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Journal of Pure and Applied Algebra 212 (2008) 1849–1851
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

Erratum

Erratum to "On Heller lattices over ramified extended orders" [J. Pure Appl. Algebra 202 (2005) 55–71]

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Received 28 January 2007; received in revised form 28 August 2007
Available online 27 December 2007

Communicated by I. Reiten

We use the same notation as in [1]. Let $(K, \varphi) \supset (K', \varphi')$ be an extension of complete discrete valuation fields, and let $\mathcal{O}$ (resp. $\mathcal{O}'$) be the valuation ring of φ (resp. φ') with unique maximal ideal (π) (resp. (π')) and the residue class field $k = \mathcal{O}/(\pi)$ (resp. $k' = \mathcal{O}'/(\pi')$). Let Λ' be an $\mathcal{O}'$-order and set $\Lambda = \mathcal{O} \otimes_{\mathcal{O}'} \Lambda'$. We denote by $\bar{\Lambda}$ and $\bar{\Lambda}'$ the k-algebra $\Lambda/\pi\Lambda$ and the k'-algebra $\Lambda'/\pi'\Lambda'$, respectively. For a $\bar{\Lambda}$-module M, the kernel Z of the projective cover P of M viewed as a Λ-module is called the *Heller lattice* of M: $0 \to Z \to P \to M \to 0$ (exact).

In the argument for proving the indecomposability of certain Heller lattices, the author carelessly estimated that $p_1(Q) \oplus p_2(Q) = \langle \alpha_i + \beta_i \mid 1 \le i \le h \rangle \Lambda$ [1, Section 2, page 62, line 12]. (We only have that $p_1(Q) \oplus p_2(Q) \supseteq \langle \alpha_i + \beta_i \mid 1 \le i \le h \rangle \Lambda$ at present.) Thus the proof of Lemma 2.8 of [1] is not valid as it is. The aim of this note is to give a corrected proof of the following [1, Theorem 2.9].

Theorem. *Assume that the ramification index of φ over φ' is greater than or equal to 3, and $\bar{\Lambda}'/\mathrm{Rad}(\bar{\Lambda}')$ is separable. Suppose that a non-projective indecomposable $\bar{\Lambda}$-module M is realizable over k'. Then the Heller Λ-lattice Z of M is indecomposable.*

Throughout this note, we assume the hypotheses in the above theorem, so $\pi' \in \pi^3\mathcal{O}$ and $k \otimes_{k'} (\bar{\Lambda}'/\mathrm{Rad}(\bar{\Lambda}'))$ is semisimple and isomorphic to $\bar{\Lambda}/\mathrm{Rad}(\bar{\Lambda})$. We keep the original notation used in Sections 1, 2 of [1]. Let I' be an $\mathcal{O}'$-pure sublattice of Λ' such that $\mathrm{Rad}(\Lambda') = \pi'\Lambda' + I'$. Define an $\mathcal{O}$-pure sublattice I of Λ as follows:

$$I = \mathcal{O} \otimes_{\mathcal{O}'} I'.$$

Note that $\mathrm{Rad}(\Lambda) = \pi\Lambda + I$ by our assumption that $\mathrm{Rad}(\bar{\Lambda}) = k \otimes_{k'} \mathrm{Rad}(\bar{\Lambda}')$. Let M be a $\bar{\Lambda}$-module and suppose that M is realizable over k'. Let P be a projective cover of M regarded as a Λ-module. Let ℓ be the Loewy length of $P/\pi P$ viewed as a $\bar{\Lambda}$-module, and let d_i be the k-dimension of the $(i+1)$-st top $(PI^i + \pi P)/(PI^{i+1} + \pi P)$ of $P/\pi P$ for $0 \le i \le \ell - 1$.

Lemma 1 (*[1, Lemma 1.2]*). *P has an $\mathcal{O}$-basis $\bigcup_{0 \le i \le \ell-1} \{x_{i,j} \mid 1 \le j \le d_i\}$ satisfying the following conditions for each $0 \le i \le \ell - 1$:*

(i) *$\{x_{i,j} + (PI^{i+1} + \pi P) \mid 1 \le j \le d_i\}$ is a k-basis for the $(i+1)$-st top of $P/\pi P$;*

DOI of original article: 10.1016/j.jpaa.2005.02.008.
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doi:10.1016/j.jpaa.2007.10.015

(ii) $\{x_{i,j} \mid 1 \le j \le d_i\} \subset \mathcal{O}\langle x_{0,j} \mid 1 \le j \le d_0\rangle I^i$;
(iii) $x_{i,j}\Lambda \subset P_{[i]} + \pi' P$ *and* $x_{i,j} I \subset P_{[i+1]} + \pi' P (\subseteq P_{[i+1]} + \pi^3 P)$,

where $P_{[i]}$ $(0 \le i \le \ell - 1)$ *are* $\mathcal{O}$*-submodules of* P *defined as follows* [1, *Section* 2]*:*

$$P_{[i]} = \bigoplus_{s=i}^{\ell-1} \mathcal{O}\langle x_{s,t} \mid 1 \le t \le d_s\rangle.$$

We recall that P has another basis $\{a_i \mid 1 \le i \le m\} \cup \{b_i \mid 1 \le i \le n\}$ such that

$$Z := \pi P + \mathcal{O}\langle b_1, \ldots, b_n\rangle = \mathcal{O}\langle \pi a_1, \ldots, \pi a_m\rangle \oplus \mathcal{O}\langle b_1, \ldots, b_n\rangle$$

is the Heller lattice of M. See Proposition 2.2 of [1]. Also, put

$$Q = \pi P \quad \text{and} \quad Y = \mathrm{Rad}(Q) + \mathcal{O}\langle b_1, \ldots, b_n\rangle.$$

Note that Y is a Λ-submodule of Z [1, Lemma 2.3].

Now, we define a subset E_Y of the endomorphism ring $\mathrm{End}_\Lambda(Z)$ of Z by

$$E_Y = \{f \in \mathrm{End}_\Lambda(Z) \mid \mathrm{Im} f \subseteq Y\}.$$

Note that $\pi \mathrm{End}_\Lambda(Z) \subset E_Y$. From the following fact, E_Y is a two-sided ideal of $\mathrm{End}_\Lambda(Z)$.

Lemma 2 (*[1, Lemma 1.4]*). *For any* Λ*-endomorphism* f *of* Z*,* $f(Y) \subseteq Y$*.*

The following easy fact will be used later.

Lemma 3. *Let* $g : Q(= \pi P) \to Z$ *be a* Λ*-homomorphism. Then* g *extends (uniquely) to a* Λ*-endomorphism of* Z *if and only if* $g(\pi b_i) \in \pi Z$ *for all* $1 \le i \le n$*. In particular, a* Λ*-endomorphism* g *of* Q *extends (uniquely) to a* Λ*-endomorphism of* Z *if and only if* $g(\pi b_i) \in \pi Z$ *for all* $1 \le i \le n$*.*

Put $Q = \bigoplus_{i=1}^{h} e_i \varepsilon_i \Lambda$, where e_i $(1 \le i \le h)$ are generators of Q and ε_i $(1 \le i \le h)$ are primitive idempotents of Λ with $e_i \Lambda_\Lambda \cong \varepsilon_i \Lambda_\Lambda$. Let f be a Λ-endomorphism of Z and suppose that $f \in E_Y$. Then each $f(e_i)$ $(1 \le i \le h)$ can be written as

$$f(e_i) = f(e_i \varepsilon_i) = y_i + \pi q_i = y_i \varepsilon_i + \pi q_i \varepsilon_i$$

for some $y_i = y_i \varepsilon_i \in (P_{[1]} + \pi^2 Q) \cap Z$ and some $\pi q_i = \pi q_i \varepsilon_i \in \pi Q$ since both $(P_{[1]} + \pi^2 Q) \cap Z$ and πQ are Λ-submodules of Z. Define Λ-endomorphisms g and h of Z by $g(e_i) = y_i$ $(1 \le i \le h)$ and by $h(e_i) = \pi q_i$ $(1 \le i \le h)$, respectively. (Indeed, $h \in \mathrm{End}_\Lambda(Z)$ by Lemma 3 and so $g = f - h \in \mathrm{End}_\Lambda(Z)$.) Then $g, h \in E_Y$ and $f = g + h$. Note that g and h satisfy the following conditions:

$(*)$ $g(\pi x_{0,j}) = \sum_{1 \le t \le d_0} x_{0,t} \sigma_{j,t}$ for some $\sigma_{j,t} \in I + \pi^3 \Lambda$ $(1 \le j \le d_0)$;
$(**)$ $h(Q) \subseteq \pi Q$.

Lemma 4. $E_Y / \pi \mathrm{End}_\Lambda(Z)$ *is a nilpotent ideal of* $\mathrm{End}_\Lambda(Z)/\pi \mathrm{End}_\Lambda(Z)$*.*

Proof. If $g \in E_Y$ satisfies the condition $(*)$, then

$$g((P_{[i]} + \pi Z) \cap Z) \subseteq (P_{[i+1]} + \pi Z) \cap Z = (P_{[i+1]} \cap Z) + \pi Z$$

for $0 \le i \le \ell - 1$. Indeed, an element $z \in P_{[i]} \cap Z$ is written as $z = \sum_{1 \le j \le d_0} x_{0,j} \delta_j$ for some $\delta_j \in \mathcal{O}\langle I^i\rangle$ by Lemma 1(ii), and so we have

$$g(z) = \sum_{1 \le j,t \le d_0} \pi^{-1} x_{0,t} \sigma_{j,t} \delta_j \in \pi^{-1}(P_{[i+1]} + \pi^3 P) \cap Z$$

by Lemma 1 (iii) since $\sigma_{j,t}\delta_j \in \mathcal{O}\langle I^{i+1}\rangle + \pi^3 \Lambda$ and $g(z) \in Z$. Hence homomorphic images of compositions of ℓ Λ-endomorphisms of Z satisfying $(*)$ are contained in πZ. Moreover, if $g \in E_Y$ satisfies $(*)$, then $g \circ h$ and $h \circ g$ also satisfy $(*)$ for any $h \in E_Y$ satisfying the condition $(**)$. If both h_1 and h_2 in E_Y satisfy $(**)$, then $[h_1 \circ h_2](Q) \subseteq \pi^2 Q$ and $[h_1 \circ h_2](Z) \subseteq \pi Z$. Thus we see that ${E_Y}^{2\ell} \subseteq \pi \mathrm{End}_\Lambda(Z)$. $\square$

With regard to the decomposition

$$Z/\pi Z = (\mathcal{O}\langle \pi a_1, \dots, \pi a_m\rangle + \pi Z)/\pi Z \oplus (\mathcal{O}\langle b_1, \dots, b_n\rangle + \pi Z)/\pi Z \cong M \oplus \Omega M$$

in Proposition 2.2(2) of [1], i_M and $i_{\Omega M}$ denote the inclusions from M and ΩM to $Z/\pi Z \cong M \oplus \Omega M$ respectively, and p_M and $p_{\Omega M}$ denote the projections from $Z/\pi Z$ onto M and ΩM respectively. For $f \in \mathrm{End}_\Lambda(Z)$, define $\bar{f} \in \mathrm{End}_{\bar{\Lambda}}(Z/\pi Z)$ by $\bar{f}(z + \pi Z) = f(z) + \pi Z (z \in Z)$. The following holds from Lemma 2.

Lemma 5. *Let $f \in \mathrm{End}_\Lambda(Z)$. Then $f \in E_Y$ if and only if $\mathrm{Im}(p_M \circ \bar{f} \circ i_M) \subseteq \mathrm{Rad}(M)$. In particular, if $p_M \circ \bar{f_1} \circ i_M = p_M \circ \bar{f_2} \circ i_M$ for $f_1, f_2 \in \mathrm{End}_\Lambda(Z)$, then $f_1 - f_2 \in E_Y$.*

Lemma 6. *Let ψ be a $\bar{\Lambda}$-endomorphism of M. Then there exists a Λ-endomorphism f of Z such that $\psi = p_M \circ \bar{f} \circ i_M$.*

Proof. Note that $M \cong Q/\pi Z$ and $\pi Z = \pi Q + \mathcal{O}\langle \pi b_1, \dots, \pi b_n\rangle$. Consider

$$\begin{array}{ccccc} Q & \xrightarrow{\varpi} & Q/\pi Z & & \\ & & \downarrow{\scriptstyle\psi} & & \\ Q & \xrightarrow{\varpi} & Q/\pi Z & \longrightarrow & 0, \end{array}$$

where ϖ is a natural surjection. Since Q ($\cong P$) is projective, there exists a Λ-endomorphism f of Q with $\varpi \circ f = \psi \circ \varpi$. Then ψ coincides with the $\bar{\Lambda}$-endomorphism of $Q/\pi Z$ ($\cong M$) defined by mapping $q + \pi Z$ to $f(q) + \pi Z (q \in Q)$. Since $f(\pi b_i) \in \pi Z$ for all $1 \le i \le n$, f extends to a Λ-endomorphism of Z by Lemma 3, and the statement holds. □

By Lemma 6, for each $\bar{\Lambda}$-endomorphism ψ of M, we can choose a Λ-endomorphism f_ψ of Z such that $p_M \circ \bar{f_\psi} \circ i_M = \psi$. By Lemma 5, a map Ψ from $\mathrm{End}_{\bar{\Lambda}}(M)$ to $\mathrm{End}_\Lambda(Z)/E_Y$ is well defined by the rule

$$\Psi(\psi) = f_\psi + E_Y \quad (\psi \in \mathrm{End}_{\bar{\Lambda}}(M)).$$

Lemma 7. *Ψ is a surjective ring homomorphism.*

Proof. Since $\Psi(p_M \circ \bar{f} \circ i_M) = \overline{f + E_Y}$ for any $f \in \mathrm{End}_\Lambda(Z)$, Ψ is surjective.

Let $\psi, \mu \in \mathrm{End}_{\bar{\Lambda}}(M)$. Then $p_M \circ \overline{(f_\mu \circ f_\psi - f_{\mu\circ\psi})} \circ i_M = p_M \circ \bar{f_\mu} \circ (i_M \circ p_M + i_{\Omega M} \circ p_{\Omega M}) \circ \bar{f_\psi} \circ i_M - p_M \circ \bar{f}_{\mu\circ\psi} \circ i_M = \mu \circ \psi + p_M \circ \bar{f_\mu} \circ i_{\Omega M} \circ p_{\Omega M} \circ \bar{f_\psi} \circ i_M - \mu \circ \psi$ and we have $\mathrm{Im}(p_M \circ \overline{(f_\mu \circ f_\psi - f_{\mu\circ\psi})} \circ i_M) \subseteq p_M \circ \bar{f_\mu}(\Omega M)$. Since $\bar{f}(\Omega M) \subseteq \mathrm{Rad}(M) \oplus \Omega M$ for any $f \in \mathrm{End}_\Lambda(Z)$ by Lemma 2, we see that $f_\mu \circ f_\psi - f_{\mu\circ\psi} \in E_Y$ by Lemma 5 and Ψ is a ring homomorphism. □

Proof of Theorem. As M is indecomposable, $\mathrm{End}_{\bar{\Lambda}}(M)$ is local and so is $\mathrm{End}_\Lambda(Z)/E_Y$ by Lemma 7. Thus $\mathrm{End}_\Lambda(Z)$ is also local by Lemma 4. □

References

[1] S. Kawata, On Heller lattices over ramified extended orders, J. Pure Appl. Algebra 202 (2005) 55–71.