).$$

This property suggests that another way to improve the algorithm is by reducing the set of feasible levels to be scanned in the various iterations of procedure "*Visit*()", that is to say, by implicitly visiting some of the feasible levels.

As a conclusion, the following procedure "*ImplicitVisit*()" can be proposed in order to improve the visit of the feasible levels. Notice that in the procedure the *lower-level sets* of function $\psi(\xi)$ have been denoted with $L(\psi, UB) = \{\xi \in \Re : \psi(\xi) \leq UB\}$.

Finally, notice that procedure "*ImplicitVisit*()" is more effective the smaller the value *UB* of the incumbent solution is. For this very reason, in order to improve the algorithm's performance it is important to initialize the method with a "good" starting incumbent solution, as has been described in the previous section.

5. Computational results

In this section, the results of a computational experience are provided in order to point out both the correctness and the performance of the proposed algorithm. All the procedures described in the previous sections have been fully implemented with the software Matlab 7.4 R2007a on a computer having 2 Gb RAM and two Xeon dual core processors at 2.66 GHz.

The following four different objective functions have been used in the computational test:

	$\phi(y_1, y_2)$	$f(x)$
P_1	$y_1 - y_2^2$	$\left(\frac{1}{2}x^TQx + q^Tx\right) - \left(d^Tx\right)^2$
P_2	$y_1 y_2^3$	$\left(\frac{1}{2}x^TQx + q^Tx\right)\left(d^Tx\right)^3$
P_3	y_1/y_2^2	$\left(\frac{1}{2}x^TQx + q^Tx\right) / \left(d^Tx\right)^2$
P_4	$y_2^2 \log(y_1)$	$\left(d^Tx\right)^2 \log\left(\frac{1}{2}x^TQx + q^Tx\right)$

Procedure **ImplicitVisit**(*inputs*: $\xi', x', \xi_{\max}$, *inside*; *outputs*: ξ', x')
  $\xi'_{old} := \xi'$;
  if $\xi' < \xi_{\max}$ *and* $\psi(\xi') > UB$ *then*
    let $\mathcal{L} := [\xi', \xi_{\max}] \cap L(\psi, UB)$;
    if $\mathcal{L} = \emptyset$ *then* $\xi' := \xi_{\max}$ *else* $\xi' := \min\{\mathcal{L}\}$ *end if*;
  end if;
  if $\xi' < \xi_{\max}$ *and inside* $=$ *true and* $O - F > \xi' - \xi'_{old}$ *then*
    $[\tilde{\theta}, \tilde{z}_{inf}] := MinRestriction(z(\theta), [F + \xi' - \xi'_{old}, \min\{O, \xi_{\max} - \xi'_{old} + F\}])$;
    if $\tilde{z}_{inf} >= UB$ *then*
      $\xi' := \xi'_{old} + O - F$;
      if $\xi' < \xi_{\max}$ *and* $\psi(\xi') > UB$ *then*
        let $\mathcal{L} := [\xi', \xi_{\max}] \cap L(\psi, UB)$;
        if $\mathcal{L} = \emptyset$ *then* $\xi' := \xi_{\max}$ *else* $\xi' := \min\{\mathcal{L}\}$ *end if*;
      end if;
    end if;
  end if;
  if $\xi' < \xi_{\max}$ *and* $\xi' > \xi'_{old}$ *then* $x' := \arg\min\{\overline{P}_{\xi'}\}$ *end if*;
end proc.

Table 1
Number of iterations.

n	num	Complete	With implicit visit			
			P_1	P_2	P_3	P_4
10	1000	24.969	3.043	1.49	5.418	3.537
15	1000	39.72	4.682	1.651	7.91	4.637
20	1000	53.748	6.467	1.88	10.159	5.264
25	1000	68.409	8.008	2.16	12.616	6.284
30	1000	84.322	9.451	2.479	15.02	7.06
35	1000	100.86	10.813	3.1842	17.279	8.005
40	1000	123.75	13.746	3.852	19.924	8.658
45	1000	165.16	16.734	4.791	22.205	9.25
50	1000	181.77	20.228	6.193	24.491	9.669
60	1000	–	27.451	10.891	29.72	10.851
70	1000	–	36.736	13.533	36.288	12.352
80	800	–	44.865	14.537	42.951	13.685
90	600	–	59.85	18.245	50.535	15.132
100	500	–	86.378	21.426	60.558	16.36

where in P_2 and P_3 function $d^T x$ is positive over the feasible region, while in P_4 function $\frac{1}{2}x^T Qx + q^T x$ is positive over the feasible region.

The problems have been randomly created; in particular, matrices and vectors $Q \in \Re^{n\times n}$, $q, d \in \Re^n$, $A \in \Re^{m\times n}$, $b \in \Re^m$, $m = 3n$, have been generated with components in the interval $[-10, 10]$ by using the "rand()" Matlab function (numbers generated with uniform distribution). Within the procedures, the linear problems and the convex quadratic problems have been solved with the "linprog()" and "quadprog()" Matlab functions, respectively.

For each amount "n" of variables a number "num" of problems have been randomly generated, and each of these problems has been solved for both the objective functions in P_1, P_2, P_3, and P_4. The average number of iterations and the CPU times spent by the algorithm to solve the problems are given as the result of the test (see Tables 1 and 2).

In order not to waste time, the complete visits of the feasible levels have been tested for dimensions up to $n = 50$, while the use of procedure "ImplicitVisit()" has been tested up to dimension $n = 100$. Clearly, when procedure "ImplicitVisit()" is not used (that is, all the feasible levels are explicitly scanned) we provide only the results related to problem P_1, since all the problems are solved in the same number of iterations (see Remark 3.1).

The results obtained show the effectiveness of the improvements proposed in Section 5; in particular, the performance is strongly improved, especially for problems P_2 and P_4, for both the number of iterations and the spent CPU time.

Finally, it is worth reasserting that in this computational test very different objective functions have been minimized by means of the same solution algorithm. Notice also that the literature does not provide any algorithm which can determine the global solution of P_1, P_2, P_3, and P_4 (the known methods are able to approach only some of the single problems by means of the use of heuristics or branch and bound techniques, thus providing in general only approximate solutions). For these very reason, it has not been possible to make comparison tests.

Table 2
CPU time.

n	num	Complete	With implicit visit			
			P_1	P_2	P_3	P_4
10	1000	3.5214	0.88849	0.58293	1.2135	0.88011
15	1000	7.69	1.6518	0.87373	2.2635	1.471
20	1000	13.94	2.9178	1.3567	3.6156	2.0976
25	1000	23.592	4.5925	1.9904	5.63	3.0767
30	1000	38.261	6.9084	2.8597	8.4231	4.2699
35	1000	68.207	11.357	4.9839	13.73	6.7145
40	1000	105.88	17.363	6.7755	19.506	8.9015
45	1000	173.48	25.543	9.7921	26.446	11.568
50	1000	243.56	37.635	14.453	35.652	14.821
60	1000	–	71.144	31.619	60.654	23.199
70	1000	–	124.48	51.64	96.727	34.411
80	800	–	193.41	70.961	143.85	47.623
90	600	–	316.53	108.29	207.84	64.279
100	500	–	553.27	155.37	302.8	84.485

6. Some applications

In the literature, d.c. problems having an objective function of the kind $f(x) = c_1(x) - c_2(x)$, where $c_1(x) = g_1(\frac{1}{2}x^TQx + q^Tx)$ and $c_2(x) = g_2\left(d^Tx\right)$ are convex functions, are generally approached by means of branch and bound methods, which inherently provide, in general, an approximated solution. In this light, the algorithm proposed in this paper allows one to determine in a finite number of steps the exact optimal solution of these problems.

The solution method proposed in this paper can be also efficiently used in portfolio/risk theory. These financial models aim to minimize the risk of a certain investment for a given level of expected return. The risk is represented by a quadratic function x^TQx, where Q is the covariance matrix for the returns on the assets in the portfolio. The expected return on the portfolio is expressed by means of a linear function of the kind d^Tx. By minimizing the quadratic convex risk function for various fixed expected returns, the so-called efficient frontier of the portfolio can be numerically determined. Then, some further criterion must be used in order to choose the "optimal" portfolio along the efficient frontier (risk aversion and risk propensity criterion). See for example [25] for numerical examples of classical portfolio problems.

A more general approach is to look for a risk that is as small as possible together with an expected return that is as big as possible. This can be done by minimizing functions such as

$$f(x) = \frac{g_1\left(\frac{1}{2}x^TQx\right)}{g_2\left(d^Tx\right)} \quad \text{or} \quad f(x) = g_1\left(\frac{1}{2}x^TQx\right) - g_2\left(d^Tx\right),$$

where g_1 and g_2 are strictly increasing functions. These objective functions can be minimized with the algorithm proposed in this paper.

7. Conclusions

The proposed algorithm allows one to solve a wide range of nonconvex problems. The computational test shows that it is possible to efficiently handle problems with up to 100 variables. In particular, the improvement criteria suggested in Section 5 resulted in being extremely effective in making the algorithm efficient.

Further improvements could be based on the study of the quasiconvexity of function $\psi(\xi)$ which makes the condition $\psi(\xi') > UB$ a global optimality condition and a concrete stopping criterion. Improvements could also be obtained by iteratively updating the underestimation function $\psi(\xi)$ over the feasible subset $X_{[\xi',\xi_{\max}]}$ which remains to be visited.

References

[1] R. Cambini, C. Sodini, An unifying approach to solve a class of parametrically-convexifiable problems, in: I.V. Konnov, D.T. Luc, A.M. Rubinov (Eds.), Generalized Convexity and Related Topics, in: Lecture Notes in Economics and Mathematical Systems, vol. 583, Springer, Berlin, 2007, pp. 149–166.
[2] I.M. Bomze, T. Csendes, R. Horst (Eds.), Developments in Global Optimization, in: Nonconvex Optimization and its Applications, vol. 18, Kluwer Academic Publishers, Dordrecht, 1997.
[3] R. Horst, P.M. Pardalos (Eds.), Handbook of Global Optimization, in: Nonconvex Optimization and its Applications, vol. 2, Kluwer Academic Publishers, Dordrecht, 1995.
[4] R. Horst, H. Tuy, Global Optimization: Deterministic Approaches, third rev., Springer, Berlin, 1996.
[5] R. Horst, P.M. Pardalos, N.V. Thoai, Introduction to Global Optimization, second ed., in: Nonconvex Optimization and Its Applications, vol. 48, Kluwer Academic Publishers, Dordrecht, 2001.
[6] H. Konno, T. Kuno, Multiplicative programming problems, in: R. Horst, P.M. Pardalos (Eds.), Handbook of Global Optimization, in: Nonconvex Optimization and Its Applications, vol. 2, Kluwer Academic Publishers, Dordrecht, 1995, pp. 369–405.
[7] H.-S. Ryoo, N.V. Sahinidis, Global optimization of multiplicative programs, Journal of Global Optimization 26 (2003) 387–418.

[8] H. Tuy, Convex Analysis and Global Optimization, in: Nonconvex Optimization and Its Applications, vol. 22, Kluwer Academic Publishers, Dordrecht, 1998.
[9] A. Cambini, L. Martein, C. Sodini, An algorithm for two particular nonlinear fractional programs, Methods of Operations Research 45 (1983) 61–70.
[10] R. Cambini, C. Sodini, A finite algorithm for a particular d.c. quadratic programming problem, Annals of Operations Research 117 (2002) 33–49.
[11] R. Cambini, C. Sodini, A finite algorithm for a class of nonlinear multiplicative programs, Journal of Global Optimization 26 (2003) 279–296.
[12] R. Cambini, C. Sodini, A sequential method for a class of box constrained quadratic programming problems, Mathematical Methods of Operations Research 67 (2008) 223–243.
[13] J.B.G. Frenk, S. Schaible, Fractional programming, in: Handbook of Generalized Convexity and Generalized Monotonicity, in: Nonconvex Optimization and Its Applications, vol. 76, Springer, New York, 2005, pp. 335–386.
[14] S. Schaible, Fractional programming, in: R. Horst, P.M. Pardalos (Eds.), Handbook of Global Optimization, in: Nonconvex Optimization and Its Applications, vol. 2, Kluwer Academic Publishers, Dordrecht, 1995, pp. 495–608.
[15] C. Sodini, Minimizing the sum of a linear function and the square root of a convex quadratic form, Methods of Operations Research 53 (1986) 171–182.
[16] A.I. Barros, Discrete and Fractional Programming Techniques for Location Models, in: Combinatorial Optimization, vol. 3, Kluwer Academic Publishers, Dordrecht, 1998.
[17] W.W. Cooper, L.M. Seiford, J. Zhu (Eds.), Handbook on Data Envelopment Analysis, in: International Series in Operations Research & Management Science, vol. 71, Kluwer Academic Publishers, Boston, 2004.
[18] K.M. Mjelde, Methods of the Allocation of Limited Resources, Wiley, New York, 1983.
[19] S. Yitzakhi, A tax programming model, Journal of Public Economics 19 (1982) 107–120.
[20] A. Cambini, L. Martein, A modified version of Martos' algorithm, Methods of Operation Research 53 (1986) 33–44.
[21] R. Cambini, A class of non-linear programs: theoretical and algorithmic results, in: S. Komlósi, T. Rapcsák, S. Schaible (Eds.), Generalized Convexity, in: Lecture Notes in Economics and Mathematical Systems, vol. 405, Springer-Verlag, Berlin, 1994, pp. 294–310.
[22] A. Ellero, The optimal level solutions method, Journal of Information & Optimization Sciences 17 (1996) 355–372.
[23] S. Schaible, C. Sodini, A finite algorithm for generalized linear multiplicative programming, Journal of Optimization Theory and Applications 87 (1995) 441–455.
[24] C. Sodini, Equivalence and parametric analysis in linear fractional programming, in: A. Cambini, E. Castagnoli, L. Martein, P. Mazzoleni, S. Schaible (Eds.), Generalized Convexity and Fractional Programming with Economic Applications, in: Lecture Notes in Economics and Mathematical Systems, vol. 345, Springer-Verlag, Berlin, 1990, pp. 143–154.
[25] P. Brandimarte, Numerical Methods in Finance and Economics, Wiley-Interscience, 2006.