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Numerical simulation of chaotic dynamical systems by the method of differential quadrature

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Abstract In this paper, the differential quadrature (DQ) method is employed to solve some nonlinear chaotic systems of ordinary differential equations (ODEs). Here, the method is applied to chaotic Lorenz, Chen, Genesio and Rössler systems. The first three chaotic systems are described by three-dimensional systems of ODEs while the last hyperchaotic system is a four-dimensional system of ODEs. It is found that the DQ method is unconditionally stable in solving first-order ODEs. But, care should be taken to choose a time step when applying the DQ method to nonlinear chaotic systems. Similar to all conventional unconditionally stable time integration schemes, the unconditionally stable DQ time integration scheme may also be possible to produce inaccurate results for nonlinear chaotic systems with an inappropriately too large time step sizes. Numerical comparisons are made between the DQ method and the conventional fourth-order Runge–Kutta method (RK4). It is revealed that the DQ method can produce better accuracy than the RK4 using larger time step sizes.

1. Introduction

It is well known that many physical and engineering phenomena can be modeled by chaotic or non-chaotic systems of ODEs. In brief, dynamical systems that exhibit chaotic behavior are sensitive to initial conditions. Although these systems are deterministic through some description by mathematical rules, the behavior of chaotic systems appears to be random due to their sensitivity to initial conditions. Chaotic behavior can be observed in a variety of systems such as electrical circuits, lasers, fluid dynamics, mechanical devices, time evolution of the magnetic field of celestial bodies, population growth in ecology, the dynamics of molecular vibrations and not forgetting the weather. One of the many chaotic systems discovered in the past is the Lorenz system. It was developed by Lorenz [1] who observes unpredictable chaotic behavior. The Lorenz dynamical system is defined as follows [1]:

$$\frac{dx}{dt} = \sigma(y - x), \tag{1}$$

$$\frac{dy}{dt} = Rx - y - xz, \tag{2}$$

$$\frac{dz}{dt} = xy + \gamma z, \tag{3}$$

where x, y, and z are dynamical variables of the Lorenz system, and σ, R, and γ are the related constants. The Lorenz system can exhibit both chaotic and non-chaotic behavior for distinct parameter values. Bifurcation studies show that with the parameters $\sigma = 10$ and $\gamma = -8/3$ chaos sets in around the critical parameter value $R = R_{cr} = 27.74$ [1–3]. In other words, Eqs. (1)–(3) exhibits non-chaotic behavior when $R < R_{cr}$ and does chaotic behavior when $R > R_{cr}$. It is also noted that the Lorenz equations govern, at lower order, the dynamics of convection in a fluid layer (or a fluid saturated porous layer) heated from below [1,2] and presents particular challenges due to its high sensitivity to small variations of the initial conditions on the threshold of transition from steady convection to weak-turbulence (chaos).

Another important chaotic dynamical system is the Chen dynamical system, which was first introduced by Chen and

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Ueta [4]. This system is defined as

$$\frac{dx}{dt} = a(y - x), \tag{4}$$

$$\frac{dy}{dt} = (c - a)x - xz + cy, \tag{5}$$

$$\frac{dz}{dt} = xy - bz, \tag{6}$$

where a, b, and c are positive constants. Bifurcation studies for the dynamical Chen system show that with the parameters $a = 35$ and $c = 28$, Eqs. (4)–(6) exhibits non-chaotic behavior and chaotic behavior when $b = 12$ and $b = 3$ respectively [5,6]. For other important aspects of this dynamical system, see, for example [7–11]. It is noted that both the Lorenz and Chen systems have nonlinear terms in the form of products of two of the dependent variables (i.e., quadratic nonlinearities). Although both systems seem to be identical and equivalent, they are not topologically equivalent [4]. In fact, the Chen system has been proved to be dual to the Lorenz system [12]. Many theoretical analysis and numerical simulation results suggest that the Chen attractor has different topological structures from those of Lorenz system [4,13,14]. It is also interesting to note that the (positive) Lyapunov exponent for the Chen system is about 2.0272, while the corresponding exponent for the Lorenz system is about 0.9056 [15,16]. In other words, the Chen system is more sensitive to initial conditions than the Lorenz system.

In this study, we also consider the chaotic Genesio system, which was first introduced by Genesio and Tesi [17]. The dynamical equations describe an uncomplicated square element and three straightforward ordinary differential equations that are dependent on three positive real parameters, as such:

$$\frac{dx}{dt} = y, \tag{7}$$

$$\frac{dy}{dt} = z, \tag{8}$$

$$\frac{dz}{dt} = -cx - by - az + x^2, \tag{9}$$

subjected to the initial conditions:

$$x(0) = 0.2, \qquad y(0) = -0.3, \qquad z(0) = 0.1, \tag{10}$$

where a, b and c are positive constants satisfying $ab < c$.

In this paper, we are also interested in the accuracy tests of the DQ method for the solution of nonlinear systems of ODEs capable of exhibiting *hyperchaotic* behavior. The system, which is of interest to us, is the hyperchaotic Rössler system [18]

$$\frac{dx}{dt} = -y - z, \tag{11}$$

$$\frac{dy}{dt} = x + ay + w, \tag{12}$$

$$\frac{dz}{dt} = b + xz, \tag{13}$$

$$\frac{dw}{dt} = -cz + dw, \tag{14}$$

where x, y, z and w are the state variables and a, b, c and d are positive constants. This system exhibits a hyperchaotic behavior when $a = 0.25$, $b = 3$, $c = 0.5$ and $d = 0.05$. This system is similar to the Lorenz system in the sense that both have nonlinear terms in the form of products of the two of the dependent variables (i.e., quadratic nonlinearities). Nevertheless, it is different from the Lorenz system as it has two positive Lyapunov exponents [19,20], $\lambda_1 = 0.109$ and $\lambda_2 = 0.024$. It is noted that the Lyapunov exponent for the Lorenz system is 0.9056.

Since analytical solutions cannot be found for the chaotic systems given in Eqs. (1)–(3), Eqs. (4)–(6), Eqs. (7)–(9) and Eqs. (11)–(14), there has been a considerable effort to solve these systems numerically. But numerical methods provide the solutions only at the discrete time points and their accuracy on long-term solutions is sometimes questionable. Besides, they often need very small time step sizes to ensure the convergence and to arrive at an accurate solution. Thus, much attention has been paid to analytical asymptotic (i.e., semi-analytic) techniques, such as the Adomian decomposition method (ADM) [15,16,20–24], Homotopy analysis method (HAM) [25], variational iteration method [26], multistage homotopy-perturbation method (MHPM) [27,28], and more recently the differential transformation method (DTM) [29]. These semi-analytic methods give some promising results, but each of these methods has its own drawbacks and weaknesses. For example, when we use the ADM or the HAM, we should then calculate some polynomials (say ADM/HAM polynomials). This procedure is often so cumbersome or the 'formula' obtained is often too complicated to understand and display clearly the principle features of the solution. VIM is relatively simple and straightforward to use but one may face longer computational time due to possible exponential coefficients in its iterations [29]. Moreover, the results of VIM are only valid for very short time spans. In the literature, some researchers have also used the HPM to handle the nonlinear dynamical systems [30,31]. However, as pointed out by Chowdhury et al. [27], this technique is only suitable for calculation of very short-term solutions. In fact, the approximate solutions obtained through using HPM, are generally not valid for long time durations [27]. For example, in [27,28], it was shown that the HPM solutions for Lorenz and Chen systems are only valid for $t \ll 1$. To overcome the difficulties of the HPM, Chowdhury et al. [27,28] proposed a multistage HPM. In this technique, the time domain of interest is first divided into some time intervals (i.e., time elements/steps/spans). Then the HPM is applied to each time interval independently. It was shown that he MHPM works very well on highly chaotic systems such as the Chen and Lorenz systems. But care has to be taken with the choice of time span, time step and the number of terms used. Although this technique produces high-accurate solutions for the chaotic systems considered, many calculations should be done to obtain the required polynomial coefficients. Thus, the major difficulty is not to overcome using this technique.

The above-mentioned difficulties can be overcome by using the differential quadrature method (DQM). The DQ method, which was first introduced by Bellman and his associates [32,33] in the early 1970s, is an alternative method for directly solving the differential equations in engineering, mathematics and physics. The DQ method is basically based on the interpolation and derivation. It was also initiated from the idea of conventional and integral quadrature. The DQ method approximates the derivative of a function at a certain discrete point by means of a weighted linear sum of the function values at all discrete points in the domain of that variable. Since its introduction, the DQ method has been successfully applied by many researchers to a variety of problems in engineering, mathematics and physics and is gradually emerging as a distinct numerical technique. Compared to the low-order methods such as the finite element and finite difference methods, the

DQ method can achieve very accurate solutions by using a considerably small number of sample points and therefore requiring relatively little computational cost [34]. Another particular advantage of the DQ method lies in its ease of use and implementation. Due to the above-mentioned favorable features, the DQ method has been applied extensively. Majority of these applications are related to statics and/or free vibrations. More recently, the DQ method has been successfully applied to initial-value problems in structural dynamics [35–49]. It has been found that the DQ time integration scheme is reliable, computationally efficient and also suitable for time integration over long time duration.

In this paper, we apply the DQ method to solve some nonlinear chaotic systems of ODEs. To the authors' best knowledge, this is the first attempt in applying the DQ method to nonlinear systems of ODEs having chaotic behaviors such as the Lorenz, Chen, Genesio and Rössler systems. Since for the systems under investigation, closed form of the analytic solutions cannot be found, the accuracy of the DQ method is tested against conventional fourth-order Runge–Kutta method (RK4). The aim of this study is to compare the effectiveness of the DQ time integration method against the classical RK4 in producing solutions for the chaotic Lorenz, Chen, Genesio and Rössler systems. It is shown that the DQ method produces much better accuracy than the RK4 using much larger time step sizes. Another particular advantage of the DQ method is its ability in providing us a continuous representation of the approximate solution, which allows better information of the solution over the time interval of interest (note that the DQ method is basically based on the interpolation (Lagrange interpolation, here) and differentiation). This characteristic distinguishes the DQ time integration scheme from the conventional single step methods. Note that the RK4 only provides solutions in discretized form (i.e., only gives the solutions at some discrete time points), thereby making it complicated in achieving a continuous representation of the approximate solution.

2. Differential quadrature method

As pointed out earlier, the DQ method is basically based on the interpolation and derivation. Let $x(t)$ be a function which is approximated by the Lagrange interpolation functions $L_j(t),\ j = 1, 2, \dots, m$, that is:

$$x(t) = \sum_{j=1}^{m} x(t_j)L_j(t), \tag{15}$$

where m is the number of sample time points in the time domain (also the order of Lagrange interpolation functions), $x(t_j)$ are the function values at these points. Differentiating Eq. (15) with respect to time, we obtain:

$$\dot{x}(t) = \frac{dx}{dt} = \sum_{j=1}^{m} x(t_j)\frac{dL_j}{dt} = \sum_{j=1}^{n} x(t_j)\dot{L}_j(t). \tag{16}$$

From Eq. (16), the first-order derivative of the function $x(t)$ with respect to time at a time point, t_i can be expressed as:

$$\dot{x}(t_i) = \sum_{j=1}^{m} x(t_j)\dot{L}_j(t_i). \tag{17}$$

Eq. (17) is, in fact, the quadrature rule

$$\dot{x}(t_i) = \sum_{j=1}^{m} A_{ij}x(t_j) \quad \text{or} \quad \dot{x}_i = \sum_{j=1}^{m} A_{ij}x_j \tag{18}$$

which gives the first-order DQ weighting coefficients, A_{ij}, as [50]:

$$A_{ik}^{(1)} = \begin{cases} \dfrac{M^{(1)}(t_i)}{(t_i - t_k)M^{(1)}(t_k)} & i \neq k,\ i, k = 1, 2, \dots, m \\ -\displaystyle\sum_{j=1, j\neq i}^{m} A_{ij}^{(1)} & i = k,\ i = 1, 2, \dots, m \end{cases} \tag{19}$$

where $M^{(1)}(t)$ is defined as:

$$M^{(1)}(t_i) = \prod_{j=1, j\neq i}^{m} (t_i - t_j). \tag{20}$$

One of the key factors in the accuracy, stability, and rate of convergence of the DQ solutions is the choice of sample time points. It is well known that the equally spaced sampling points are not very desirable [51]. It has been suggested that non-uniformly spaced sample points perform consistently better than the equally spaced sample points [51]. The zeros of some orthogonal polynomials are commonly adopted as the sampling points. In this work, the Chebyshev–Gauss–Lobatto sample points are used for the calculation of weighting coefficients. These points are given by:

$$t_i = T/2\left[1 - \cos\left(\frac{(i-1)\pi}{m-1}\right)\right], \quad i = 1, 2, \dots, m \tag{21}$$

where T is the time span.

3. Numerical solutions by the DQ method

For the DQ solution of chaotic dynamical Lorenz, Chen, Genesio and Rössler systems (see systems of ODEs given in Eqs. (1)–(3), Eqs. (4)–(6), Eqs. (7)–(9) and Eqs. (11)–(14)), first the required quadrature rules for the first-order derivative of the functions $x(t), y(t), z(t)$ and $w(t)$ are written from Eq. (18) as:

$$\begin{aligned} \dot{x}_i &= \sum_{j=1}^{m} A_{ij}x_j, & \dot{y}_i &= \sum_{j=1}^{m} A_{ij}y_j, \\ \dot{z}_i &= \sum_{j=1}^{m} A_{ij}z_j, & \dot{w}_i &= \sum_{j=1}^{m} A_{ij}w_j. \end{aligned} \tag{22}$$

The initial conditions are assumed to be:

$$\begin{aligned} &x(t=0) = x(t_1) = x_1 = x_0, \qquad y_1 = y_0, \\ &z_1 = z_0, \qquad w_1 = w_0. \end{aligned} \tag{23}$$

Substituting Eqs. (22) in systems of ODEs given in Eqs. (1)–(3), Eqs. (4)–(6), Eqs. (7)–(9) and Eqs. (11)–(14), and applying the corresponding initial conditions results in the following nonlinear systems of algebraic equations:

(I) The Lorenz system

$$\begin{cases} \displaystyle\sum_{j=2}^{m} A_{ij}x_j + A_{i1}x_0 = \sigma(y_i - x_i) \\ \displaystyle\sum_{j=2}^{m} A_{ij}y_j + A_{i1}y_0 = Rx_i - y_i - x_iz_i \\ \displaystyle\sum_{j=2}^{m} A_{ij}z_j + A_{i1}z_0 = x_iy_i + \gamma z_i \end{cases},$$

$$i = 2, 3, \dots, m. \tag{24}$$

(II) The Chen system

$$\begin{cases} \sum_{j=2}^{m} A_{ij}x_j + A_{i1}x_0 = a(y_i - x_i) \\ \sum_{j=2}^{m} A_{ij}y_j + A_{i1}y_0 = (c-a)x_i - x_iz_i + cy_i, \\ \sum_{j=2}^{m} A_{ij}z_j + A_{i1}z_0 = x_iy_i - bz_i \end{cases}$$

$$i = 2, 3, \ldots, m. \tag{25}$$

(III) The Genesio system

$$\begin{cases} \sum_{j=2}^{m} A_{ij}x_j + A_{i1}x_0 = y_i \\ \sum_{j=2}^{m} A_{ij}y_j + A_{i1}y_0 = z_i \\ \sum_{j=2}^{m} A_{ij}z_j + A_{i1}z_0 = -cx_i - by_i - az_i + x_i^2 \end{cases},$$

$$i = 2, 3, \ldots, m. \tag{26}$$

(IV) The hyperchaotic Rössler system

$$\begin{cases} \sum_{j=2}^{m} A_{ij}x_j + A_{i1}x_0 = -y_i - z_i \\ \sum_{j=2}^{m} A_{ij}y_j + A_{i1}y_0 = x_i + ay_i + w_i \\ \sum_{j=2}^{m} A_{ij}z_j + A_{i1}z_0 = b + x_iz_i \\ \sum_{j=2}^{m} A_{ij}w_j + A_{i1}w_0 = -cz_i + dw_i \end{cases},$$

$$i = 2, 3, \ldots, m. \tag{27}$$

The resulting nonlinear systems of algebraic equations, given in Eqs. (24)–(27), can be easily and efficiently solved by using various iterative methods. In this work, we use the Newton–Raphson method to solve systems (24)–(27). Our numerical experiment for the present problems shows that the Newton method with only 3–5 iterations produces accurate solutions.

4. A step-by-step DQ in time

If the whole time domain of interest is discretized simultaneously, the size of systems (25)–(28) then becomes large and many unknowns have to be solved simultaneously. This increases the CPU time, especially when the size of resultant system is too large. This problem also becomes a crucial one when very long-term solutions are required. As a result, it is more convenient to apply the DQ method as a step-by-step time integration scheme to advance the solutions progressively over the time domain of interest [45–48]. In this technique, the time domain of interest is first divided into several time elements (i.e., time steps). Since the time domain in not bounded, the DQ method can then be applied to each time element independently. The solutions at the end of each DQM time element will be used as the initial conditions for the next time element. This technique reduces considerably the computational cost, since the DQ method is applied to each time element independently, and a smaller system of nonlinear algebraic equations should be solved at each step. Note that each step in this technique itself consists of several sub-steps.

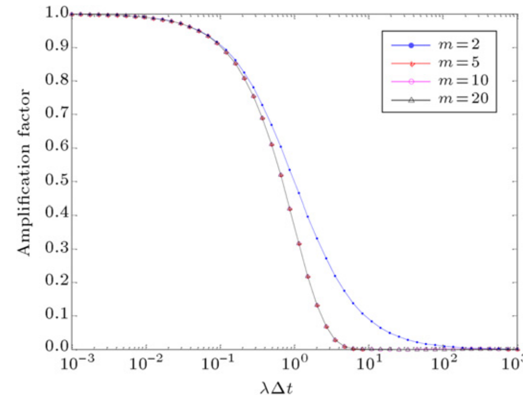

Figure 1: Stability of the DQ time integration scheme for the solution of first-order ODEs.

5. Numerical accuracy

Consider a function $f(t)$ which is approximated by the Lagrange interpolation polynomial of degree $(m-1)$. The error for the first- order derivative approximation of this function at point t_i can be obtained as [34]:

$$E\left[f(t_i)\right] = \frac{KM^{(1)}(t_i)}{m!}, \quad i = 1, 2, \ldots, m, \tag{28}$$

where K is a positive constant and:

$$M^{(1)}(t_i) = \prod_{j=1, j\neq i}^{m} (t_i - t_j). \tag{29}$$

It may be seen that very high accuracy can be achieved even if the number of sample points, m, is not too large. Also the accuracy of DQ method is proportional to m or its power.

6. Numerical stability

In addition to accuracy, another important aspect to be considered for the integration of ODEs is stability. One can loosely define stability as the property of an integration method to keep the errors resulting in the integration process of a given equation bounded at subsequent time steps. An unstable method will make the integration errors increase exponentially, and an arithmetic overflow can be expected even after just a few time steps. Since stability depends not only on the given method but also on the type of problem, the test equation $\dot{x}(t) = \lambda x(t)$, where λ is a complex valued constant, is customarily used to characterize the stability properties of a given method. This characterization is performed by defining the set of values of λ and Δt for which the corresponding method is stable. An algorithm is said to be A-stable if the solution to $\dot{x}(t) = \lambda x(t)$ tends to zero as $\lambda\Delta t \to \infty$ when $\mathrm{Re}(\lambda) < 0$, which means that the numerical solution decays to zero whenever the corresponding exact solution decays to zero. The most important consequence of the A-stability property is that there is no limitation on the size of $\lambda\Delta t$ for the stability of the integration process. A-stable algorithms have also been called unconditionally stable in the

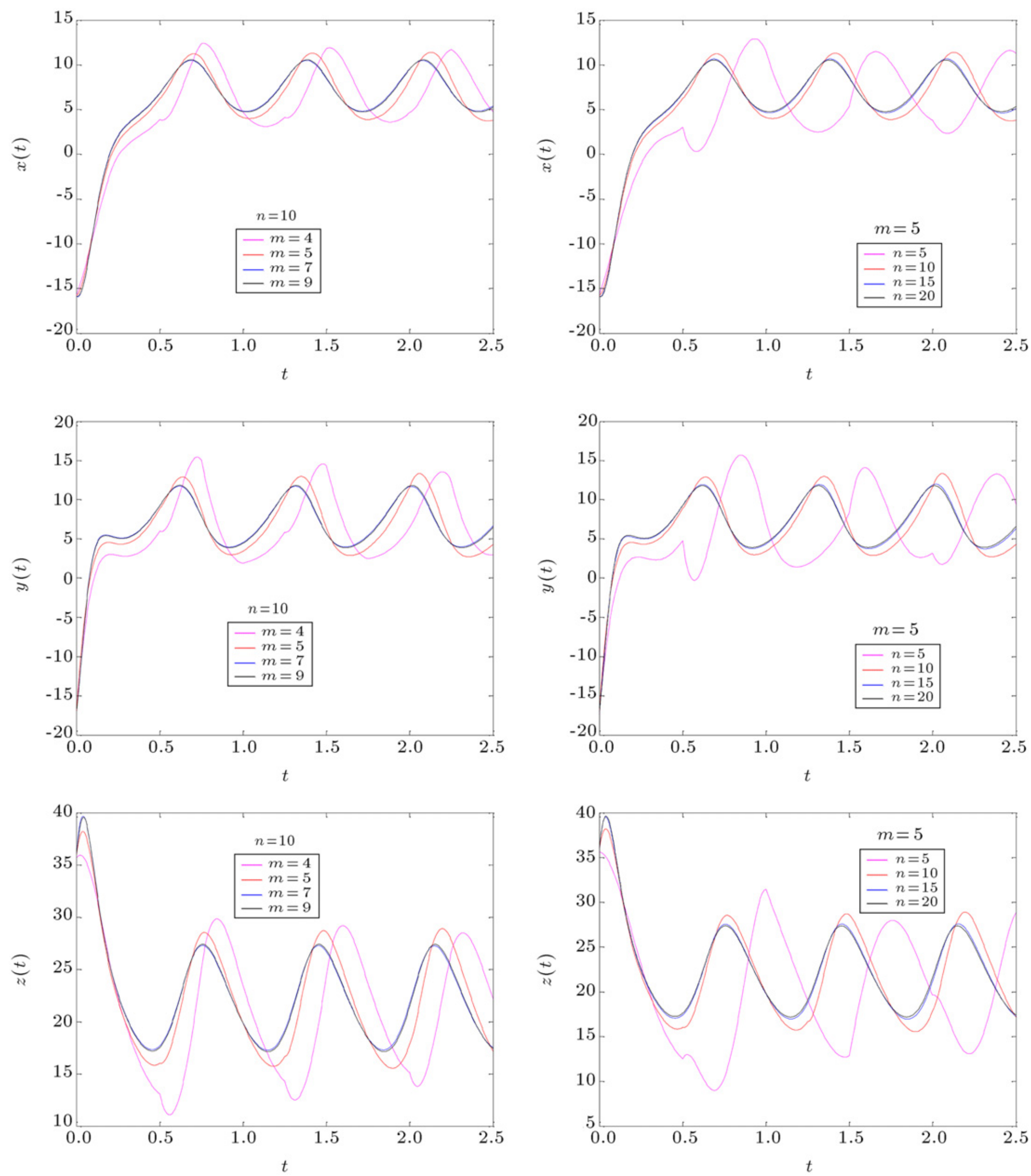

Figure 2: Convergence and accuracy of DQ time integration method for the solution of non-chaotic Lorenz system ($R = 23.5$).

linear setting. It is apparent that this property is very important and generally desired in the integration of nonlinear systems of ODEs, since the analyst would only have to be concerned with the step size for accuracy purposes and not for stability.

To investigate the stability property of the DQ time integration scheme, we consider the following differential equation

$$\dot{x}(t) = \lambda x(t), \tag{30}$$

with initial condition $x(t = 0) = x_0$. Eq. (30) can be normalized as:

$$\dot{x}(\tau) = \lambda \Delta t x(\tau), \quad \tau = t/\Delta t. \tag{31}$$

The stability is evaluated by calculating the amplification factor which relates the final state at the end of a time interval to the given initial condition at the beginning of the time interval, i.e.,

$$x(t = \Delta t) = A(\lambda \Delta t)x_0. \tag{32}$$

The algorithm is said to be A-stable if the amplification factor $|A(\lambda\Delta t)| \rightarrow 0$ as $\lambda\Delta t \rightarrow \infty$. The amplification factor is evaluated for different number of DQ sampling time points (m) and $\lambda\Delta t$ values. Figure 1 illustrates the variation of amplification factor versus $\lambda\Delta t$ for different number of DQ sample time points (m). The converging trend of DQ solutions with increasing number of sample time points is obvious in Figure 1. It can be observed that the amplification factor $|A(\lambda\Delta t)|$ tends to zero as $\lambda\Delta t \rightarrow \infty$. Therefore, the DQ time integration scheme is unconditionally stable (i.e., A-stable) in solving first-order ODEs.

7. Numerical results and discussion

In this section, the accuracy of the DQ method is tested against the traditional fourth-order Runge–Kutta (RK4) technique for the solution of the dynamical Lorenz, Chen, Genesio and Rössler systems. To accurately obtain the long-term

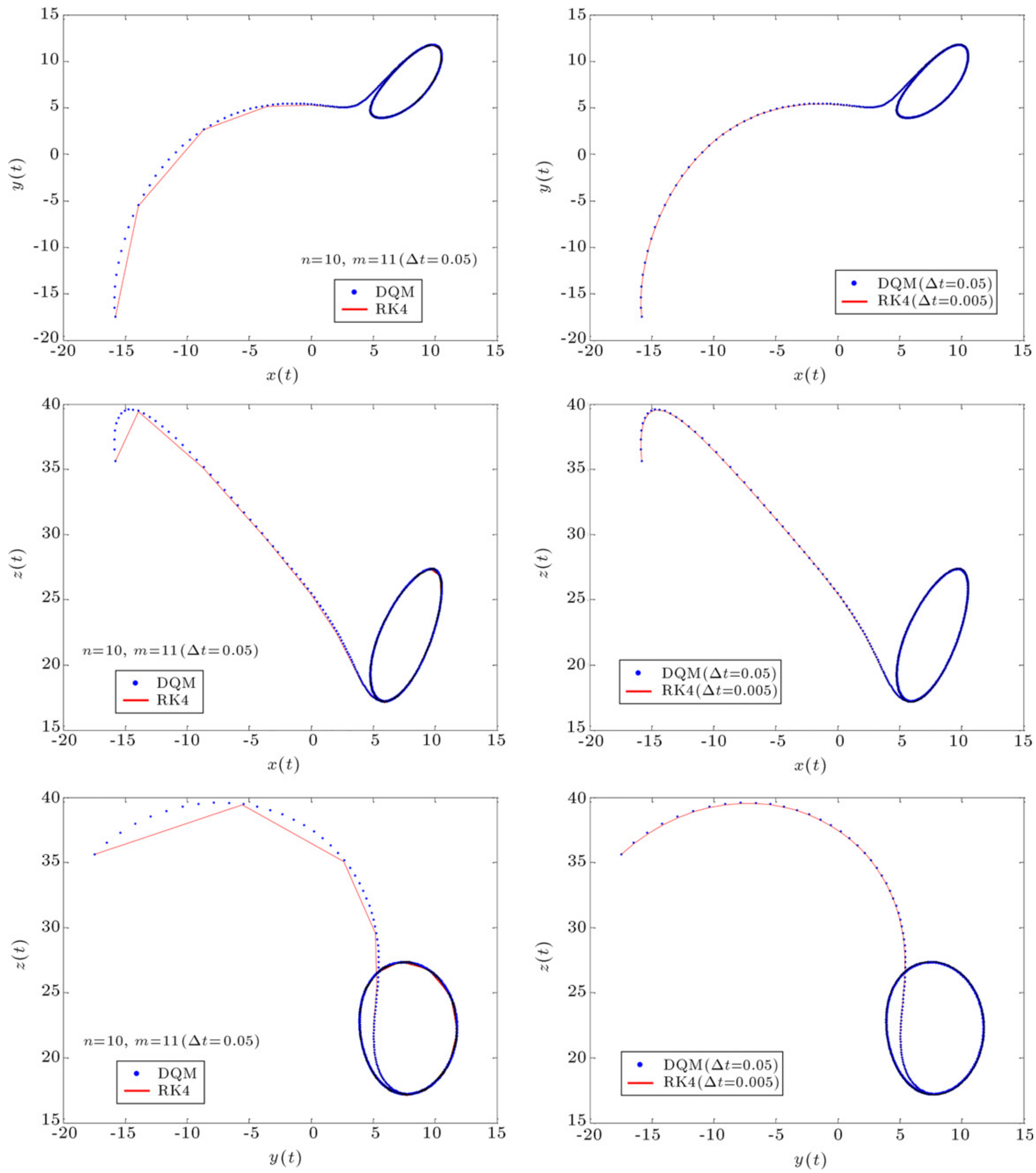

Figure 3: Phase portraits of the non-chaotic Lorenz system obtained using DQM and RK4 ($R = 23.5$).

solutions, the DQ method is employed as a step-by-step scheme (say DQEM: differential quadrature element method, see Section 4). To do so, the time domain of interest is divided into n equal DQM time element with m sample time points (per each DQM time element). The total number of sample time points and the average time step can be obtained as:

$$M_{\text{tot}} = n(m-1) + 1 \tag{33}$$

$$\Delta t = T/(M_{\text{tot}} - 1) = T/(n(m-1)). \tag{34}$$

Respectively, where T is the time domain of interest (i.e., time span). It is noted that the time step given in Eq. (34) is an average time step since the sample time points are taken non-uniformly spaced in the time domain (see Eq. (21)).

7.1. Numerical results for the dynamical Lorenz system

To test the accuracy and efficiency of the DQ time integration method and to provide a comparison of the results with those previously obtained by Guellal et al. [21], Hashim et al. [24] and Chowdhury et al. [27], the parameters of the problem are chosen as: $\sigma = 10$ and $b = -8/3$. The initial conditions of the problem are also considered as $x_0 = -15.8$, $y_0 = -17.48$ and $z_0 = 35.64$. Similar to the analysis done in [24,27], we also demonstrate the accuracy and convergence of the DQ method for the solutions of both non-chaotic and chaotic systems. For the purpose of comparison, we will consider two cases: $R = 23.5$ where the system is non-chaotic and $R = 28$ where the system exhibits chaotic behavior. In addition to the above cases, we also consider two cases $R = 50$ and $R = 100$, corresponding

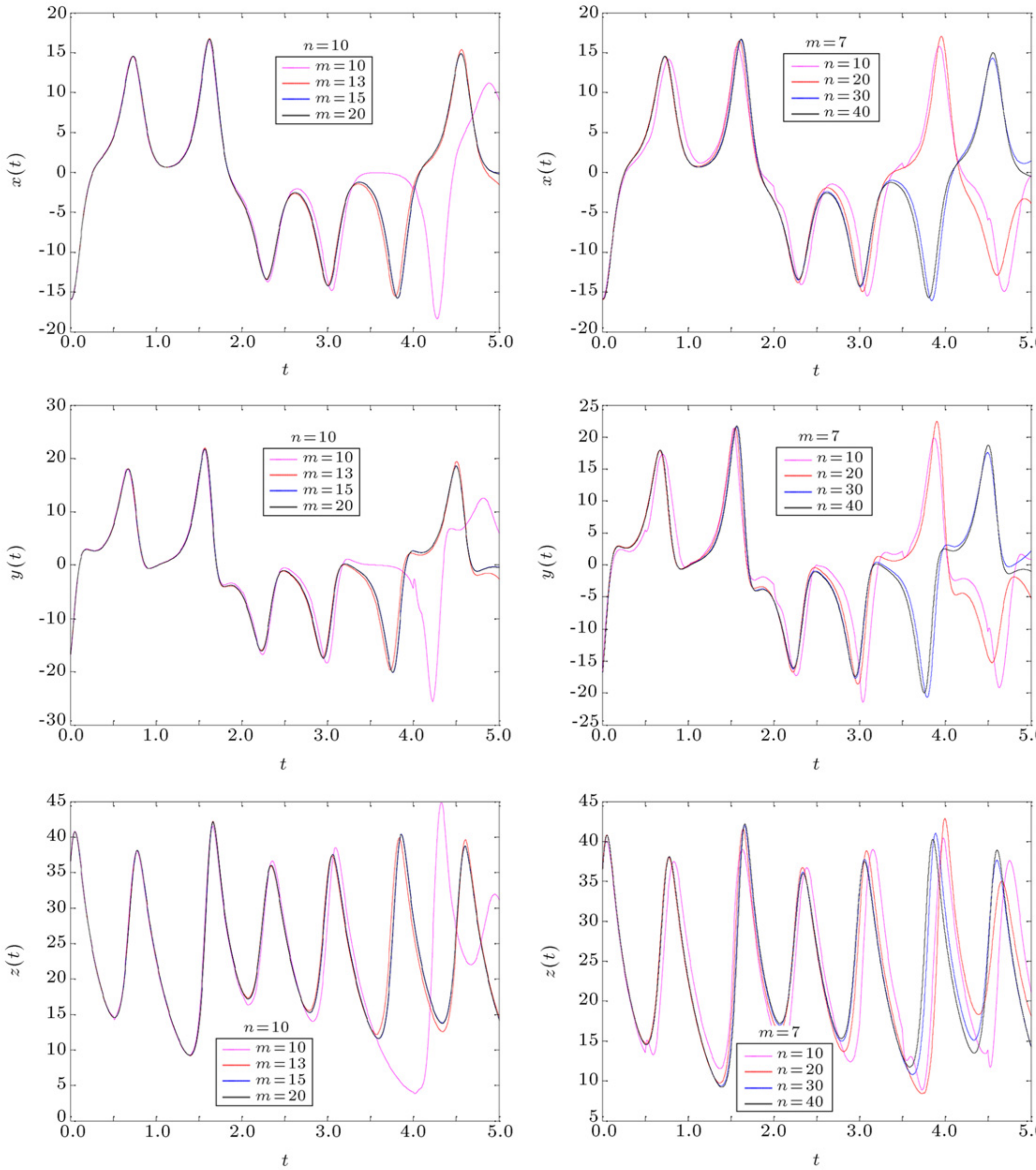

Figure 4: Convergence and accuracy of DQ time integration method for the solution of chaotic Lorenz system ($R = 28$).

to chaotic systems, in our attempt to show the applicability of DQ time integration method in prediction of behavior of chaotic systems.

7.1.1. Non-chaotic solutions

First, we consider the non-chaotic case where $\sigma = 10$, $b = -8/3$ and $R = 23.5$. Figure 2 presents the convergence of DQ solutions for this case. The use of Lagrange interpolation polynomials (in each DQM time element) enables us to reach a continuous representation of the approximate solutions. A good convergence trend of solutions can be observed. However, when the total number of sample time points is too small (i.e., when the size of time steps are too large), a visible phase shift can be observed. On the other hand, the accuracy of solutions can be controlled by choosing the proper values of n and m. In other words, the accuracy of solutions can be improved by increasing n and/or m. Figure 3 presents the phase planes obtained using the DQ method and the RK4. The numerical simulations are done in the time interval $0 \le t \le 5$. By comparing the DQ solutions with those of RK4, one can conclude that the DQ method gives more accurate solutions than the RK4 using a considerably larger time step sizes.

7.1.2. Chaotic solutions

As pointed out earlier, the system (1)–(3) with $R > 27.74$, and the other parameters given above, exhibits chaotic behavior. For chaotic behavior of Lorenz system, we consider three cases: $R = 28$, $R = 50$ and $R = 100$. When the system is chaotic, care should be taken in choosing a time step because the solutions are highly sensitive to time step size. Figure 4 shows the convergence of the DQ time integration method for the solutions of chaotic Lorenz system against the number of time elements, n, and the number of DQ sample time points per each time element, m, when $R = 28$. It can be observed that the DQ method encounters some large attenuation of amplitude and overshoot for long-term solutions when the time step is so

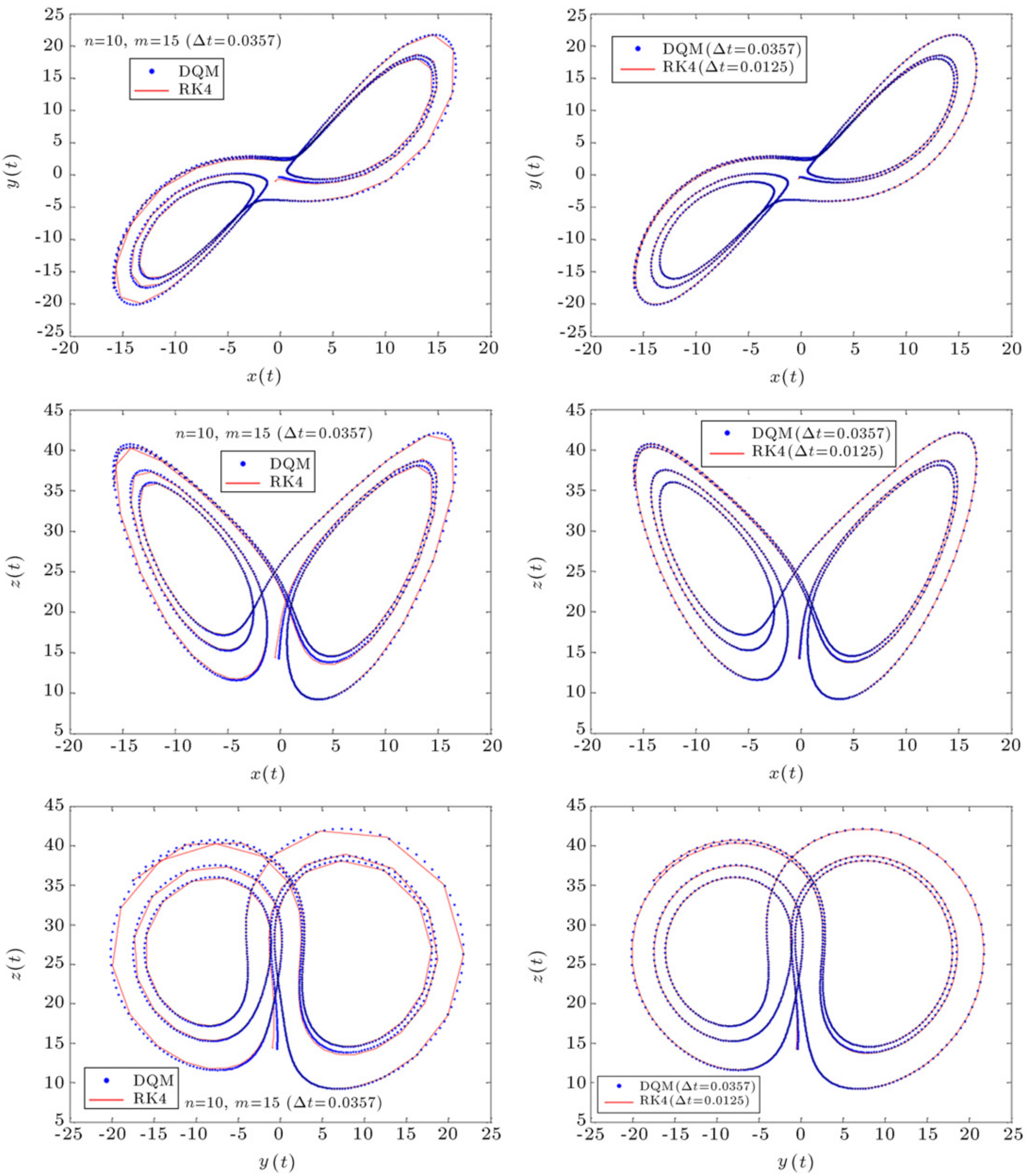

Figure 5: Phase portraits of chaotic Lorenz system obtained using DQM and RK4 ($R = 28$).

large (i.e., when n or m is too small). However, by decreasing the time step (i.e., by increasing n or m), the solutions converge to the true solutions. In conclusion, the DQ time integration scheme may be possible to yield inaccurate solutions for chaotic systems with an inappropriately too large time step sizes. Figure 5 shows the phase portraits of the Lorenz system, obtained by the RK4 and the DQmethod. It can be seen that the results of the DQ method with $\Delta t = 0.0357$ are comparable in accuracy to those of RK4 with $\Delta t = 0.0125$. This demonstrates the superiority of DQ time integration method over the conventional RK4 for the solution of chaotic Lorenz system.

Figure 6 presents the results for the chaotic Lorenz system with $R = 50$. Significant differences in numerical accuracy, amplitude attenuation and phase shift are easily observed from Figure 6 when using RK4 with $\Delta t = 0.0277$. It is also found that the DQ method confront some small attenuation of amplitude and overshoot for long-term solutions when $\Delta t = 0.0277$. However, the DQ solutions are better than those of RK4 in this case (i.e., when $\Delta t = 0.0277$). It is observed that both the DQ method and RK4 provide true solutions using sufficiently small time step $\Delta t = 0.0104$. By comparing the DQ solutions shown in Figures 2, 4 and 6, one can conclude that as the parameter R increases, the size of time step required to achieve accurate solutions decreases (i.e., the total number of sample time points required to accurately obtain the solutions, increases). This is actually due to the chaotic behavior of the Lorenz system. As the parameter R increases, the shape of dynamic responses becomes non-smoother, and thus, a larger number of sample time points (i.e., a smaller size of time steps) are required to obtain accurate solutions as the DQ method is basically based on the interpolation and derivation. The phase portraits of the Lorenz system obtained using DQ method and RK4 are given in Figure 7. The DQ solutions are obtained using $\Delta t = 0.01786$ while those of RK4 are calculated using $\Delta t = 0.01786$ and $\Delta t = 0.0104$. It can be seen that the DQ solutions with $\Delta t = 0.01786$ are comparable in accuracy to RK4 solutions

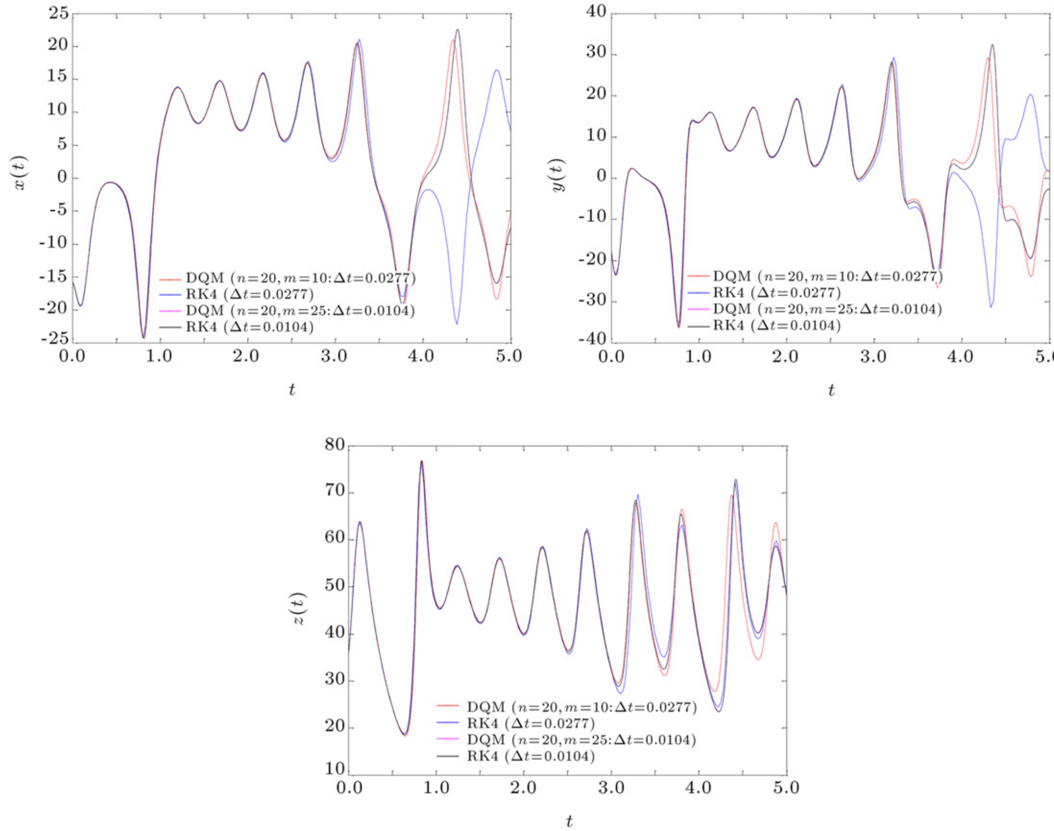

Figure 6: Accuracy of DQ time integration method for the solution of chaotic Lorenz system and comparisons with those of RK4 ($R = 50$).

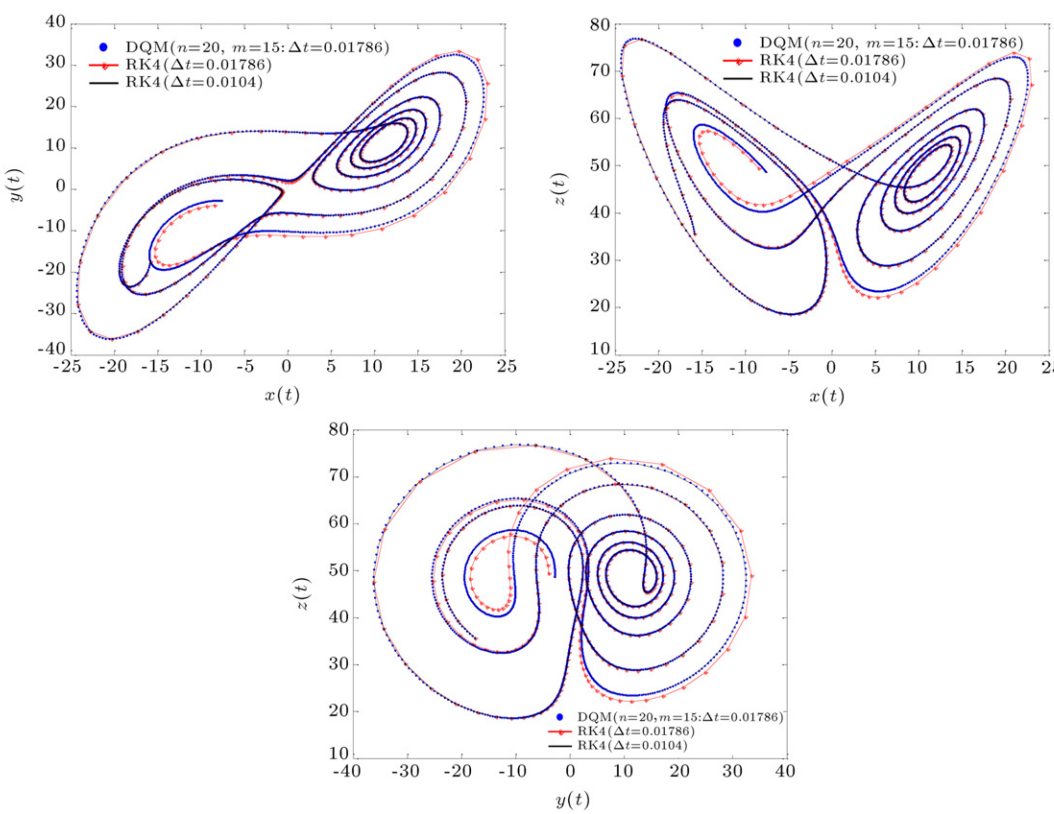

Figure 7: Phase portraits of chaotic Lorenz system obtained using DQM and RK4 ($R = 50$).

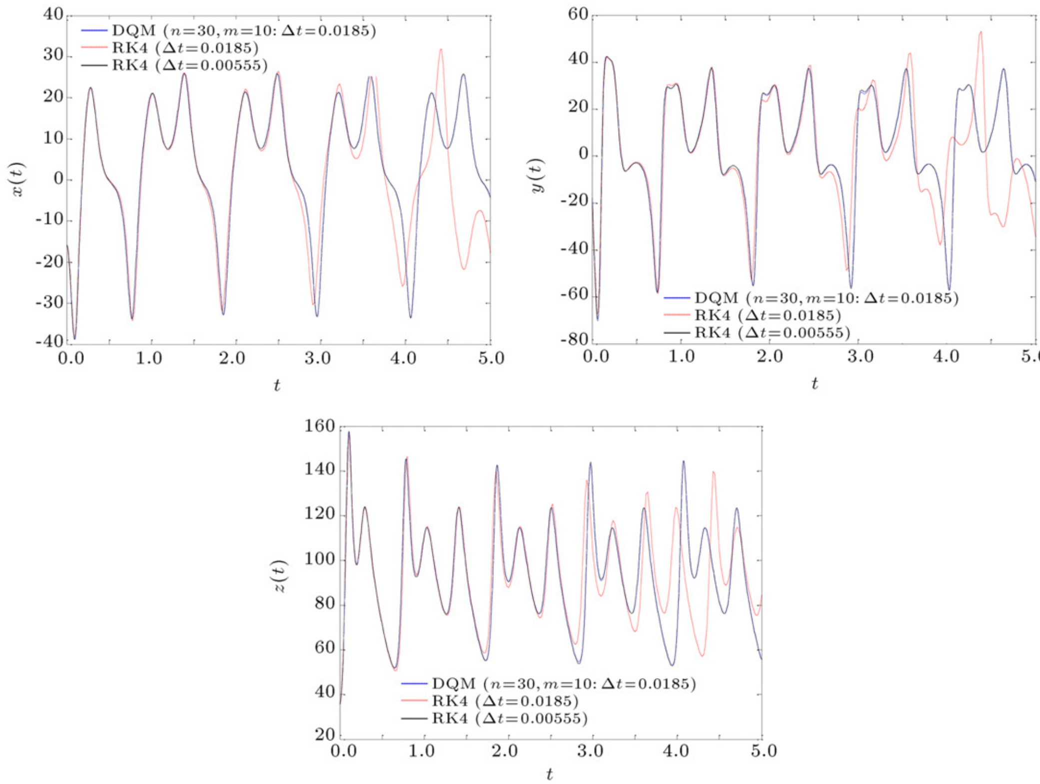

Figure 8: Accuracy of DQ time integration method for the solution of chaotic Lorenz system and comparisons with those of RK4 ($R = 100$).

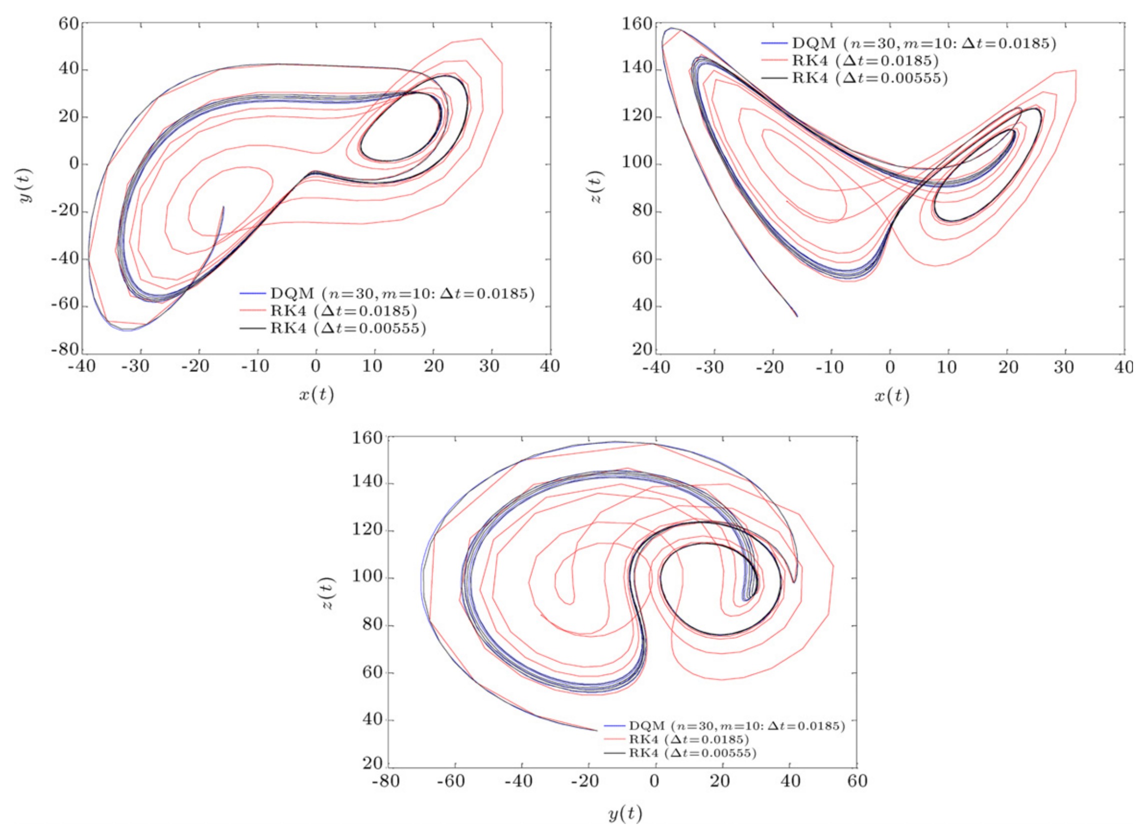

Figure 9: Phase portraits of chaotic Lorenz system obtained using DQM and RK4 ($R = 100$).

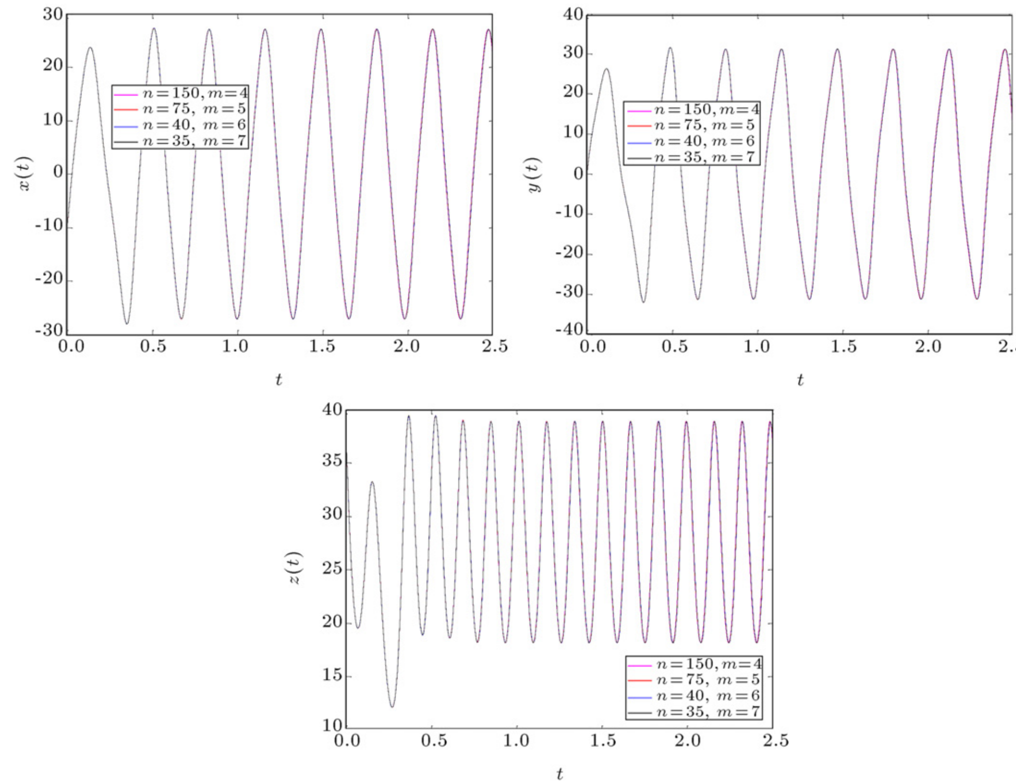

Figure 10: Convergence of the DQ time integration method for the solution of non-chaotic Chen system.

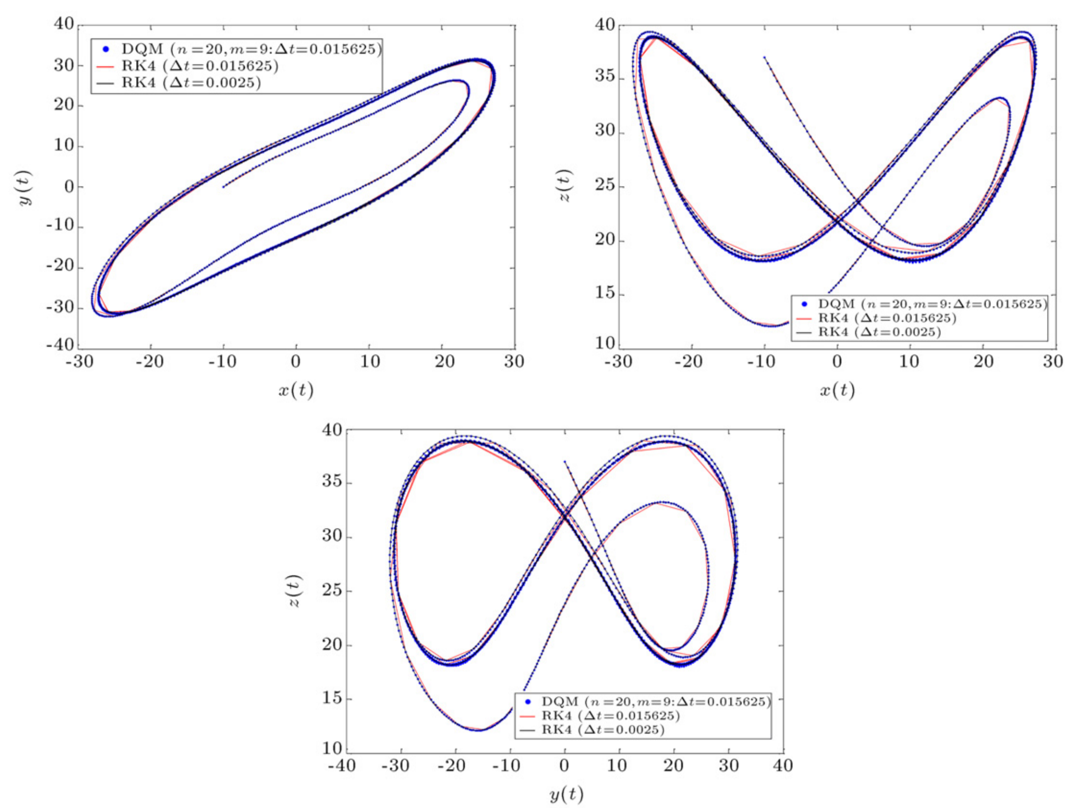

Figure 11: Phase portraits of the non-chaotic Chen system obtained using DQM and RK4.

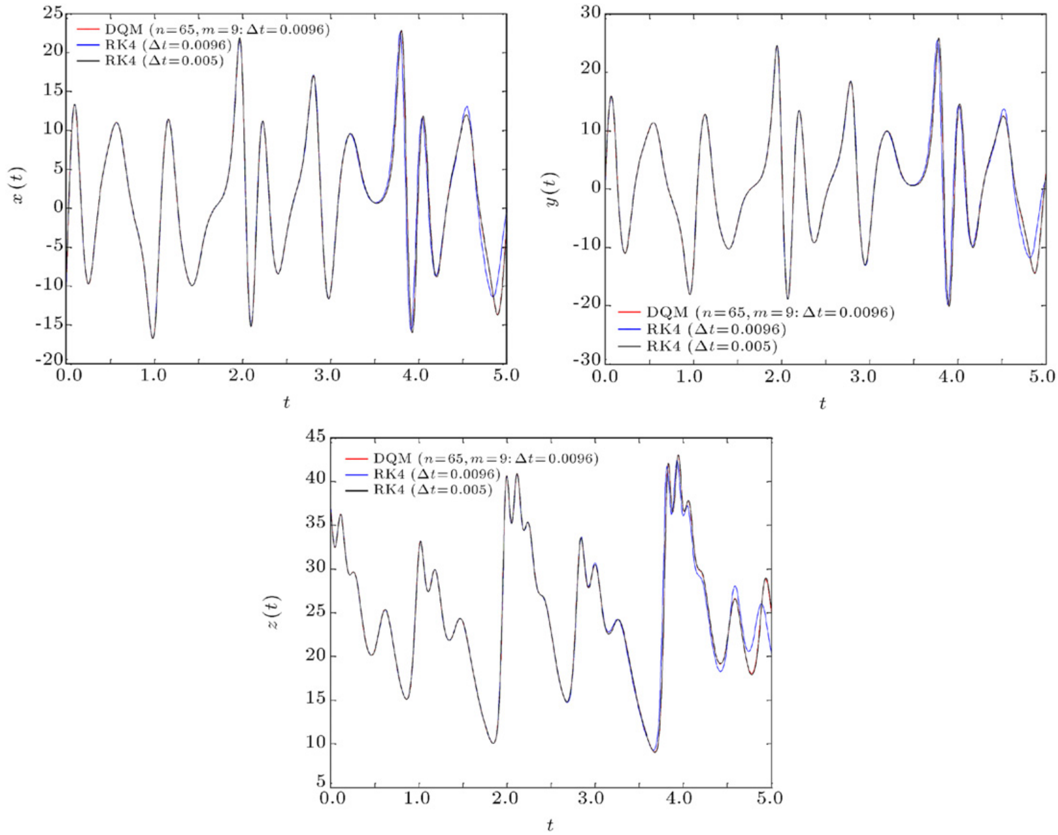

Figure 12: Accuracy of DQ time integration method for the solution of chaotic Chen system and comparisons with the RK4 solutions.

with $\Delta t = 0.0104$. It can also be observed that the RK4 solutions have visible phase shift when $\Delta t = 0.01786$.

Figures 8 and 9 illustrate the results for the chaotic Lorenz system with $R = 100$. From Figure 8, it can be observed that the DQ solutions with a rather large time step $\Delta t = 0.0185$ are comparable in accuracy to the RK4 solutions with a small time step $\Delta t = 0.00555$. Again, the solutions of the RK4 encounter a sharp drop of accuracy for the long-term responses when the size of time step is large, i.e., when $\Delta t = 0.0185$, as seen in Figure 8. As pointed out earlier and as it is illustrated in Figures 2, 4, 6 and 8, as the parameter R increases a smaller time steps should be used to ensure the convergence and to reach accurate solutions. The phase portraits of the DQ solution with $\Delta t = 0.0185$ and the RK4 with $\Delta t = 0.0185$ and $\Delta t = 0.00555$ are given in Figure 9. It can be observed that as compared to the RK4, the DQ method produces better results using a much larger time step size. Note that the RK4 solutions with $\Delta t = 0.0185$ are not acceptable in accuracy in this case.

7.2. Numerical results for the dynamical Chen system

In this sub-section, the applicability of the DQ method for the solution of the Chen system is investigated. In this analysis, we attempt to illustrate the accuracy and efficiency of the DQ method for the solutions of both non-chaotic and chaotic systems. We also fix the values of parameters $a = 35$ and $c = 28$ with $b = 12$ (for non-chaotic case) and $b = 3$ (for chaotic case). The initial conditions are set to be $x_0 = -10$, $y_0 = 0$ and $z_0 = 37$.

Figure 10 presents the convergence of the DQ time integration scheme for the solution of non-chaotic Chen system. An excellent convergency trend with the increase in the number of DQM time elements (n) and DQM sample time points (m) can be observed. It may be seen that by increasing the number of time elements, a smaller number of sample time points per element are required to achieve converged solutions. Figure 11 illustrates the $x-y$, $x-z$ and $y-z$ phase portraits of the non-chaotic Chen system obtained using the DQ method. The results of the RK4 are also shown for comparison. It can be observed that the results of the DQ method with $\Delta t = 0.015625$ are comparable in accuracy to those of RK4 with $\Delta t = 0.0025$.

Figure 12 displays the solutions of the chaotic Chen system. When a rather large time step $\Delta t = 0.0096$ is employed, it is found from Figure 12 that the RK4 method confronts some small attenuation of amplitude and overshoot for long-term response, whereas the DQ method gives very accurate solutions. It is also noted that the DQ solutions with the time step $\Delta t = 0.0096$ and those of RK4 with the time step $\Delta t = 0.005$ seem to coincide and overlap each other on the curves. The results for the phase portraits of the chaotic Chen system are shown in Figure 13. Again, one sees that the DQ method needs larger time steps than the RK4 to obtain accurate solutions. These results illustrate the capability of the DQ method in the solution of the chaotic Chen system and its superiority over the traditional RK4 method.

7.3. Numerical results for the dynamical Genesio system

The Genesio system, given in Eqs. (7)–(9), with the parameters $a = 1.2$, $b = 2.92$ and $c = 6$ exhibits chaotic behavior [17,26]. This case is considered in the present analysis. Since the system is chaotic, its solutions are expected to be very

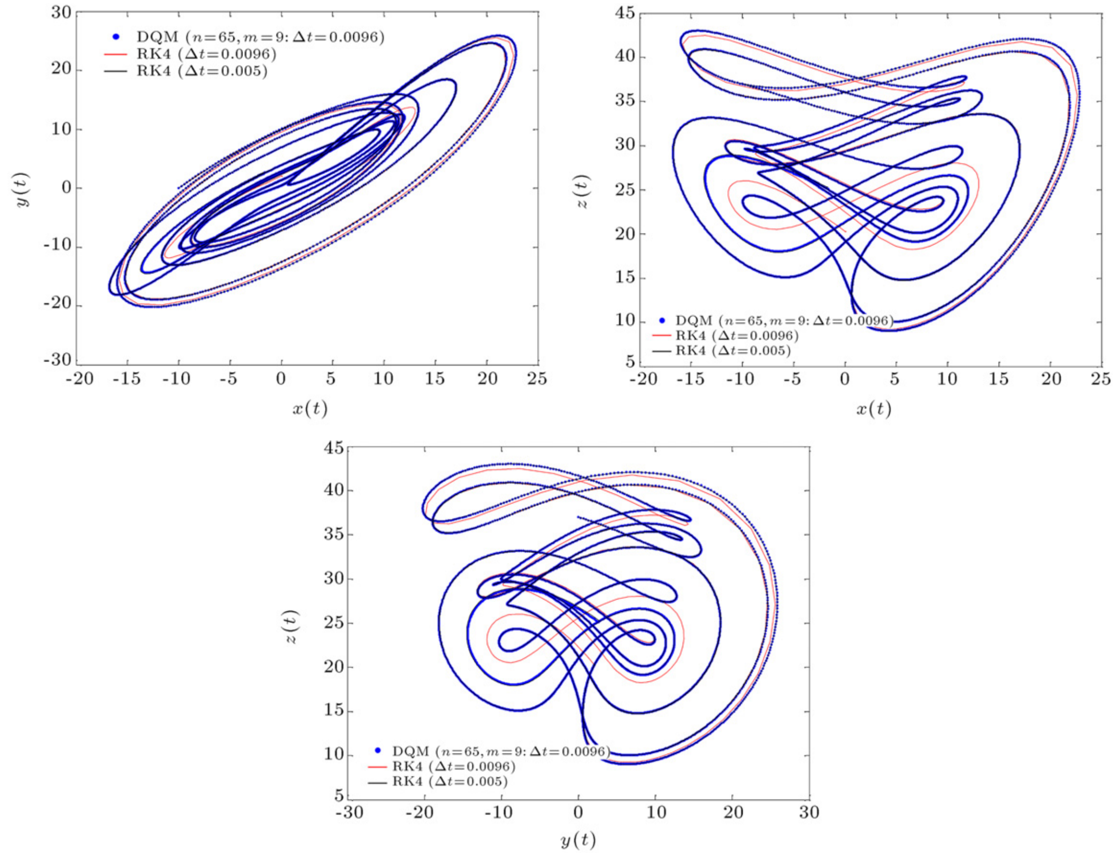

Figure 13: Phase portraits of the chaotic Chen system obtained using DQM and RK4.

sensitive to time step sizes. The initial conditions are set to be $x_0 = 0.2, y_0 = -0.3$ and $z_0 = 0.1$. The simulation is done in the time domain $t \in [0, 12]$.

Goh et al. [26] also solved the present problem using the variational iteration method (VIM) and multistage variational iteration method (MVIM). The key element of VIM is to use linearization for the mathematical problem in order to find its initial approximation or trial functions. Then a well-accurate approximation at some special point can be determined. VIM is relatively simple and straightforward to use, but one may face long computational time due to possible exponential coefficients in its iterations.

Moreover, VIM is only reliable and applicable on a small domain of time. In other words, VIM is a conditionally stable method and the validity domain of its solution (say stability domain) is often an issue. For example, in solving the present problem, Goh et al. [26] has shown that the VIM solutions are only valid for $t \ll 2$. The results of the VIM tend to deviate after that. To overcome the above-mentioned difficulty, Goh et al. [26] proposed a multistage variational iteration method (MVIM). MVIM can give considerably accurate results on a longer time spans compared to VIM. However, similar to VIM, MVIM is also a conditionally stable method (i.e, its solution is not valid for long time duration) and an attempt should be made to obtain its stability/validity domain. The results of Goh et al. [26] for the present problem have shown that MVIM gives rather accurate solutions on a longer time span of $t \in [0, 11]$. But its solutions tend to deviate after $t = 11$. Therefore, the major question is not responded using the MVIM.

Since the DQ method is a high-order method, it can easily tackle the above-mentioned difficulty. In other words, the DQ method can accurately obtain the long-term solutions. Moreover, similar to analytical methods such as the VIM and MVIM, the DQ method has the ability to provide a continuous representation of the approximate solution. This characteristic distinguishes the DQ method from the conventional single time step methods such as the RK4.

Figure 14 presents the convergence of the DQ method for the solution of chaotic Genesio system. An excellent converging trend of DQ solutions with increasing n (number of DQM time element) and m (number of DQM sample time points per DQM time element) can be observed. In Figures 15 and 16 the results of the DQ method are compared with those of RK4. It can be observed that the DQ method can accurately predict the long-term solutions of the Genesio system. Again, the DQ method gives more accurate results than the RK4 using larger time step sizes.

7.4. Numerical results for the dynamical Rössler system

As pointed out earlier, the Rössler system exhibits a hyperchaotic behavior when $a = 0.25, b = 3, c = 0.5$ and $d = 0.05$. Thus, this case is considered in the numerical simulation. The initial conditions are also set to be $x_0 = -20, y_0 = 0, z_0 = 0$ and $w_0 = 15$. To demonstrate the capability of the DQ method for computing the long-term solutions of the hyperchaotic Rössler system, the numerical simulation is done in the time domain $t \in [0, 50]$.

The results of the present problem are shown in Figure 17. The converging trend of DQ solutions with increasing m and n is obvious in Figure 17. It can be seen that as m increases, a larger time steps are required to achieve accurate solutions.

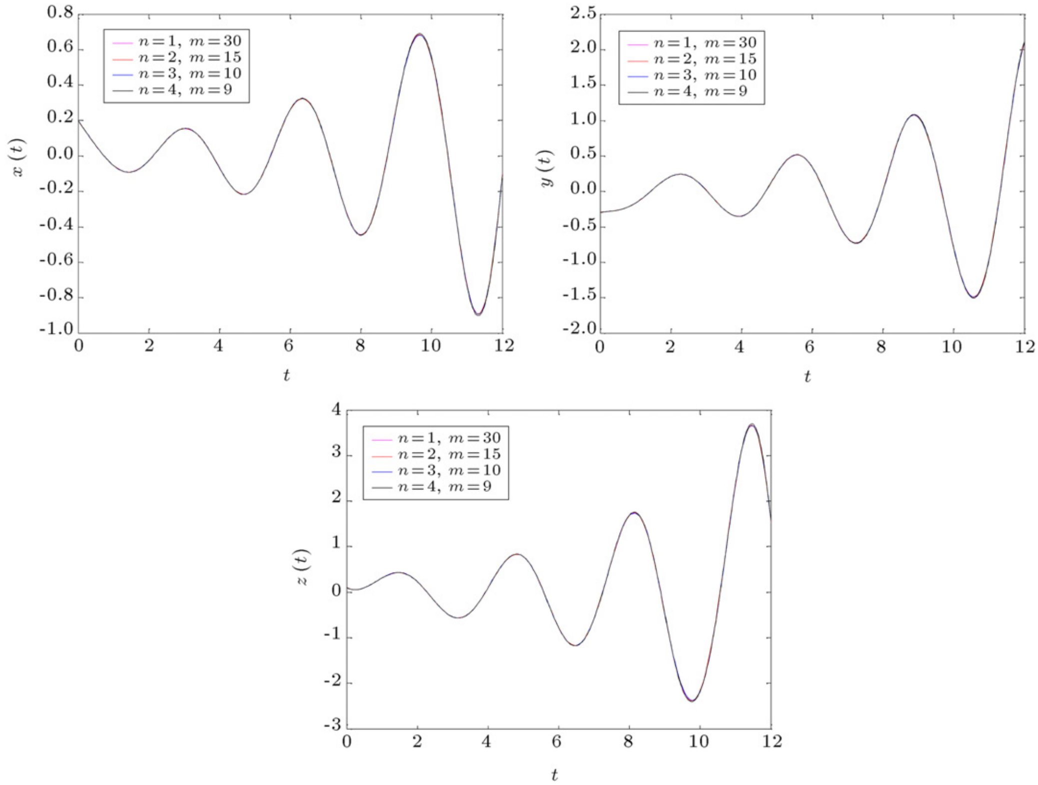

Figure 14: Convergence of the DQ time integration method for the solution of chaotic Genesio system.

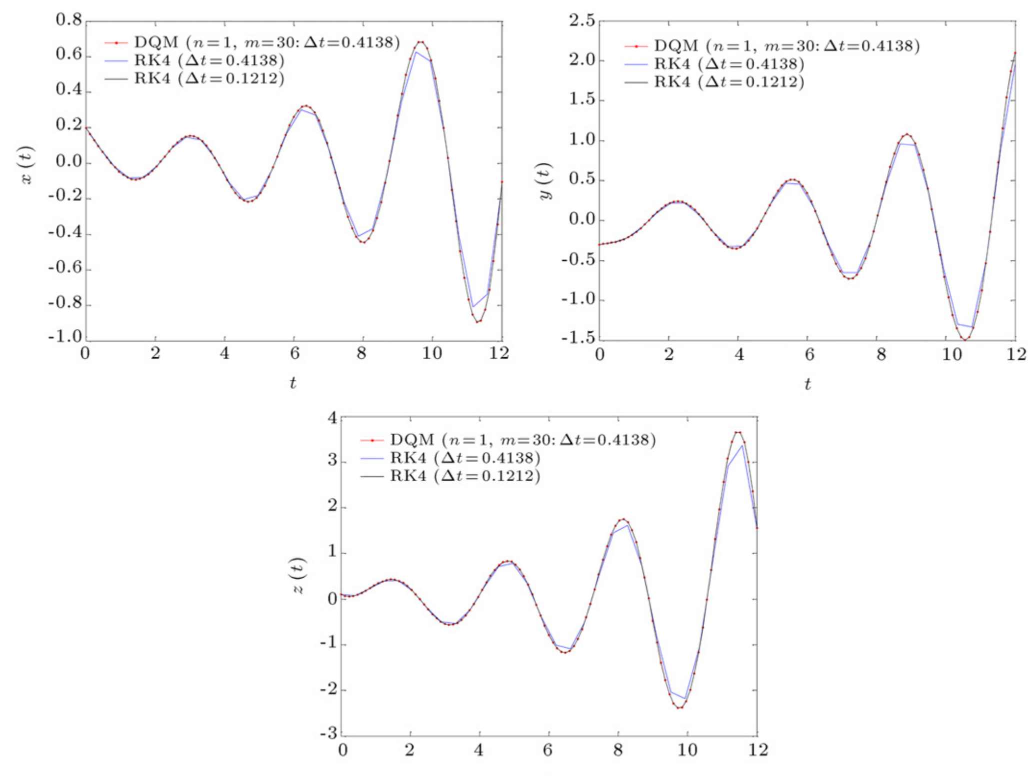

Figure 15: Accuracy of DQ time integration method for the solution of chaotic Genesio system and comparisons with the RK4 solutions.

Therefore, to obtain accurate results using the DQ method with a reasonably large time steps, the number of sample time points per DQM time element (i.e., m) should be rather large. If m is too small, then we should use a very large number

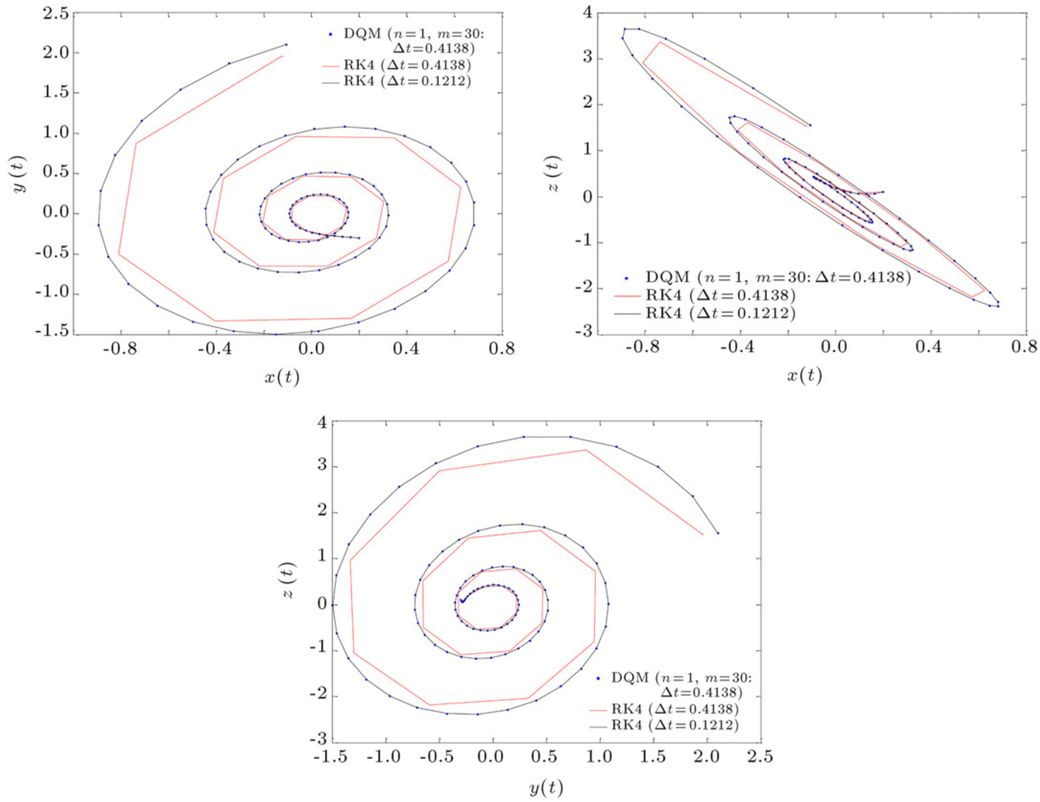

Figure 16: Phase portraits of the chaotic Genesio system obtained using DQM and RK4.

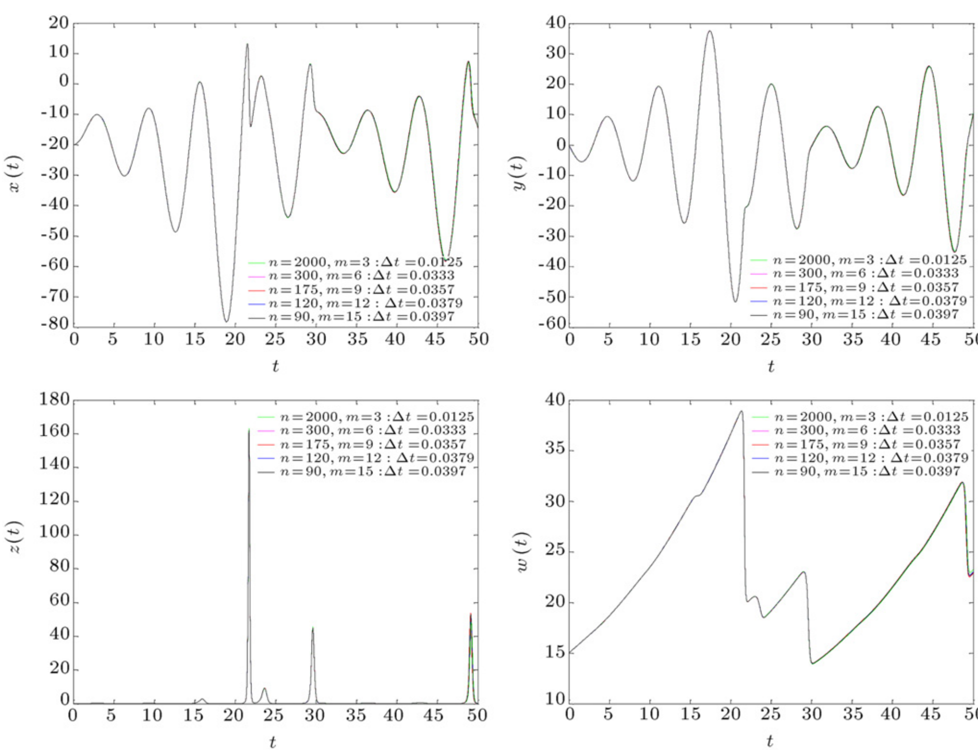

Figure 17: Convergence of the DQ time integration method for the solution of chaotic Rössler system.

of time elements to achieve accurate solutions and this may increase the CPU time considerably. In Figure 18, the results of the DQ method are compared with those of the RK4. By comparing the present results with those of other chaotic systems considered in this paper (see Sections 7.1–7.3), one sees that the differences between the results of the DQ method

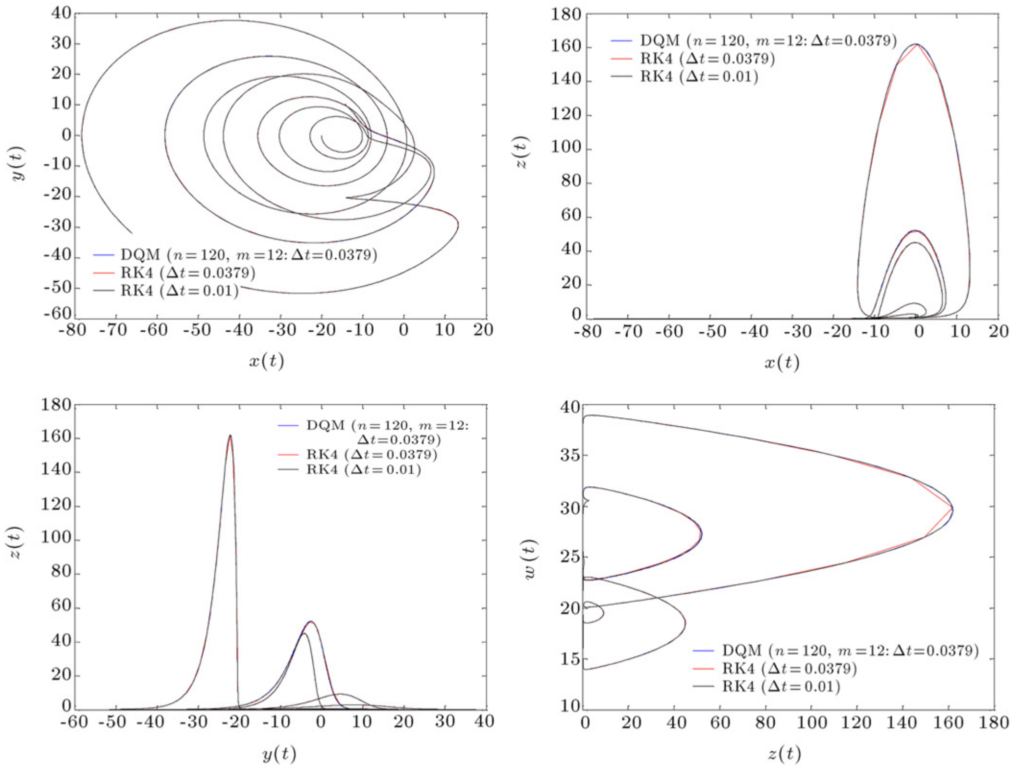

Figure 18: Phase portraits of the chaotic Rössler system obtained using DQM and RK4.

and RK4 solutions for large time steps for hyperchaotic Rössler system are less than those for other chaotic systems (i.e., Lorenz, Chen and Genesio systems). In other words, the RK4 for the hyperchaotic Rössler system is much more accurate than the RK4 for other chaotic systems. This may be due to the lower Lyapunov exponents of the hyperchaotic Rössler system than those of Lorenz, Chen and Genesio systems. However, the DQ method always shows the high accuracy and efficiency. Concluding the above four examples (given in Sections 7.1–7.4), there are essential differences in the accuracy, amplitude attenuation and phase shift behaviors between the DQ method and the classical RK4 if large time step is chosen for computing economy.

8. Conclusion

In this paper, the DQ method is used for solving some chaotic dynamical systems, namely, Lorenz, Chen, Genesio and Rössler systems. Based on the numerical results reported herein, one can conclude that the DQ time integration scheme is reliable, computationally efficient and also suitable for time integrations over long time duration. But care should be taken when applying the DQ method to chaotic systems. For the dynamical Lorenz and Chen system, the numerical results are given for both non-chaotic and chaotic cases. It is found that the simulation of chaotic cases needs smaller time steps than the non-chaotic cases. It is also found that the unconditionally stable DQ time integration scheme may be also possible to yield inaccurate results for chaotic systems with an inappropriately too large time step. Comparisons are made between the solutions of the DQ time integration method and those of RK4. It is revealed that the DQ method produces much better accuracy than the RK4 using much larger time step sizes. Thus, the DQ time integration method seems to be an effective and promising tool for handling the chaotic systems.

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