Characterizing LR-visibility polygons and related problems

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Abstract

A simple polygon $P$ is said to be LR-visibility polygon if there exists two points $s$ and $t$ on the boundary of $P$ such that every point of the clockwise boundary of $P$ from $s$ to $t$ (denoted as $L$) is visible from some point of the counterclockwise boundary of $P$ from $s$ to $t$ (denoted as $R$) and vice versa. In this paper we derive properties of shortest paths in LR-visibility polygons and present a characterization of LR visibility polygons in terms of shortest paths between vertices. This characterization suggests a simple algorithm for the following recognition problem. Given a polygon $P$ with distinguished vertices $s$ and $t$, the problem is to determine whether $P$ is a LR-visibility polygon with respect to $s$ and $t$. Our algorithm for this problem checks LR-visibility by traversing shortest path trees rooted at $s$ and $t$ in DFS manner and it runs in linear time.

Using our characterization of LR-visibility polygons, we show that the shortest path tree rooted at a vertex or a boundary point can be computed in linear time for a class of polygons which contains LR-visibility polygons as a subclass. As a result, this algorithm can be used as a procedure for computing the shortest path tree in our recognition algorithm as well as in the recognition algorithm of Das, Heffernan and Narasimhan. Our algorithm computes the shortest path tree by scanning the boundary of the given polygon and it does not require triangulation as a preprocessing step. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Characterizing, recognizing and computing visibility polygons under various criteria are central issues in visibility problems in computational geometry and related application areas [1]. The notion of visibility of a polygon from an internal segment arose when Avis and Toussaint [2] considered variations of the following art gallery problem: to place minimum number of stationary guards in an art gallery so that, together they can see every point in the interior of the gallery. In formal setting, the art gallery can be

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viewed as a simple polygon and guards as some points in the polygon. Avis and Toussaint [2] considered the case when number of guards is restricted to one but the guard is allowed to move along an edge of the polygon. Formally, this corresponds to finding an edge of the polygon such that every point in the polygon is visible from some point on the edge. Avis and Toussaint referred to visibility of a polygon from an edge as weak visibility of the polygon. A more general notion of (weak) visibility is one which allows visibility of the polygon from an internal segment, not necessary an edge. We refer to polygons, which have such an internal segment as weak visibility polygons. Weak visibility polygons have several interesting geometric properties, which allow simple and efficient algorithms for the class of weak visibility polygons [5,6,8–10,14,16].

A weak visibility polygon $P$ from a chord can be viewed as a polygon such that there exist two points $s$ and $t$ on the boundary of $P$ such that (i) $s$ and $t$ are mutually visible and (ii) every point of the clockwise boundary from $s$ to $t$ (denoted as $L$) is visible from some point of the counterclockwise boundary from $s$ to $t$ (denoted as $R$) and vice versa. If we remove the restriction (i), then it defines a new class of polygons, called $LR$-visibility polygons which contains weak visibility polygons as a subclass.

In this paper we derive properties of shortest paths in $LR$-visibility polygons and present a characterization of $LR$-visibility polygons in terms of shortest paths between vertices. This characterization is similar to the characterization of weak visibility polygons given by Ghosh et al. [10]. Das et al. [7] also characterized $LR$-visibility polygons and their characterization is in terms of non-redundant components.

Our characterization of $LR$-visibility polygons suggests a simple algorithm for the following recognition problem. Given a polygon $P$ with distinguished vertices $s$ and $t$, the problem is to determine whether $P$ is a $LR$-visibility polygon with respect to $s$ and $t$. Our algorithm for this recognition problem checks $LR$-visibility by traversing shortest path trees rooted at $s$ and $t$ in DFS manner and it runs in linear time. The shortest path trees rooted at $s$ and $t$ can be computed in linear time by the algorithm of Guibas et al. [11]. This algorithm computes the shortest path tree by splitting funnels using a finger search tree and needs a triangulation of the given polygon which can be done in linear time by the algorithm of Chazelle [4].

Instead of using the algorithm of Guibas et al. as a procedure for our recognition algorithm, we show that using our characterization of $LR$-visibility polygons, the shortest path tree rooted at a vertex or a boundary point can also be computed in linear time for a class of polygons which contains $LR$-visibility polygons as a subclass. So, our algorithm for computing shortest path tree may terminate by reporting that the given polygon is not a $LR$-visibility polygon. If it computes the shortest path trees rooted at $s$ and $t$, then the trees can be traversed in DFS manner to check $LR$-visibility as stated earlier. Our algorithm computes the shortest path tree by scanning the boundary of the given polygon and it does not require triangulation as a preprocessing step.

Let us now look at the previous results on characterizing and recognizing $LR$-visibility polygons. Heffernan [12] presented a linear time algorithm for recognizing $LR$-visibility polygons with respect to the given pair of points. Tseng et al. [17] proposed an $O(n \log n)$ time algorithm and Das et al. [7] proposed a linear time algorithm for recognizing $LR$-visibility polygons by locating all pairs of points such that the given polygon is a $LR$-visibility polygon with respect to each of these pairs of points. However, all these algorithms require triangulation of $P$ as a preprocessing step.

The algorithm of Das et al. needs a procedure for computing the shortest path tree rooted at some vertex as a preprocessing step in order to compute non-redundant components that are used later for checking $LR$-visibility. Their algorithm uses the linear time algorithm of Guibas et al. for computing the shortest path tree inside a polygon. Instead of using the algorithm of Guibas et al. as a procedure...
for their recognition algorithm, our algorithm for computing shortest path tree can be used. As a result, their overall recognition algorithm becomes simple as it does not require triangulation of the polygon as a preprocessing step and can also avoid using intricate data structure like finger search trees.

Above discussions claim that one of the merits of our algorithm for computing the shortest path tree is that our algorithm does not need triangulation of a polygon as a preprocessing step. Since a polygon can be triangulated in linear time by the algorithm of Chazelle [4], there is no advantage from the time complexity point of view to avoid the preprocessing step of triangulation. However, let us have a look at the algorithm of Chazelle [3,4]. The algorithm is very intricate and uses involved tools and notions such as a planar separator theorem, polygon cutting theorem, conformality, etc. Though the algorithm is asymptotically optimal, it is conceptually difficult and too complex to be considered practical. Chazelle [3] mentions that the existence of a truly simple linear time algorithm remains an open question and conjectures that such an algorithm is unlikely. Since the triangulation of a polygon can easily be obtained from the shortest path tree in linear time, our algorithm for computing the shortest path tree establishes that a truly simple asymptotically optimal algorithm for triangulating a simple polygon is possible for the class of LR-visibility polygons. Since the class of LR-visibility polygons contains many special classes of polygons such as spiral polygons, star-shaped polygons, weak visibility polygons, monotone polygons, etc., our algorithm can compute the shortest path tree in linear time in any of these special classes of polygons and, therefore, triangulating these special classes of polygons is also possible in linear time.

The paper is organized as follows. In Section 2, we characterize LR-visibility polygons and present the recognition algorithm. In Section 3, we present the algorithm for computing the shortest path tree rooted at some vertex. In Section 4, we conclude the paper with a few remarks.

2. Characterizing and recognizing LR-visibility polygons

Let $SP(u, v)$ denote the Euclidean shortest path inside $P$ from a point $u$ to another point $v$. For any vertex $u$ of $P$ the shortest path tree of $P$ rooted at $u$, denoted as $SPT(u)$, is the union of the shortest paths from $u$ to all vertices of $P$. $L$ (or $R$) is referred to as the same chain of every point of $L$ (respectively $R$). Similarly, $L$ (or $R$) is referred to as the opposite chain of every point of $R$ (respectively $L$). In the following theorem we characterize LR-visibility polygons.

**Theorem 2.1.** Let $P$ denote a simple polygon. The following statements are equivalent.

(i) $P$ is a LR-visibility polygon with respect to vertices $s$ and $t$.

(ii) $u, x$ and $v$ be any three vertices of $L$ (or $R$) (including $s$ and $t$) such that they are in clockwise (respectively counterclockwise) order while traversing $L$ (respectively $R$) from $s$ to $t$. $SP(u, x)$ and $SP(v, x)$ meet only at $x$.

(iii) For any vertex $x \in L$ (or $x \in R$), $SP(s, x)$ makes a left turn (respectively a right turn) at every vertex of $L$ (respectively $R$) in the path. Analogously, for any vertex $x \in L$ (or $x \in R$), $SP(t, x)$ makes a right turn (respectively a left turn) at every vertex of $R$ (respectively $L$) in the path.

**Proof.** Firstly, we show that (i) implies (ii). Since $P$ is a LR-visibility polygon with respect to $s$ and $t$, then $x$ is visible from some point $y$ of the opposite chain of $x$ (Fig. 1). So, the line segment $xy$ partitions the polygon into two simple polygons, one containing $s$ and $u$, and the other containing $t$ and $v$. So, $SP(u, x)$ and $SP(v, x)$ cannot cross the line segment $xy$ and therefore, they meet only at $x$. 
Fig. 1. $SP(u, x)$ and $SP(v, x)$ meet only at $x$ and $x$ is visible from the opposite chain of $x$.

Fig. 2. $SP(u, x)$ and $SP(v, x)$ meet at a vertex $y$ other than $x$.

Secondly, we show (ii) implies (iii). We prove only for the case when $x$ belongs to $L$ (Fig. 2). Assume on the contrary that $SP(s, x)$ makes a right turn at some vertex $y \in L$. Consider the convex angle formed by $SP(s, x)$ at $y$. If the angle is facing towards the interior of $P$, then by triangle inequality $SP(s, x)$ does not pass through $y$, a contradiction. If the angle is facing towards the exterior of $P$, then $SP(s, x)$ and $SP(t, x)$ meet at the vertex $y$ other than $x$ contradicting (ii) (Fig. 2). Hence, $SP(s, x)$ makes a right turn at $y$. Analogous arguments show that $SP(t, x)$ makes a right turn at every vertex of $L$ in the path.

Thirdly, we show that (iii) implies (i). It suffices to show that any vertex $x$ of $P$ is visible from some point of the opposite chain of $x$ (Fig. 1). We prove only for the case when $x$ belongs to $L$. Let $z_c$ and $z_{cc}$ be the vertices preceding $x$ on $SP(s, x)$ and $SP(t, x)$, respectively. If $z_c$ or $z_{cc}$ belongs to the opposite chain of $x$, then the claim holds. So we assume that $z_c$, $z_{cc}$ and $x$ belong to the same chain. From the condition (iii) we know that $SP(s, x)$ makes a left turn at $z_c$ and $SP(t, x)$ makes a right turn at $z_{cc}$. It means that extensions of $xz_c$ and $xz_{cc}$ from $z_c$ and $z_{cc}$ respectively cannot meet the same chain of $x$. So, both extensions meet at points of the opposite chain of $x$. Therefore, $x$ is visible for some point of the opposite chain of $x$. Hence, $P$ is a $LR$-visibility polygon with respect to $s$ and $t$.  

Theorem 2.1 suggests the following simple procedure for recognizing $LR$-visibility polygons given $s$ and $t$.

1. Compute $SPT(s)$ and $SPT(t)$.
2. Starting from $s$, traverse $SPT(s)$ using DFS traversal and for every vertex $x$ in $SPT(s)$, check the turn at $x$. If the path in $SPT(s)$ makes a right turn at $x \in L$ or makes a left turn at $x \in R$, then goto Step 5.
3. Starting from $t$, traverse $SPT(t)$ using DFS traversal and for every vertex $x$ in $SPT(t)$, check the turn at $x$. If the path in $SPT(t)$ makes a left turn at $x \in L$ or makes a right turn at $x \in R$, then goto Step 5.
4. Report “The given polygon is a $LR$-visibility polygon with respect to $s$ and $t$” and goto Step 6.
5. Report “The given polygon is not a $LR$-visibility polygon with respect to $s$ and $t$”.
6. Stop.

We now analyze the time complexity of the algorithm. $SPT(s)$ and $SPT(t)$ can be computed in linear time by the algorithm mentioned in Section 3. Steps 2 and 3 require the traversal of $SPT(s)$ and $SPT(t)$ in DFS fashion which takes linear time. Hence the overall time complexity of the algorithm is linear. We summarize the result in the following theorem.
Theorem 2.2. Given a simple polygon $P$ with distinguished vertices $s$ and $t$, it can be determined in linear time whether or not $P$ is a LR-visibility polygon with respect to $s$ and $t$.

3. An algorithm for computing the shortest path tree

In this section, we present a linear time algorithm for computing $SPT(a)$ rooted at some vertex $a$ inside a given simple polygon $P$. If the given $P$ is not a LR-visibility polygon with respect to any pair of boundary points of $P$, then the algorithm may terminate before computing the entire shortest path tree. If $P$ is a LR-visibility polygon, then algorithm always succeeds in computing $SPT(a)$.

The visibility polygon of $P$ from some internal point $z$ of $P$ is the set of all points of $P$ that are visible for $z$ and is denoted by $VP(z)$. Our algorithm first computes $VP(a)$ by the algorithm of Lee [15]. If $VP(a)$ is removed from $P$ (Fig. 3), then $P$ is split into disjoint regions of $P$ called pockets of $VP(a)$. Since the vertices of $VP(a)$ are visible from $a$, they are children of $a$ in $SPT(a)$. So, the remaining task for computing $SPT(a)$ is to compute the shortest path from $a$ to each vertex of all pockets. Since the shortest path from $a$ to any two vertices $b$ and $c$ of different pockets are disjoint, i.e., $SP(a, b)$ and $SP(a, c)$ meet only at $a$, the shortest path from $a$ to the vertices of one pocket can be computed independent of other pockets of $VP(a)$. So, it is enough to state the procedure for computing $SPT(a)$ to all vertices of one pocket. We have the following lemma.

Lemma 3.1. If $P$ is a LR-visibility polygon with respect to some pair of boundary points $s$ and $t$, then at most one of $s$ and $t$ can lie in a pocket of $VP(a)$.

Proof. If both $s$ and $t$ lie in the same pocket of $VP(a)$, then the entire opposite chain of $a$ lies in the pocket of $VP(a)$. Since no point of the pocket is visible from $a$, $a$ is not visible from any point of the opposite chain of $a$. Hence $P$ is not a LR-visibility polygon.

Keeping the above lemma in mind, we proceed to develop the procedure for computing the shortest path tree in a pocket. Let $yy'$ be a non-polygonal edge of $VP(a)$ where $y$ is a vertex of $P$ and $y'$ is some boundary point of $P$ (Fig. 3). Note that $a$, $y$ and $y'$ are collinear. We assume that if the boundary of $P$ is traversed from $a$ to $y'$ in counterclockwise order, then $y$ is encountered before reaching $y'$. So, the boundary of the pocket consists of the counterclockwise boundary of $P$ from $y$ to $y'$ and the segment $yy'$. The counterclockwise (or clockwise) boundary from a boundary point $z$ to another boundary point $z'$ is denoted by $bd_{cc}(z, z')$ (respectively $bd_{c}(z, z')$). Since the shortest path from $a$ to any vertex of this pocket passes through $y$, it is enough to compute $SPT(y)$ to the vertices of the pocket.

Before we present the algorithm for computing $SPT(y)$ in the pocket, we discuss the overall structure of the algorithm. It can be seen that for any vertex $x$ in the pocket, $SP(y, x)$ satisfies one of the following properties.

1. $SP(y, x)$ passes through only the vertices of $bd_{cc}(y, x)$.
2. $SP(y, x)$ passes through only the vertices of $bd_{c}(y, x)$.
3. $SP(y, x)$ passes through vertices of both $bd_{c}(y, x)$ and $bd_{cc}(y, x)$.

If $SP(y, x)$ passes through only the vertices of $bd_{cc}(y, x)$, it means that $bd_{c}(y, x)$ does not interfere $SP(y, x)$ and therefore, $SP(y, x)$ can be computed by considering only $bd_{cc}(y, x)$. Similarly, if $SP(y, x)$ passes through only the vertices of $bd_{c}(y, x)$, it means that $bd_{cc}(y, x)$ does not interfere $SP(y, x)$ and
therefore, \( SP(y, x) \) can be computed by considering only \( bd_c(y, x) \). However, if \( SP(y, x) \) passes through vertices of both \( bd_c(y, x) \) and \( bd_{cc}(y, x) \), it is necessary to consider both \( bd_c(y, x) \) and \( bd_{cc}(y, x) \) for computing \( SP(y, x) \). Instead of considering \( bd_c(y, x) \) and \( bd_{cc}(y, x) \) simultaneously, two convex paths from \( y \) to \( x \) can be constructed in \( bd_c(y, x) \) and \( bd_{cc}(y, x) \) separately and then merge these two convex paths to construct \( SP(y, x) \).

Our algorithm scans \( bd_{cc}(y, y') \) in counterclockwise order to construct convex paths from \( y \) to vertices of \( bd_{cc}(y, y') \). During the scan, if the current vertex \( x \) interferes the convex path from \( y \) to the previous scanned vertex of \( x \) (say, \( u \)), then a partial tree consisting of convex paths from \( y \) to vertices of \( bd_{cc}(y, u) \) is constructed. If no such vertex is encountered, convex paths from \( y \) to vertices of \( bd_{cc}(y, y') \) form \( SPT(y) \) because for all \( x \), \( SP(y, x) \) passes through only the vertices of \( bd_{cc}(y, x) \). If \( u \) is located, \( bd_c(y', y) \) is scanned in clockwise order to construct convex paths from \( y \) to vertices of \( bd_c(y, y') \). During the scan, if the current vertex \( x \) interferes the convex path from \( y \) to the previous scanned vertex of \( x \) (say, \( v \)), then a partial tree consisting of convex paths from \( y \) to vertices of \( bd_c(y, v) \) is constructed. If no such vertex is encountered, convex paths from \( y \) to vertices of \( bd_c(y', y) \) form \( SPT(y) \) because for all \( x \), \( SP(y, x) \) passes through only the vertices of \( bd_c(y, x) \). If two partial trees from \( y \) to vertices of \( bd_{cc}(y, u) \) and \( bd_c(y, v) \) are constructed, it means that \( SP(y, u) \) passes through vertices of \( bd_c(y, v) \) or \( SP(y, v) \) passes through vertices of \( bd_{cc}(y, u) \). So, these two trees are merged to make one tree rooted at \( y \) such that paths from \( y \) to vertices of \( bd_{cc}(y, u) \) now pass through vertices of \( bd_c(y, v) \) and vice versa, i.e., paths in the merged tree are \textit{outward convex}. Properties of \( LR\)-visibility polygons stated in Theorem 2.1 ensure that the outward convex path from \( y \) to \( u \) in the merged tree is \( SP(y, u) \) or the outward convex path from \( y \) to \( v \) in the merged tree is \( SP(y, v) \). The process of scanning and merging are repeated till \( SPT(y) \) is constructed or the algorithm terminates by reporting that the given polygon \( P \) is not a \( LR\)-visibility polygon with respect to any pair of boundary points.

We now present the algorithm for computing \( SPT(y) \) to the vertices of this pocket. Let \( SP_{cc}(y, u_k) = (y = u_0, u_1, \ldots, u_k) \) denote the convex path restricted to \( bd_{cc}(y, u_k) \) from \( y \) to a vertex \( u_k \).
Fig. 5. Figure shows that \( \text{SP}_{\text{cc}}(y, x) = (u_0, u_1, \ldots, u_k, x) \).

Fig. 6. Figure shows a reverse turn at \( u_k \).

that (i) vertices \( u_1, \ldots, u_k \) are vertices of \( bd_{\text{cc}}(y, u_k) \) and (ii) for any \( 0 < i < k, u_{i-1}, u_i \) and \( u_{i+1} \) is a right turn. Note that \( \text{SP}(y, u_k) \) and \( \text{SP}_{\text{cc}}(y, u_k) \) may be different as \( \text{SP}(y, u_k) \) may pass through vertices of \( bd_{\text{c}}(y', u_k) \) and therefore, \( \text{SP}_{\text{cc}}(y, u_k) \) may not lie totally inside \( P \). Union of \( \text{SP}_{\text{cc}}(y, x) \) for all \( x \) of \( bd_{\text{cc}}(y, y') \) is denoted by \( \text{SPT}_{\text{cc}}(y) \). Analogously, union of \( \text{SP}_c(y, x) \) for all \( x \) of \( bd_c(y', y) \) is denoted by \( \text{SPT}_c(y) \).

The algorithm for computing \( \text{SPT}_{\text{cc}}(y) \) scans \( bd_{\text{cc}}(y, y') \) in counterclockwise order starting from \( y \) and computes \( \text{SP}_{\text{cc}}(y, x) \) for every vertex \( x \) of \( bd_{\text{cc}}(y, y') \) unless it encounters some special vertex where the path from \( y \) is no longer a right turning path (Fig. 6). Assume that \( \text{SP}_{\text{cc}}(y, u_k) = (y = u_0, u_1, \ldots, u_k) \) has been computed and the algorithm wants to compute \( \text{SP}_{\text{cc}}(y, x) \) where \( x \) is the next counterclockwise vertex of \( u_k \). The following three cases can arise. To make the presentation simple, we assume from now on that no three vertices of \( P \) are collinear.

**Case 1.** \((u_{k-1}, u_k, x)\) is a left turn (Fig. 4). Scan \( \text{SP}_{\text{cc}}(y, u_k) \) from \( u_k \) till \( y \) is reached or a vertex \( u_i \) is reached such that \((u_{i-1}, u_i, x)\) is a right turn. So, \( \text{SP}_{\text{cc}}(y, x) = (u_0, u_1, \ldots, u_i, x) \). For every vertex \( u_j \) where \( i < j < k \), extend \( u_j u_{j-1} \) from \( u_j \) to \( xu_k \) and insert the point of intersection \( w_j \) on \( xu_k \).

**Case 2.** \((u_{k-1}, u_k, x)\) is a right turn and \((x', u_k, x)\) is a right turn (Fig. 5), where \( x' \) is the next clockwise vertex of \( u_k \). So, \( \text{SP}_{\text{cc}}(y, x) = (u_0, u_1, \ldots, u_k, x) \).

**Case 3.** \((u_{k-1}, u_k, x)\) is a right turn and \((x', u_k, x)\) is a left turn where \( x' \) is the next clockwise vertex of \( u_k \) (Fig. 6). The algorithm returns \( \text{SPT}_{\text{cc}}(y) \) up to the vertex \( u_k \) (denoted by \( \text{SPT}_{\text{cc}}(y, u_k) \)). We refer Case 3 as a reverse turn. The above procedure can reach \( y' \) if there is no reverse turn. In that situation, entire \( \text{SPT}_{\text{cc}}(y) \) (i.e., \( \text{SPT}(y) \)) has been computed and the algorithm terminates. Otherwise, for some vertex \( u_k \), \( \text{SP}_{\text{cc}}(y, u_k) \) has been computed and there is a reverse turn at \( u_k \). In order to overcome the reverse turn at \( u_k \), the algorithm tries to compute \( \text{SPT}_c(y') \) by the analogous procedure mentioned as Cases 1–3 by scanning \( bd_c(y', y) \) in clockwise order starting from \( y' \). So, the following three cases can arise depending upon whether or not there is a reverse turn in \( bd_c(y', y) \).

**Case 3.1.** Entire \( \text{SPT}_c(y') \) has been computed.

**Case 3.2.** \( \text{SPT}_c(y', v_l) \) has been computed and there is a reverse turn at \( v_l \) where \( v_l \in bd_{\text{cc}}(y, u_k) \) (Fig. 7).

**Case 3.3.** \( \text{SPT}_c(y', v_l) \) has been computed and there is a reverse turn at \( v_l \) where \( v_l \in bd_c(y', u_k) \) (Fig. 8).
Fig. 7. Figure shows a reverse turn at \( v_l \) where \( v_l \in bd_{cc}(y, u_k) \).

Fig. 8. Figure shows a reverse turn at \( v_l \) where \( v_l \in bd_{c}(y', u_k) \).

The remaining part of the algorithm consists of procedures for computing \( SPT(y) \) in Cases 3.1–3.3 and they are stated in the following three subsections. In the last subsection, we analyze the time complexity of the entire algorithm.

3.1. Procedure for computing \( SPT(y) \) in Case 3.1

In this subsection, we consider Case 3.1. Since entire \( SPT_c(y') \) has been computed, \( SPT_c(y') \) is the same as \( SPT(y') \) as shown in the following lemma.

**Lemma 3.2.** If no reverse turn is encountered during the clockwise scan of \( bd_c(y', y) \), then \( SPT_c(y') \) is the same as \( SPT(y') \).

**Proof.** In order to prove the lemma, we have to show that the parent of every vertex \( x \in bd_c(y', y) \) in \( SPT(y') \) is the same as that of \( x \) in \( SPT_c(y') \). Since no reverse turn is encountered during the clockwise scan of \( bd_c(y', y) \), the parent of every vertex \( x \in bd_c(y', y) \) in \( SPT_c(y') \) is a vertex of \( bd_c(y', x) \). If the parent of \( x \) in \( SPT_c(y') \) is not the same as that of \( x \) in \( SPT(y') \), then the parent \( z \) of \( x \) in \( SPT(y') \) is a vertex of \( bd_{cc}(y, x) \). Let \( z' \) be the next counterclockwise vertex of \( z \). So, \( z \) must be the parent of \( z' \) in \( SPT(y') \) and by Theorem 2.1, the path from \( y' \) to \( z' \) in \( SPT(y') \) makes a right turn at \( z \). So, there is a reverse turn at \( z' \) while scanning \( bd_c(y', y) \) in clockwise order. \( \square \)

The algorithm computes \( SPT(y) \) from \( SPT(y') \) by scanning \( bd_{cc}(y, y') \) in counterclockwise order starting from \( y \). Though in this situation shortest paths in \( SPT(y') \) make only left turns, we present the procedure for computing \( SPT(y) \) for general situation when \( SPT(y') \) is known and there are shortest paths in \( SPT(y') \) that make right turns. Observe that the parent of a vertex \( x \) in \( SPT(y') \) and \( SPT(y) \) are different vertices if and only if \( x \) is visible from some point of the segment \( yy' \). So, the algorithm has to compute shortest paths from \( y \) to only those vertices which are visible from some point of the segment \( yy' \) and for the remaining vertices, the parents are the same in both \( SPT(y') \) and \( SPT(y) \). For
details of this concept, see Hershberger [13]. Before we use this concept in our algorithm, we need the following definition. Let \( b_1, b_2, \ldots, b_j \) be the children of a vertex \( b \) in \( SPT_c(y') \) (or \( SPT_{cc}(y) \)) in clockwise (respectively counterclockwise) angular order with respect to \( b \) (Fig. 9), then \( b_1 \) is called the first child of \( b \), \( b_i \) is called the next sibling of \( b_{i-1} \) for all \( i \), and \( b_j \) is the last child of \( b \). Observe that the edge connecting the first child and its parent is always a polygonal edge.

Assume that \( SP(y, u_k) = (y = u_0, u_1, \ldots, u_k) \) has been computed for some vertex \( u_k \) where \( u_k \) is visible from some point of the segment \( yy' \). Now, the algorithm wants to compute \( SP(y, x) \) where \( x \) is the next counterclockwise vertex of \( u_k \). Since \( u_k \) is visible from some point of \( yy' \), \( SP(y', u_k) \) makes only left turns and \( SP(y, u_k) \) makes only right turns and they meet only at \( u_k \). Let \( SP(y', x) = (y' = v_0, v_1, \ldots, v_q, x) \). If \( v_q \) and \( u_k \) are the same vertex (Fig. 10), then the subtree of \( SPT(y') \) rooted at \( v_q \) becomes the subtree rooted at \( u_k \) in \( SPT(y) \). Let \( z \) be a descendent in the subtree rooted at \( u_k \) such that each vertex in the path \( SP(u_k, z) \) is the last child of its parent. Now the algorithm treats the next counterclockwise vertex of \( z \) as \( x \) and \( z \) as \( u_k \). If \( v_q \) and \( u_k \) are not the same (Fig. 11), observe that \( yy' \), \( SP(y', x) \), \( xu_k \) and \( SP(y, u_k) \) form a hourglass where \( SP(y', x) \) and \( SP(y, u_k) \) are two sides of the hourglass. The algorithm locates the tangent \( ab \) in the hourglass between \( SP(y', x) \) and \( SP(y, u_k) \) where \( a \in SP(y', x) \) and \( b \in SP(y, u_k) \). So, \( SP(y, x) = (y = u_0, u_1, \ldots, b, a, \ldots, v_q, x) \). Note that the tangent \( ab \) can be the edge \( xu_k \). Observe that if \( a \) and \( x \) are not the same vertex, then the subtree of \( SPT(y') \) rooted at \( a \) becomes the subtree rooted at \( x \) in \( SPT(y) \) with \( b \) as the parent of \( a \). Now the algorithm treats the next counterclockwise vertex of \( a \) as \( x \) and \( a \) as \( u_k \). The procedure is repeated till \( y' \) is reached. Once the procedure reaches \( y' \), \( SPT(y) \) has been computed.

3.2. Procedure for computing \( SPT(y) \) in Case 3.2

In this subsection, we consider Case 3.2. The algorithm has computed \( SPT_c(y', v_l) \) and \( SPT_{cc}(y, u_k) \) where \( v_l \) belongs to \( bd_{cc}(y, u_k) \) (Fig. 7). Our algorithm computes \( SPT(y') \) by merging \( SPT_c(y', v_l) \) and \( SPT_{cc}(y, u_k) \). Once \( SPT(y') \) is computed, then \( SPT(y) \) can be computed from \( SPT(y') \) by the procedure.
Fig. 11. Figure shows that $v_q$ is not the same as $u_k$ in $SP(y', u_k)$.

stated in Case 3.1. Before we state the procedure for merging $SPT_c(y', v_l)$ and $SPT_{cc}(y, u_k)$, we discuss the overall approach in the merging step. We start with the following lemma.

**Lemma 3.3.** Let $u_p$ and $v_q$ be the parents of a vertex $x \in bd_c(v_l, u_k)$ in $SPT_{cc}(y, u_k)$ and $SPT_c(y', v_l)$, respectively. If $(u_p, x, v_q)$ is a left turn, then $SP_{cc}(y, x)$ is the same as $SP(y, x)$ and $SP_c(y', x)$ is the same as $SP(y', x)$ (Fig. 7).

**Proof.** If $(u_p, x, v_q)$ is a left turn, it means that $SP_{cc}(y, x)$ and $SP_c(y', x)$ meet only at $x$ because $SP_{cc}(y, x)$ makes only right turns and $SP_c(y', x)$ makes only left turns. So, $x$ is visible from some point of $yy'$. By Theorem 2.1, $SP_{cc}(y, x)$ is the same as $SP(y, x)$ and $SP_c(y', x)$ is the same as $SP(y', x)$. $\square$

The above lemma suggests that if there exists such $x$, we can use the same procedure stated in Case 3.1 for computing $SP(y)$ starting from $x$. If no such $x$ exists, for every vertex $x \in bd_c(v_l, u_k)$, both $SP(y, x)$ and $SP(y', x)$ pass through vertices of $bd_{cc}(y, x)$ as well as vertices of $bd_c(y', x)$. In order to identify vertices of $SP(y, x)$ and $SP(y', x)$, we need the following lemma.

**Lemma 3.4.** For any vertex $x \in bd_c(y', y)$, all vertices of $SP_c(y', x)$ (or $SP_{cc}(y, x)$) belong to $SP(y', x)$ (respectively $SP(y, x)$).

**Proof.** The lemma follows from the fact that the line segment joining any two consecutive vertices of $SP_c(y', x)$ (or $SP_{cc}(y, x)$) is not intersected by any edge of $bd_c(y', x)$ (respectively $bd_{cc}(y, x)$). $\square$

Though the above lemma suggests that all vertices of $SP_c(y', x)$ belong to $SP(y', x)$, all edges of $SP_c(y', x)$ do not belong to $SP(y', x)$. So, there exists at least one edge $uw$ of $SP_c(y', x)$ intersected by $bd_{cc}(y, x)$. In order to compute $SP(y', x)$, the problem is to replace each such edge $uw$ by $SP(u, w)$. Observe that since $uw$ is intersected by an edge of $bd_{cc}(y, x)$, $SP(u, w)$ must pass through some vertices of $bd_{cc}(y, x)$. Moreover, $SP(u, w)$ may pass through some vertices of $bd_c(u, w)$ (excluding $u$ and $w$) where $w$ is assumed to be the next vertex of $u$ in $SP_c(y', x)$. By testing the intersection between each edge of $SP_c(y', x)$ with every edge of $bd_{cc}(y, x)$, we can locate all such edges. However, this costly
Lemma 3.5. Let $P$ be a $LR$-visibility polygon with respect to boundary points $s$ and $t$ (Fig. 12). Let $yy'$ be a segment inside $P$ connecting two boundary points $y$ and $y'$. Let $u_1w_1$ and $u_2w_2$ be two edges of $SP_c(y', x)$ for some vertex $x \in bd_c(y, y')$ where $u_1$, $w_1$, $u_2$ and $w_2$ occur in this order in $SP_c(y', x)$. Let $u_1'w_1'$ be an edge of $bd_c(y, x)$ such that $u_1'w_1'$ intersects $u_1w_1$. If any edge $u_2'w_2' \in bd_c(y, x)$ intersects $u_2w_2$, then $u_2'w_2'$ belongs to $bd_c(u_1', x)$.

Proof. Without loss of generality, we assume that $u_1'$ is the next counterclockwise vertex of $w_1'$ and $u_1'$ lies in the region enclosed by $u_1w_1$ and $bd_c(u_1, w_1)$. Since $u_1'w_1'$ intersects $u_1w_1$, $w_1$ is visible only from $bd_c(u_1', u_1)$. Therefore, $u_1$ and $u_1'$ must belong to opposite chain because $P$ is a $LR$-visibility polygon. So, $s$ or $t$ (say, $t$) belongs to $bd_c(u_1', u_1)$ and $s$ belongs to $bd_c(u_1', u_1)$. If $t \in bd_c(w_1, u_2)$ and any edge $u_2'w_2'$ of $bd_c(y, x)$ intersects $u_2w_2$, then $w_2$ is not visible from any point of the opposite chain $bd_c(s, t)$. Therefore, either $u_2'w_2'$ does not intersect $u_2w_2$ or $t$ belongs to $bd_c(u_1', w_2)$. So, we assume that $t \in bd_c(u_1', w_2)$ and $u_2'w_2'$ intersects $u_2w_2$. If $u_2'w_2' \in bd_c(y, w_1')$, it means that both segments $u_1'u_1$ and $u_1'w_1$ are intersected by $bd_c(y, w_1')$ and therefore, $u_1'$ or $w_1$ is not visible from its opposite chain. Hence, $u_2'w_2'$ belongs to $bd_c(u_1', x)$.  

Lemma 3.6. Let $P$ be a $LR$-visibility polygon with respect to boundary points $s$ and $t$ (Fig. 12). Let $yy'$ be a segment inside $P$ connecting two boundary points $y$ and $y'$. Let $y'u_1$ and $u_1w_1$ be two consecutive edges of $SP_c(y', x)$ for some vertex $x \in bd_c(y, y')$. Let $z$ be the first point of intersection between the ray drawn from $y'$ through $u_1$ and $bd_c(y, x)$, where $bd_c(y, x)$ is traversed from $y$ to $x$. Then no edge of $bd_c(z, x)$ intersects $y'u_1$. 

Fig. 12. The edge $u_1'w_1'$ intersects $u_1w_1$ and $u_2'w_2'$ intersects $u_2w_2$. 

Fig. 13. The vertex \( y \) lies inside the clockwise pocket induced by the segment \( y' \) and its last child in \( SPT_c(y', v_l) \).

Fig. 14. The inward edge \( zx \) from \( bd_{cc}(y, u_k) \) is entering in the clockwise pocket induced by \( uw \).

Proof. If any edge of \( bd_{cc}(z, x) \) intersects \( y'u_1 \), then some edge of \( bd_{cc}(z, x) \) also intersects \( zu_1 \). It means that \( z \) cannot be visible from any point of the opposite chain of \( z \). Therefore, the given polygon is not a \( LR \)-visibility polygon, a contradiction. \( \square \)

The above lemmas suggest a method for locating all edges of \( bc_{cc}(y, x) \) intersecting \( SP_c(y', x) \) as follows. Scan \( bd_{cc}(y, x) \) from \( y \) till an edge intersects \( y'u_1 \) or the ray drawn from \( y' \) through \( u_1 \). If the ray is intersected, repeat the process of checking the intersection treating \( u_1 \) as \( y' \). If an edge of \( bd_{cc}(y, x) \) intersects \( y'u_1 \), then continue the scan till another edge intersects \( y'u_1 \). Once another edge intersecting \( y'u_1 \) is found, continue the scan as before to check for intersection with \( y'u_1 \) or the ray drawn from \( y' \) through \( u_1 \). We use this method in our merging procedure for locating every edge of \( bd_{cc}(y, u_k) \) intersecting any edge of \( SPT_c(y', v_l) \). A lid is a line segment connecting two boundary points such that the segment contains two vertices. Note that a segment connecting two vertices is also called a lid. A lid always divides the polygon into two pockets, called the clockwise pocket and the counterclockwise pocket of the lid.

The merging procedure starts by initializing an edge of \( bd_{cc}(y, u_k) \) as an inward edge \( zx \) and an edge \( uw \) of \( SPT_c(y', v_l) \) as follows. If \( y \) lies inside the clockwise pocket induced by the lid connecting \( y' \) and the last child of \( y' \) in \( SPT_c(y', v_l) \) (Fig. 13), traverse the children of \( y' \) in \( SPT_c(y', v_l) \) starting from the last child till a child \( w \) is located such that \( y \) does not lie in the clockwise pocket of \( y'w \). Otherwise, the last child of \( y' \) in \( SPT_c(y', v_l) \) is assigned to \( w \). Now, \( z \) is initialized to \( y, x \) is initialized to the next counterclockwise vertex of \( z, u \) is initialized to \( y' \).

Intuitively, the situation after initialization can be viewed as the inward edge \( zx \) is entering into the pocket induced by \( uw \). So, in general, there are two situations: either the inward edge \( zx \) from \( bd_{cc}(y, u_k) \) is entering in the clockwise pocket induced by \( uw \) in \( bd_c(y', v_l) \) (Fig. 14) or the inward edge \( zx \) from \( bd_c(y', v_l) \) is entering in the counterclockwise pocket induced by \( uw \) in \( bd_{cc}(y, u_k) \) (Fig. 15). Since two
cases are analogous, we present here only the case when inward edge $zx$ from $bd_{ce}(y', vl)$ is entering the clockwise pocket induced by $uw$ in $bd_{ce}(y', vl)$ (Fig. 14).

Step 1. While $zx$ intersects $uw$, remove the edge $uw$ from $SPT_c(y', vl)$ and $w :=$ the last child of $u$ in $SPT_c(y', vl)$ (Fig. 16);

Step 2. If $zx$ intersects the extension $ww'$ (if it exists) of $uw$ (Fig. 17), then begin $u := w$; $w :=$ the last child of $w$ in $SPT_c(y', vl)$; goto Step 1 end;

Step 3. If $uw$ is an edge of $SP_c(y', vl)$ and $zx$ intersects the ray drawn from $u$ through $w$ (Fig. 18), then begin $u := w$; $w :=$ the last child of $w$ in $SPT_c(y', vl)$; goto Step 1 end;

Step 4. Let $x'$ be the next counterclockwise vertex of $x$. If $(u, x, x')$ is a left turn (Fig. 19), then begin $z := x$; $x := x'$; goto Step 1 end;

Step 5. If $(z, x, x')$ is a left turn (Fig. 20), then begin scan the boundary in counterclockwise order starting from $x'$ till an edge $bc$ intersecting $ux$ is found; assign $bc$ as the inward edge $zx$; goto Step 1 end;

Step 6. $[(z, x, x')$ is not a left turn.] Let $q$ be the point obtained by extending $ux$ from $x$ to the boundary. If $q$ belongs to an edge of $bd_{ce}(y, vl)$ (Fig. 21), then begin assign $u$ as the parent of $x$ in $SPT_c(y', vl)$; by scanning $bd_{ce}(x, q)$, compute the descendants of $x$ by using the procedure stated as Case 1 and Case 2; assign the edge containing $q$ as the inward edge $zx$; goto Step 1 end;

Step 7. If $q$ belongs to an edge of $bd_{ce}(u_k, vl)$ (Fig. 22), then begin assign $u$ as the parent of $x$ in $SPT_c(y', vl)$; by scanning $bd_{ce}(x, q)$, compute the descendants of $x$ by using the procedure stated earlier as Case 1 and Case 2; goto Step 9 end;

Step 8. If $q$ belongs to an edge of $bd_{ce}(y', u_k)$ (Fig. 23), then begin assign $u$ as the parent of $x$ in $SPT_c(y', vl)$; locate the child $x''$ of $x$ in $SPT_{ce}(y, u_k)$ such that the ray drawn from $u$ through $x$ passes between $x''$ and its next sibling; $w := x''$; $u := x$; assign the edge containing $q$ as the inward edge $zx$; call the analogous procedure for checking the intersection of inward edge $zx$ entering into the counterclockwise pocket induced by $uw$ end;

Step 9. STOP.
Observe that after the above procedure is executed, the computation of $SPT_c(y')$ is not complete as it may not contain paths from $y'$ to all vertices of $bd_c(y', y)$. If there is no path in $SPT_c(y')$ from $y'$ to some vertex, then the vertex must be a vertex of some inward edge. Let $z_1z_2, z_2z_3, \ldots, z_{j-1}z_j$ be the consecutive inward edges such that the parents of $z_1$ and $z_j$ in $SPT(y')$ are $x_1$ and $x_j$, respectively (Fig. 24). We wish to compute the parents of $z_2, z_3, \ldots, z_{j-1}$ in $SPT(y')$. We know that either $x_j$ is a descendant of $x_1$ or $x_1$ is a descendant of $x_j$. Without loss of generality we assume that $x_j$ is
Fig. 21. The extension point $q$ belongs to an edge of $bd_{cc}(y, v_l)$.

Fig. 22. The extension point $q$ belongs to an edge of $bd_{c}(u_k, v_l)$.

Fig. 23. The extension point $q$ belongs to an edge of $bd_{c}(y', u_k)$.

Fig. 24. Computing the parents of vertices of inward edges.

a descendant of $x_1$. Consider the polygon consisting of $SP(x_1, x_j), x_jz_j, z_j-1z_j, \ldots, z_2z_1, z_1x_1$. In this polygon, $SPT(x_1)$ can be computed using the procedure mentioned as Case 1 and Case 2 if the chain of inward edges belongs to $bd_{c}(y', u_k)$ or by the analogous procedure of Case 1 and Case 2 if the chain of inward edges belongs to $bd_{cc}(y, v_l)$. Observe that if Case 3 (i.e., a reverse turn) arises, it contradicts Lemma 3.1. Hence, the procedure mentioned as Case 1 and Case 2 can be used to compute parents of
vertices of all inward edges. Once the parent of each vertex of inward edges is located, entire $SPT_c(y')$ has been computed. Since all edges of $SPT_c(y')$ lie inside the polygon and paths in $SPT_c(y')$ are outward convex, $SPT_c(y')$ has become $SPT(y')$. Once $SPT(y')$ is computed, $SPT(y)$ can be computed from $SPT(y')$ as stated in Case 3.1.

Let us now explain how the extension point $q$ in Step 6 (Fig. 21) or Step 7 (Fig. 22) or Step 8 (Fig. 23) can be located on an edge. Let $z_c$ be a leaf in $SPT_c(y', v_l)$ such that $SP_c(u, z_c)$ passes through $w$ and every vertex in the path is the last child of its parent. Let $z'$ be the next clockwise vertex of $z_c$. Let $q_c$ be the point of intersection of the ray drawn from $u$ through $x$ and the polygonal edge $z'z_c$. Now, the child $x''$ of $x$ is located in $SPT_{cc}(y, u_k)$ such that the ray drawn from $u$ through $x$ passes between $x''$ and its next sibling. Let $z_{cc}$ be a leaf in $SPT_{cc}(y, u_k)$ such that every vertex in $SPT_{cc}(x'', z_{cc})$ is the last child of its parent. Let $z''$ be the next counterclockwise vertex of $z_{cc}$ such that the ray drawn from $u$ through $x$ and the polygonal edge $z''z_c$. If $q_c$ lies on the segment $xq_{cc}$ then $q$ is assigned to $q_c$ (Fig. 23). Otherwise, $q$ is assigned to $q_{cc}$ (Fig. 21).

3.3. Procedure for computing $SPT(y)$ in Case 3.3

In this subsection, we consider Case 3.3. $SPT_c(y', v_l)$ and $SPT_{cc}(y, u_k)$ have been computed and there is a reverse turn at $v_l$ as well as at $u_k$ where $v_l \in bd_c(y', u_k)$ (Fig. 8). We have the following lemma.

**Lemma 3.7.** If $P$ is a LR-visibility polygon with respect to two boundary points $s$ and $t$, then the parent of $v_l$ in $SPT(y')$ is a vertex of $bd_{cc}(y, u_k)$ or the parent of $u_k$ in $SPT(y)$ is a vertex of $bd_c(y', v_l)$.

**Proof.** Let $b_l$ and $c_l$ be the parents of $v_l$ and $u_k$ in $SPT(y')$ and $SPT(y)$, respectively (Fig. 26). If $b_l \in bd_c(y', v_l)$, then one of $s$ and $t$, say $s$, must lie on $bd_c(b_l, v_l)$. If $b_l \in bd_c(v_l, u_k)$, then one of $s$ and $t$, say $s$, must lie on $bd_c(v_l, b_l)$. So, if the parent of $v_l$ in $SPT(y')$ is not a vertex of $bd_{cc}(y, u_k)$, then $s$ must lie on $bd_c(y', v_l)$. Analogously, if the parent of $u_k$ in $SPT(y)$ is not a vertex of $bd_c(y', v_l)$, then $t$ must lie on $bd_{cc}(y, u_k)$. Since both $s$ and $t$ cannot lie on $bd_c(y', y)$ by Lemma 3.1, the parent of $v_l$ in $SPT(y)$ is a vertex of $bd_{cc}(y, u_k)$ or the parent of $u_k$ in $SPT(y)$ is a vertex of $bd_c(y', v_l)$.

The above lemma suggests that if $P$ is a LR-visibility polygon, then $bd_{cc}(y, u_k)$ has intersected the edge $b_lv_l$ or $bd_c(y', v_l)$ has intersected the edge $c_lu_k$. Since these two situations are analogous, we state the procedure when $bd_{cc}(y, u_k)$ has intersected $b_lv_l$ and $bd_c(y', v_l)$ has not intersected the edge $c_lu_k$. Since the order of intersections satisfies Lemmas 3.5 and 3.6, intersections can be checked by the merging procedure stated as Steps 1–9 till $v_l$ is reached as a vertex of an inward edge. If $u_k$ is reached before $v_l$ is reached as a vertex of an inward edge, the algorithm reports that the given polygon is not a LR-visibility polygon. So, we assume that the merging procedure has reached $v_l$ as a vertex of an inward edge before reaching $u_k$. Let $x$ and $x'$ be the next clockwise and counterclockwise vertices of $v_l$. If $(b_l, v_l, x)$ is a left turn and $(x', v_l, x)$ is a right turn, i.e., the reverse turn at $v_l$ remains (Fig. 25), then the parent of $b_l$ in $SPT(y')$ belongs to $bd_c(v_l, u_k)$. In that case $P$ is not a LR-visibility polygon by Lemma 3.7. So, we assume that $(b_l, v_l, x)$ is a right turn (Fig. 26). Treating $b_l$ as $y'$ and $v_l$ as the last vertex scanned so far, scan the clockwise boundary from $v_l$ using the analogous procedure mentioned as Cases 1–3 till another reverse turn is encountered at some vertex $v_m$ in Case 3. Now there are reverse turns at $v_m$ and $u_k$ which is Case 3.3 itself. The entire process of scanning and merging is repeated till it becomes Case 3.2 or $P$ is found not to be a LR-visibility polygon.
3.4. Analysis of the algorithm

Since $SPT_c(x')$ and $SPT_{cc}(y)$ are computed by considering each vertex of $bd_c(x', y)$ at most twice and each edge of $SPT_c(x')$ and $SPT_{cc}(y)$ is considered at most twice by the merging procedure, the overall complexity of the algorithm is linear. We summarize the result in the following theorem.

**Theorem 3.1.** The shortest path tree from a point inside a simple polygon can be computed in linear time if the given polygon is a LR-visibility polygon.

4. Concluding remarks

As stated earlier, our algorithm computes the shortest path tree from a point $a$ in each pocket of $VP(a)$ separately. Suppose the given polygon is a not a LR-visibility polygon but each pocket of $VP(a)$ satisfies the LR-visibility property. Then our algorithm succeeds in computing the shortest path tree. It will be interesting to identify the class of polygons for which our algorithm always succeeds in computing the shortest path tree from any vertex.

As mentioned in the introduction, one of the aims for developing the algorithm for computing the shortest path tree for the class of LR-visibility polygons is to show that the shortest path tree can be computed in linear time without the preprocessing step of triangulating the given polygon. Suppose it is possible to decompose a polygon into LR-visibility polygons in linear time. In that case, it is possible to triangulate each LR-visibility polygon by using our algorithm for computing the shortest path tree. So, the entire polygon can be triangulated once the polygon is decomposed into LR-visibility polygons. We feel that an arbitrary polygon can be decomposed into some number of LR-visibility polygons in linear time.
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References