Invariant sublattices for positive operators

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ABSTRACT

There are, by now, many results which guarantee that positive operators on Banach lattices have non-trivial closed invariant sublattices. In particular, this is true for every positive compact operator. Apart from some results of a general nature, in this paper we present several examples of positive operators on Banach lattices which do not have non-trivial closed invariant sublattices. These examples include both AM-spaces and Banach lattices with an order continuous norm and which are and are not atomic. In all these cases we can ensure that the operators do possess non-trivial closed invariant subspaces.

1. INTRODUCTION

After a long history of partial results, examples of operators which do not have non-trivial invariant closed subspaces were first produced by Enflo [7] and later examples on $\ell_1$ were produced by Read [11,12] and [13]. Following on from seminal work by de Pagter [4], Abramovich, Aliprantis and Burkinshaw in about 1992 commenced a series of papers devoted to the study of the conjecture that every positive operator on a Banach lattice of dimension at least two has a non-trivial closed invariant subspace. An account of their work may be found in Chapter 10 of [1].

There are many positive results to be found there, but as motivation for what follows we will restrict ourselves to one of the simplest examples, namely positive

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compact operators on Banach lattices. If such an operator has strictly positive spectral radius then the spectral radius is an eigenvalue and the corresponding eigenvector is positive. The linear span of such an eigenvalue is an invariant subspace and is a sublattice. If, on the other hand, the spectral radius is zero then the operator has a non-trivial closed invariant ideal. So, in particular, every positive compact operator on a Banach lattice has a non-trivial invariant closed sublattice.

This work arose out of the question of whether or not every positive operator on a Banach lattice does possess a non-trivial closed invariant sublattice. We had no hopes of proving their existence in general, but the examples that we have seem to put paid to any positive result that involves only a condition on the Banach lattice involved, apart from the obvious finite-dimensional case and the almost trivial case when the Banach lattice is non-separable, which we present in Section 2. The bulk of the material that we have consists of several examples in Sections 3 and 4. We present an example of a positive operator which possesses non-trivial invariant closed sublattices, even though it has no positive eigenvector and no non-trivial closed invariant ideal. In this case we can actually give a complete description of all the non-trivial invariant closed sublattices. After that we present several examples of positive operators which possess no non-trivial invariant closed sublattices. Those in Section 3 are all atomic whilst those in Section 4 have no atoms at all. Although the examples that we present are set in the context of real scalars, they remain valid for complex scalars. Finally, in Section 5 we propose some open problems.

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2. SOME "TRIVIA"

In the theory of invariant subspaces of bounded operators on Banach spaces, there are results that are virtually trivial. It is obvious that any bounded linear operator on a non-separable Banach space has a non-trivial closed invariant subspace as if $x \neq 0$ then the closed linear span of $\{T^nx: n = 0, 1, 2, \ldots\}$ is invariant and separable so non-trivial. This is still true for invariant sublattices of positive operators, although it needs a little more proof. Even the fact that a countably generated sublattice of a Banach lattice must be separable does not seem to have been previously recorded. Recall that if $A$ is a subset of a vector lattice $X$, then $^\wedge A$ is the set of all finite infima from $A$ and $^\vee A$ is the set of all finite suprema from $A$.

**Proposition 2.1.** If $X$ is a Banach lattice then the vector sublattice generated by any countable subset is separable.

**Proof.** If $\{f_m: n \in \mathbb{N}\}$ is a countable subset of $X$, let $A$ be the $\mathbb{Q}$-vector subspace generated by that set, which is countable. The set $^\wedge (^\vee A)$ is countable and, by arguments used in the proof of 2.2.11 of [9], is a $\mathbb{Q}$-vector space and sublattice. Its norm closure is an $\mathbb{R}$-vector sublattice of $X$ which certainly contains the vector sublattice generated by $\{f_m: n \in \mathbb{N}\}$, which is therefore separable. \[\Box\]
Corollary 2.2. If $X$ is any Banach lattice, $T$ a positive operator on $X$ and \{${f_n}: n \in \mathbb{N}$\} is any countable family in $X$ then the smallest $T$-invariant vector sublattice generated by that family is separable.

Proof. Let $H_1$ be the vector sublattice of $X$ generated by \{${f_n}: n \in \mathbb{N}$\} which, as we have just seen, is separable. If $H_n$ is defined and separable, let $A_n$ be a countable dense subset of $H_n$. Define $H_{n+1}$ to be the closed vector sublattice of $X$ generated by $A_n \cup TA_n$, which the preceding proposition guarantees to be separable. Clearly $TA_n \subseteq H_{n+1}$ and the facts that positive operators on Banach lattices are continuous and that $H_{n+1}$ is closed guarantees that $T(H_n) \subseteq H_{n+1}$. The set $\bigcup_{n=1}^{\infty} H_n$ is a vector sublattice of $X$ which is $T$-invariant and clearly separable, from which the result is immediate. \[\square\]

Corollary 2.3. If $X$ is a non-separable Banach lattice and $T$ a positive operator on $X$ then $T$ has a non-trivial closed invariant sublattice.

Proof. Pick any non-zero $x \in X_+$ then by the preceding corollary the smallest $T$-invariant sublattice containing $x$ is separable, as is its closure which is the smallest $T$-invariant closed sublattice containing $x$. This is proper as $X$ is non-separable. \[\square\]

We next look at the question of not-necessarily-closed sublattices. Again, in the linear case, every linear operator $T$ on a Banach space $X$ has a non-trivial linear subspace, not necessarily closed, simply because if we take any non-zero $x \in X$ then the linear span of the family $\{T^n x: n = 0, 1, 2, \ldots \}$ is of countable Hamel dimension whilst it follows from the Baire category theorem that Banach spaces are not of countable Hamel dimension. In fact Schaefer [15] has shown that this is true for every linear operator on any infinite-dimensional vector space. Neither Schaefer’s argument nor that outlined above can be used when dealing with invariant sublattices. For example the smallest sublattice generated even by two elements of a vector lattice need not have countable Hamel dimension. For instance in $C([0, 1])$ the lattice generated by $1$, the constantly one function, and $x$, defined by $x(t) = t$, consists of all continuous piecewise linear functions on $[0, 1]$. This contains the uncountable linearly independent family $\{x \wedge \lambda 1: \lambda \in (0, 1)\}$ (note that $x \wedge \lambda 1$ is non-differentiable precisely at $\lambda$).

In spite of this, it is possible to show that in any Banach lattice the smallest vector sublattice generated by a countable set is proper. Of course an attempt to apply this to a sequence $x, Tx, T^2x, \ldots$ will fail to produce an invariant sublattice in general. Things work out well if $T$ is a lattice homomorphism, and in fact all is well simultaneously for a finite family of such operators.

We extract part of the proof as a separate result.

Proposition 2.4. If $K$ is an infinite compact Hausdorff space, and \{${f_n}: n \in \mathbb{N}$\} a countable family in $C(K)$, then the vector sublattice generated in $C(K)$ by this family is a proper sublattice of $C(K)$. 

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Proof. Pick a non-isolated point \( p \in K \). Write each \( f_n = g_n + c_n1 \) where \( c_n \in \mathbb{R} \), \( g_n(p) = 0 \) and 1 is the constantly one function on \( K \). Let \( U = \{ k \in K : \exists n \in \mathbb{N} \text{ with } g_n(k) \neq 0 \} \). If \( p \notin \overline{U} \) then all the functions \( g_n \) are zero on the non-empty closed set \( K \setminus \overline{U} \). The sublattice \( H \) consisting of all functions which are constant on \( K \setminus \overline{U} \) is a proper vector sublattice containing each \( f_n \).

In the case that \( p \in \overline{U} \) let us form the sum

\[
e = \sum_{n=1}^{\infty} \frac{|g_n|}{\|g_n\|_\infty 2^n}
\]

(ignoring those \( g_n \) which are zero – if all are zero then all the \( f_n \) are constant and the result is trivial) and let \( J \) denote the principal ideal in \( C(K) \) generated by \( e \).

This ideal is not the whole of \( I = \{ f \in C(K) : f(p) = 0 \} \). To see this, consider the function \( \sqrt{e} \) which clearly lies in \( I \). As \( U = \{ k \in K : e(k) > 0 \} \), we see that it is possible to find a net \( u_\gamma \in U \) converging to \( p \) and with \( e(u_\gamma) \to 0 \). If \( \sqrt{e} \in J \) then there would be \( \lambda \in \mathbb{R} \) with \( \sqrt{e} \leq \lambda e \). Hence \( \sqrt{e(u_\gamma)} \leq \lambda e(u_\gamma) \) from which it follows that \( e(u_\gamma) \geq \lambda^{-2} > 0 \) which contradicts \( e(u_\gamma) \to 0 \).

Let \( M \) denote the linear span in \( C(K) \) of 1 and \( J \). We claim that \( M \) is a sublattice of \( C(K) \). Indeed, if \( j \in J \) and \( \alpha \in \mathbb{R} \) then consider the function \( |j + \alpha 1| \). Without loss of generality we may suppose that \( \alpha \geq 0 \). If \( j(k) \geq -\alpha \) then \( |j + \alpha 1|(k) = j(k) + \alpha \) so that \( (|j + \alpha 1| - \alpha 1)(k) = j(k) \). If, on the other hand, \( j(k) < -\alpha \) then \( |j + \alpha 1|(k) = -j(k) - \alpha \) so that \( (|j + \alpha 1| - \alpha 1)(k) = -j(k) - 2\alpha \) and hence \( |(j + \alpha 1| - \alpha 1)(k)| \leq |j(k)| + 2\alpha \leq 3|j(k)| \). Thus \( |j + \alpha 1| - \alpha 1| \leq 3|j| \) so that \( |j + \alpha 1| - \alpha 1 \in J \) and hence \( |j + \alpha 1| \in M \). As \( M \cap I = J \neq I \), \( I \not\subseteq M \) and therefore \( M \neq C(K) \). As each \( g_n \in J \), every \( f_n \in M \). Thus \( M \) is a proper vector sublattice of \( C(K) \) containing each \( f_n \), so the vector sublattice of \( C(K) \) generated by the \( f_n \) is not the whole of \( C(K) \).

Corollary 2.5. If \( X \) is an infinite-dimensional Banach lattice and \( \{ f_n : n \in \mathbb{N} \} \) a countable family in \( X \), then the vector sublattice generated in \( X \) by this family is a proper sublattice of \( X \).

Proof. Clearly we may assume that each \( f_n \neq 0 \). Take \( e = \sum_{n=1}^{\infty} \frac{f_n}{2^n \|f_n\|} \). The ideal generated by \( e \) contains each \( f_n \) and hence the sublattice generated by them. That ideal may be identified with some space \( C(K) \) so that Proposition 2.4 tells us that this sublattice is proper.

Recall that if \( A \) is a vector subspace of \( X \) then the vector sublattice generated by \( A \) is equal to \( \vee (\wedge A) = \wedge (\vee A) \), by 2.2.11 of [9].

Theorem 2.6. If \( T_k \) \((1 \leq k \leq m)\) are lattice homomorphisms on a Banach lattice \( X \), of dimension greater than 1, then there is a non-trivial not-necessarily-closed sublattice \( H \) of \( X \) which is invariant under each \( T_k \) and hence under any operator in the algebra generated by these \( T_k \).
Proof. Write \( T = \sum_{k=1}^{m} T_k \), pick any non-zero \( x \in X_+ \) and form the sum (omitting zero terms)

\[
y = \sum_{n=0}^{\infty} \frac{T^n x}{2^n \|T\|^n} \in X_+.
\]

Let \( J \) denote the principal lattice ideal generated in \( X \) by \( y \). Note that

\[
T_{n+1} x = 2^n \sum_{n=0}^{\infty} \frac{T^n x}{2^n \|T\|^n} = 2^n \sum_{n=0}^{\infty} \frac{T^n x}{2^n \|T\|^n} = 2^n \sum_{n=1}^{\infty} \frac{T^n x}{2^n \|T\|^n} \leq 2 \|T\| y,
\]

so the positivity of \( T \) makes it clear that \( J \) is invariant under \( T \) and hence under each \( T_k \). If \( J \neq X \) then our proof is complete.

As \( y \) is a strong order unit for \( J \), if \( J = X \) then the Kakutani representation theorem tells that, as a vector lattice, we may identify \( X \) with \( C(K) \) for some compact Hausdorff space \( K \). If \( K \) is a finite set then \( X \) is finite-dimensional and therefore \( T \) has a positive eigenvector \( x_0 \) with the corresponding eigenvalue being \( r(T) \), the spectral radius of \( T \), [1, Theorem 8.11]. The linear span of \( x_0 \) is an invariant sublattice of \( X \) and is proper as \( X \) has dimension greater than 1.

In the case that \( K \) is an infinite set, let \( \Pi \) denote the family of all finite products of the operators \( T_k \), which is certainly a countable set. Pick any \( 0 \neq x \in X_+ \) and let \( A \) be the linear span of the set \( \{\pi x: \pi \in \Pi\} \). By Proposition 2.4 the vector lattice generated by the set \( \{\pi x: \pi \in \Pi\} \) is not the whole of \( C(K) \). We claim that it is \( T_k \)-invariant for \( 1 \leq k \leq m \).

If \( a \in A \) then there are reals \( \alpha_j \) and \( \pi_j \in \Pi \), for \( 1 \leq j \leq n \) with \( a = \sum_{j=1}^{n} \alpha_j \pi_j x \). For each \( j \) and \( k \), \( T_k \circ \pi_j \in \Pi \) so that \( T_k(a) \in A \). I.e. \( T_k(A) \subset A \). As each \( T_k \) is a lattice homomorphism, if \( a_1, a_2, \ldots, a_n \in A \) then

\[
T_k(a_1 \lor a_2 \lor \cdots \lor a_n) = T_k(a_1) \lor T_k(a_2) \lor \cdots \lor T_k(a_n) \in \bigvee (T_k(A)) \subset \bigvee A
\]

so that \( T_k(\bigvee A) \subset \bigvee A \). Similarly we see that \( T_k(\bigwedge (\cdot) A) \subset \bigwedge (\cdot) A \).

Clearly this sublattice will be invariant under the action of any operator (not necessarily positive) in the algebra generated by the \( T_k \).

The class of operators for which we can guarantee the existence of a proper invariant (non-necessarily-closed) sublattice is quite small, but in some cases at least it will contain some familiar operators. For example, in the case of \( \ell_p \) \((1 \leq p < \infty) \) or \( c_0 \), it will contain all operators which when represented as a matrix have all non-zero entries restricted to a band (of arbitrary finite width) containing
the main diagonal, which is reminiscent of results in [5] and [8]. Of course, it will also contain matrices with non-zero entries restricted to a finite number of columns, or sums of these two types etc.

3. SOME DISCRETE EXAMPLES

We start with an example to show that in infinite dimensions it is possible for a positive operator to have a non-trivial invariant closed sublattice even when it has no positive eigenvector and no non-trivial invariant closed ideal. This shows at least that there is a slightly higher chance of finding non-trivial invariant closed sublattices than might have been expected. We remind the reader that closed sublattices of \( \ell_p(\mathbb{Z})\) (\(1 \leq p < \infty\)), or of \(c_0(\mathbb{Z})\) are all defined by a family of constraints of the form \(x_{m_i} = \alpha_i x_{n_i}\), with \(m_i, n_i \in \mathbb{Z}\) and \(\alpha_i \geq 0\), for some \(i\). In the case of \(c_0(\mathbb{Z})\) this follows from Theorem 3 of [10]. In the \(\ell_p\)-case, it follows from the fact that a closed sublattice of \(\ell_p(\mathbb{Z})\) also has a \(p\)-additive norm and is atomic. If all its atoms are also atoms in the original space it is the whole space, otherwise the constraint is clear on looking at an atom in the sublattice.

We precede the example with some lemmas.

**Lemma 3.1.** Let \(H\) be a closed sublattice of \(\ell_p(\mathbb{Z})\) (\(1 \leq p < \infty\)), or of \(c_0(\mathbb{Z})\) with the only constraints on its elements being of the form \(x_m = x_n\) for certain \(m, n \in \mathbb{Z}\). Suppose \(p, q, r, s \in \mathbb{Z}\) are distinct integers and \(x_p + x_q = x_r + x_s\) for all \(x \in H\) then one of the following holds for all \(x \in H\):

1. \(x_p = x_q = x_r = x_s\).
2. \(x_p = x_r\) and \(x_q = x_s\).
3. \(x_p = x_s\) and \(x_q = x_r\).

**Proof.** Let \(P : \ell_p(\mathbb{Z}) \to \mathbb{R}^4\) map \(x\) to \((x_p, x_q, x_r, x_s)\), which is a lattice homomorphism. \(P(H)\) is a vector sublattice of \(\mathbb{R}^4\) and the only possible constraints on its elements are of the form \(x_m = x_n\).

If \(x_p = x_q\) then \(2x_p = x_r + x_s\). This does not define a sublattice of \(\mathbb{R}^4\) so there must be a further constraint. If, for example, \(x_p = x_r\) then we also have \(x_p = x_s\), which implies we are in case (1). If \(x_r = x_s\) then \(2x_p = 2x_s\) so that again we are in case (1).

If there however there is a constraint of the form \(x_p = x_r\) then it follows that \(x_q = x_s\), so we are in case (2). Case (3) arises in a similar way. \(\square\)

**Lemma 3.2.** Suppose that \(x \in \ell_p(\mathbb{Z})\) (\(1 \leq p < \infty\)), or that \(x \in c_0(\mathbb{Z})\) and that there are \(m, n \in \mathbb{Z}\) with \(m > n\) such that \(x_{m+k} = x_{n+k}\) for all integers \(k \geq 0\) then \(x = 0\).

**Proof.** The sequence of values \(x_m, x_{m+1}, \ldots, x_{n-1}\) repeats indefinitely. Unless all these entries are zero, which is what we assert, this means that \(x\) has infinite norm! \(\square\)
Lemma 3.3. Suppose that $T$ is defined on $\ell_p(\mathbb{Z}) \ (1 \leq p < \infty)$, or on $c_0(\mathbb{Z})$ by $(Tx)_k = x_{k-1} + x_{k+1}$ and that $H$ is a closed $T$-invariant sublattice such that for some $m, n \in \mathbb{Z}$, with $m > n$ say, $x_m = x_n$ and $x_{m+1} = x_{n+1}$ then $H = \{0\}$.

Proof. Consider the statement that $x_{m+j} = x_{n+j}$ for all $x \in H$ and for $0 \leq j \leq k$. We know that this holds for $k = 1$. If we assume that it holds for some particular $k$ then $(TX)_{m+k} = (TX)_{n+k}$ so that $x_{m+k-1} + x_{m+k+1} = x_{n+k-1} + x_{n+k+1}$. We know that $x_{m+k-1} = x_{n+k-1}$ so that $x_{m+k+1} = x_{n+k+1}$. We thus have $x_{m+k} = x_{n+k}$ for all $k \in \mathbb{N}$. This can be similarly proved for negative $k$, after which we may appeal to Lemma 3.2. □

Example 3.4. Let $X = \ell_p(\mathbb{Z}) \ (1 \leq p < \infty)$ or $c_0(\mathbb{Z})$ and $T$ be the operator on $X$ defined by $Tx = (x_{n-1} + x_{n+1})$ then

1. $T$ has no positive eigenvector.
2. $T$ has no non-trivial closed invariant ideals.
3. The $T$-invariant closed sublattices of $X$ are precisely the sublattices

$$H_q = \{x \in X : x_m = x_n \text{ if } m + n = 2q\}$$

for either $q \in \mathbb{Z}$ or $q - \frac{1}{2} \in \mathbb{Z}$.

Proof. Let $S(x_n) = (x_{n+1})$, so that $S$ is the shift operator. Note that if $Sx = \lambda x$ then clearly $\lambda \neq 0$ whilst the fact that $x_{n+1} = \lambda x_n$ shows that (whether $|\lambda| \leq 1$ or $|\lambda| \geq 1$) $|x_n| \neq 0$ and hence $x \notin X$. Thus $S$ has no eigenvalues. Our operator $T$ is precisely $S + S^{-1}$. If $Tx = \lambda x$ then $STx = \lambda Sx$ so that $S^2x - \lambda Sx + x = 0$. I.e. $(S - \alpha_1)(S - \alpha_2)x = 0$ where $\alpha_1 = \frac{\lambda + \sqrt{\lambda^2 - 4}}{2}$ and $\alpha_2 = \frac{\lambda - \sqrt{\lambda^2 - 4}}{2}$. Either $x$ is an eigenvector for $S$ (with corresponding eigenvalue $\alpha_2$) or $(S - \alpha_2)x$ is an eigenvector for $S$ (with corresponding eigenvalue being $\alpha_1$). Either possibility is impossible, so that $T$ has no eigenvalues at all, let alone positive ones.

If $J$ is a non-trivial closed lattice $T$-invariant ideal in $X$ then there is $p \in \mathbb{Z}$ such that $x_p = 0$ for all $x \in J$. Let $A = \{n \in \mathbb{Z} : x_n = 0 \forall x \in J\}$, so that certainly $p \in A$. If $n \in A$ and $x \in J_+$ then $Tx \in J$ so that $Tx_n = x_{n-1} + x_{n+1} = 0$. As $x_{n-1}, x_{n+1} \geq 0$ this forces $x_{n-1} = x_{n+1} = 0$. This also holds for any $x = x^+ - x^- \in J$ so that $n - 1, n + 1 \in A$. By induction, $A = \mathbb{Z}$ so that $J = \{0\}$ and $J$ is trivial after all.

Each of the closed sublattices $H_p$ are $T$-invariant as if $x \in H_q, m > n$ and $m + n = 2q$ then also $(m + 1) + (n - 1) = \frac{m - 1}{m + 1}$ so that 

$$(Tx)_m = x_{m+1} + x_{m-1} = x_{n+1} + x_{n-1} = (Tx)_n$$

so that $Tx \in H_q$.

Now suppose that $H$ is a proper $T$-invariant closed sublattice of $X$ then there are $m, n \in \mathbb{Z}$ ($m \neq n$) and $\alpha > 0$ such that $x_m = \alpha x_n$ for all $x \in H$ (we know from the lack of proper closed invariant ideals that we do not need to worry about the possibility that $\alpha = 0$). We claim that for all $x \in H$ and for all integers $k \geq 0$ we
have $\sum_{j=m-k}^{n+k} x_j = \alpha \sum_{j=n-k}^{n+k} x_j$. This is clear for $k = 0$. If we assume that this does hold for $k$ then it holds in particular for $T x$. Thus

$$
\sum_{j=m-k}^{m+k} (T x)_j = \sum_{j=m-k}^{m+k} x_{j-1} + x_{j+1} = \sum_{j=m-k}^{m+k-1} x_j + \sum_{j=m-k+1}^{m+k} x_j = x_{m-k} + 2 \sum_{j=m-k}^{m+k} x_j + x_{m+k+1} = \alpha \left( x_{n-k} + 2 \sum_{j=n-k}^{n+k} x_j + x_{n+k+1} \right) = \alpha \sum_{j=n-k}^{n+k} (T x)_j.
$$

Subtracting the assumed equality for $k$ gives the required equality for $k + 1$.

Assume $n < m$ and that $k$ is large enough that $n - k < m - k < n + k < m + k$. Set $A_k = \sum_{j=m-k}^{n+k} x_j$, $B_k = \sum_{j=n+k+1}^{m+k} x_j$ and $C_k = \sum_{j=n-k}^{m-k} x_j$ so that $\sum_{j=m-k}^{m+k} x_j = A_k + B_k + C_k$ and $\sum_{j=n-k}^{n+k} x_j = A_k + C_k$. Thus $A_k + B_k = \alpha (A_k + C_k)$. In the case that $x \in \ell_p(\mathbb{Z})$ $(1 \leq p < \infty)$ for all $\eta > 0$ there is certainly $n_0$ such that $|x_j|^p < \eta$ for all $j > n_0$ whilst if $x \in c_0(\mathbb{Z})$ then we can find $n_0$ such that $|x_j| < \eta$ for all $j < n_0$. In either case, given $\epsilon > 0$ we can choose $n_0$ such that $|x_j| < \epsilon$ for $n < n_0$. If $k$ is large enough then $B_k$ consists of $m - n$ terms each with modulus at most $\epsilon$. If we take $\epsilon$ small enough then we can ensure that $|B_k|$ is arbitrarily small. Thus $B_k \to 0$ (and similarly $C_k \to 0$) as $k \to \infty$. Hence $A_k (1 - \alpha) = \alpha C_k - B_k \to 0$ as $k \to \infty$. If we take $x \in H_+$ then $A_k \uparrow$ as $k \to \infty$, so unless $\alpha = 1$ we have $|A_k (1 - \alpha)|$ increasing as well as tending to 0, so that $A_k = 0$ for all $k$. In that case $x = 0$ so that $H_+ = \{0\}$ and hence $H = \{0\}$.

Consider the collection $Q$ of all pairs of integers $(m, n)$ such that $m > n$ and $x_m = x_n$ for all $x \in H$. We claim that there is a pair $(m, n) \in Q$ with $n + 2 > m \geq n$. There is clearly a smallest difference $m - n$. We suppose that this smallest difference is attained at $(m, n)$ and that $m - n > 2$ and hence that $m - 1 > n + 1$. From the fact that $(Tx)_m = (Tx)_n$ for all $x \in H$ we see that $x_{m-1} + x_{m+1} = x_{n-1} + x_{n+1}$ for all $x \in H$. By Lemma 3.1 we have three possibilities. The first is that $x_{m-1} = x_{m+1} = x_{n-1} = x_{n+1}$ which would imply that $(m - 1, n + 1) \in Q$ contradicting the choice of $(m, n)$. In the second case $x_{m-1} = x_{n-1}$ and $x_{m+1} = x_{n+1}$ so that again $(m - 1, n + 1) \in Q$ giving a contradiction. In the final case, $x_{m-1} = x_{n-1}$ and $x_{m+1} = x_{n+1}$. Consider the claim that $x_{m+j} = x_{n+j}$ for $0 \leq j \leq k$, which we know for $k = 1$. If we assume this for $k$, then as $Tx \in H$,

$$(Tx)_{m+j} = x_{m+j+1} + x_{m+j-1} = x_{n+j+1} + x_{n+j-1} = (Tx)_{n+j}.$$
As \( x_{m+j-1} = x_{n+j-1} \) we have \( x_{m+j+1} = x_{n+j+1} \). This proves our claim for \( k + 1 \). A similar proof for negative \( k \) shows that \( x_{m+k} = x_{n+k} \) for all integers \( k \).

Now Lemma 3.2 tells us that \( x = 0 \) for all \( x \in H \) which contradicts the assumption that \( H \) is proper.

If we pick a pair \((m, n) \in Q\) for which \( m - n \) is minimal then there are two cases to consider. We look first at the case that \( m - n = 2 \). In this case, set \( q = (m + n)/2 \) then we have \( x_{q-1} = x_{q+1} \). We claim that \( x_{q-k} = x_{q+k} \) for all \( k \in \mathbb{N} \). Consider the statement that for all \( x \in H \), \( x_{q-j} = x_{q+j} \) for \( 1 \leq j \leq k \). We are starting with this statement for \( k = 1 \). If we assume this statement for \( k \) then using the fact that \( T x \in H \) we see that

\[
x_{q-k-1} + x_{q-k+1} = (Tx)_{q-k} = (Tx)_{q+k} = x_{q+k-1} + x_{q+k+1}
\]

as we also have \( x_{q-k+1} = x_{q+k-1} \) we see that \( x_{q-k-1} = x_{q+k+1} \), establishing the claim for \( k + 1 \). This shows that \( H \subseteq H_q \) in this case. If, on the other hand, \( m - n = 1 \) then \( x_{m+1} = x_m \) and an argument similar to the case for \( m - n = 2 \) shows that \( x_{m-k} = x_{m+1+k} \) for all \( x \in H \), showing that \( H \subseteq H_{m+1} \).

It remains to show that there are no other constraints on the elements of \( H \). We know that all constraints are of the form \( x_m = x_n \). We first deal with the case that \( H \subseteq H_{q+1} \). We have constraints of the form \( x_q = x_{q+1} = x_n \) with (without loss of generality) \( n > q + 1 \). As \((Tx) \in H\) we have \((Tx)_q = x_{q-1} + x_{q+1} = (Tx)_n = x_{n-1} + x_{n+1} \). If \( n > q + 2 \) then consider the possibilities given by Lemma 3.1. If \( x_{q-1} = x_{q+1} = x_{n-1} = x_{n+1} \) then \( x_q = x_{q+1} = x_n = x_{n+1} \) so Lemma 3.3 gives a contradiction. If \( x_{q-1} = x_{n-1} \) and \( x_{q+1} = x_{n+1} \) as well as \( x_q = x_n \) then again Lemma 3.3 gives a contradiction. If \( x_{q-1} = x_{n+1} \) and \( x_{q+1} = x_{n-1} \) as well as \( x_q = x_n \) then applying Lemma 3.3, using \( x_q = x_{q+1} = x_n = x_{n-1} \), gives a contradiction again. If \( n = q + 2 \) then we have \( x_q = x_{q+1} = x_{q+2} \) for all \( x \in H \), and a routine inductive argument now shows that \( x \) is a constant vector which is impossible unless \( x = 0 \). Thus \( H = H_{q+1} \) in this case.

In the case that \( H \subseteq H_q \), let us assume that we have \( x_{q-1} = x_{q+1} = x_n \) for all \( x \in H \). Assume first (again without loss of generality) that \( n > q + 2 \) then as \((Tx)_{q+1} = (Tx)_n \) we have \( x_q + x_{q+1} = x_{n-1} + x_{n+1} \) for all \( x \in H \). Using Lemma 3.1 we have three possibilities, (1) that \( x_q = x_{q+1} = x_{n-1} = x_{n+1} \), and as \( x_{q+1} = x_n \) we have \( x_q = x_{q+1} = x_n = x_{n+1} \) and we obtain a contradiction using Lemma 3.3. In case (2), \( x_q = x_{n-1} \) as well as \( x_{q+1} = x_n \) we can apply Lemma 3.3 again. In case (3) we have \( x_q = x_{n+1} \) as well as \( x_{q-1} = x_n \) so we can again apply Lemma 3.3. Finally, if \( n = q + 2 \) then we have \( x_{q-1} = x_{q+1} = x_{q+2} \) for all \( x \in H \). The arguments of the preceding paragraph have already eliminated this possibility. We now see that \( H = H_q \).

Of course, if we had taken \( X \) to be either \( \ell_{\infty}(\mathbb{Z}) \) or \( c(\mathbb{Z}) \) then \( T \) would have had a positive eigenvector, namely any constant sequence. Similarly \( c_0(\mathbb{Z}) \) would be a closed invariant ideal.
Note that if \( p \in \mathbb{Z} \) then elements of \( H_q \) are of the form
\[
(\ldots, x_2, x_1, x_0, x_1, x_2, \ldots)
\]
whilst if \( p - \frac{1}{2} \in \mathbb{Z} \) then they are of the form
\[
(\ldots, x_2, x_1, x_0, x_0, x_1, x_2, \ldots).
\]

It is not difficult to see that the intersection of any two of these invariant sublattices is \( \{0\} \). It follows that, for example, the restriction of \( T \) to \( H_0 \) has no invariant closed non-trivial sublattice. \( H_0 \) may be identified with \( \ell_\rho \) (where we index the entries by the non-negative integers) and \( T \) with the operator
\[
(Tx)_n = \begin{cases} 
  x_{n-1} + x_{n+1} & \text{if } n > 0, \\
  2x_1 & \text{if } n = 0.
\end{cases}
\]

This is almost the more familiar operator \( S + S^* \), where \( S \) is the unilateral shift.

We can give a direct self-contained proof of the non-existence of non-trivial closed invariant sublattices in that case.

**Example 3.5.** Let \( X = \ell_\rho \) (\( 1 \leq p < \infty \)) or \( c_0 \) and \( T \) be the operator on \( X \) defined by
\[
(Tx)_n = \begin{cases} 
  x_{n-1} + x_{n+1} & \text{if } n > 0, \\
  x_1 & \text{if } n = 0
\end{cases}
\]

then there is no non-trivial closed \( T \)-invariant sublattice of \( X \).

**Proof.** We start by showing that \( T \) does not have a positive eigenvector. Each element \( x \in X \) defines a function \( f(z) = \sum_{n=0}^{\infty} x_n z^n \) that is analytic on the open unit disk in the complex plane. The action of \( T \) on \( X \) corresponds to the mapping taking \( f \) to \( zf(z) + (f(z) - f(0))/z \). If \( x \) is positive and \( Tx = \lambda x \) then \( 2 \geq \lambda \geq 0 \), noting that \( \|T\| = 2 \). We have
\[
zf(z) + (f(z) - f(0))/z = \lambda f(z)
\]
so that
\[
f(z) = \frac{f(0)}{z^2 - \lambda z + 1}.
\]

Thus, taking \( f(0) = 1 \), the coefficients of the Taylor expansion about 0 of \( (z^2 - \lambda z + 1)^{-1} \) lie in \( X \). If \( \lambda = 2 \) then this expansion is \( \sum_{n=0}^{\infty} (n + 1) z^n \) and the coefficients are unbounded so certainly do not lie in \( X \). If \( \lambda < 2 \) then if we define \( \theta \) by \( \cos(\theta) = \lambda/2 \) and \( \sin(\theta) = \sqrt{1 - \cos(\theta)^2} \) and set \( w = \cos(\theta) + i \sin(\theta) \) then \( z^2 - \lambda z + 1 = (z - w)(z - \overline{w}) \). We then see that
\[
\frac{1}{z^2 - \lambda z + 1} = \frac{-i}{2\sin(\theta)} \left( \frac{1}{(z - w)} - \frac{1}{(z - \bar{w})} \right)
\]
\[
= \frac{i}{2\sin(\theta)} \left( \frac{\bar{w}}{1 - \bar{w}z} - \frac{w}{1 - wz} \right)
\]
\[
= \frac{i}{2\sin(\theta)} \left( \sum_{n=0}^{\infty} \bar{w}^n z^n - \sum_{n=0}^{\infty} w^n z^n \right)
\]
\[
= \frac{i}{2\sin(\theta)} \sum_{n=0}^{\infty} (\bar{w}^{n+1} - w^{n+1}) z^n
\]
\[
= \frac{i}{2\sin(\theta)} \sum_{n=0}^{\infty} \left[ (\cos((n+1)\theta) + i \sin((n+1)\theta)) - (\cos((n+1)\theta) - i \sin((n+1)\theta)) \right] z^n
\]
\[
= \frac{i}{2\sin(\theta)} \sum_{n=0}^{\infty} 2i \sin((n+1)\theta) z^n
\]

so the Taylor coefficients are \( \sin((n+1)\theta)/\sin(\theta) \) which (as \( \theta \neq 0 \)) does not converge to 0 and hence do not lie in \( X \). This establishes the absence of positive eigenvectors.

Now suppose that \( H \) is a closed non-trivial sublattice of \( X \) that is \( T \)-invariant. First we show that there is no \( m \) such that \( x_m = 0 \) for all \( x \in H \). If \( m = 0 \) then \( (Tx)_0 = x_1 = 0 \) for all \( x \in H \) so we may suppose that \( m > 0 \). If there were such an \( m \), then take \( x \in H_+ \) and observe that \( (Tx)_m = x_{m-1} + x_{m+1} = 0 \) so (as \( x \geq 0 \)) \( x_{m-1} = x_{m+1} = 0 \). This must also hold for all \( x = x^- = x^+ \in H \). Proceeding inductively we see that \( x_p = 0 \) for all non-negative integers \( p \) so that \( H = \{0\} \). Otherwise, if \( H \neq X \), there are \( m, n \geq 0 \) with \( m > n \) and \( \alpha > 0 \) such that \( x_m = \alpha x_n \) for all \( x \in H \). We claim that if \( x \in H \) and \( x_1, x_2, \ldots, x_m \) are known then \( x \) is specified uniquely. Consider the statement \( P(p) \) that we can express \( x_k \) uniquely as a linear combination of terms \( x_j \), with \( 1 \leq j \leq m \), for all \( k \leq p \). This is trivially true for \( p = m \). Let us assume \( P(p) \). Note that \( T^{p+1-m} x \in H \) as \( H \) is \( T \)-invariant and that \( (T^{p+1-m}(x))_m \) is a linear combination of \( x_k \) for \( k \leq p + 1 \) with the coefficient of \( x_{p+1} \) being 1. Similarly \( (T^{p+1-m}(x))_n \) is a linear combination of \( x_k \) for \( k \leq p + n + 1 - m \). As \( (T^{p+1-m}(x))_m = \alpha (T^{p+1-m}(x))_n \) we can solve for \( x_{p+1} \) in terms of \( x_k \) for \( k \leq p \) and hence express \( x_{p+1} \) as a linear combination of \( x_k \) for \( 1 \leq k \leq m \). I.e. we have proved \( P(p+1) \). It follows that \( H \) is finite-dimensional. But now Theorem 8.11 of [1] tells us that \( T \) has a positive eigenvector, which we already know to be false. \( \Box \)

Note that when \( p = 2 \) the operator \( T \) is self-adjoint so certainly has a plentiful supply of non-trivial closed invariant subspaces.

Again, the argument breaks down if we take \( X = c \) or \( X = \ell_\infty \) as \( c_0 \) is a non-trivial invariant closed ideal in that case. This is of special interest as it follows from a theorem of M.G. Krein [1, Corollary 9.46], which asserts that for any positive operator \( T \) on a \( C(K) \)-space the adjoint \( T^* \) has a positive eigenvector, from which it is immediate that \( T \) has a non-trivial closed invariant subspace, namely the kernel
of such an eigenvector. Thus it might have been conjectured that positive operators on $C(K)$-spaces had non-trivial closed invariant sublattices. Here is an example to show that this is not so. Like our previous example, the next one has many atoms.

**Example 3.6.** Let $X = c$ and $T$ be the operator on $X$ defined by

$$(Tx)_n = \begin{cases} 
  x_{n-1} + x_{n+1} + x_0 & \text{if } n > 0, \\
  x_1 + x_0 & \text{if } n = 0
\end{cases}$$

then there is no non-trivial closed $T$-invariant sublattice of $X$.

**Proof.** Suppose, first, that $x \in X_+$ is an eigenvector of $T$ and that $\lambda \geq 0$ is the corresponding eigenvalue. As in the preceding proof, if we set $f(z) = \sum_{n=0}^{\infty} x_n z^n$ then $T_x$ similarly corresponds to the mapping taking $z$ to $zf(z) + (f(z) - f(0))/z + f(0)/(1 - z)$ so that

$$zf(z) + (f(z) - f(0))/z + f(0)/(1 - z) = \lambda f(z),$$

from which it follows that

$$f(z) = \frac{(1 - 2z)f(0)}{(1 - z)(z^2 - \lambda z + 1)}.$$ 

If $\lambda > 2$ then $f$ has a pole at $(\lambda - \sqrt{\lambda^2 - 4})/2 < 1$ which conflicts with $f$ being analytic on the open unit disk.

Having established that $\lambda \leq 2$ let us revert to the sequence viewpoint. Note that we already know that $x_0 = f(0) \neq 0$ else $f = 0$. From the fact that $Tx = \lambda x$ we see that $(Tx)_0 = x_1 + x_0 = \lambda x_0$ from which $x_1 = (\lambda - 1)x_0$. From $(Tx)_1 = x_2 + 2x_0 = \lambda x_1 = \lambda (\lambda - 1)x_0$ we see that $x_2 = (\lambda^2 - \lambda - 2)x_0$. From $(Tx)_2 = x_3 + x_1 + x_0 = x_3 + \lambda x_0 = \lambda x_2 = \lambda (\lambda^2 - \lambda - 2)x_0$ we have $x_3 = \lambda (\lambda^2 - \lambda - 3)x_0$. As $\lambda \leq 2$, $\lambda^2 - \lambda - 3 < 0$ so that $x_3 < 0$ which contradicts $x \geq 0$. There are thus no positive eigenvectors for $T$.

Suppose that $H$ is a $T$-invariant closed sublattice of $X$. We first show that there is no $m$ such that $x_m = 0$ for all $x \in H$. If $m > 1$ and we take any $x \in H_+$ then $(Tx)_m = x_0 + x_{m-1} + x_{m+1} = 0$ so (by positivity) $x_0 = x_{m-1} = x_{m+1} = 0$ for all $x \in H_+$ and hence for all $x \in H$. Proceeding inductively we see that $x_p = 0$ for all $p$ (the fact that $x_2 = 0$ will give us $x_0 = x_1 = x_3 = 0$). If $m = 1$ then the fact that $(Tx)_1 = 2x_0 + x_2$ will give us $x_0 = x_2 = 0$ for all $x \in H$ in a similar way. We may now revert to the $m > 1$ case. Finally if $m = 0$ then $(Tx)_0 = x_0 + x_1$ showing that $x_1 = 0$ for all $x \in H$ and again we may revert to a previous case. It is also impossible that $\lim_{n \to \infty} x_n = 0$ for all $x \in H$ as in that case we would have, for each $x \in H_+$,

$$\lim_{n \to \infty} (Tx)_n = \lim_{n \to \infty} (x_{n-1} + x_{n+1} + x_0) = x_0$$

so that $x_0 = 0$ for all $x \in H_+$ and hence for all $x \in H$. Again, we have already seen that this is impossible.
The only possibility left, if \( H \neq X \) is that there are \( m, n \) and \( \alpha > 0 \) such that 
\[ x_m = \alpha x_n \]
for all \( x \in H \) or that there are \( m \) and \( \alpha > 0 \) such that 
\[ x_m = \alpha \lim_{n \to \infty} x_n. \]
In the first of these cases, the proof that \( H \) must be finite-dimensional proceeds exactly as in the preceding example and we obtain a contradiction. In the second case, notice that there can only be one such constraint as if we also have 
\[ x_p = \beta \lim_{n \to \infty} x_n \]
for all \( x \in H \) then 
\[ x_m = (\alpha/\beta)x_p \]
for all \( x \in H \) which we have already established is impossible. This means that the constraint that 
\[ x_m = \alpha \lim_{n \to \infty} x_n \]
is the only possible restriction on \( H \).

If \( m = 0 \), set \( b \) to be the sequence starting \( \alpha, 0 \) and then having all its terms 1 so that \( b \in H \). Note that 
\[ \lim_{n \to \infty} (Tb)_n = 3 \]
whilst \( (Tb)_0 = \alpha \) so that \( \alpha = 3\alpha \) which contradicts \( \alpha > 0 \). If \( m > 0 \) then let \( a_m = \alpha, a_n = 1 \) for \( n > m + 1 \) and with all other \( a_n = 0 \). This time, \( (Ta)_m = 0 \) whilst \( \lim_{n \to \infty} (Ta)_n = 3 \) so that \( 0 \times \alpha = 3 \) which is impossible. We have now eliminated all possible constraints which can be possibly hold on a proper closed sublattice of \( X \), so we have to admit that there can be no such closed \( T \)-invariant sublattice.

Note that this operator is also defined on \( \ell_\infty \) where it certainly has an invariant closed sublattice by Corollary 2.3. Of course, we can see directly that \( c \) is such an invariant sublattice.

4. SOME EXAMPLES WITHOUT ATOMS

Lest the reader go away with the impression that it is the existence of atoms which prevents the existence of non-trivial closed invariant sublattices, we present further examples of positive operators, with no non-trivial closed invariant sublattices, defined on a \( C(K) \)-space with no atoms at all and on a non-atomic \( L^p \)-space.

In what follows \( \Gamma \) is the unit circle and \( C(\Gamma) \) the space of all continuous real-valued functions on \( \Gamma \). We will identify \( C(\Gamma) \) and the set \( \{ f \in C[0, 2\pi]: f(0) = f(2\pi) \} \). Let us fix an \( \alpha \) from \( (0, 2\pi) \) such that \( \alpha/\pi \) is an irrational number. \( H \) is a hyperplane in \( C(\Gamma) \) defined in the following way.

\[
H = \left\{ f \in C(\Gamma) : \int_0^{2\pi} f(\theta) \, d\theta = 0 \right\}.
\]

Let us recall that a function \( f \in C(\Gamma) \) is called an additive coboundary for \( \alpha \) if there is a \( g \in C(\Gamma) \) such that 
\[ f(\theta) = \sum_{n=0}^{\infty} g(\theta + n\alpha) \]
for all \( \theta \in [0, 2\pi) \) where \( + \) means addition modulo \( 2\pi \). \( f \) is a trivial cocycle for \( \alpha \) if for some \( c \in \mathbb{R} \) the function \( f - c \) is an additive coboundary. Let \( C_\alpha \) be the set of all \( \alpha \)-coboundaries, then clearly \( C_\alpha \subset H \). The next lemma is surely well known but it is difficult to provide an exact reference.

Lemma 4.1. \( C_\alpha \) cannot be a set of the second category in \( H \).

Proof. Let \( D_\alpha g(\theta) = g(\theta + \alpha) - g(\theta), \theta \in [0, 2\pi) \). Clearly \( D_\alpha \) is a bounded linear operator from \( H \) into \( C_\alpha \). We claim that the operator \( D_\alpha : H \to C_\alpha \) is injective
and that \( D_{\alpha}H = C_{\alpha} \). Indeed, if \( D_{\alpha}g = 0 \) then \( g(n\alpha) = g(0), \ n \in \mathbb{N} \), where \( n\alpha = (n-1)\alpha + \alpha, \ n \in \mathbb{N} \). The set \( \{n\alpha; \ n \in \mathbb{N}\} \) is dense in \([0, 2\pi)\) because \( \alpha/\pi \) is an irrational number whence \( g(\theta) = g(0), \ \theta \in [0, 2\pi) \). Recalling that \( \int_0^{2\pi} g(\theta) \, d\theta = 0 \) we have \( g = 0 \).

To prove that \( D_{\alpha}H = C_{\alpha} \) notice that if \( h \in C_{\alpha} \) then there is a \( g \in C(\Gamma) \) such that \( h(\theta) = g(\theta + \alpha) - g(\theta), \ \theta \in [0, 2\pi) \). Let \( \tilde{g}(\theta) = g(\theta) - \int_0^{2\pi} g(\lambda) \, d\lambda, \ \theta \in [0, 2\pi) \), then clearly \( \tilde{g} \in H \) and \( D_{\alpha}\tilde{g} = h \).

The operator \( D_{\alpha} \) is therefore invertible but the inverse operator \( D_{\alpha}^{-1} : C_{\alpha} \to H \) is not bounded. Indeed, let \( \mu_n = \sum_{j=0}^{\infty} \delta_{j\alpha} \), where \( \delta_{j\alpha}(f) = f(j\alpha), \ f \in H \). For any \( n \in \mathbb{N} \) we can find an \( f_n \in H \) such that \( \|f_n\| = 1 \) and \( f_n(j\alpha) = 1, \ j \in [0; n] \) whence \( \|\mu_n\|_{H^*} = n + 1 \). On the other hand \( \|D_{\alpha}^*\mu_n\|_{C_{\alpha}^*} = \|\delta_0 - \delta_{(n+1)\alpha}\|_{C_{\alpha}^*} \leq 2 \) whence \( (D_{\alpha}^*)^{-1} \) is not bounded.

By the classical Banach theorem, see for example [14, Theorem 2.11], \( C_{\alpha} = D_{\alpha}H \) cannot be of the second category in \( H \). \( \Box \)

**Lemma 4.2.** There is a subset \( M \) of \( H \) such that \( M \) is of second category in \( H \) and for any \( f \in M \) and for any \( p, q \in \mathbb{N} \) such that \( p < 2q \) we have

\[
\sup_{n \in \mathbb{N}} \left| \sum_{j=0}^{n} f(j\alpha) - \sum_{j=0}^{n} f(p\pi/q + j\alpha) \right| = \infty.
\]

**Proof.** For any \( m \in \mathbb{N} \) we define the subset \( H_m \) of \( H \) in the following way.

\( f \in H_m \) if \( f \in H \) and there are \( p, q \in \mathbb{N} \) such that \( p < 2q \leq m \) and

\[
\sup_{n \in \mathbb{N}} \left| \sum_{j=0}^{n} f(j\alpha) - \sum_{j=0}^{n} f(p\pi/q + j\alpha) \right| \leq m.
\]

Clearly \( H_m \) is a closed subset of \( H \). We claim that \( H_m \) is nowhere dense in \( H \). Let us fix an \( f \in H_m \) and a positive scalar \( \varepsilon \). We proceed with construction of a function \( g \in H \) such that \( \|g\| \leq \varepsilon \) and \( f + g \notin H_m \). To this end let us consider the set \( Q_m = \{(p, q); \ p, q \in \mathbb{N}, \ p < 2q \leq m\} \) and the partition of this set \( Q_m = Q_m^{(1)} \cup Q_m^{(2)} \) where

\[
Q_m^{(1)} = \left\{ (p, q) \in Q_m; \ \sup_{n \in \mathbb{N}} \left| \sum_{j=0}^{n} f(j\alpha) - \sum_{j=0}^{n} f(p\pi/q + j\alpha) \right| \leq m \right\}
\]

and \( Q_m^{(2)} = Q_m \setminus Q_m^{(1)} \). Then for any \( (p, q) \in Q_m^{(2)} \) there is an \( n = n(p, q) \in \mathbb{N} \) such that \( \left| \sum_{j=0}^{n} f(j\alpha) - \sum_{j=0}^{n} f(p\pi/q + j\alpha) \right| > m \).

Let us fix \( N \in \mathbb{N} \) such that \( N\varepsilon > 2m \) and \( N > \max_{(p, q) \in Q_m^{(2)}} n(p, q) \). We can find \( g \in H \) such that

- \( \|g\| = \varepsilon; \)
- \( g(j\alpha) = \varepsilon, \ j = 0, \ldots, N; \)
- if \( (p, q) \in Q_m^{(1)} \) then \( g(p\pi/q + j\alpha) = 0, \ j = 0, \ldots, N; \)
- if \( (p, q) \in Q_m^{(2)} \) then \( g(p\pi/q + j\alpha) = \varepsilon, \ j = 0, \ldots, N. \)
Notice that \( g \) is well defined because \( \alpha/\pi \) is irrational and it is easy to see that 
\[ f + g \notin H_m. \]

Finally it remains to notice that if \( f \in M = H \setminus \bigcup_{m \in \mathbb{N}} H_m \) then \( f \) satisfies (1). \( \square \)

**Corollary 4.3.** The set \( M \setminus C_\alpha \) is not empty.

**Proof.** Indeed, otherwise \( M \subseteq C_\alpha \) in contradiction with Lemma 4.1. \( \square \)

**Example 4.4.** Let \( \alpha \in [0, 2\pi) \) be such that \( \alpha/\pi \) is an irrational number. Let \( f \in M \setminus C_\alpha \), let \( w = \exp(f) \) and let \( T \) be the positive operator on \( C(\Gamma) \) defined as 
\[ (Th)(x) = w(x)h(x + \alpha). \]
Then the operator \( T \) does not have a non-trivial closed invariant sublattice in \( C(\Gamma) \).

**Proof.** We will prove first that \( T \) does not have positive eigenvectors. Because 
\[ \int_0^{2\pi} f(\theta) \, d\theta = 0 \] we have (see e.g. [1, Theorem 10.52]), \( \rho(T) = \rho(T^{-1}) = 1 \), where \( \rho(T) \) means the spectral radius of \( T \), so that \( \sigma(T) \subseteq \Gamma \). Therefore, if \( g \) is a positive eigenvector for \( T \) then \( Tg = g \) whence \( g \) is strictly positive on \( \Gamma \) and \( w(x) = g(x)/g(x + \alpha) \), \( x \in [0, 2\pi) \). Therefore \( f(x) = \ln(g(x)) - \ln(g(x + \alpha)) \), \( x \in [0, 2\pi) \), in contradiction with our assumption that \( f \notin C_\alpha \).

Assume that \( X \) is a non-trivial \( T \)-invariant norm-closed sublattice of \( C(\Gamma) \). By Theorem 3 of [10] there are two distinct points \( x, y \in [0, 2\pi) \) and a real \( \lambda \geq 0 \) such that 
\[ g(y) = \lambda g(x), \quad \text{for any } g \in X. \]

First notice that \( \lambda > 0 \). Indeed, otherwise we would have \( T^n g(y) = 0 \) for all \( n \in \mathbb{N} \). But 
\[ T^n g(y) = w_n(y)g(y + n\alpha) \]
where 
\[ w_n(y) = w(y)w(y + \alpha)\cdots w(y + (n-1)\alpha). \]
Therefore \( g(y + n\alpha) = 0 \), \( n \in \mathbb{N} \) whence \( g = 0 \).

We can assume without loss of generality that \( x > y \). Note first that the Banach lattice \( X \) contains a strictly positive function \( g \). Suppose \( f \geq 0 \) but \( f \neq 0 \). Let 
\[ g = \sum_0^\infty c_n T^n f \]
where \( c_n > 0 \) and the scalars \( c_n \) are small enough for the series to converge. Then, recalling that the weight \( w \) is strictly positive and that \( \alpha/\pi \) is irrational, \( g \) is strictly positive. We now have to consider separately two cases.

(a) \( \frac{x-y}{\pi} \) is an irrational number. We know that \( X \) contains a strictly positive element \( g \). Suppose that \( h \) is another positive element from \( X \). We will prove that the functions \( h \) and \( g \) are proportional. Considering, if needed, \( g + h \) instead of \( h \) we can assume that \( h \) is also strictly positive. It follows from (2) that for any \( n \in \mathbb{N} \) we have 
\[ w_n(y)g(y + n\alpha) = \lambda w_n(x)g(x + n\alpha) \]
and 
\[ w_n(y)h(y + n\alpha) = \lambda w_n(x)h(x + n\alpha), \]

whence
\[ \frac{g}{h}(y + n\alpha) = \frac{g}{h}(x + n\alpha), \quad n \in \mathbb{N}. \]

Recalling that the set \( \{n\alpha: n \in \mathbb{N}\} \) is dense in \([0, 2\pi]\) we obtain that
\[ (3) \quad \frac{g}{h}(y + \theta) = \frac{g}{h}(x + \theta), \quad \theta \in [0, 2\pi). \]

But \( \frac{x - y}{\pi} \) is irrational whence the set \( \{n(x - y): n \in \mathbb{N}\} \) is dense in \([0, 2\pi]\), and therefore (3) implies that the function \( \frac{g}{h} \) is constant on \( \Gamma \).

Because all positive elements of the Banach lattice \( X \) are proportional \( X \) is one-dimensional, in contradiction to the already proved fact that \( T \) does not have positive eigenvectors.

(b) \( x - y = \frac{p}{q}\pi \) where \( p, q \in \mathbb{N} \) and \( p < 2q \). Again, let \( g \) be a strictly positive function from \( X \). Then for any \( m, n \in \mathbb{N} \) such that \( m > n \) we have
\[ w_m(y)g(y + m\alpha) = \lambda w_m(y + \frac{p}{q}\pi)g(y + \frac{p}{q}\pi + m\alpha) \]
and
\[ w_{m-n}(y)(y + (m - n)\alpha) = \lambda w_{m-n}(y + \frac{p}{q}\pi)(y + \frac{p}{q}\pi + (m - n)\alpha) \]
whence
\[ (4) \quad \frac{w_n(y + (m - n)\alpha)}{w_n(y + \frac{p}{q}\pi + (m - n)\alpha)} = \frac{g(y + \frac{p}{q}\pi + m\alpha)g(y + (m - n)\alpha)}{g(y + \frac{p}{q}\pi + (m - n)\alpha)g(y + m\alpha)}. \]

For any fixed \( n \in \mathbb{N} \) the set \( \{(m - n)\alpha: m \in \mathbb{N}, \ m > n\} \) is dense in \([0, 2\pi]\). Therefore (4) implies that
\[ (5) \quad \frac{1}{C} \leq \frac{w_n(\theta)}{w_n(\theta + \frac{p}{q}\pi)} \leq C, \]
for all \( \theta \in [0, 2\pi) \) and \( n \in \mathbb{N} \), where \( C = \|g\|^2\|1/g\|^2 \).

It remains to notice that (5) is in obvious contradiction with the fact that \( w = \exp(f) \) and \( f \notin M \). □

Of course, Krein’s theorem will again guarantee that \( T \) has a non-trivial invariant subspace. Note also that Theorem 2.6 shows that, since \( T \) is a lattice isomorphism, \( T \) has a non-trivial invariant sublattice which will not, of course, be closed.

If we take \( f \in M \cap C_{\alpha} \) and \( w = \exp(f) \) then the corresponding weighted rotation operator \( T \) will have only a single one-dimensional non-trivial norm-closed sublattice, whilst if we take \( f \notin C_{\alpha} \) but such that \( f(\theta + \pi) = f(\theta), \ \theta \in [0, 2\pi) \), then the corresponding operator \( T \) will have neither positive eigenvectors nor
non-trivial invariant ideals but it will have an invariant norm-closed sublattice
\( X = \{ g \in C(\Gamma) : g(\theta + \pi) = g(\theta), \ \theta \in [0, 2\pi) \} \).

We conclude our examples with one to show that it is possible for a purely
non-atomic Banach lattice with an order continuous norm to support a positive
operator which has no non-trivial invariant sublattices. It will simplify some of our
calculations to work with both real and complex scalars from now on. Again, we
need some preliminaries before giving the example. As above, we take \( \Gamma \) to be the
unit circle. If \( 1 \leq p < \infty \) and \( X \) is a subspace of \( L^p(\Gamma) \) we define
\[
\mathcal{M}(X) = \{ f \in L^\infty(\Gamma) : fX \subseteq X \}.
\]

**Lemma 4.5.** If \( X \) is a closed sublattice of \( L^p(\Gamma) \), then \( \mathcal{M}(X) \) is a \( \sigma(L^\infty, L^1) \)-
closed subalgebra and sublattice of \( L^\infty(\Gamma) \).

**Proof.** Clearly \( \mathcal{M}(X) \) is a norm-closed subalgebra of \( L^\infty(\Gamma) \) containing the
constants. Therefore in case when the field of scalars is \( \mathbb{R} \), \( \mathcal{M}(X) \) is automatically a
sublattice of \( L^\infty \). If the field of scalars is \( \mathbb{C} \) and \( f \in \mathcal{M}(X) \) then for any \( x \in X \)
we have \( f^*x = f^*X \) in \( \mathcal{M}(X) \).

To prove that \( \mathcal{M}(X) \) is \( \sigma(L^\infty, L^1) \)-closed let us consider a net \( f_\alpha \in \mathcal{M}(X) \) such
that \( f_\alpha \xrightarrow{\sigma(L^\infty, L^1)} f \), \( f \in L^\infty(\Gamma) \). Let \( g \in L^q(\Gamma) \), where \( L^q(\Gamma) \) is the conjugate to
\( L^p(\Gamma) \), be such that \( g(x) = 0 \) for all \( x \in X \). If we take any \( x \in X \) and any \( x \)
we have \( \int_0^{2\pi} f_\alpha x^* \bar{\phi} \ d\theta = 0 \). But \( x \bar{\phi} \in L^1(\Gamma) \) so that \( \int_0^{2\pi} f \bar{\phi} \ d\theta = 0 \) and therefore
\( fx \in X \). \( \square \)

**Lemma 4.6.** Let \( X, Y \) be norm-closed vector sublattices of \( L^p(\Gamma) \), \( 1 \leq p < \infty \),
such that \( Y \subseteq X \) and \( Y^dd = L^p(\Gamma) \), where \( Y^dd \) is the band generated by \( Y \) in \( L^p(\Gamma) \). Then \( \mathcal{M}(Y) \subseteq \mathcal{M}(X) \).

**Proof.** Because \( Y \) is a separable \( L^p \) space there is a \( y \in Y_+ \) such that \( \{ y \}^dd = Y^dd = \)
\( L^p(\Gamma) \). Let \( f \in \mathcal{M}(Y) \) and \( x \in X_+ \). Let \( z = x + y \) and \( I, J \) be the principal ideals
in \( X \), respectively in \( Y \), generated by \( z \), respectively by \( y \). By Kakutani’s theorem
we can assume without loss of generality that \( I \) is a uniformly closed sublattice of
\( C(Q) \), where \( Q \) is the hyperstonean compact of \( L^\infty(\Gamma) \), and that \( z \) is represented
by the function \( 1 \). Let us notice that if \( g \in L^\infty(\Gamma) \) then \( gI \subseteq I \) if and only if for
any two points \( p, q \in Q \) such that \( u(p) = u(q) \), \( u \in I \) we have \( g(p) = g(q) \). Let us fix \( n \in \mathbb{N} \)
and let \( Q_n = cl\{ t \in Q : \frac{1}{n} < y(t) < n \} \). Because \( I \) has the projection
property we have \( z_n = x_{Q_n}z \in I \). If \( t_1, t_2 \in Q_n \) and \( u(t_1) = u(t_2) \) for all \( u \in I \) then
\( v(t_1) = v(t_2) \) for all \( v \in J \) and because \( y(t_1) = y(t_2) \) \( \neq 0 \) we have \( f(t_1) = f(t_2) \)
(recall that \( f \in \mathcal{M}(Y) \)). Therefore \( f z_n \in X \). But \( \| z_n - z \| \to 0 \) so that \( f z \in X \) and
therefore \( fx = f z - fy \in X \). \( \square \)

**Lemma 4.7.** Let \( T_x(\theta) = w(\theta)x(\theta + \alpha) \) be a weighted rotation operator on \( L^p(\Gamma) \),
\( 1 \leq p < \infty \), where \( w \in L^\infty_+ \) is invertible and \( \alpha/\pi \) is an irrational number. If \( X \) be
a norm-closed $T$-invariant vector sublattice of $L^p(\Gamma)$ then $\mathcal{M}(X)$ is a $\sigma(\ell^\infty, L^1)$-closed, rotation invariant subalgebra and a sublattice of $L^\infty$.

**Proof.** By Lemma 4.5 we only have to prove that $\mathcal{M}(X)$ is rotation invariant. Write $T_\alpha x(\theta) = x(\theta + \alpha)$, so that $T = wT_\alpha$. Notice that $X \subseteq T^{-1}X$ and that because $T$ is a lattice isomorphism $T^{-1}X$ is a closed vector sublattice of $L^p(\Gamma)$. Notice also that if $x \in X_+$, $x \neq 0$ then $z = \sum_{n=0}^{\infty} \frac{n\pi}{2\pi ||T_x||} \in X$ and $\{z\} = L^p(\Gamma)$. Therefore by Lemma 4.6 $\mathcal{M}(X) \subseteq \mathcal{M}(T^{-1}X)$. Let $f \in \mathcal{M}(X)$ and $x \in X$; then $TfT^{-1}x \in X$. Note that $T^{-1}x(\theta) = w^{-1}(\theta + (-\alpha))x(\theta + (-\alpha))$, so that

$$TfT^{-1}x(\theta) = w(\theta)f(\theta + \alpha)w^{-1}(\theta + (-\alpha) + \alpha)x(\theta + (-\alpha) + \alpha)$$

$$= w(\theta)f(\theta + \alpha)w^{-1}(\theta)x(\theta)$$

$$= f(\theta + \alpha)x(\theta)$$

so that $T_\alpha f \in \mathcal{M}(X)$. As $\mathcal{M}(X)$ is an algebra, $T_\alpha^n f \in \mathcal{M}(X)$ for all $n \in \mathbb{N}$. It remains to notice that the set $n\alpha$, for $n \in \mathbb{N}$, is dense in $\Gamma$ and that the mapping $\beta \mapsto T_\beta f : \Gamma \to (L^\infty, \sigma(\ell^\infty, L^1))$ is continuous. \(\square\)

**Lemma 4.8.** Consider $L^\infty(\Gamma)$ over the field of complex numbers $\mathbb{C}$. Let $\mathcal{M}$ be a $\sigma(\ell^\infty, L^1)$-closed subalgebra of $L^\infty$. Assume additionally that $\mathcal{M}$ is closed under complex conjugation and that $\mathcal{M}$ is rotation invariant. Then there is a non-negative integer number $m$ such that $\mathcal{M}$ coincides with the closed linear span of the set $\{e^{i\alpha}n: n \in \mathbb{Z}\}$ in $L^\infty(\sigma(\ell^\infty, L^1))$.

**Proof.** Let $f \in X$ and $c_n$ are the complex Fourier coefficients of $f$ then the Cesàro means

$$\sigma_n(\theta) = \sum_{j=-n}^{n} c_j \left(1 - \frac{|j|}{n+1}\right)e^{ij\theta}$$

converge to $f$ in $L^\infty(\sigma(\ell^\infty, L^1))$, by Theorem 6.1.1 in [6]. Notice that for any $\beta \in [0, 2\pi)$

$$T_\beta f = \lim_{n \to \infty} T_\beta \sigma_n \quad \text{in} \quad \sigma(\ell^\infty, L^1)$$

and that

$$T_\beta \sigma_n(\theta) = \sum_{j=-n}^{n} c_j \left(1 - \frac{|j|}{n+1}\right)e^{ij\beta}e^{ij\theta}.$$

For any $j \in \mathbb{Z}$ we have

$$\int_{0}^{2\pi} e^{-ij\beta} T_\beta \sigma_n \, d\beta = \int_{0}^{2\pi} e^{-ij\beta} \left(\sum_{k=-n}^{n} c_k \left(1 - \frac{|k|}{n+1}\right)e^{ik\beta}e^{ik\theta}\right) d\beta$$

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and note that the integral certainly lies in $\mathcal{M}$. We now have
\[
\int_0^{2\pi} e^{-ij\beta} T_{\beta} f \, d\beta = \lim_{n \to \infty} \int_0^{2\pi} e^{-ij\beta} T_{\beta} e_n \, d\beta = 2\pi c_j e^{ij\theta},
\]
so that if $c_j \neq 0$ (for any $f \in \mathcal{M}$) then the function $\theta \mapsto e^{ij\theta}$ lies in $\mathcal{M}$. It follows immediately that $\mathcal{M}$ is the closed linear span in $(L^\infty, \sigma(L^\infty, L^1))$ of the set \{e^{in\theta}: n \in \mathbb{Z}, e^{in\theta} \in \mathcal{M}\}.

If $\mathcal{M}$ consists only of the constant functions, the result is obvious. Otherwise, there is an $n \in \mathbb{Z}$, $n \neq 0$ such that $e^{in\theta} \in \mathcal{M}$. Because $\mathcal{M}$ is closed under complex conjugation $e^{-in\theta} \in \mathcal{M}$. Let $m = \text{GCF}\{n \in \mathbb{N}: e^{in\theta} \in \mathcal{M}\}$. Recalling that $\mathcal{M}$ is an algebra we see, using the Euclidean algorithm, that $e^{im\theta} \in \mathcal{M}$ whence the statement of the lemma follows. \qed

Lemma 4.9. Let $X$ be a closed non-trivial $T$-invariant sublattice of $L^p(\Gamma)$, $1 \leq p < \infty$. Then either $\dim(\mathcal{M}(X)) = 1$ or there is an $m \in \mathbb{N}$ such that $m > 1$ and $T_{2\pi/m} f = f$ for all $f \in \mathcal{M}(X)$.

Proof. In case when $L^p(\Gamma)$ and $X$ are considered over the field of complex numbers $\mathbb{C}$ then the statement follows directly from Lemmas 4.7 and 4.8. If the field of scalars is $\mathbb{R}$ then we obtain the result applying these lemmas to the standard complexifications of $L^p(\Gamma)$ and $X$. \qed

Corollary 4.10. Let $X$ be a closed non-trivial $T$-invariant sublattice of $L^p(\Gamma)$ where $1 \leq p < \infty$. Then either $\dim(X) = 1$ or there are a measurable function $\lambda \in L^0(\Gamma)$ and a positive integer $m$ such that $x(\theta + 2\pi/m) = \lambda(\theta) x(\theta)$ for all $x \in X$ and $\theta \in [0, 2\pi)$.

Proof. For each $j \in \mathbb{N}$, let $x_j$ be a positive element in $X$ such that their linear combinations are dense in $X$. Let
\[
z = \sum_{j \in \mathbb{N}} \sum_{n = 0}^{\infty} \frac{T^n x_j}{2^{j+n} \|T^n x_j\|}.
\]
Let $I$ be the principal ideal generated in $X$ by $z$. Then clearly $TI \subseteq I$ and the center of $I$, $Z(I)$, coincides with $\mathcal{M}(X)$. By Lemmas 4.8 and 4.9 there are three possibilities.

1. $\dim(\mathcal{M}(X)) = 1$ whence $\dim(I) = 1$ and hence $\dim(X) = 1$.
2. $Z(I) = L^\infty(\Gamma)$ whence $X = L^p(\Gamma)$ in contradiction with the assumption of the corollary.
3. There is an $m \in \mathbb{N}$, $m > 1$, such that $T_{2\pi/m} f = f$ for all $f \in Z(I)$.  

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From now on we have to deal only with case (3). Let \( x \in X \) be such that \( 0 \leq x \leq z \). Because \( I \) is Dedekind complete there is an \( h \in Z(I) \) such that \( x = hz \). Therefore 
\[
T_{2\pi/m}x = T_{2\pi/m}hT_{2\pi/m}z = hT_{2\pi/m}z
\]
whence
\[
x(\theta + 2\pi/m) = \frac{z(\theta + 2\pi/m)}{z(\theta)}x(\theta).
\]

The function \( \lambda(\theta) = \frac{z(\theta + 2\pi/m)}{z(\theta)} \) is measurable because \( \{z\}_d \) is \( L^p(\Gamma) \). Now the statement of the corollary follows from the fact that \( I \) is dense in \( X \). \( \square \)

Next we need a result from [2].

**Theorem 4.11** [2, Theorem 1]. Let \( v \) be an integrable, real analytic function on the open interval \((0, 1)\), which is not a trigonometric polynomial. Then the set \( U \), of all irrationals for which \( v \) is a trivial cocycle, is of the first category.

The notational difference between our setting (on the interval \([0, 2\pi]\)) and that in [2] is obvious unimportant. Now we are ready to provide an example of a bounded positive linear operator on \( L^p(\Gamma) \), \( 1 \leq p < \infty \), without non-trivial closed invariant sublattices.

**Example 4.12.** Let \( v \) be a real analytic function from \( L^\infty(\Gamma) \) which is not a trigonometric polynomial. Let \( w = \exp(v) \). There is a subset \( V \) of the second category in \([0, 2\pi)\) such that for any \( \alpha \in V \) the number \( \frac{\alpha}{2\pi} \) is irrational and the operator \( T = wT_\alpha \) has no non-trivial closed invariant sublattices in \( L^p(\Gamma) \), \( 1 \leq p < \infty \).

**Proof.** By Theorem 4.11 there is a set \( V \) of irrationals (modulo \( 2\pi \)) such that \( V \) is of second category in \([0, 2\pi] \) and for any \( \alpha \in V \) none of the functions \( v, v_2, v_3, \ldots \), where \( v_m(\theta) = v(\theta + \frac{2\pi}{m}) - v(\theta) \), is a trivial cocycle for \( \alpha \).

Let us fix an \( \alpha \in V \), let \( T = wT_\alpha \) and let \( X \) be a non-trivial closed \( T \)-invariant sublattice of \( L^p(\Gamma) \). By Corollary 4.10 there are two possibilities.

1. \( \dim(X) = 1 \). Then \( v \) is a trivial cocycle in contradiction with our choice of \( \alpha \).
2. There are a measurable function \( \lambda \in L^0(\Gamma) \) and a positive integer \( m \) such that 
\[
x(\theta + 2\pi/m) = \frac{x(\theta) + \alpha}{\lambda(\theta + \alpha)}
\]
so that
\[
\frac{\lambda(\theta + \alpha)w(\theta + \alpha)}{\lambda(\theta + \alpha)} = x(\theta + \alpha)
\]

so that

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\[ v_m(\theta) = v(\theta + 2\pi/m) - v(\theta) = \log(w(\theta + 2\pi/m)) - \log(w(\theta)) = \log(\lambda(\theta)) - \log(\lambda(\theta + \alpha)) \]

showing that \( v_m \) is a coboundary and hence a trivial cocycle contradicting the choice of \( \alpha \). \( \Box \)

Note that instead of \( L^p(\Gamma) \) we could have considered any symmetric ideal in \( L^0(\Gamma) \) with order continuous norm and obtained a similar result.

Note that in Example 4.12 our \( \alpha \) is chosen from a set in \( \Gamma \) of second category. The Liouville numbers form a set of first category in \( \Gamma \) so that we can also ensure that \( \alpha \) is not a Liouville number. It follows from [3] that our operator \( T \) will have a closed non-trivial invariant subspace even though it does not have a closed non-trivial invariant sublattice.

Finally, we note that there is an important difference between \( \ell_p \) and \( L^p \) here. In \( L^p \) we have examples of lattice isomorphisms without non-trivial closed invariant sublattices but we cannot have such an example in \( \ell^p \). Indeed, such an operator would be a weighted composition and the composition should be transitive so that such an operator would be similar to a weighted shift on \( \ell_p(\mathbb{Z}) \). If \( T(x_n) = (\alpha_n x_{n+1}) \) is such an operator then it actually has a family of non-trivial closed invariant ideals \( J_m = \{ x : x_k = 0 \text{ for all } k \geq m \} \) for each \( m \in \mathbb{Z} \).

5. SOME OPEN PROBLEMS

We saw above that quite a large class of positive operators have non-trivial invariant (not-necessarily-closed) sublattices. However, that family is far from being all positive operators.

**Question 5.1.** Is it true that any positive operator on a Banach lattice \( X \) has a non-trivial invariant (not-necessarily-closed) vector sublattice?

By working in a principal ideal containing a sequence \( x, Tx, T^2x, \ldots \) where \( x \in X_+ \), we can reduce the problem to the case that \( X = C(K) \) and by Corollary 2.3 we may assume that \( K \) is a compact metric space.

Our examples seem to leave little room for even a reasonable conjecture as to a class of Banach lattices on which non-trivial invariant closed sublattices will exist for all positive operators. We therefore pose the following question.

**Question 5.2.** Is there an infinite-dimensional separable Banach lattice \( X \) such that every positive operator \( T \) on \( X \) has a non-trivial invariant closed sublattice?

**REFERENCES**


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