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A Combinatorial Analysis of Topological Dissections

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From a topological space remove certain subspaces (cuts), leaving connected components (regions). We develop an enumerative theory for the regions in terms of the cuts, with the aid of a theorem on the Möbius algebra of a subset of a distributive lattice. Armed with this theory we study dissections into cellular faces and dissections in the d-sphere. For example, we generalize known enumerations for arrangements of hyperplanes to convex sets and topological arrangements, enumerations for simple arrangements and the Dehn–Sommerville equations for simple polytopes to dissections with general intersection, and enumerations for arrangements of lines and curves and for plane convex sets to dissections by curves of the 2-sphere and planar domains.

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INTRODUCTION

A problem which arises from time to time in combinatorial geometry is to determine the number of pieces into which a certain geometric set is divided by given subsets. Think for instance of a plane cut by prescribed curves, as in a map of a land area crisscrossed by roads and hedgerows, or of a finite set of hyperplanes slicing up a convex domain in a Euclidean space.

Such problems we call "topological dissections." They have a very general definition. Let X be a topological space, $H_1, \ldots, H_n$ a finite set of subspaces, which we remove, and $R_1, \ldots, R_m$ the connected components of the remainder of the space. This construction we call a *topological dissection* of X by the *cut spaces* $H_1, \ldots, H_n$ into the *regions* $R_1, \ldots, R_m$. The fundamental enumerative problem for topological dissections is to count the regions by means of properties of the cuts.

Suppose the cuts cover the whole space. Since the number of regions is 0, a solution to the fundamental problem becomes a condition on the cuts. This situation is illustrated by the partition of a topological space into open topological cells of various dimensions. The condition turns out to be just the Euler relation.

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A second enumerative problem concerns the *faces* of a dissection. The cut spaces induce a partition of X: two points are equivalent when every cut that contains one contains both. The faces of the dissection are the connected components of the blocks of this partition. The second problem is to count the faces of each dimension.

There is a large literature on these questions for particular kinds of dissections, going back at least to Steiner in 1826. Indeed much of the study of arrangements of hyperplanes has been concerned with counting faces and bounded faces. But most solutions have been only for straight cuts in two dimensions or in general position—such as the well-known enumerations for arrangements of lines in the projective plane, or the often-rediscovered maximum number of faces determined by n hyperplanes in Euclidean or projective space. There have been exceptions to the general run, notably Steiner's interest in circular and spherical cuts, Roberts' intricate investigation of bounded regions in three dimensions, and Winder's and our work on arbitrary arrangements of hyperplanes. Yet the only general method available has been that of algebraic topology, which, as we explain in Section 6, does not lend itself to finding useful enumerations. In this paper we develop a unified theory based on combinatorial ideas that applies to any topologically nice dissection and in a variety of important cases gives exact answers.

Our method is twofold. The general part is a fundamental relation involving the partially ordered set of intersections of cuts and the combinatorial Euler numbers of the regions and the cut intersections. The proof is essentially combinatorial: it is a corollary of a theorem concerning the Möbius algebra of a subset of a distributive lattice. Topology becomes involved in the applications, where known properties of a dissection may permit an explicit solution.

We consider in detail two types of dissection. If all the faces are open cells, we obtain a generating polynomial for the numbers of k-dimensional faces in terms of the cut intersections. This polynomial generalizes the enumerations in [25] of the faces of arrangements of hyperplanes to, for instance, convex open sets and arrangements of topological hyperplanes. For dissections with general intersection we find a common generalization of formulas of Steiner, Schläfli, and many others about simple (that is, generally positioned) arrangements of hyperplanes and of the Dehn–Sommerville equations for simple polytopes.

Second, we consider dissections of a subspace of a d-sphere. By incorporating the lower Betti numbers of the complement of the regions into the enumerating expression we can avoid any assumptions about the regions, except that they are open sets. We make no assumptions either about the cuts. (It might, however, be fruitful to examine specifically dissections by embedded or immersed $(d - 1)$-spheres.)

Specializing to the 2-sphere, we obtain generalizations of classical formulas for arrangements of straight lines in the Euclidean and projective planes. Indeed we can count the faces of a plane, a sphere, or a domain in the Euclidean plane, dissected by open and closed curves with self-intersections allowed.

We hope the detailed arguments and many examples presented below illuminate the techniques of dissection theory for geometers with similar dissection problems.

1. The Fundamental Relations for Dissections and Covers

All the results of our dissection theory are relations between the regions of a dissection and intersection sets determined by the cuts.

If $\mathscr{H}$ is a finite set of subspaces of X, let

$\mathscr{I}_0 =$ the set consisting of X and all intersections $H_1 \cap \cdots \cap H_k$ of members of $\mathscr{H}$;

$\mathscr{I}_0^c =$ the set consisting of X and all connected components of intersections $H_1 \cap \cdots \cap H_k$ $(k \geqslant 1)$;

and let $\mathscr{I}$ be $\mathscr{I}_0$ (and $\mathscr{I}^c$ be $\mathscr{I}_0^c$) but excluding the null set. We may write $\mathscr{I}(\mathscr{H})$, etc., for clarity.

The intersection sets we use are $\mathscr{I}$ (or occasionally $\mathscr{I}_0$) and its "meet refinements," of which $\mathscr{I}^c$ is one. A class $\mathscr{L}$ of subspaces of X is a meet refinement of $\mathscr{I}$ if it has the three properties:

(1) $\mathscr{L}$ is finite.

(2) $\mathscr{L}$ refines $\mathscr{I}$: $\mathscr{L}$ contains X but not $\varnothing$, and any member of $\mathscr{I}$ is a union of members of $\mathscr{L}$.

(3) $\mathscr{L}$ has meet support: any nonvoid intersection of members of $\mathscr{L}$ is also a union of members of $\mathscr{L}$.

(The definition of a meet refinement $\mathscr{L}_0$ of $\mathscr{I}_0$ is similar, except that $\varnothing$ is not excluded.)

A fourth property is fundamental. Let $\mathscr{D}$ be the lattice of sets generated by $\mathscr{L} \cup \{R_1, ..., R_m\}$ through unions and intersections. Call φ, a real-valued function on $\mathscr{D}$, a *valuation* of $\mathscr{L}$ if, for all $S, T \in \mathscr{D}$:

$$\varphi(S) + \varphi(T) = \varphi(S \cup T) + \varphi(S \cap T).$$

For dissection theory the important valuation is the *combinatorial Euler number* κ, which may be defined as

$$\kappa(X) = \chi(\hat{X}) - 1, \tag{1.1}$$

where $\chi(\hat{X})$ is the Euler characteristic of the one-point compactification of X. (If X is itself compact, $\kappa(X) = \chi(X)$.) Let us call a meet refinement of $\mathscr{I}$ *Eulerian* if it satisfies:

(4) κ is a valuation on $\mathscr{L}$.

It is the Eulerian meet refinements to which our theorems apply.

The crucial nature of κ, already apparent in Buck's paper [4] on dissections of Euclidean space by hyperplanes, is due to the well-known *Euler relation*: if X is representable as a disjoint union of n_0 points, n_1 open 1-cells, n_2 open 2-cells,..., and n_d open d-cells, then

$$\kappa(X) = n_0 - n_1 + n_2 - \cdots \pm n_d . \tag{1.2}$$

Based on (1.2) we can present a criterion for $\mathscr{L}$ to be Eulerian. It will help to have an alternative definition of the faces of $\mathscr{H}$. On any subspace Y there is an induced dissection whose cuts are

$$\mathscr{H}_Y = \{H \cap Y : H \in \mathscr{H} \text{ and } \varnothing \neq H \cap Y \neq Y\}.$$

The faces of $\mathscr{H}$ are the regions of the cut intersections $T \in \mathscr{I}$ as dissected by the induced dissections $\mathscr{H}_T$. Now, any $\mathscr{L}$ which is a meet refinement of $\mathscr{I}$ can be regarded as the set of cut spaces of a new dissection of X, so one can speak of the faces of $\mathscr{L}$. The unions of faces of $\mathscr{L}$ form a ring of sets which contains the lattice $\mathscr{D}$ generated by $R_1, \ldots, R_m$ and the components of all members of $\mathscr{L}$. That, with Euler's relation, proves:

LEMMA 1.1. *If every face of a dissection $\mathscr{H}$ is a finite, disjoint union of open topological cells, then κ is a valuation on any meet refinement of $\mathscr{I}$ which is refined by $\mathscr{I}^c$. For any meet refinement $\mathscr{L}$ of $\mathscr{I}$, if every face of $\mathscr{L}$ is a finite, disjoint union of open topological cells, then κ is a valuation on $\mathscr{L}$.* ∎

We now state the fundamental theorem of dissection theory.

THEOREM 1.2. *Let X be a topological space dissected by a set $\mathscr{H} = \{H_1, \ldots, H_n\}$ (possibly empty) of cut spaces into regions $R_1, \ldots, R_m$ ($m \geq 0$). Assume every H_i is a proper subspace of X. Let $\mathscr{L}$ be a meet refinement of $\mathscr{I}(\mathscr{H})$, such as $\mathscr{I}$ or $\mathscr{I}^c$. If $\mathscr{L}$ is Eulerian,*

$$\sum_{j=1}^{m} \kappa(R_j) = \sum_{T \in \mathscr{L}} \mu_{\mathscr{L}}(T, X)\, \kappa(T), \tag{1.3}$$

where $\mu_{\mathscr{L}}$ is the Möbius function of $\mathscr{L}$.

We still have to define the combinatorial Möbius function. The theory of this essential tool is mainly due to Rota [18]. Let P be a finite partially ordered set. Its *Möbius function* μ is defined on pairs of elements of P by $\mu(y, x) = 0$ if $y \not\leq x$, and by either of the equivalent recursions

$$\left.\begin{aligned} \sum_{y \leq t \leq x} \mu(y, t) &= \delta_{xy} \\ \sum_{y \leq t \leq x} \mu(t, x) &= \delta_{xy} \end{aligned}\right\} \quad \text{if } y \leq x,$$

where $\delta_{xy} = 1$ if $x = y$, $\delta_{xy} = 0$ if $x \neq y$. Evidently μ is determined by the partial order structure of P. Some useful expressions for μ are given in Section 5.

To define the Möbius function of $\mathscr{L}$ or any other class of subsets of X, we order the class by inclusion.

Proof of the fundamental theorem. First note that the case $m = 0$ is basic. For we can replace $\mathscr{H}$ by $\mathscr{H}' = \mathscr{H} \cup \{R_1, ..., R_m\}$. Then $\mathscr{L}' = \mathscr{L} \cup \{R_1, ..., R_m\}$ is a meet refinement of $\mathscr{I}(\mathscr{H}')$ and is Eulerian because $\mathscr{L}$ is. Using the fact that $\mu_{\mathscr{L}'}(T, X) = \mu_{\mathscr{L}}(T, X)$ for $T \in \mathscr{L}$, and calculating $\mu_{\mathscr{L}'}(R_j, X) = -1$, Eq. (1.3) for $\mathscr{L}$ follows from (1.3) for $\mathscr{L}'$.

Second, note that, because $\kappa(\varnothing) = 0$, (1.3) for $\mathscr{L}$ follows from (1.3) for $\mathscr{L}_0$.

Now let us devote our attention to (1.3) with $m = 0$, summed over $\mathscr{L}_0$. It brings out the essentials of the proof to rewrite the equation for an arbitrary valuation φ instead of κ, whose special properties are no longer needed. We obtain:

$$\sum_{T \in \mathscr{L}_0} \mu_{\mathscr{L}_0}(T, X)\, \varphi(T) = 0. \tag{1.4}$$

In Section 2 we use the general theory of valuations to prove a result, Corollary 2.2, whose conclusion looks just like (1.4). Moreover, by Theorem 2.3, φ and $\mathscr{L}_0$ satisfy the hypotheses of the corollary: for we assumed $\mathscr{L}_0$ to be a meet refinement of $\mathscr{I}_0$ and φ to be a valuation on it. Therefore Eq. (1.4) follows, completing the proof. ∎

This proof of (1.4) requires elaborate machinery, yet the formula itself is relatively elementary. Is there a more elementary proof? I have found it possible to avoid valuation theory by applying instead a theorem on Galois connections [18, p. 347, Theorem 1], but only for refinements of $\mathscr{I}$ which are coarser than $\mathscr{I}^c$. That is not sufficient for dissection theory. Evidently there is work to be done.

2. Algebraic Combinatorics: Valuations and Möbius Algebras

The abstract properties that underlie the fundamental relations are those of valuations on distributive lattices. We prove (1.4) in that setting.

First we require the valuation ring of a distributive lattice D, devised by Rota [19, Sect. 3]. Let R be a unitary ring; e.g., the integers. In the free R-module $M(D, R)$ whose basis is the elements of D, define multiplication by $xy = x \wedge y$ on basis elements, extended by linearity to the whole space. This makes M into an algebra. In M the set $N(D, R)$ of all linear combinations of elements of the form

$$x \vee y + x \wedge y - x - y$$

is an ideal.

A linear functional φ on M which obeys the law

$$\varphi(x \vee y) + \varphi(x \wedge y) = \varphi(x) + \varphi(y) \qquad \text{for} \quad x, y \in D$$

is a *valuation* of D. A functional φ is a valuation if and only if it is zero on N. Thus we call M/N the *valuation ring* of D over R. We can study valuations, which are what matter for the fundamental relations, by looking at M/N.

Second, we need the *Möbius algebra* $M(P, R)$ of a finite partially ordered set P over the unitary ring R. This ingenious construction is due to Solomon [21]. (See [8] for a slick treatment.) $M(P, R)$ is the free R-module whose basis is the elements of P, with a product defined by

$$xy = \sum_{t: t \leqslant x, t \leqslant y} e_t(P) \qquad \text{for} \quad x, y \in P,$$

and extended to $M(P, R)$ by linearity, where we define

$$e_t(P) = \sum_{s \in P} \mu_P(s, t) s \qquad \text{for} \quad t \in P.$$

In case x and y have a greatest lower bound in P, it equals their product. Thus for a distributive lattice the two definitions of $M(D, R)$ agree.

By Möbius inversion [18, Sect. 3],

$$x = \sum_{t \in P; t \leqslant x} e_t(P) \qquad \text{for} \quad x \in P.$$

Therefore the $e_t(P)$ generate $M(P, R)$. But more: they are orthogonal idempotents: $e_x{}^2 = e_x$ and $e_x e_y = 0$ if $x \neq y$; hence they are a basis.

The main theorem of this section states a relationship between the valuation ring of D and the canonical idempotents $e_t(P)$ of a subset of D.

THEOREM 2.1. *Let D be a finite distributive lattice and let P be a subset of D that contains 0_D and every join irreducible element of D. If $t \in P$ is not 0_D or a join irreducible, then in $M(D, R)$,*

$$e_t(P) \equiv 0 \qquad \text{mod } N.$$

Proof. First some more theory. An injection of partially ordered sets, i: $Q \to S$, induces two homomorphisms of the Möbius algebras. One is i, extended to $M(Q, R)$ by linearity. The other is a reverse map, i^*: $M(S, R) \to M(Q, R)$, which on the canonical basis has the values

$$\begin{aligned} i^*(e_t(S)) &= e_t(Q) \qquad \text{if} \quad t \in Q, \\ &= 0 \qquad\qquad \text{if} \quad t \notin Q. \end{aligned} \tag{2.1}$$

Both derived maps are functorial, so composition is preserved. (The definition and properties of i^* are from Geissinger [6, Proposition 13].) By (2.1), $i^* \circ i = id_Q$ and $i \circ i^* \equiv id_S \bmod \ker i^*$. Therefore i and i^* are inverse isomorphisms $M(Q, R) \cong M(S, R)/\ker i^*$.

Now we prove the theorem. Let $I =$ the set consisting of 0_D and the join irreducibles of D. The commutative diagram of injections

$$\begin{tikzcd} D & & I \arrow[ll, "i_0"'] \arrow[dl, "i_1"] \\ & P \arrow[ul, "i_2"] & \end{tikzcd}$$

is transformed by the functor $*$ into

$$\begin{tikzcd} M(D, R) \arrow[rr, "i_0^*"] \arrow[dr, "i_2^*"'] & & M(I, R) \\ & M(P, R) \arrow[ur, "i_1^*"'] & \end{tikzcd}$$

Let $t \in P - I$. By (2.1), $e_t(P) = i_2^*(e_t(D))$ and $i_0^*(e_t(D)) = 0$, whence $e_t(P) = i_2(e_t(P)) \in \ker i_0^*$. Davis [5](see [6, Theorem 2]) proved in effect that

$$i_0 : M(I, R) \cong M(D, R)/N(D, R).$$

Hence $\ker i_0^* = N$, which proves the theorem.

An elementary proof is possible, in two steps. One first proves the theorem for P closed under meets, then does an induction on card$(Q - P')$ for $P \subseteq P' \subsetneq Q =$ the set of all meets of members of P. We omit the details. ∎

COROLLARY 2.2. *Let φ be a valuation of the finite distributive lattice D, and let P be a subset of D containing 0_D and every join irreducible. Then for any $t \in P$ which is not 0_D or a join irreducible of D,*

$$\sum_{s \in P;\, s \leq t} \mu_P(s, t)\, \varphi(s) = 0. \quad ∎$$

There is another way of stating the condition on P of Theorem 2.1 and Corollary 2.2. We say that P has *meet support* if the meet of any nonempty subset of P is the join of elements of P. (Meet and join are performed in D.)

THEOREM 2.3. *If P is a subset of a distributive lattice D, then P contains 0_D and every join irreducible if and only if it has meet support and generates D through meets and joins.* ∎

Since the "meet refinements" defined in Section 1 have meet support, the proof of Eq. (1.4) from Corollary 2.2 is justified.

3. Dissections into Cells and Properly Cellular Faces

Let us suppose that X, a d-dimensional space, is dissected by $\mathscr{H}$ into regions, every one an open d-cell. The number of regions is then

$$m(\mathscr{H}) = (-1)^d \sum_U \mu(U, X)\,\kappa(U). \tag{3.1}$$

The range of summation may be any Eulerian meet refinement $\mathscr{L}$ of the set $\mathscr{I}$ defined in Section 1; μ is the Möbius function of $\mathscr{L}$.

With a further assumption we can find f_k, the number of k-faces of $\mathscr{H}$. Say $\mathscr{H}$ has *properly cellular faces* when, for every $T \in \mathscr{I}^c$, every region of $\mathscr{H}_T$ is an open dim T-cell. Lemma 1.1 guarantees that $\mathscr{I}^c$ is Eulerian. Using (3.1) we can substitute for $m(\mathscr{H}_T)$ in

$$f_k = \sum \{m(\mathscr{H}_T) \colon T \in \mathscr{I}^c \text{ and } \dim T = k\},$$

sum $x^{d-k} f_k$, and collect terms, noting that

$$\mathscr{I}^c(\mathscr{H}_T) = \{U \in \mathscr{I}^c : U \subseteq T\}.$$

Theorem 3.1. *Let X be a d-dimensional space, dissected by $\mathscr{H}$ into properly cellular faces. Then the numbers of k-faces are given by the generating polynomial*

$$\sum_{k=0}^{d} x^{d-k} f_k = (-1)^d \sum_{U \in \mathscr{I}^c} \kappa(U) \sum_{U \subseteq T \in \mathscr{I}^c} \mu(U, T)(-x)^{d-\dim T}. \quad \blacksquare$$

The most interesting dissections into properly cellular faces have a property which we term *geometric intersection*: if $T \in \mathscr{I}^c$ and $H \in \mathscr{H}$,

$$H \cap T = \varnothing \qquad \text{or} \qquad H \supseteq T \qquad \text{or} \qquad \dim H \cap T = \dim T - 1.$$

Then one can show every interval of $\mathscr{I}^c$ is an inverted geometric lattice.

Theorem 3.2. *Let $\mathscr{H}$ dissect the d-space X into properly cellular faces. Assume $\mathscr{H}$ has geometric intersection.*

(A) *If every $T \in \mathscr{I}^c - \{X\}$ is an open cell,*

$$\sum_{k=0}^{d} x^{d-k} f_k = \sum_{U, T \in \mathscr{I}^c} \sum |\mu(U, T)|\, x^{d-\dim T} + (-1)^d \kappa(X) - 1.$$

(B) *If every $T \in \mathscr{I}^c - \{X\}$ has the same Euler number, κ_0,*

$$\sum_{k=0}^{d} x^{d-k} f_k = (-1)^d \left(\kappa_0 \sum_{U, T \in \mathscr{I}^c} \sum \mu(U, T)(-x)^{d-\dim T} + \kappa(X) - \kappa_0\right).$$

Let $Z = \bigcap \mathscr{H}$, $\mathscr{I}_0 = \mathscr{I} \cup \{Z\}$, *and* $i = \dim Z$. *Put* $f_{-1} = 1$ *if* $Z = \varnothing$, $f_{-1} = 0$ *if* $Z \neq \varnothing$.

(C) *If every* $T \in \mathscr{I}^c - \{X\}$ *is a topological projective space,*

$$\sum_{k=0}^{d} x^{d-k} f_k = \sum_{\substack{U,T \in \mathscr{I}^c \\ \dim U \text{ even}}} \sum |\mu(U, T)|\, x^{d-\dim T} + (-1)^d(\kappa(X) - \tfrac{1}{2}[1 + (-1)^d]).$$

If every nonvoid cut intersection is a topological projective space,

$$\sum_{k=-1}^{d} x^{d-k} f_k = \frac{1}{2}\left(x^{d-i} + \sum_{U,T \in \mathscr{I}_0} \sum |\mu(U, T)|\, x^{d-\dim T}\right).$$

(D) *If every* $T \in \mathscr{I}^c - \{X\}$ *is a topological sphere or a point,*

$$\sum_{k=0}^{d} x^{d-k} f_k = 2 \sum_{\substack{U,T \in \mathscr{I}^c \\ \dim U \text{ even} \geq 2}} \sum |\mu(U, T)|\, x^{d-\dim T}$$

$$+ \sum_{\substack{U,T \in \mathscr{I}^c \\ \dim U = 0}} \sum |\mu(U, T)|\, x^{d-\dim T} + (-1)^d(\kappa(X) - [1 + (-1)^d]).$$

If every nonvoid cut intersection is a topological sphere,

$$\sum_{k=-1}^{d} x^{d-k} f_k = \sum_{U,T \in \mathscr{I}_0} \sum |\mu(U, T)|\, x^{d-\dim T}.$$

Proof. In Theorem 3.1 substitute the appropriate Euler numbers. For (C) and (D), the second recursion for the Möbius function in Section 1 must be used to get sums over $\mathscr{I}_0$; and the cases $Z = \varnothing$ and $Z \in \mathscr{I}$ must be distinguished.

The absolute values arise from a theorem of Rota, [18, Sect. 7, Theorem 4], which implies that in a geometric lattice, $(-1)^{\dim T - \dim U} \mu(U, T) > 0$ if $U \leqslant T$. Observe that in (C) and (D) $\mathscr{I}_0$ is an inverted geometric lattice. ▌

Examples. (A, C) generalize the enumerations in [25, Theorems A, B] of the faces of an arrangement of hyperplanes. An arrangement is the dissection of a Euclidean or projective space due to a finite set of hyperplanes. Any reasonable topological generalization would also fall under Theorem 3.2 since it should have properly cellular faces and geometric intersection. (The model is the arrangements of pseudolines studied by Levi [14], Ringel [16], and Grünbaum [10, Sect. 18.3; 12].)

(A) includes as well the dissection of an open convex set $K \subseteq \mathbf{E}^d$ by hyperplanes. Note that $\mathscr{I}$ consists only of the hyperplane intersections that meet K, and that $\kappa(K) = (-1)^d$. For an application see [27].

(B) includes dissections in which every $T \in \mathscr{I}^c$ is a closed cell ($\kappa_0 = 1$); in particular, a convex polytope P dissected by hyperplanes among which are numbered the supporting hyperplanes of facets of P.

In particular when P is dissected by its facets, $\mathscr{I}$ is the lattice of faces of the polytope. By [19, Theorem 4], if $F \subseteq F'$ are faces of P, $\mu(F, F') = (-1)^{\dim F' - \dim F}$. This case of Theorem 3.2(B) generalizes the Dehn–Sommerville equations (see Sect. 4), to cubical polytopes, for instance (see [10, Sect. 9.4] for a statement and another proof).

Also under (B) comes the enumeration of the bounded faces of a Euclidean arrangement of hyperplanes, $\mathscr{E}$. Let $X =$ the union of all bounded faces and $\mathscr{H} = \{h \cap X : h \in \mathscr{E}\}$. The faces of $\mathscr{H}$ are precisely the bounded faces of $\mathscr{E}$. If it is shown that $\kappa(X) = 1$ when X is not void, the bounded faces can be counted by means of (B). Unfortunately no direct proof that avoids technical problems with topology has yet been formulated. The known proof that $\kappa(X) = 1$ depends on counting the faces first, then deducing (B) and setting $x = -1$. See [25, Sects. 2C, 3C, 4C, 5A] for further discussion and details.

4. General Position

Often when there is some restriction on a dissection the largest possible number of regions occurs when the cuts have *relative general position* in X: the intersection of any k cuts, $k \geqslant 1$, is either void or $(d-k)$-dimensional; or even *absolute general position*: every k cuts, $1 \leqslant k \leqslant d+1$, have $(d-k)$-dimensional intersection. For instance a maximal dissection of Euclidean space by n hyperplanes has absolute general position. Relative general position is illustrated by the maximal dissections of a simplex introduced by Alexanderson and Wetzel [1, 2].[1]

THEOREM 4.1. *Suppose X is a d-dimensional space, dissected into properly cellular faces by cuts having relative general position. Let $a_k =$ the number of k-dimensional $T \in \mathscr{I}^c$, $\delta_{kj} =$ the Kronecker delta.*

(A) *If every $T \in \mathscr{I}^c - \{X\}$ is an open cell,*

$$\sum_{k=0}^{d} x^{d-k} f_k = \sum_{j=0}^{d} (x+1)^{d-j} a_j + (-1)^d \kappa(X) - 1,$$

and

$$f_k = \sum_{j=0}^{k} \binom{d-j}{d-k} a_j + \delta_{kd}[(-1)^d \kappa(X) - 1].$$

[1] I am grateful to John Wetzel for stimulating and clarifying many of the ideas in this section.

(B) *If every $T \in \mathscr{I}^c - \{X\}$ has the same Euler number*, κ_0,

$$\sum_{k=0}^{d} x^{d-k} f_k = \kappa_0 \sum_{j=0}^{d} (-1)^j (x+1)^{d-j} a_j + (-1)^d [\kappa(X) - \kappa_0],$$

and

$$f_k = \kappa_0 \sum_{j=0}^{k} (-1)^j \binom{d-j}{d-k} a_j + \delta_{kd} (-1)^d [\kappa(X) - \kappa_0].$$

(C) *If every $T \in \mathscr{I}^c - \{X\}$ is a topological projective space*,

$$\sum_{k=0}^{d} x^{d-k} f_k = \sum_{\substack{j=0 \\ \text{even}}}^{d} (x+1)^{d-j} a_j + (-1)^d (\kappa(X) - \tfrac{1}{2}[1 + (-1)^d]),$$

and

$$f_k = \sum_{\substack{j=0 \\ \text{even}}}^{k} \binom{d-j}{d-k} a_j + \delta_{kd} (-1)^d (\kappa(X) - \tfrac{1}{2}[1 + (-1)^d]).$$

(D) *If every $T \in \mathscr{I}^c - \{X\}$ is a topological sphere or a point*,

$$\sum_{k=0}^{d} x^{d-k} f_k = 2 \sum_{\substack{j=2 \\ \text{even}}}^{d} (x+1)^{d-j} a_j + (x+1)^d a_0 + (-1)^d (\kappa(X) - [1 + (-1)^d]),$$

and

$$f_k = 2 \sum_{\substack{j=2 \\ \text{even}}}^{k} \binom{d-j}{d-k} a_j + \binom{d}{k} a_0 + \delta_{kd} (-1)^d (\kappa(X) - [1 + (-1)^d]).$$

Proof. As general position entails geometric intersection, Theorem 3.2 can be applied. It also follows from general position that every interval of $\mathscr{I}^c$ is a Boolean algebra, so the Möbius function is $\mu(U, T) = (-1)^{\dim T - \dim U}$. The rest of the proof is simple manipulations. ∎

A principal example of Theorem 4.1(A) is an open convex subset $K \subseteq \mathbf{E}^d$ dissected by hyperplanes which have general intersection in K. Since $\kappa(K) = (-1)^d$, the extra terms on the right of (A) sum to zero and one obtains the simplified formula

$$f_k = \sum_{j=0}^{k} \binom{d-j}{d-k} a_j . \tag{4.1}$$

For some applications see [13; 26, Sect. 5], and the end of this section.

Similarly (B) may be applied to the bounded faces of a dissection of $\mathbf{E}^d$ by hyperplanes which have relatively general position (that is, no multiple intersections) and whose smallest-dimensional nonvoid intersection is a point (which rules out "degenerate" exceptions). Let X be the ***bounded part*** (the union of all the bounded faces) of the dissection, as described in Section 3. From [25, Corollary 5.2] it is known that $\kappa(X) = 1$; also $\kappa(T) = \kappa_0 = 1$ since every T is the bounded part of an induced dissection. This gives us formulas for b_k, the number of bounded k-faces of an arrangement of hyperplanes with relatively general position:

$$b_k = \sum_{j=0}^{k} (-1)^j \binom{d-j}{d-k} a_j . \tag{4.2}$$

The method of "sweep hyperplanes," an elegant idea of several authors that shows geometrically why (4.1) and (4.2) hold true, is described in articles by Alexanderson, Kerr, and Wetzel [3, 13] and Greene and Zaslavsky [9].

There are two special cases of particular interest. Suppose there are n cut hyperplanes, which have absolutely general position in K; then $a_j = \binom{n}{d-j}$ in (4.1) and (4.2). Letting $K = \mathbf{E}^d$, we have well-known formulas for the numbers of faces of a simple arrangement of hyperplanes—which have been periodically rediscovered since Steiner found them for $d \leqslant 3$ in 1826 and Schläfli obtained them for all dimensions a quarter century later. (For lists of references see Grünbaum, [10, Chap. 18; 11].)

A less restricted situation was also considered by Steiner (for $d = 2$ and 3, as usual). In a Euclidean arrangement with relatively general position the hyperplanes fall into parallel pencils of $n_1, n_2, \ldots, n_p$ cuts. Suppose a set consisting of one cut from each pencil has absolutely general position. Then $a_j = \sigma_{d-j}$, the $(d-j)$th symmetric polynomial in $n_1, \ldots, n_p$. The numbers of faces and bounded faces are again given by (4.1) and (4.2). These numbers are the maxima for Euclidean arrangements composed of p pencils with $n_1, \ldots, n_p$ cuts.

Steiner went on to discuss maximal dissections of the plane by circles and straight lines, of the sphere by circles, and of space by spheres and planes. One could generalize his results by means of (4.1) and (4.2), but there are geometrical problems, concerning the existence of intersections and the topology of faces, which it would be out of place to try to deal with here.

Theorem 4.1(B) may be considered a generalization to dissections of the Dehn–Sommerville equations for simple polytopes (cf. [10, Sect. 9.2]). A polytope is called "simple" when each vertex lies in exactly d facets. The Dehn–Sommerville equations state that, if P is simple and has f_k k-faces,

$$f_k = \sum_{j=0}^{k} (-1)^j \binom{d-j}{d-k} f_j , \qquad 0 \leqslant k \leqslant d. \tag{4.3}$$

Since the dissection of P whose cuts are the closed facets has for cut intersections the closed faces of P, $a_k = f_k$. Also $\kappa(P) = \kappa_0 = 1$. With these substitutions

(B) becomes precisely (4.3). And from the generating function in 4.1(B) we obtain the generating-function form of the Dehn–Sommerville equations (cf. [15, p. 101]).

5. Counting Low-Dimensional Faces

When one wants to solve a dissection problem it is convenient to have the Möbius function worked out beforehand in terms of the incidences of the cut intersections. Thus we give here two expansions of μ (not requiring geometric intersection) which are especially useful in low dimensions and lead to previously published enumerations.

The first expansion is a theorem of P. Hall's [18, Sect. 3, Proposition 6], which relates $\mu(S, T)$ to the number of ordered chains between S and T. Let $c_i(S, T) =$ the number of such chains of length i (including S and T) in $\mathscr{L}$. Then

$$\mu(S, T) = \sum_{i \geqslant 1} (-1)^{i-1} c_i(S, T).$$

The enumerations that result are written out in full for arrangements of lines and planes in [25, Sect. 5C].

The second expression is tractable only for small d. One expands the recursive definition of μ. Let S_k, T_k, U_k denote k-dimensional members of $\mathscr{L}$. Let $a_j(U_l, T_k) =$ the number of S_j which lie between U_l and T_k. Assume that $U \subsetneqq T$ implies $\dim U < \dim T$. Then, for $U_l \subseteq T_k$,

$$\begin{aligned}
\mu(T_k, T_k) &= 1, \\
\mu(U_{k-1}, T_k) &= -1, \\
\mu(U_{k-2}, T_k) &= a_{k-1}(U_{k-2}, T_k) - 1, \\
\mu(U_{k-3}, T_k) &= a_{k-1}(U_{k-3}, T_k) - 1 - \sum_{U_{k-3} \subseteq S_{k-2} \subseteq T_k} [a_{k-1}(S_{k-2}, T_k) - 1].
\end{aligned} \tag{5.1}$$

From (5.1) one can compute $f_0, \ldots, f_3$ for a dissection if the Euler numbers are known. An instance is a hyperplanar dissection of a convex open subset of the Euclidean plane or 3-space: one substitutes (5.1) in Theorem 3.2(A). Letting $k = d = 2$ and 3, one obtains the formulas of Brousseau and Alexanderson, Kerr, and Wetzel for the number of regions [3, 13].

6. Spheres and Their Subspaces

The regions of a dissected d-sphere or a subspace can be counted without requiring them to be cells. The price is an added complexity in the enumeration.

The formula involves two kinds of properties: the incidences of the cuts, as reflected in the Möbius function and the Euler numbers of an Eulerian meet refinement of $\mathscr{I}$; and the intricacy with which they intertwine, indicated by the Betti numbers $\beta_q(A)$ which measure the connectivity of the complement A of the regions.

THEOREM 6.1. *Let a subspace X of S^d, $d \geqslant 2$, be dissected by a finite set $\mathscr{H}$ of subspaces such that $R = X - \bigcup \mathscr{H}$ is nonempty and open in S^d, and let $A = S^d - R$. Assume A is a topological polyhedron (i.e., triangulable). Then the number of regions into which $\mathscr{H}$ dissects X is*

$$m = (-1)^d \Big(\sum_U \mu(U, X)\, \kappa(U) - 1 + \beta_0(A) - \beta_1(A) + \cdots \pm \beta_{d-2}(A)\Big),$$

where the summation and Möbius function may be taken over any Eulerian meet refinement of $\mathscr{I}(\mathscr{H})$.

We assume A is a polyhedron in order to avoid topological complications. By assuming also that $S^d - X$ and the faces of the dissection are each a disjoint union of finitely many open topological cells, we can assure that $\mathscr{I}$, $\mathscr{I}^c$, and any meet refinement of $\mathscr{I}$ which is coarser than $\mathscr{I}^c$ are Eulerian (Lemma 1.1). In practice it should be clear when these conditions are met. For instance they are satisfied by a dissection of S^d by spheres and ellipsoids, or of $\mathbf{E}^d$ or a convex open subset by hyperplanes and spheres, and so on.

The *proof*, like the formula, has a combinatorial and a homological part. First we note by Theorem 1.2 that

$$\sum_U \mu(U, X)\, \kappa(U) = \kappa(R) = \kappa(S^d) - \kappa(A) = 1 + (-1)^d - \chi(A).$$

Since $m =$ the number of components of R, which is $\beta_0(R)$, Theorem 6.1 is equivalent to the assertion

$$\beta_0(R) - 1 = (-1)^{d-1}[\chi(A) - \beta_0(A) + \cdots \pm \beta_{d-2}(A)].$$

Since A is a proper subspace of S^d, the Betti numbers β_d and higher are all zero. Thus the right-hand side reduces to $\beta_{d-1}(A)$. When $d \geqslant 2$, Alexander duality and the triangulability of A give isomorphisms of augmented rational homology (cf. [7; 22, 6.2.17 and 6.1.11]):

$$H_0^{\#}(R) \cong H_1(S^d, R) \cong H^{d-1}(A) \cong \bar{H}^{d-1}(A) \cong H_{d-1}(A).$$

From this we conclude that $\beta_0(R) - 1 = \beta_{d-1}(A)$, proving the theorem. ∎

There are homology isomorphisms for other dimensions as well. In fact $H_i^{\#}(R) \simeq H_{d-1-i}^{\#}(A)$ for $0 \leqslant i \leqslant d-1$; consequently

$$\begin{aligned} \beta_0(A) &= \beta_{d-1}(R) + 1, \\ \beta_k(A) &= \beta_{d-1-k}(R) \qquad \text{for} \quad 1 \leqslant k \leqslant d-2. \end{aligned} \tag{6.1}$$

From (6.1) we can draw conclusions about the Betti numbers in Theorem 6.1 if we know something about the connectivities of the regions. If, in particular, the regions are known to be d-cells—as we assumed in Section 3—all the Betti numbers on the right sides of Eq. (6.1) are 0 and Theorem 6.1 reduces to the formula (3.1). (Nevertheless (3.1) is not a corollary, as we did not there assume $X \subseteq S^d$.)

Theorem 6.1 indicates why algebraic topology alone is not suitable for counting regions. It tells us that their number equals the rank of $H_{d-1}(A)$, the $(d-1)$st homology group of the union of the cuts (plus the complement of the dissected subspace of S^d), but not how to compute the group. And that is precisely where the greatest difficulty lies; for as the preceding paragraph suggests, one can be sure the group $H_{d-1}(A)$ is complicated, no matter what other simplifications may be possible. What is needed, and is provided by our combinatorial analysis, is a way of evaluating the rank of $H_{d-1}(A)$. One might ask whether the group itself is computable in a similar way. It appears to be so, but we do not pursue the matter here.

7. Dissections by Curves

Turning now to the two-dimensional case, we consider a Euclidean or projective plane, a sphere, or a domain, dissected by curves. Our results generalize classical enumerations for arrangements of lines and results of Steiner [23, Sects. 11, 12] on the maximum number of regions of a dissection of $\mathbf{E}^2$ or S^2 by straight lines and circles.

A *system of curves* in a 2-manifold X means a set of curves which satisfy two regularity conditions. First, each curve must be a closed subset of X, homeomorphic either to an interval (an *open curve*) or a circle (a *closed curve*) except that it may intersect itself at a finite number of points. Also, two curves of the system, if they meet at all, may do so only at finitely many points.

Each open curve has two ends. Let us look at an end. If it has an end point lying in X, we call it a *free end* of the curve. If it has no end point in X, we call it a ***bound end***. A curve looks like a closed interval at a free end and like an open interval at a bound end. (The terminology becomes clearer if one thinks of X as embedded in a compact manifold like the sphere, so a bound end has its end point, as it were, stapled fast to the complement of X while a free end is "at liberty" in X.)

A free end of an open curve, or any point where a curve intersects itself or another curve in the system, is called a *node*. A curve which touches no other in the system and either is closed or has both ends free is said to be *isolated*.

Let $\mathscr{C}$ be a system of curves. We need the following indices for $\mathscr{C}$:

l = the number of open curves,

c_0 = the number of isolated simple, closed curves,

e = the number of connected components of $\bigcup \mathscr{C}$,

e' = the number that contain only curves without bound ends,

and for $j \geqslant 0$,

v_j = the number of nodes which lie in exactly j branches of curves, not counting branches that end at the node.

By a *domain* we mean any open subset of $\mathbf{E}^2$ or S^2, except the whole sphere. For a domain D we need two additional indices. First, regard D, if planar, as a subspace of the Riemann sphere. Call a *hole* any component of $S^2 - D$. Given a system of curves in D, two holes are *equivalent* if they are connected by arcs of curves in the system. Define

p = the number of holes — the number of equivalence classes.

In the calculation we can ignore any isolated hole (which no curves touch). So p is well defined even if D has an infinite number of holes.

The second index is elementary. Put

k = the number of components of D.

If D is simply connected, $p = 0$. For the sphere, $k = 1$, $e' = e$, and $l = p = 0$. Note that an isolated curve is an edge.

THEOREM 7.1. *The number of regions into which a domain D or the sphere S^2 is dissected by a finite system of curves is*

$$m = k + e' + l + \sum_{j=2}^{\infty} (j - 1)\, v_j - v_0 - p.$$

The number of edges is $l + c_0 + \sum_1^\infty j v_j$.

Proof. First, the only domains we need to look at are the connected ones, where $k = 1$. The general case follows by summing the values of m for each component of D. All the indices add up correctly.

Next, for a connected domain D, we redefine it by incorporating all isolated holes, so D is finitely connected. None of the indices is altered.

By Theorem 6.1 with $X = D$ or S^2,

$$m = \sum_P \mu(P, X) - \sum_C \kappa(C) + \kappa(X) - 1 + \beta_0(A), \tag{7.1}$$

summed over points $P \in \mathscr{I}^c$ and curves $C \in \mathscr{C}$. Virtually by definition,

$$p = \beta_0(S^2 - X) - [\beta_0(A) - e'].$$

Since X is connected and open, $\beta_0(S^2 - X) = 2 - \kappa(X)$. Substituting this in the expression for p, and that in (7.1), yields

$$m = \sum_{P \in \mathscr{I}^c} \mu(P, X) - \sum_C \kappa(C) + 1 + e' - p. \tag{7.2}$$

Let us define four new indices for a node P and a curve C. First:

$c(P) =$ the number of curves through P, including curves that end at P,

whence $c(P) - 1 = \mu(P, X)$ by (5.1) if $P \in \mathscr{I}^c$ and $= 0$ by inspection if $P \in \bigcup \mathscr{C}$ but $\notin \mathscr{I}^c$. Next:

$d(P, C) =$ the number of branches of C through P, not counting branches that end at P,

$f(C) =$ the number of free ends of C,

$\gamma(C) = 0$ if C is closed, 1 if it is open.

We prove that

$$-\kappa(C) = \gamma(C) + \sum_{P \in C} [d(P, C) - 1], \tag{7.3}$$

summed over all nodes on C. Regard C as the image of a circle C' if C is a closed curve, of an interval C' closed at the free ends of C if C is an open curve. If we pull the nodes on C back into C', then C' is partitioned into $f_1(C') = f_1(C)$ edges and $f_0(C') = \sum_{P \in C} d(P, C) + f(C)$ nodes. Now we can calculate, using Euler's relation in C', $\kappa(C') = f(C) - \gamma(C)$ and

$$f_1(C) = f_0(C') - \kappa(C') = \sum_{P \in C} d(P, C) + \gamma(C), \tag{7.4}$$

from which (7.3) is evident by Euler's relation in C. We should note that (7.4) is not valid when C is an isolated simple, closed curve. Then (7.3) may be verified by inspection.

Let us substitute for $\mu(P, X)$ and $\kappa(C)$ in (7.2). We obtain

$$m = \sum_P [c(P) - 1] + l + \sum \sum_{P \in C} [d(P, C) - 1] + 1 + e' - p, \tag{7.5}$$

summed over nodes $P \in \bigcup \mathscr{C}$ and curves $C \in \mathscr{C}$. The sum of the summations in (7.5) is $\sum (j-1) v_j$. Thus we have the desired expression for m.

The number of edges is computed by summing (7.4) over all curves except the isolated closed, simple ones, then adding c_0 to count the latter. ∎

Besides such obvious special cases as forbidding free ends, which makes $v_0 = 0$, or having a connected, simply connected domain like $\mathbf{E}^2$, so $k - p = 1$, we might look at the case where there are no closed curves and no intersections except at the end points of curves. The system of curves is then merely a planar graph in D, whose edges are the curves and whose vertices are the nodes of the system plus the bound end points. The formula of Theorem 7.1 then simplifies immensely. Indeed there are v_0 nodes and l edges in the system and the number of faces of the graph is

$$m = l - v_0 + k + e' - p.$$

In the projective plane a closed curve not contractible within the plane we call *pseudo-open*. A projective line is pseudo-open; generally a pseudo-open simple curve has been called a *pseudoline* [14; 10, Sect. 18.3; 12]. For a system of curves in $\mathbf{P}^2$, let

$$e_0 = \begin{cases} 0 & \text{if there are no pseudo-open curves,} \\ 1 & \text{if there are any.} \end{cases}$$

THEOREM 7.2. *The number of regions into which the projective plane is dissected by a finite system of closed curves is*

$$m = e + 1 + \sum_{j=2}^{\infty} (j-1) v_j - e_0.$$

The number of edges is $c_0 + \sum_2^\infty j v_j$.

Proof. By pulling $\mathbf{P}^2$ back to S^2 and using Theorem 7.1. A pseudo-open curve pulls back to a single curve, a contractible curve to an antipodal pair. ∎

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