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On the diameters of commuting graphs

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Abstract

The commuting graph of a ring $\mathfrak{R}$, denoted by $\Gamma(\mathfrak{R})$, is a graph whose vertices are all non-central elements of $\mathfrak{R}$ and two distinct vertices x and y are adjacent if and only if $xy = yx$. Let D be a division ring and $n \geqslant 3$. In this paper we investigate the diameters of $\Gamma(M_n(D))$ and determine the diameters of some induced subgraphs of $\Gamma(M_n(D))$, such as the induced subgraphs on the set of all non-scalar non-invertible, nilpotent, idempotent, and involution matrices in $M_n(D)$. For every field F, it is shown that if $\Gamma(M_n(F))$ is a connected graph, then diam $\Gamma(M_n(F)) \leqslant 6$. We conjecture that if $\Gamma(M_n(F))$ is a connected graph, then diam $\Gamma(M_n(F)) \leqslant 5$. We show that if F is an algebraically closed field or n is a prime number and $\Gamma(M_n(F))$ is a connected graph, then diam $\Gamma(M_n(F)) = 4$. Finally, we present some applications to the structure of pairs of idempotents which may prove of independent interest.

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Keywords: Commuting graph; Diameter; Division ring; Idempotent

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1. Introduction

For a ring $\mathfrak{R}$, we denote the *center* of $\mathfrak{R}$ by $Z(\mathfrak{R})$. If X is either an element or a subset of $\mathfrak{R}$, then $C_{\mathfrak{R}}(X)$ denotes the *centralizer* of X in $\mathfrak{R}$. For each non-commutative ring $\mathfrak{R}$, we associate a graph, with the vertex set $\mathfrak{R}\backslash Z(\mathfrak{R})$ and join two vertices x and y if and only if $x \neq y$ and $xy = yx$. This graph has been introduced in [2], is called the *commuting graph* of $\mathfrak{R}$, and is denoted by $\Gamma(\mathfrak{R})$. If $\mathfrak{X}$ is a subset of $\mathfrak{R}$, then $\Gamma(\mathfrak{X})$ denotes the induced subgraph of $\Gamma(\mathfrak{R})$ on $\mathfrak{X}\backslash Z(\mathfrak{R})$; that is the subgraph of $\Gamma(\mathfrak{R})$ with vertex set $\mathfrak{X}\backslash Z(\mathfrak{R})$. If D is a division ring and m, n are natural numbers, then we denote the set of all $m \times n$ matrices over D and the ring of all $n \times n$ matrices over D by $M_{m\times n}(D)$ and $M_n(D)$, respectively, and for simplicity we put $D^n = M_{n\times 1}(D)$. We denote the group of all invertible matrices in $M_n(D)$ by $GL_n(D)$. For any i, j, $1 \leqslant i, j \leqslant n$, we denote by E_{ij}, that element in $M_n(D)$ whose (i, j)-entry is 1 and whose other entries are 0. Also 0, I, 0_r, and I_r denote the zero matrix, the identity matrix, the zero matrix of size r, and the identity matrix of size r, respectively. A matrix $E \in M_n(D)$ is called *idempotent* if $E^2 = E$. Also a matrix $T \in M_n(D)$ is called an *involution* if $T^2 = I$. For any matrix $X \in M_{m\times n}(D)$, we denote the transpose of X by X^t. Moreover, for any two matrices $X \in M_{m\times n}(D)$ and $Y \in M_{r\times s}(D)$, we define

$$X \oplus Y = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \in M_{(m+r)\times(n+s)}(D).$$

For any field F and matrices $A, B, A', B' \in M_n(F)$, a pair $\{A, B\}$ is said to be *similar* to a pair $\{A', B'\}$ if there is a matrix $P \in GL_n(F)$ such that $A' = PAP^{-1}$ and $B' = PBP^{-1}$. We say that $\{A, B\}$ is *triangularizable* if there exists a matrix $P \in GL_n(F)$ such that PAP^{-1} and PBP^{-1} are upper triangular. Also a pair $\{A, B\}$ is said to be *irreducible* if every invariant subspace of $\{A, B\}$ is equal to $\{0\}$ or F^n. In this paper, a matrix $A \in M_n(D)$ is called *cyclic* if there is a vector $\alpha^t \in D^n$ such that $\{\alpha, \alpha A, \ldots, \alpha A^{n-1}\}$ is a basis for $M_{1\times n}(D)$ as a left vector space over D. Indeed, the representation of A in the above basis has the following form:

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ a_1 & a_2 & \ldots & a_{n-1} & a_n \end{bmatrix} \qquad (\dagger)$$

for some $a_1, \ldots, a_n \in D$. If $a_1 = \cdots = a_n = 0$, then the above matrix is denoted by J. For any matrix $A \in M_n(D)$, $\mathscr{L}_A$ and $\mathscr{R}_A$ denote the left multiplication and the right multiplication transformations of D^n and $M_{1\times n}(D)$ by A, respectively. We use nullity$A$ for $\dim \operatorname{Ker} \mathscr{L}_A = \dim \operatorname{Ker} \mathscr{R}_A$. Let D be a division ring with center F. Then for any matrix $A \in M_n(D)$, $F[A]$ denotes the F-subalgebra generated by A.

In a graph G, a *path* $\mathscr{P}$ is a sequence of distinct vertices $v_1 - v_2 \cdots - v_{k+1}$ in which every two consecutive vertices are adjacent. The number k is called the *length* of $\mathscr{P}$. For two vertices u and v in a graph G, the distance between u and v, denoted by $d(u, v)$, is the length of the shortest path between u and v, if such a path exists; otherwise we define $d(u, v) = \infty$. The *diameter* of a graph G is defined

$$\operatorname{diam} G = \sup \big\{ d(u, v) \mid u \text{ and } v \text{ are distinct vertices of } G \big\}.$$

Moreover, a graph G is called *connected* if there exists a path between every two distinct vertices of G.

In this article, we denote the set of all non-invertible, nilpotent, idempotent, and involution matrices in $M_n(D)$ by $\mathscr{A}_n, \mathscr{N}_n, \mathscr{E}_n$, and $\mathscr{I}_n$, respectively. In [3] it is shown that the graphs $\Gamma(\mathscr{A}_n)$, $\Gamma(\mathscr{N}_n)$, $\Gamma(\mathscr{E}_n)$, $\Gamma(\mathscr{I}_n)$ are connected. Here we find the diameters of these graphs as follows:

(i) diam $\Gamma(\mathscr{A}_n) = 4$ for any $n \geqslant 3$;
(ii) diam $\Gamma(\mathscr{N}_3) = 5$ and diam $\Gamma(\mathscr{N}_n) = 4$ for each $n \geqslant 4$;
(iii) diam $\Gamma(\mathscr{E}_n) = 3$ for any $n \geqslant 3$;
(iv) diam $\Gamma(\mathscr{I}_n) = 3$ for every $n \geqslant 3$, if char $D \neq 2$; otherwise, diam $\Gamma(\mathscr{I}_3) \leqslant 5$ and diam $\Gamma(\mathscr{I}_n) \leqslant 4$ for every $n \geqslant 4$.

Note that according to Remarks 2–5 of [3], all the aforementioned commuting graphs for the case $n = 2$, fail to be connected for every division ring D.

2. Non-invertible matrices

In this section we would like to obtain the diameter of the induced subgraph on all non-invertible matrices in $M_n(D)$. We begin with the following lemma.

Lemma 1. *Let D be a division ring and $n \geqslant 2$. If $A \in M_n(D)$ is a cyclic matrix of the form (†), then for any matrix $B \in C_{M_n(D)}(A)$, there exists a polynomial $f(x) \in D[x]$ such that $B = f(A)$.*

Proof. Let $\alpha = [1\ 0\ \cdots\ 0]$ and B be an element of $C_{M_n(D)}(A)$. Since $\{\alpha, \alpha A, \ldots, \alpha A^{n-1}\}$ is a basis for $M_{1\times n}(D)$ as a left vector space over D, there are $d_0, \ldots, d_{n-1} \in D$ such that $\alpha B = \sum_{i=0}^{n-1} d_i(\alpha A^i)$. We show that $B = \sum_{i=0}^{n-1} d_i A^i$. Since $AB = BA$,

$$(\alpha A^j)B = (\alpha B)A^j = \sum_{i=0}^{n-1} d_i(\alpha A^j)A^i$$

for any $j, 0 \leqslant j \leqslant n-1$. But all entries of αA^j are contained in $Z(D)$ for each $j, 0 \leqslant j \leqslant n-1$, so we have $(\alpha A^j)B = (\alpha A^j)\sum_{i=0}^{n-1} d_i A^i$. This completes the proof. □

Lemma 2. *Let D be a division ring and $n \geqslant 3$. Then $d(J, J^t) = 4$ in $\Gamma(\mathscr{A}_n)$.*

Proof. We show that if two non-invertible matrices $A \in C_{M_n(D)}(J)$ and $B \in C_{M_n(D)}(J^t)$ commute, then at least one of them is scalar. By Lemma 1, there exist $\alpha_0, \ldots, \alpha_{n-1} \in D$ such that

$$A = \begin{bmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_{n-2} & \alpha_{n-1} \\ 0 & \alpha_0 & \alpha_1 & \cdots & \alpha_{n-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & \alpha_0 & \alpha_1 \\ 0 & 0 & \ldots & 0 & \alpha_0 \end{bmatrix}.$$

Since A is a non-zero non-invertible matrix, there exists the minimum integer $r \geqslant 1$ such that $\alpha_r \neq 0$. So we may assume that

$$A = \begin{bmatrix} 0 & U \\ 0 & 0 \end{bmatrix}$$

for some matrix $U \in GL_{n-r}(D)$. Assume that $r \geqslant n/2$. If the matrix

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix} \in M_n(D),$$

where $X_{11}, X_{33} \in M_{n-r}(D)$, and $X_{22} \in M_{2r-n}(D)$, commutes with A, then by an easy calculation, using the invertibility of U, we find that X has the form

$$\text{(i)} \quad \begin{bmatrix} \bigstar & \bigstar & \bigstar \\ 0_{(2r-n)\times(n-r)} & \bigstar & \bigstar \\ 0_{n-r} & 0_{(n-r)\times(2r-n)} & \bigstar \end{bmatrix}.$$

If $r \leqslant n/2$, then using a similarity we obtain that any element of $C_{M_n(D)}(A)$ has the form

$$\text{(ii)} \quad \begin{bmatrix} \bigstar & \bigstar & \bigstar \\ 0_{(n-2r)\times r} & \bigstar & \bigstar \\ 0_r & 0_{r\times(n-2r)} & \bigstar \end{bmatrix}.$$

On the other hand, B^t commutes with J, so Lemma 1 yields that there exist $\beta_0, \ldots, \beta_{n-1} \in D$ such that

$$B = \begin{bmatrix} \beta_0 & 0 & \cdots & 0 & 0 \\ \beta_1 & \beta_0 & \cdots & 0 & 0 \\ \vdots & \beta_1 & \ddots & \vdots & \vdots \\ \beta_{n-2} & \vdots & \ddots & \beta_0 & 0 \\ \beta_{n-1} & \beta_{n-2} & \cdots & \beta_1 & \beta_0 \end{bmatrix}.$$

Now, if B has one of the forms (i) or (ii), then we have $\beta_1 = \cdots = \beta_{n-1} = 0$. This shows that $d(J, J^t) \geqslant 4$ in $\Gamma(\mathscr{A}_n)$. Since $J - E_{1n} - E_{22} - E_{n1} - J^t$ is a path in $\Gamma(\mathscr{A}_n)$, the proof is complete. □

Theorem 3. *Let F be a field and $n \geqslant 3$. If $\Gamma(M_n(F))$ is a connected graph, then* diam $\Gamma(M_n(F)) \geqslant 4$.

Proof. We show that $d(J, J^t) = 4$. To get a contradiction, assume that there is a path $J - A - B - J^t$ in $\Gamma(M_n(F))$. So A and B have the forms given in the proof of Lemma 2. Hence two matrices $A - \alpha_o I \in C_{M_n(F)}(J)$ and $B - \beta_0 I \in C_{M_n(F)}(J^t)$ commute. By Lemma 2, one of them is a scalar matrix, a contradiction. □

Lemma 4. *Let D be a division ring and $n \geqslant 2$. Suppose $A, B \in M_n(D)$ are two matrices such that* $\operatorname{Ker} \mathscr{L}_A \cap \operatorname{Ker} \mathscr{L}_B \neq \{0\}$ *and* $\operatorname{Ker} \mathscr{R}_A \cap \operatorname{Ker} \mathscr{R}_B \neq \{0\}$. *Then $C_{M_n(D)}(\{A, B\})$ contains at least one matrix with* rank 1.

Proof. By the hypothesis, there are non-zero elements $X, Y \in D^n$ such that $AX = BX = 0$ and $Y^t A = Y^t B = 0$. If we put $M = XY^t$, then we have $AM = MA = 0$ and $BM = MB = 0$. Since X and Y are non-zero, rank $M = 1$ and the proof is complete. □

Theorem 5. *Let D be a division ring and $n \geqslant 3$. If $\mathscr{A}_n$ is the set of all non-invertible matrices in $M_n(D)$, then* diam $\Gamma(\mathscr{A}_n) = 4$.

Proof. Suppose that A and B are two non-zero matrices in $\mathscr{A}_n$. Since A is non-invertible, there exist non-zero elements $X, Y \in D^n$ such that $AX = Y^t A = 0$. Let $A_1 = XY^t$. We have rank

$A_1 = 1$ and $AA_1 = A_1A = 0$. Similarly, we find a matrix $B_1 \in M_n(D)$ such that rank $B_1 = 1$ and $BB_1 = B_1B = 0$. Since A_1 and B_1 are rank 1 matrices, nullityA_1 + nullity$B_1 = 2n - 2 > n$. This implies that Ker $\mathscr{L}_{A_1} \cap$ Ker $\mathscr{L}_{B_1} \neq \{0\}$ and Ker $\mathscr{R}_{A_1} \cap$ Ker $\mathscr{R}_{B_1} \neq \{0\}$. By Lemma 4, there is a matrix $M \in C_{M_n(D)}(\{A_1, B_1\})$ with rank 1. Therefore $A - A_1 - M - B_1 - B$ is a path in $\Gamma(\mathscr{A}_n)$. Now Lemma 2 completes the proof. □

Theorem 6. *Let F be a field and $n \geqslant 3$. If $\mathscr{T}_n$ is the set of all triangularizable matrices in $M_n(F)$, then* diam $\Gamma(\mathscr{T}_n) = 4$.

Proof. Suppose that A and B are two non-scalar matrices in $\mathscr{T}_n$. Since A and B are triangularizable matrices, each of them has at least one eigenvalue in F. It means that there are scalars α, $\beta \in F$ such that $A - \alpha I$ and $B - \beta I$ are non-zero non-invertible matrices. Using the proof of Theorem 5, there is a path in $\Gamma(M_n(F))$ of length at most 4 between $A - \alpha I$ and $B - \beta I$ whose intermediate vertices are rank 1 matrices. Since each matrix of rank 1 is triangularizable, noting Theorem 3, the assertion is proved. □

Corollary 7. *Let F be an algebraically closed field and $n \geqslant 3$. Then* diam $\Gamma(M_n(F)) = 4$.

3. Nilpotent matrices

Theorem 8. *Let D be a division ring. If $\mathscr{N}_n$ is the set of all nilpotent matrices in $M_n(D)$, then* diam $\Gamma(\mathscr{N}_3) = 5$ *and* diam $\Gamma(\mathscr{N}_n) = 4$, *for any $n \geqslant 4$.*

Proof. Suppose that A and B are two non-zero matrices in $\mathscr{N}_n$. There are two matrices P, $Q \in GL_n(D)$ such that PAP^{-1} and QBQ^{-1} are upper triangular matrices whose diagonal entries are 0. Clearly, $E_{1n}(PAP^{-1}) = (PAP^{-1})E_{1n} = 0$ and $E_{1n}(QBQ^{-1}) = (QBQ^{-1})E_{1n} = 0$. Hence if we put $A' = P^{-1}E_{1n}P$ and $B' = Q^{-1}E_{1n}Q$, then we have $AA' = A'A = 0$ and $BB' = B'B = 0$. Furthermore, rank A' = rank $B' = 1$ imply that dim (Ker $\mathscr{L}_{A'} \cap$ Ker $\mathscr{L}_{B'}) \geqslant n - 2$. Assume that $n \geqslant 3$. Hence there is a matrix $T \in GL_n(D)$ such that the first columns of two matrices $TA'T^{-1}$ and $TB'T^{-1}$ are zero. So we have $(TA'T^{-1})E = (TB'T^{-1})E = 0$, where $E = [1\ \ 0\ \ \cdots\ \ 0]^t \in D^n$.

First, assume that $n \geqslant 4$. Since dim (Ker $\mathscr{R}_{A'} \cap$ Ker $\mathscr{R}_{B'}) \geqslant 2$, there is an element $X \in D^n$ whose first component is 0 and $X^t(TA'T^{-1}) = X^t(TB'T^{-1}) = 0$. Let $S = EX^t$. We have $(TA'T^{-1})S = S(TA'T^{-1}) = 0$ and $(TB'T^{-1})S = S(TB'T^{-1}) = 0$. Note that S is a non-zero nilpotent matrix, so $A - A' - T^{-1}ST - B' - B$ is a path in $\Gamma(\mathscr{N}_n)$. Now Lemma 2 shows that diam $\Gamma(\mathscr{N}_n) = 4$.

Next, suppose that $n = 3$. Since nullity $TA'T^{-1}$ = nullity $TB'T^{-1} = 2$, using the method used for Ker $\mathscr{R}_{A'} \cap$ Ker $\mathscr{R}_{B'}$ in the previous case, we find two elements $Y, Z \in D^3$ whose first components are 0, $Y^t(TA'T^{-1}) = 0$ and $Z^t(TB'T^{-1}) = 0$. Let $M = EY^t$ and $N = EZ^t$. We have $(TA'T^{-1})M = M(TA'T^{-1}) = 0$ and $(TB'T^{-1})N = N(TB'T^{-1}) = 0$. On the other hand, it is not hard to see that M and N are non-zero nilpotent matrices and $MN = NM = 0$. Hence

$$A - A' - T^{-1}MT - T^{-1}NT - B' - B$$

is a path in $\Gamma(\mathscr{N}_3)$. Now, we claim that $d(J, J^t) = 5$ in $\Gamma(\mathscr{N}_3)$. By the proof of Lemma 2, every nilpotent matrix that commutes with a matrix $H_1 \in C_{M_n(D)}(J)$ is strictly upper triangular and every nilpotent matrix that commutes with a matrix $H_2 \in C_{M_n(D)}(J^t)$ is strictly lower triangular.

This implies that $d(J, J^t) \geqslant 5$ in $\Gamma(\mathcal{N}_3)$, so the claim is established. Therefore diam $\Gamma(\mathcal{N}_3) = 5$, and the proof is complete. □

Theorem 9. *Let D be a division ring and $n \geqslant 3$. If $M, N \in M_n(D)$ are two non-zero matrices such that $M^2 = N^2 = 0$, then $d(M, N) \leqslant 2$ in $\Gamma(M_n(D))$.*

Proof. Clearly, nullity M and nullity N are more than or equal to $n/2$. If $\operatorname{Ker} \mathscr{L}_M \cap \operatorname{Ker} \mathscr{L}_N \neq \{0\}$ and $\operatorname{Ker} \mathscr{R}_M \cap \operatorname{Ker} \mathscr{R}_N \neq \{0\}$, then Lemma 4 establishes the assertion. So without loss of generality, suppose that $\operatorname{Ker} \mathscr{L}_M \cap \operatorname{Ker} \mathscr{L}_N = \{0\}$ (if $\operatorname{Ker} \mathscr{R}_M \cap \operatorname{Ker} \mathscr{R}_N = \{0\}$, then we consider M^t and N^t instead of M and N). It implies that $n = 2r$ for some integer $r \geqslant 2$, and nullity $M =$ nullity $N = r$. If $\mathscr{W}_1$ and $\mathscr{W}_2$ are two bases for $\operatorname{Ker} \mathscr{L}_M$ and $\operatorname{Ker} \mathscr{L}_N$, respectively, then $\mathscr{W}_1 \cup \mathscr{W}_2$ is a basis for D^n. Since $M^2 = N^2 = 0$, then using the basis $\mathscr{W}_1 \cup \mathscr{W}_2$, we find a matrix $P \in GL_n(D)$ such that

$$PMP^{-1} = \begin{bmatrix} 0 & M_1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad PNP^{-1} = \begin{bmatrix} 0 & 0 \\ N_1 & 0 \end{bmatrix}$$

for some $M_1, N_1 \in GL_r(D)$. Now for any non-scalar matrix $X \in C_{M_r(D)}(M_1N_1)$, we have $P^{-1}(X \oplus M_1^{-1}XM_1)P \in C_{M_n(D)}(\{M, N\}) \backslash FI$, as desired. □

4. Idempotent and involution matrices

Theorem 10. *Let D be a division ring and $n \geqslant 3$. If $\mathscr{E}_n$ is the set of all idempotent matrices in $M_n(D)$, then* diam $\Gamma(\mathscr{E}_n) = 3$.

Proof. First we prove the assertion for $n = 3$. Let A, B be two non-scalar matrices in $\mathscr{E}_3$. Without loss of generality, replacing an idempotent P by $I - P$ if necessary, assume that nullity A and nullity B are equal to 2. Hence dim $(\operatorname{Ker} \mathscr{L}_A \cap \operatorname{Ker} \mathscr{L}_B) \geqslant 1$. There exists a matrix $Q \in GL_3(D)$ such that

$$QAQ^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad QBQ^{-1} = \begin{bmatrix} 0 & R \\ 0 & S \end{bmatrix},$$

where $S \in M_2(D)$ is a non-scalar idempotent. Clearly, $RS = R$ and we have the path

$$A - Q^{-1}E_{11}Q - Q^{-1}(I_1 \oplus S)Q - B.$$

Now, suppose that $n \geqslant 4$ and A and B are two non-scalar matrices in $\mathscr{E}_n$. There are two matrices $P, Q \in GL_n(D)$ such that $A_1 = PAP^{-1} = I_r \oplus 0_{n-r}$ and $QBQ^{-1} = I_s \oplus 0_{n-s}$ for some $r, s \geqslant 1$. Thus B and $Q^{-1}E_{11}Q$ commute. So it is enough to prove that $C_{M_n(D)}(\{A_1, B_1\})$ contains at least one non-central idempotent, where $B_1 = P(Q^{-1}E_{11}Q)P^{-1}$. Assume that

$$B_1 = \begin{bmatrix} X & Y \\ Z & T \end{bmatrix},$$

where Y is an $r \times (n - r)$ matrix. Since rank $B_1 = 1$, rank X and rank T are at most 1. First assume that both of X and T are nilpotent. Hence $X^2 = 0$ and $T^2 = 0$. Idempotency of B_1 implies that $XY + YT = Y$. Thus $XY = Y(I - T)$ and since $I - T$ is invertible, $Y = XY(I - T)^{-1} = X^2Y(I - T)^{-2}$. Now since $X^2 = 0$, we have $Y = 0$. Similarly we obtain that $Z = 0$. Therefore

$B_1^2 = 0$, a contradiction. Without loss of generality, we may assume that X is not nilpotent. First, suppose that $r \geqslant 2$. Since rank $X = 1$, there is a matrix $U \in GL_r(D)$ which $UXU^{-1} = \lambda E_{rr}$ for some $\lambda \in D\backslash\{0\}$. Using similarity with the matrix $V = U \oplus I_{n-r}$, it is enough to show that there exists a non-scalar idempotent matrix which commutes with both A_1 and B_1', where

$$B_1' = VB_1V^{-1} = \begin{bmatrix} \lambda E_{rr} & UY \\ ZU^{-1} & T \end{bmatrix}.$$

Since rank $B_1' = 1$, the first row of UY and the first column of ZU^{-1} are zero. This implies that $V^{-1}E_{11}V \in C_{M_n(D)}(\{A_1, B_1\})$, as desired. Now, assume that $r = 1$. If T is not nilpotent, then since $n - 1 \geqslant 2$ by a similar argument we prove the assertion. Thus suppose that T is nilpotent. Since rank $T \leqslant 1$, there is a matrix $U' \in GL_{n-1}(D)$ which $U'TU'^{-1} = \mu E_{1(n-1)}$ for some $\mu \in D$. Using similarity with the matrix $V' = I_1 \oplus U'$, it is enough to show that there exists a non-scalar idempotent matrix which commutes with both A_1 and B_1'', where

$$B_1'' = V'B_1V'^{-1} = \begin{bmatrix} X & YU'^{-1} \\ U'Z & \mu E_{1(n-1)} \end{bmatrix}.$$

If $\mu = 0$, since rank $B_1'' = 1$, at most one of two matrices Y and Z is non-zero. Without loss of generality, suppose that $Y = 0$. Now, if $S \in M_{n-1}(D)$ is a non-zero idempotent matrix such that $SU'Z = 0$, then $0_1 \oplus S \in C_{M_n(D)}(\{A_1, B_1''\})$, as desired. If not, since rank $B_1'' = 1$, it is not hard to see that the third row and the third column of B_1'' are zero. This yields that $V'^{-1}E_{33}V' \in C_{M_n(D)}(\{A_1, B_1\})$, as desired.

To complete the proof, for each $n \geqslant 3$ we should find two matrices A and B whose distance in $\Gamma(\mathscr{E}_n)$ is equal to 3. Let

$$R = \sum_{i \text{ is odd}} E_{ii}, \quad S_1 = \sum_{i<n \text{ is odd}} E_{i(i+1)}, \quad \text{and} \quad S_2 = \sum_{i<n \text{ is even}} E_{i(i+1)}.$$

If we put $A = R + S_1$ and $B = R - S_2$, then with an easy calculation we find that A and B are idempotents and $A - B = S_1 + S_2 = J$. Assume that M is an idempotent matrix commutes with both A and B. Then M is also commutes with J and by Lemma 1, M is a polynomial in J. Thus M is an upper triangular matrix with the same diagonal entries. Hence all eigenvalues of M are the same and so $M = 0$ or I. This shows that $C_{M_n(D)}(\{A, B\})$ contains no non-scalar idempotent, so the proof is complete. □

Theorem 11. *Let D be a division ring and $n \geqslant 3$. If $A, B \in M_n(D)$ are two non-scalar idempotent matrices, then $d(A, B) \leqslant 2$ in $\Gamma(M_n(D))$.*

Proof. We have $A(I - A) = (I - A)A = 0$, so one of nullity A or nullity $(I - A)$ is at least $n/2$. Since $I - A$ is idempotent, without loss of generality, we may assume that nullity $A \geqslant n/2$ and similarly nullity $B \geqslant n/2$. If Ker $\mathscr{L}_A \cap$ Ker $\mathscr{L}_B \neq \{0\}$ and Ker $\mathscr{R}_A \cap$ Ker $\mathscr{R}_B \neq \{0\}$, then using Lemma 4, we find a non-scalar matrix in $C_{M_n(D)}(\{A, B\})$, as desired. So without loss of generality, suppose that Ker $\mathscr{L}_A \cap$ Ker $\mathscr{L}_B = \{0\}$ (if Ker $\mathscr{R}_A \cap$ Ker $\mathscr{R}_B = \{0\}$, then we consider A^t and B^t instead of A and B). This implies that $n = 2r$ for some integer $r \geqslant 2$, and nullity A = nullity $B = r$. If $\mathscr{W}_1$ and $\mathscr{W}_2$ are two bases for Ker $\mathscr{L}_A$ and Ker $\mathscr{L}_B$, respectively, then $\mathscr{W}_1 \cup \mathscr{W}_2$ is a basis for D^n. Since D^n = Ker $\mathscr{L}_A \oplus$ Im $\mathscr{L}_A$, then for any $\omega \in \mathscr{W}_2$, there are vectors $a \in$ Ker $\mathscr{L}_A$ and $a' \in$ Im $\mathscr{L}_A$ such that $\omega = a + a'$. So $A\omega = a' = -a + \omega$. Using the representation of A in the basis $\mathscr{W}_1 \cup \mathscr{W}_2$, we find a matrix $P \in GL_n(D)$ such that

$$PAP^{-1} = \begin{bmatrix} 0 & A' \\ 0 & I_r \end{bmatrix}$$

for some $A' \in M_r(D)$, and by a similar method, we conclude that

$$PBP^{-1} = \begin{bmatrix} I_r & 0 \\ B' & 0 \end{bmatrix}$$

for some $B' \in M_r(D)$. Now, if $A'B' \neq B'A'$, then $P^{-1}(A'B' \oplus B'A')P$ is a non-scalar element of $C_{M_n(D)}(\{A, B\})$. So assume that $A'B' = B'A'$. Hence there is a non-scalar matrix $S \in M_r(D)$ commuting with A' and B' and therefore $P^{-1}(S \oplus S)P$ is a non-scalar element of $C_{M_n(D)}(\{A, B\})$, and the proof is complete. □

Remark 12. The previous theorem shows that if D is a division ring with center F and $n \geqslant 3$, then $M_n(D)$ cannot be generated by any two idempotents as an F-algebra. This fact has been proved in [6, Theorem 4] and the above gives a new proof for it.

Theorem 13. *Let D be a division ring and $n \geqslant 3$. If $\mathscr{I}_n$ is the set of all involutions in $M_n(D)$, then the following hold*:

(i) *If* char $D \neq 2$, *then* diam $\Gamma(\mathscr{I}_n) = 3$.
(ii) *If* char $D = 2$, *then* diam $\Gamma(\mathscr{I}_3) \leqslant 5$ *and* diam $\Gamma(\mathscr{I}_n) \leqslant 4$ *for any $n \geqslant 4$.*

Proof. First, assume that char $D \neq 2$. Indeed, the matrix $A \in M_n(D)$ is a non-scalar involution if and only if $(A + I)/2$ is a non-scalar idempotent matrix. Hence by Theorem 10, the assertion given in (i) is proved.

Next, suppose that char $D = 2$. For any non-scalar $B \in \mathscr{I}_n$, we have $(B + I)^2 = 0$ and so $B + I$ is a non-scalar nilpotent matrix. Moreover, for any non-zero nilpotent matrix N, we know that there is a natural number k such that $N^k = 0$ and $N^{k-1} \neq 0$. If s is the least integer such that $2^s \geqslant k$, then $(N^{2^{s-1}} + I)^2 = I$. Therefore if we have a path in $\Gamma(\mathscr{N}_n)$, then we can find a path in $\Gamma(\mathscr{I}_n)$. Hence Theorem 8 completes the proof. □

5. Invertible matrices

The following theorems have been proved in [1] and [3], respectively.

Theorem A. *Let F be a field and $n \geqslant 3$. The graph $\Gamma(M_n(F))$ is connected if and only if for each cyclic matrix $A \in M_n(F)$, $F[A]\backslash FI$ contains at least one non-cyclic matrix.*

Theorem B. *Let D be a division ring with center F and $|F| \geqslant 3$, and let n be a natural number. Then $\Gamma(M_n(D))$ is a connected graph if and only if $\Gamma(GL_n(D))$ is a connected graph.*

Let D be a division ring with center F, and let n a natural number. The matrix $A \in M_n(D)$ is called *totally transcendental* over F if for any non-zero polynomial $f(t) \in F[t]$, $f(A)$ is an invertible matrix.

Now, we would like to obtain some relations between the diameter of commuting graph of invertible matrices and the diameter of commuting graph of the full matrix ring.

Theorem 14. *Let D be a division ring with center F such that $|F| \geqslant 3$ and n be a natural number. Then*

$$\operatorname{diam} \Gamma(GL_n(D)) \leqslant \operatorname{diam} \Gamma(M_n(D)) \leqslant \operatorname{diam} \Gamma(GL_n(D)) + 2.$$

Furthermore, if $D = F$ and $n \geqslant 3$, then

$$4 \leqslant \operatorname{diam} \Gamma(GL_n(F)) \leqslant \operatorname{diam} \Gamma(M_n(F)) \leqslant \operatorname{diam} \Gamma(GL_n(F)) + 1.$$

Proof. If $n = 1$, then there is nothing to prove. So we may assume that $n \geqslant 2$. By Theorem B, if $\Gamma(M_n(D))$ is non-connected, then so is $\Gamma(GL_n(D))$. In this case $\operatorname{diam} \Gamma(GL_n(D)) = \operatorname{diam} \Gamma(M_n(D)) = \infty$ and the result follows. So we may suppose that both of them are connected graphs. We show that for any non-invertible matrix A, there exists a polynomial $f(x)$ over F such that $f(A)$ is a non-scalar invertible matrix. First, suppose that A is not algebraic over F, then by [5, Proposition 8.3.1], A is similar to a matrix with form $A_0 \oplus A_1$, where A_0 is algebraic and A_1 is totally transcendental over F. By the fact that A is a non-invertible matrix, we have $A_0 \oplus A_1 \neq A_1$, and since A is not algebraic over F, $A_0 \oplus A_1 \neq A_0$. Let $g(x)$ be the minimal polynomial of A_0 over F. Thus $g(A_1) + I$ and so $g(A) + I$ is a non-scalar invertible matrix. Now, suppose that A is algebraic over F. Thus $F[A]$ is an Artinian ring. If there exists a nilpotent matrix $C \in F[A]$, then $I + C$ is a non-scalar invertible matrix. Otherwise, since the Jacobson radical of $F[A]$ is a nilpotent ideal, it is zero. Therefore by [4, Theorem 8.7, p. 90], $F[A]$ is a direct product of finitely many fields. Since $|F| \geqslant 3$, it is easily seen that there exists a non-scalar invertible matrix in $F[A]$, as desired. This shows that $\operatorname{diam} \Gamma(GL_n(D)) \leqslant \operatorname{diam} \Gamma(M_n(D))$. Now, suppose that $B,\ C \in M_n(D)\backslash FI$ are arbitrary. There are $h_1(x),\ h_2(x) \in F[x]$ such that $h_1(B)$ and $h_2(C)$ are non-scalar invertible matrices. Therefore $d(B, C) \leqslant \operatorname{diam} \Gamma(GL_n(D)) + 2$.

Next, suppose that D is commutative. By the proof of Theorem 3, $d(I + J, I + J^t) \geqslant 4$ in $\Gamma(GL_n(F))$. So, by the first part of the theorem, to prove the second part it is suffices to show that $\operatorname{diam} \Gamma(M_n(F)) \leqslant \operatorname{diam} \Gamma(GL_n(F)) + 1$. Assume that $E, G \in M_n(F)\backslash FI$ are arbitrary. If both of them are non-invertible, then by Theorem 5, $d(E, G) \leqslant 4$. If both of them are invertible, then the result clearly follows. So we may assume that one of them, for example E, is non-invertible. Since $\Gamma(M_n(F))$ is a connected graph, by Theorem A, there exists $H \in F[G]$ which is a non-cyclic non-scalar matrix. Assume that $H_1 \oplus \cdots \oplus H_k$ is the rational form of H, where for any i, $1 \leqslant i \leqslant k$, $H_i \in M_{n_i}(F)$ and $n_1 \geqslant \cdots \geqslant n_k$. Since $0 \oplus \cdots \oplus 0 \oplus I_k$ commutes with $H_1 \oplus \cdots \oplus H_k$, we find a matrix $K \in M_n(F)$ such that $\operatorname{rank} K \leqslant n/2$ and $d(G, K) \leqslant 2$. On the other hand, since E is a non-invertible matrix, by the proof of Theorem 5, it commutes with a rank 1 matrix, say L. Since $n \geqslant 3$, $\operatorname{Ker} \mathscr{L}_L \cap \operatorname{Ker} \mathscr{L}_K \neq \{0\}$ and $\operatorname{Ker} \mathscr{R}_L \cap \operatorname{Ker} \mathscr{R}_K \neq \{0\}$. Hence by Lemma 4, there exists a matrix $M \in C_{M_n(F)}(K) \cap C_{M_n(F)}(L)$ such that $\operatorname{rank} M = 1$. So we have the path $G - H - K - M - L - E$ and the proof is complete. □

Theorem 15. *Let D be a division ring with center F and $n \geqslant 2$. If $|F| > n$, then*

$$\operatorname{diam} \Gamma(GL_n(D)) = \operatorname{diam} \Gamma(M_n(D)).$$

Proof. Since $|F| > n$, [5, Theorem 8.2.3, p. 377] implies that for any matrix $X \in M_n(D)$, there is a scalar $\lambda_X \in F$ such that $X - \lambda_X I$ is invertible.

Now, suppose that R and S are two arbitrary distinct vertices of $\Gamma(M_n(D))$. If $\mathscr{P}$ is a path between $R - \lambda_R I$ and $S - \lambda_S I$ in $\Gamma(GL_n(D))$, then by replacing the vertices $R - \lambda_R I$ and $S - \lambda_S I$ in $\mathscr{P}$ with R and S, respectively, we conclude that $\operatorname{diam} \Gamma(M_n(D)) \leqslant \operatorname{diam} \Gamma(GL_n(D))$ and Theorem 14 completes the proof. □

6. Full matrix rings

The following theorem has been proved in [1].

Theorem C. *Let F be a field and $n \geqslant 3$. The graph $\Gamma(M_n(F))$ is connected if and only if every field extension of degree n over F contains at least one proper intermediate field.*

Lemma 16. *Let $A \in M_n(F)$ and $B \in M_m(F)$ be two matrices such that the minimal polynomial of A divides the minimal polynomial of B. Then the equation $AX = XB$ has at least one non-zero solution in $M_{n\times m}(F)$.*

Proof. Suppose that E is the algebraic closure of F. Since the minimal polynomial of A divides the minimal polynomial of B, A and B have at least one common eigenvalue in E. Since $X = 0$ is a solution of the equation $AX = XB$, by [7, Theorem 27.5.1], this equation has infinitely many solutions over E. Now, since $AX - XB = 0$ is a system of linear equations with coefficients in F which has a non-zero solution over E, it should have a non-zero solution over F. The proof is complete. □

Theorem 17. *Let F be a field and $n \geqslant 3$. If $\Gamma(M_n(F))$ is a connected graph, then* $\operatorname{diam} \Gamma(M_n(F)) \leqslant 6$.

Proof. By Theorem 9, it is enough to show that for every vertex A of $\Gamma(M_n(F))$, there is a vertex C such that $C^2 = 0$ and $d(A, C) \leqslant 2$. Since $\Gamma(M_n(F))$ is a connected graph, by Theorem A, there exists a non-cyclic matrix B in $F[A]\backslash FI$. Assume that $B_1 \oplus \cdots \oplus B_k$ is the rational form of B, where for any i, $1 \leqslant i \leqslant k$, B_i is a cyclic matrix of size n_i and $n_1 \geqslant \cdots \geqslant n_k$. Since the minimal polynomial of B_2 divides the minimal polynomial of B_1, by Lemma 16, there exists a non-zero matrix $B' \in M_{n_1\times n_2}(F)$ such that $B_1 B' = B' B_2$. So the matrix

$$C = \begin{bmatrix} 0 & B' \\ 0 & 0 \end{bmatrix} \oplus 0_{n-n_1-n_2}$$

commutes with B and its square is zero, as desired. □

Conjecture 18. *Let F be a field. If $\Gamma(M_n(F))$ is a connected graph, then its diameter is at most* 5.

In the next theorem we show that the conjecture is true when n is a prime number.

Theorem 19. *Let F be a field and $p \geqslant 3$ a prime number. If $\Gamma(M_p(F))$ is a connected graph, then* $\operatorname{diam} \Gamma(M_p(F)) = 4$.

Proof. Let M be an arbitrary matrix in $M_p(F)\backslash FI$. We show that M is adjacent to a matrix whose nullity is at least $(p+1)/2$. If M is a non-cyclic matrix, then using the rational form of M, we find a matrix with the desired property. So we may assume that M is a cyclic matrix. Let $f(x)$ be the minimal polynomial of M. Since $\Gamma(M_p(F))$ is a connected graph, by Theorem C, $f(x)$ is reducible. So there are non-scalar polynomials $f_1(x)$ and $f_2(x)$ in $F[x]$ such that $f(x) = f_1(x)f_2(x)$. Since $f_1(M)f_2(M) = 0$ and $f_1(M)$ and $f_2(M)$ are non-zero matrices, the nullity of at least one of them is not less than $(p+1)/2$.

Suppose that $A, B \in M_p(F)\backslash FI$ are two arbitrary matrices. There are $A', B' \in M_p(F)\backslash FI$ such that $AA' = A'A$ and $BB' = B'B$ and their nullities are at least $(p+1)/2$. Then Ker $\mathscr{L}_{A'} \cap$ Ker $\mathscr{L}_{B'} \neq \{0\}$ and Ker $\mathscr{R}_{A'} \cap$ Ker $\mathscr{R}_{B'} \neq \{0\}$. By Lemma 4, we find a common neighbor for A' and B', say S. So $A - A' - S - B' - B$ is a path in $\Gamma(M_p(F))$. Now Theorem 3 completes the proof. □

Theorem 20. *Let* $\mathbb{H}$ *be the division ring of real quaternions. Then* diam $\Gamma(M_2(\mathbb{H})) \leqslant 6$ *and* diam $\Gamma(M_n(\mathbb{H})) \leqslant 4$, *for all* $n \geqslant 3$.

Proof. Suppose that A and B are two vertices of $\Gamma(M_n(\mathbb{H}))$. By [9, Theorem 1], there are two matrices P, Q in $M_n(\mathbb{H})$ such that PAP^{-1} and QBQ^{-1} are contained in $M_n(\mathbb{C})$. Using the proof of Theorem 5, the vertices PAP^{-1} and QBQ^{-1} have neighbors of rank 1 in $\Gamma(M_n(\mathbb{C}))$. Hence there are two matrices $A_1, B_1 \in M_n(\mathbb{H})$ with rank 1 that commute with A, B, respectively. If Ker $\mathscr{L}_{A_1} \cap$ Ker $\mathscr{L}_{B_1} \neq \{0\}$ and Ker $\mathscr{R}_{A_1} \cap$ Ker $\mathscr{R}_{B_1} \neq \{0\}$, then by Lemma 4, there is a non-scalar matrix M that commutes with both A_1 and B_1. Therefore $A - A_1 - M - B_1 - B$ is a path in $\Gamma(M_n(\mathbb{H}))$, as desired. So without loss of generality, assume that Ker $\mathscr{L}_{A_1} \cap$ Ker $\mathscr{L}_{B_1} = \{0\}$ (if Ker $\mathscr{R}_{A_1} \cap$ Ker $\mathscr{R}_{B_1} = \{0\}$, then we consider A_1^t and B_1^t instead of A_1 and B_1). Since rank $A_1 =$ rank $B_1 = 1$, dim (Ker $\mathscr{L}_{A_1} \cap$ Ker $\mathscr{L}_{B_1}) \geqslant n-2$ and hence $n = 2$. Moreover, there is a matrix $U \in GL_2(\mathbb{H})$ such that

$$UA_1U^{-1} = \begin{bmatrix} 0 & a_1 \\ 0 & a_2 \end{bmatrix} \quad \text{and} \quad UB_1U^{-1} = \begin{bmatrix} b_1 & 0 \\ b_2 & 0 \end{bmatrix}$$

for some $a_1, a_2, b_1, b_2 \in \mathbb{H}$. Let $D = d_1 I$ for some $d_1 \in C_{\mathbb{H}}(\{a_1, a_2\})\backslash\mathbb{R}$, if $a_1a_2 = a_2a_1$; and otherwise, let $D = \mathrm{diag}(a_1a_2a_1^{-1}, a_2)$. Also let $D' = d_2 I$ for some $d_2 \in C_{\mathbb{H}}(\{b_1, b_2\})\backslash\mathbb{R}$, if $b_1b_2 = b_2b_1$; and otherwise, let $D' = \mathrm{diag}\,(b_1, b_2b_1b_2^{-1})$. Now,

$$A - A_1 - U^{-1}DU - U^{-1}E_{11}U - U^{-1}D'U - B_1 - B$$

is a path in $\Gamma(M_2(\mathbb{H}))$. This completes the proof. □

7. On the structure of pairs of idempotents

In this section we would like to obtain simple representations for pairs of idempotents in $M_n(F)$, for any field F and each integer $n \geqslant 2$. We start with three well-known results; we include short proofs for completeness.

Lemma 21. *Let F be an algebraically closed field and $n \geqslant 3$. Then every pair of idempotents in $M_n(F)$ has a non-trivial common invariant subspace.*

Proof. Assume that $\{A, B\}$ is a pair of idempotents in $M_n(F)$. By Theorem 11, there exists a non-scalar matrix M that commutes with both A and B. Since F is algebraically closed, there is $\lambda \in F$ such that $M - \lambda I$ is not invertible. Clearly, Ker $\mathscr{L}_{M-\lambda I}$ is an invariant subspace under A and B. This completes the proof. □

Lemma 22. *Let F be a field and $\{A, B\}$ an irreducible pair of idempotents in $M_2(F)$. Then there is an element $t \in F\backslash\{0, 1\}$ such that $\{A, B\}$ is similar to*

$$\left\{\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}t & t\\ 1-t & 1-t\end{bmatrix}\right\}.$$

Proof. Without loss of generality, we may assume that

$$A = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

for some $a, b, c, d \in F$. By irreducibility, we have $bc \neq 0$. Since B is not scalar, rank $B = 1$. This implies that $a + d = 1$ and $ad \neq 0$. Using the similarity effected by diag (a, b), we obtain that

$$B = \begin{bmatrix}a & a\\ c' & 1-a\end{bmatrix}$$

for some $c' \in F$. Since rank $B = 1$, we have $c' = 1 - a$, and the proof is complete. □

Corollary 23. *Let F be an algebraically closed field and $n \geqslant 2$. If A and B are two idempotents in $M_n(F)$, then there exists an integer $k \geqslant 0$ such that $\{A, B\}$ is similar to a pair of block upper triangular form matrices with diagonal blocks $\{A_i, B_i\}$, where for any i, $1 \leqslant i \leqslant k$,*

$$A_i = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} \quad \text{and} \quad B_i = \begin{bmatrix}t_i & t_i\\ 1-t_i & 1-t_i\end{bmatrix}$$

are matrices in $M_2(F)$ for some scalars $t_i \neq 0, 1$, and $\{A_i, B_i\} \subseteq \{0, 1\}$ for each $i \geqslant k + 1$.

Proof. If $n = 2$, then using Lemma 22, we are done. So assume that $n \geqslant 3$. By Lemma 21, $\{A, B\}$ has a non-trivial invariant subspace. Thus there are the idempotents A_1, A_2, B_1, B_2 whose sizes are less than n and $\{A, B\}$ is similar to

$$\left\{\begin{bmatrix}A_1 & \bigstar\\ 0 & A_2\end{bmatrix}, \begin{bmatrix}B_1 & \bigstar\\ 0 & B_2\end{bmatrix}\right\}.$$

Now, by induction the proof is complete. □

Lemma 24. *Let F be an algebraically closed field and $n \geqslant 2$. If $\{A, B\}$ is a pair of idempotents in $M_n(F)$ such that $d(A, B) = 3$ in $\Gamma(\mathscr{E}_n)$, then there exist a scalar $\lambda \in F$ and a nilpotent matrix N such that $(A - B)^2 = \lambda I + N$.*

Proof. Clearly, $S = (A - B)^2$ commutes with both A and B. Since $F[S] \subseteq C_{M_n(F)}(\{A, B\})$ and $d(A, B) = 3$ in $\Gamma(\mathscr{E}_n)$, $F[S]$ has no non-trivial idempotent. By [4, Theorem 8.7, p. 90], $F[S]$ is a local ring. Since F is an algebraically closed field, there exists a scalar $\lambda \in F$ such that $S - \lambda I$ is not invertible. Because $F[S]$ is an Artinian local ring and $S - \lambda I$ is contained in the Jacobson radical of $F[S]$, by [4, Corollary 8.2, p. 89], $S - \lambda I$ is a nilpotent matrix. This implies that $S = \lambda I + N$ for some nilpotent matrix N. □

Lemma 25. *Let F be a field and $\{A, B\}$ a pair of idempotents in $M_n(F)$, where $n \geqslant 2$. If $A - B$ is a nilpotent matrix, then $\{A, B\}$ is triangularizable.*

Proof. By McCoy's Theorem [8, Theorem 1.3.4, p. 8], we may assume that F is an algebraically closed field. For any $t \in F\backslash\{1\}$, the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} t & t \\ 1-t & 1-t \end{bmatrix}$$

is not nilpotent. Since $A - B$ is a nilpotent matrix, so all of the diagonal blocks A_i and B_i appearing in Corollary 23, are 0 or 1. This yields that $\{A, B\}$ is triangularizable, as desired. □

Corollary 26. *Let F be an algebraically closed field and $n \geqslant 3$ an odd integer. Then every pair of idempotents in $M_n(F)$ with distance 3 in $\Gamma(\mathscr{E}_n)$ is triangularizable.*

Proof. Without loss of generality, we may assume that nullity A and nullity B are at least $(n + 1)/2$. Thus $\operatorname{Ker} \mathscr{L}_A \cap \operatorname{Ker} \mathscr{L}_B \neq \{0\}$ and therefore $A - B$ is not invertible. By Lemma 24, $A - B$ is nilpotent and so Lemma 25 completes the proof. □

Theorem 27. *For every field F, the following are equivalent:*

(i) *F is an algebraically closed field.*
(ii) *For any $n \geqslant 3$, every pair of idempotents in $M_n(F)$ has a non-trivial common invariant subspace.*
(iii) *For any $n \geqslant 1$, every non-triangularizable pair of idempotents in $M_{2n}(F)$ with distance 3 in $\Gamma(\mathscr{E}_{2n})$, is similar to*

$$\left\{\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} M & M \\ I-M & I-M \end{bmatrix}\right\},$$

where $M = \lambda I + J \in M_n(F)$ and $\lambda \neq 0, 1$.

Proof. By Lemma 21, (i) implies (ii). For the other direction, suppose that F is not algebraically closed. Thus there is an irreducible polynomial $p(x)$ of degree $m \geqslant 2$ in $F[x]$. Let $S \in M_m(F)$ be the companion matrix of $p(x)$. Then we claim that the following pair of idempotents:

$$E_1 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad E_2 = \begin{bmatrix} S & S \\ I-S & I-S \end{bmatrix}$$

has no non-trivial common invariant subspace. Assume that $V \subseteq F^{2m}$ is a non-trivial common invariant subspace of E_1 and E_2. Let

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

be a non-zero vector in V, where $\alpha, \beta \in F^m$. We know that

$$E_1 \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \end{bmatrix} \quad \text{and} \quad (I - E_1) \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ \beta \end{bmatrix}$$

are two vectors in V. Thus without loss of generality, we may assume that $\alpha \neq 0$. For any $f(x) \in F[x]$, we have

$$f(E_1E_2) \begin{bmatrix} \alpha \\ 0 \end{bmatrix} = \begin{bmatrix} f(S)\alpha \\ 0 \end{bmatrix} \in V,$$

and since F^{2m} is irreducible as $F[S]$-module,

$$\begin{bmatrix} x \\ 0 \end{bmatrix} \in V,$$

for each $x \in F^m$. Now for any $x, y \in F^m$, we have

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - S(I-S)^{-1}y \\ 0 \end{bmatrix} + E_2 \begin{bmatrix} (I-S)^{-1}y \\ 0 \end{bmatrix} \in V,$$

a contradiction.

Next, we prove that (i) implies (iii). Suppose that $\{A, B\}$ is a pair of non-triangularizable idempotents in $M_{2n}(F)$ such that $d(A, B) = 3$ in $\Gamma(\mathscr{E}_{2n})$. We claim that $\operatorname{Ker} \mathscr{L}_A \cap \operatorname{Ker} \mathscr{L}_B = \{0\}$. To get a contradiction assume that $\operatorname{Ker} \mathscr{L}_A \cap \operatorname{Ker} \mathscr{L}_B \neq \{0\}$. So, Lemma 24 implies that $(A-B)^2$ is nilpotent. Now by Lemma 25, $\{A, B\}$ is triangularizable, a contradiction. Therefore $\operatorname{Ker} \mathscr{L}_A \cap \operatorname{Ker} \mathscr{L}_B = \{0\}$ and similarly $\operatorname{Ker} \mathscr{L}_A \cap \operatorname{Ker} \mathscr{L}_{I-B} = \{0\}$, $\operatorname{Ker} \mathscr{L}_{I-A} \cap \operatorname{Ker} \mathscr{L}_B = \{0\}$ and $\operatorname{Ker} \mathscr{L}_{I-A} \cap \operatorname{Ker} \mathscr{L}_{I-B} = \{0\}$. So we conclude that rank $A =$ rank $B = n$. Without loss of generality, we can assume that

$$A = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} X & Y \\ Z & T \end{bmatrix},$$

where $X, Y, Z, T \in M_n(F)$. We claim that Y is an invertible matrix. Suppose otherwise. Since B is idempotent, $XY + YT = Y$ and so $\operatorname{Ker} \mathscr{L}_Y$ is invariant under T. Thus $\{A, B\}$ is similar to

$$\left\{ \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} X & Y_1 & 0 \\ Z_1 & T_{11} & 0 \\ Z_2 & T_{21} & T_{22} \end{bmatrix} \right\},$$

where T_{22} is an idempotent matrix. Thus at least one of the subspace $\operatorname{Ker} \mathscr{L}_A \cap \operatorname{Ker} \mathscr{L}_B$ and $\operatorname{Ker} \mathscr{L}_A \cap \operatorname{Ker} \mathscr{L}_{I-B}$ is non-zero, a contradiction. By a similar argument, one can prove that Z is invertible. Now, using similarity by the matrix $Y^{-1} \oplus I_n$, we may assume that $Y = I$. Since $B^2 = B$, we find that

$$B = \begin{bmatrix} X & I \\ X - X^2 & I - X \end{bmatrix}.$$

Also, noting that X is necessarily invertible, and using similarity by the matrix $X \oplus I_n$, we may assume that B is equal to the matrix

$$\begin{bmatrix} X & X \\ I - X & I - X \end{bmatrix}. \qquad (*)$$

By Lemma 24, there are $\lambda \in F$ and nilpotent matrix N such that

$$(A-B)^2 = \begin{bmatrix} I - X & 0 \\ 0 & I - X \end{bmatrix} = \lambda I + N.$$

This yields that $X = (1-\lambda)I + N'$ for some nilpotent matrix $N' \in M_n(F)$. Since $\{A, B\}$ is not triangularizable and X is invertible, by Lemma 25, we have $\lambda \neq 0, 1$. To complete the proof, we must show that N' is a cyclic matrix. If N' is not cyclic, then using the rational form of N', we find a non-scalar idempotent $E \in C_{M_n(F)}(N')$. Now, by the form of B given in $(*)$, it is easily seen that $E \oplus E$ is a non-scalar idempotent matrix which commutes with both A and B, and this contradicts $d(A, B) = 3$ in $\Gamma(\mathscr{E}_{2n})$.

Finally, we prove that (iii) implies (i). To get a contradiction, suppose that F is not algebraically closed. We show that $\{E_1, E_2\}$ which was defined in the first step of the proof, is not triangularizable and $d(E_1, E_2) = 3$ in $\Gamma(\mathscr{E}_{2m})$. Since $F[S]$ is a field,

$$(E_1E_2 - E_2E_1)^2 = \begin{bmatrix} S^2 - S & 0 \\ 0 & S^2 - S \end{bmatrix}$$

is not a nilpotent matrix and so $\{E_1, E_2\}$ is not triangularizable. Moreover, if $d(E_1, E_2) \leqslant 2$ in $\Gamma(\mathscr{E}_{2m})$, then there exists a non-scalar idempotent matrix with the form $X \oplus Y$ commuting with E_2. Since S is a cyclic matrix, $C_{M_{2m}(F)}(S) = F[S]$ is a field. Thus X, Y are two idempotents in $F[S]$ and so $\{X, Y\} = \{0, I_m\}$. Since $XS = SY$, we conclude that $X = Y$, a contradiction. Now, we show that $\{E_1, E_2\}$ is not similar to

$$\left\{\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} M & M \\ I - M & I - M \end{bmatrix}\right\},$$

where $M = \lambda I + J \in M_m(F)$ and $\lambda \in F\backslash\{0, 1\}$. Indeed if it is, then there exists a matrix $P \in GL_n(F)$ such that $M = PSP^{-1}$. This yields that $x - \lambda$ divides $p(x)$, which contradicts the fact that $p(x)$ is an irreducible polynomial of degree $m \geqslant 2$. □

Remark 28. Note that each of the three statements in the previous theorem is equivalent to the assertion that every matrix $A \in M_n(F)$ has a non-trivial invariant subspace.

Theorem 29. *Let F be an algebraically closed field and $n \geqslant 2$. If A and B are two idempotents in $M_n(F)$, then there is an integer $k \geqslant 1$ such that $\{A, B\}$ is similar to*

$$\{A_1 \oplus \cdots \oplus A_k, B_1 \oplus \cdots \oplus B_k\},$$

such that for every i, $1 \leqslant i \leqslant k$, the pair $\{A_i, B_i\}$ is either upper triangularizable or equal to

$$\left\{\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} M_i & M_i \\ I - M_i & I - M_i \end{bmatrix}\right\},$$

where for each i, $n_i \geqslant 1$, $I = I_{n_i}$, $M_i = \lambda_i I + J \in M_{n_i}(F)$ and $\lambda_i \in F\backslash\{0, 1\}$.

Proof. By Lemma 22, the case $n = 2$ is easily verified. Assume that $n \geqslant 3$. If $d(A, B) = 1$ in $\Gamma(\mathscr{E}_n)$, then $\{A, B\}$ is triangularizable and we are done. Also if $d(A, B) = 3$ in $\Gamma(\mathscr{E}_n)$, then by Corollary 26 and Theorem 27, the assertion is proved. Thus assume that $d(A, B) = 2$ in $\Gamma(\mathscr{E}_n)$. This means that there exists a non-scalar idempotent matrix E commuting with both A and B. Indeed, E is similar to $I_r \oplus 0_{n-r}$ for some $r \geqslant 1$. Hence A and B are similar to $A_1 \oplus A_2$ and $B_1 \oplus B_2$, respectively, where $A_1, B_1 \in M_r(F)$ and $A_2, B_2 \in M_{n-r}(F)$ are idempotents. Now by induction, the proof is complete. □

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