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FINITE INJURY ARGUMENTS IN INFINITE COMPUTATION THEORIES*

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0. Introduction

A significant part of post-Friedberg recursion theory has been successfully generalized to recursion theory on an admissible ordinal α. Such a recursion theory has two properties seemingly important for priority arguments: it is an "infinite" theory and its domain is recursively wellordered. Kreisel ([5], pp. 172–173) has asked (with some persistence — see his reviews of [6] and [13] in Zentralblatt 1973 and 1976 respectively) whether these properties are significant for the existence of incomparable r.e. degrees. Recently Sy Friedman [4] has considered the first property by doing recursion theory over an arbitrary limit ordinal β, thus dropping the admissibility criterion. His main result is the existence for many β of a pair of sets Σ_1 over $L(\beta)$ such that neither is β-recursive in the other. We, on the other hand, are keeping admissibility while relaxing the requirement of a wellordered domain to that of a prewellordered domain, that is we are essentially studying recursion theory over resolvable admissible sets with urelements.

However, rather than restricting our attention to resolvable admissible sets, our approach in this paper is axiomatic. Starting with a precomputation theory in the sense of Moschovakis [6] with a computable selection operator, we add two axioms to obtain an infinite computation theory. The first asserts the existence of a prewellorder whose initial segments are uniformly "finite", while the second insures that all "computations" can be effectively generated and that this generation is matched up with the complexity of the domain as expressed by the prewellorder. The class of infinite computation theories coincides with the class of Friedberg theories as defined in [6].

It is doubtful (see Simpson [14]) whether the axioms for an infinite theory are quite adequate for giving a positive solution to Post's problem.[1] A trivial but

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[1] L. Harrington has recently shown the following: Con(ZF)$\Rightarrow$Con(ZFC + Post's problem has a negative solution for $H(\aleph_2) = \{X : |TC(X)| < \aleph_2\}$). It is still open (not assuming A' ') whether there is a resolvable admissible set with a negative solution to Post's problem.

significant observation for α-recursion theory (or for any recursively wellordered infinite theory) is that any α-r e set bounded strictly below α^*, the projectum of α, is α-finite We call an infinite computation theory adequate whenever the analogous theorem holds For adequate theories we prove a strong form of Sacks' splitting theorem [7, 10], thereby supporting the conjecture that any of the usual finite injury priority arguments can be carried out for such theories

In Section 1 we give the axioms for an infinite computation theory and prove some elementary results Section 2 introduces different notions of relative computability and gives sufficient conditions in terms of regularity and hyperregularity for the notions to coincide Shore's blocking technique using Σ_2 functions is developed in Section 3 while the proof of the splitting theorem, along with some of the usual corollaries, is given in Section 4

S G Simpson (see [14] and [13]) has independently studied recursion theory over resolvable admissible sets In particular, he was the first to note that Shore's blocking technique could be used to obtain a version of the Friedberg–Muchnik theorem for what he calls thin admissible sets

1. Infinite computation theories

We will be dealing with partial multivalued functions and functionals on some set U An n-ary partial multivalued function is just an $n+1$-ary relation Following the notation of Moschovakis [6] we mean by $f(x_1, \quad, x_n) \to z$ that the partial multivalued function f has z as one of its values at $x_1, \quad, x_n$, i e $(x_1, \quad, x_n, z)$ is an element of the defining relation for f In case f is singlevalued we may without confusion write $f(x_1, \quad, x_n) = z$ for $f(x_1, \quad, x_n) \to z$ In this paper partial multivalued functions on U will simply be called *functions*, whereas a total singlevalued function will be called a *mapping*

The notation used should easily be understood from the context keeping the following *loosely* defined conventions in mind Functions on U are denoted by f $g, h, \quad, p, q, r, \qquad \alpha, \beta$ are reserved for ordinals and i, j, m, n for elements in N Remaining lower case latin and greek letters (except λ μ and ν which will have their usual meanings) denote elements of U

A *computation domain* is a structure $\mathfrak{A} = \langle U, N, s, M, K, L\rangle$ where U is a set, $N \subseteq U$, $\langle N, s \restriction N\rangle$ is isomorphic to the natural numbers with the successor function, M is a pairing function and K and L are inverses to M The latter means that if $M(x, y) = z$, then $K(z) = x$ and $L(z) = y$ From M, K and L we define the tupling function $\langle \quad \rangle$ and its ith inverse $(\quad)_i$ in the usual fashion

A set $\Theta \subseteq \bigcup \{U^n \quad n \geq 2\}$ is called a *computation set* on $\mathfrak{A}$ For a computation set Θ we define the relation

$$\{\varepsilon\}^n_\Theta(\boldsymbol{x}) \to z \quad \text{iff} \quad \mathrm{lh}(\boldsymbol{x} = n \quad \& \quad (\varepsilon \ \boldsymbol{x}, z) \in \Theta$$

where $\mathrm{lh}(\boldsymbol{x})$ denotes the length of the sequence $\boldsymbol{x}$ Thus $\{\varepsilon\}^n_\Theta$ defines an n-ary function for each $\varepsilon \in U$ and $n \in N$. An n-ary function f on U is *Θ-computable* if there is $\varepsilon \in U$ such that $f = \{\varepsilon\}^n_\Theta$, in which case ε is a *Θ-index* for f. An n-ary relation R on U is *Θ-semicomputable* (Θ-s.c.) with a *Θ-s.c. index* ε if R is the domain of a Θ-computable function with Θ-index ε. R is *Θ-computable* with Θ-index ε in case its characteristic mapping c_R is Θ-computable with Θ-index ε. Finally we say that a consistent functional $F(f_1, \ldots, f_k, \boldsymbol{x})$, where f_i varies over n_i-ary functions, is *Θ-computable* with *Θ-index* δ if

$$\forall \varepsilon_1, \ldots, \varepsilon_k, \boldsymbol{x}, z\big(F(\{\varepsilon_1\}^{n_1}_\Theta, \ldots, \{\varepsilon_k\}^{n_k}_\Theta, \boldsymbol{x}) \to z \Leftrightarrow \{\delta\}^{k+n}_\Theta(\varepsilon_1, \ldots, \varepsilon_k, \boldsymbol{x}) \to z\big)$$

The first step in putting some structure on a computation set Θ is to require Θ to be a *precomputation theory* in the sense of Moschovakis. For a precise definition we refer to [6]. Roughly speaking, Θ is a precomputation theory if the constant mappings, the identity mapping, M, K, L and s are Θ-computable. Furthermore the Θ-computable functions must satisfy the usual closure and enumeration conditions in a uniform way. A basic fact of precomputation theories is the second recursion theorem.

The existence of a Θ-computable selection operator is normally assumed in order for the Θ-s.c. relations to behave nicely. A *selection operator* for Θ is a function q such that (henceforth dropping n and Θ from $\{\varepsilon\}^n_\Theta$ whenever possible)

$$q(\varepsilon)\downarrow \Leftrightarrow \exists x\{\varepsilon\}(x)\downarrow$$

and

$$\forall z(q(\varepsilon) \to z \Rightarrow \{\varepsilon\}(z))\downarrow)$$

where "$\downarrow$" means "is defined". Note that the existence of a selection operator implies the existence of a uniform selection operator. That is there exists a Θ-computable mapping $p(n)$ such that for each $n \in N$, $p(n)$ is a Θ-index for an n-ary selection operator q^n. The usual ν notation for a uniform selection operator will be used, namely

$$\nu z(\{\varepsilon\}^{n+1}_\Theta(z, \boldsymbol{x}\downarrow) = q^n(\varepsilon, \boldsymbol{x})$$

For a precomputation theory Θ with a Θ-computable selection operator, the Θ-s.c. relations are closed under disjunctions and existential quantification and a relation is Θ-computable iff it and its complement are Θ-s.c. Furthermore Θ-computable functions can be defined by cases in a general way.

In this setting one can define a well behaved notion of "finite". Following Moschovakis we say that a set K is *Θ-finite* if the consistent functional

$$E_K(f) \to \begin{cases} 0 & \text{if} \quad \exists x \in K\,(f(x) \to 0), \\ 1 & \text{if} \quad \forall x \in K\,(f(x) \to 1) \end{cases}$$

is Θ-computable. A Θ-index for E_K is said to be a *canonical Θ-index* for the

Θ-finite set K. The usual properties (see [6]) of a generalized notion of finite hold uniformly.

Having asserted the existence of a selection operator it is too restrictive to require Θ to be a single-valued theory as this would exclude some of the intended models. However when considering functions whose values are (canonical Θ-indices for) Θ-finite sets, then Θ is essentially single-valued. In the lemma stated below let K_η denote the Θ-finite set with canonical Θ-index η in case η is such an index.

Lemma 1.1. *Suppose r is a Θ-computable function whose values are canonical Θ-indices such that* $\forall x,\xi,\eta(r(x)\to\xi\ \&\ r(x)\to\eta\Rightarrow K_\xi = K_\eta)$.

Then there is a Θ-computable mapping q obtained uniformly from r such that $\forall x,\eta(r(x)\to\eta\Rightarrow K_\eta = K_{q(x)})$.

We now list two additional axioms making Θ into an infinite computation theory.

Axiom 1. There is a Θ-computable prewellorder $\leqslant$ on U such that initial segments of $\leqslant$ are uniformly Θ-finite.

Given a prewellorder $\leqslant$ we let $x<y$ denote $\neg(y\leqslant x)$ and $x\sim y$ denote $x\leqslant y\ \&\ y\leqslant x$.

Definition 1.2. A $(\leqslant)$-*enumeration* of a set W is a Θ-computable mapping $\lambda\sigma W^\sigma$ (whose values are canonical Θ-indices for the Θ-finite sets W^σ) such that

(i) $\tau\leqslant\sigma\Rightarrow W^\tau\subseteq W^\sigma$,

(ii) $W=\bigcup\{W^\sigma : \sigma\in U\}$.

Axiom 2. There is a Θ-computable mapping $p(n)$ such that for each $n\in N$, $p(n)$ is a Θ-index for a $(\leqslant)$-enumeration of the set

$$T_n=\{\langle\varepsilon,\boldsymbol{x},y\rangle : \{\varepsilon\}(\boldsymbol{x})\to y\quad\&\quad \mathrm{lh}(\boldsymbol{x})=n\}$$

Definition 1.3. Let Θ be a computation set over a computation domain $\mathfrak{A}$. Θ is an *infinite computation theory* if

(i) Θ is a precomputation theory.

(ii) Equality on U is a Θ-computable relation.

(iii) Θ has a computable selection operator.

(iv) Axiom 1 and Axiom 2 hold for some prewellorder $\leqslant$ on U.

A basic fact of infinite recursion theories, e.g. recursion on an admissible ordinal α, is that computations can be coded effectively into the domain in such a way that the complexity of the domain corresponds to the complexity of the coded computations It therefore seems reasonable to assert the existence of a Θ-computable partial order $\leqslant$ whose initial segments are well-founded and uniformly Θ-finite Here we restrict ourselves to the case where $<$ is a prewellorder Axiom 2 then stipulates that all "computations" $\{\varepsilon\}(\boldsymbol{x}) \to y$ can be effectively generated and that this generation is matched up with the complexity of the domain Note that U is not Θ-finite for an infinite computation theory

Following the notation of Barwise [1], let $\mathscr{A}_u$ be a resolvable admissible set with urelements relative to a language $L^* = L(\in, \quad)$ By combining Moschovakis' characterization theorem for Friedberg theories with Gandy's theorem for Σ_1 inductive definitions over admissible sets (see Barwise [1] p 208), it is easily verified that $\mathscr{A}_u$ constitutes an infinite computation theory

J Stavi (unpublished) has shown the converse of Gandy's theorem to be false for some transitive sets However, for resolvable transitive structures $\mathscr{A}$ (closed under pairing and satisfying Δ_0-separation) the converse is true, i e for such $\mathscr{A}$, $\mathscr{A}$ is admissible iff every Σ_1 inductive operator over $\mathscr{A}$ has a Σ_1 fixed point This result, due to A Nyberg [16], gives some justification for Axiom 1

In the sequel Θ will always denote an infinite computation theory over some computation domain $\mathfrak{A}$

Lemma 1.4. *Suppose f is a Θ-computable function Then there is a Θ-computable function g obtained uniformly from f such that* $\operatorname{dom} g = \operatorname{dom} f$, $g \subseteq f$, *and for each Θ-finite set $K \subseteq \operatorname{dom} f$ there is a Θ-finite set N obtained uniformly from K and f such that $g(K) \subseteq N \subseteq f(K)$*

Before proceeding with the proof we need to introduce the μ-operator By $\mu z R(z, \boldsymbol{x})$ we mean a function whose values for $\boldsymbol{x}$ are some minimal z such that $R(z, \boldsymbol{x})$ In particular, if R is Θ computable we set

$$\mu z R(z, \boldsymbol{x}) = \nu z (R(z, \boldsymbol{x}) \quad \& \quad (\forall y < z) \neg R(y, \boldsymbol{x}))$$

Proof of Lemma 1.4. Let $\lambda\sigma W^\sigma$ be a $(\leqslant)$-enumeration of $T_1 = \{\langle\varepsilon, x, y\rangle \; \{\varepsilon\}(x) \to y\}$ and let $h(x) = \mu\sigma[(\exists y < \sigma)(\langle\varepsilon, x, y\rangle \in W^\sigma)]$ where ε is a Θ-index for f Whenever $f(x)$ is defined let $N_x = \{y < h(x) \; \langle\varepsilon, x, y\rangle \in W^{h(x)}\}$ Note that N_x is well-defined since $h(x) \to \sigma \; \& \; h(x) \to \tau \Rightarrow \sigma \sim \tau$ It follows from Lemma 1 1 that a canonical Θ-index for N_x is obtained uniformly and single-valuedly from ε and x Let $g(x) = \nu y(y \in N_x)$ If Θ-finite $K \subseteq \operatorname{dom} f$, let $N = \bigcup\{N_x \; x \in K\}$

An immediate corollary to Lemma 1 4 is the existence of a "selection operator" which single-valuedly chooses a canonical Θ-index for a non-empty subset of a non-empty Θ-s e set It is such a "selection operator", rather than the multivalued one we assumed, which is needed for our arguments In [3] Fenstad gives axioms for infinite computation theories which do not assert the existence of a selection operator, but where the existence of a "selection operator" as above nonetheless is a theorem. Thus Fenstad may and does restrict himself to single-valued theories.

Definition 1.5. A *($\leqslant$)-parametrization* of Θ-s.c. sets is a Θ-computable mapping $\lambda\varepsilon\sigma W_\varepsilon^\sigma$ such that

(i) $\forall\varepsilon, \tau, \sigma(\tau \leqslant \sigma \Rightarrow W_\varepsilon^\tau \subseteq W_\varepsilon^\sigma)$,

(ii) for each Θ-s.c. set W there is an ε such that $W = \bigcup\{W_\varepsilon^\sigma : \sigma \in U\}$.

Axiom 2 asserts the existence of a ($\leqslant$)-parametrization of Θ-s.c. sets. Considering a fixed ($\leqslant$)-parametrization, we let W_ε denote $\bigcup\{W_\varepsilon^\sigma : \sigma \in U\}$. Note that a Θ-s.c. index for W_ε is obtained uniformly from ε, using the selection operator. Indices from a ($\leqslant$)-parametrization can therefore be used in explicit definitions of Θ-computable functions.

Definition 1.6. (i) A *projection into* W is a total Θ-computable function p whose range is a subset of W such that if $x \neq y$ then $p(x) \cap p(y) = \emptyset$. (Here $p(x)$ denotes the set $\{z : p(x) \to z\}$.)

(ii) Θ is *projectible into* W if there is a projection into W.

Lemma 1.7. (i) *Let* $W = \{\varepsilon : W_\varepsilon \neq \emptyset\}$ *for a given ($\leqslant$)-parametrization of* Θ-s.c. *sets. Then* Θ *is projectible into* W.

(ii) *Suppose* p *is a projection. Then there is a ($\leqslant$)-parametrization of* Θ-s.c. *sets such that* $\{\varepsilon : W_\varepsilon \neq \emptyset\} \subseteq \operatorname{ran} p$.

Proof. (i) Define

$$f(x) = \mu\sigma[(\exists\varepsilon < \sigma)(x \in W_\varepsilon^\sigma \quad \& \quad (\forall y \in W_\varepsilon^\sigma)(y = x))]$$

and let

$$p(x) = \nu\varepsilon[x \in W_\varepsilon^{f(x)} \quad \& \quad (\forall y \in W_\varepsilon^{f(x)})(y = x)]$$

Then p is clearly a projection into W.

(ii) Let $\lambda\varepsilon\sigma V_\varepsilon^\sigma$ be any ($\leqslant$)-parametrization of Θ-s.c. sets. Using Lemma 1.4 we have a collection of Θ-finite sets K_x, each obtained uniformly from x, such that $\emptyset \neq K_x \subseteq p(x)$. Let $W = \bigcup\{K_x : x \in U\}$ and let $\lambda\sigma W^\sigma$ be a ($\leqslant$)-enumeration of W. Define

$$r(\varepsilon, \sigma) = \begin{cases} \nu x[\varepsilon \in K_x] & \text{if} \quad \varepsilon \in W^\sigma, \\ 0 & \text{if} \quad \varepsilon \notin W^\sigma. \end{cases}$$

Letting

$$W_\varepsilon^\sigma = \begin{cases} V_{r(\varepsilon,\sigma)}^\sigma & \text{if} \quad \varepsilon \in W^\sigma, \\ \emptyset & \text{if} \quad \varepsilon \notin W^\sigma \end{cases}$$

we obtain a ($\leqslant$)-parametrization with the required property.

Due to the negative result in Simpson [14] it is reasonable to formulate yet another condition which isolates a subclass of infinite computation theories for which the priority argument can be carried out. The problem in the general case is that for any $(\leqslant)$-parametrization the set $\{e: W_e \neq \emptyset\}$ may be too "wide". Lemma 1.7 reduces the problem of finding a "narrow" $(\leqslant)$-parametrization to that of finding a "narrow" projection.

A Θ-finite set K is said to be *strongly Θ-finite* if every Θ-s.c. subset of K is Θ-finite.

Definition 1.8. An infinite computation theory Θ is said to be *adequate* if Θ is projectible into the field of a Θ-s.c. prewellorder whose initial segments are uniformly strongly Θ-finite.

In the sequel we assume that the prewellorder of Definition 1.8 is $\leqslant$ or an initial segment of $\leqslant$, for some $\leqslant$ satisfying Axiom 1 and Axiom 2. The modifications necessary for the general case are left to the reader.

Let $\rho_\leqslant$ be the unique order-preserving map from U onto the ordinal $|\leqslant|$. Often we will be imprecise and write x when we mean $\rho_\leqslant(x)$. Thus $x < \alpha$ where α is an ordinal and stands for $\rho_\leqslant(x) < \alpha$. Throughout the paper we use the following

Convention. $L^\beta = \{x \in U: x < \beta\}$.

Definition 1.9. (i) *The projectum* $(\leqslant)$, denoted $|\leqslant|^*$, is the least ordinal β such that Θ is projectible into L^β.

(ii) The *r.e.-projectum* $(\leqslant)$, denoted $|\leqslant|^+$, is the least ordinal β for which there is a Θ-s.c. non-Θ-finite set $W \subseteq \mathrm{L}^\beta$.

Since the range of a projection is a Θ-s.c. non-Θ-finite set it follows that $|\leqslant|^+ \leqslant |\leqslant|^*$. Thus, modulo our assumption after Definition 1.8, Θ is adequate if and only if $|\leqslant|^+ = |\leqslant|^* =$ limit ordinal.

Every computably wellordered Θ is adequate since for such theories every Θ-s.c. non-Θ-finite set is the range of an injective Θ-computable mapping. Any (choiceless) standard model of ZF constitutes a (non-wellorderable) adequate theory relative to the power set operator $\mathcal{P}$. We may also use urelements to give some further examples of non-wellorderable adequate theories. Let $\mathcal{M} = \langle M \rangle$ be an infinite structure without relations or let $\mathcal{M} = \langle M, < \rangle$ be a dense linear ordering. Then $\mathrm{HYP}_{\mathcal{M}}$, the smallest admissible set above $\mathcal{M}$ (defined in [1]), as well as $\mathrm{HYP}(\mathrm{HYP}_{\mathcal{M}})$, $\mathrm{HYP}(\mathrm{HYP}(\mathrm{HYP}_{\mathcal{M}}))$ and so on, can be shown to be adequate.

2. Relative computability

Equivalent notions of Turing reducibility for ordinary recursion theory become distinct when considering recursion theory on an arbitrary admissible ordinal α. As Kreisel [5] emphasizes, the different notions fall into essentially two categories: those concerned with computability and those concerned with definability. Below, a notion from each will be defined (along with some auxiliary notions) corresponding to $\leq_\alpha$ and $\leq_{w\alpha}$ for α-recursion theory. We will then show that, as in the case of α-recursion theory, the notions agree on regular hyperregular sets.

By an *enumeration of Θ-finite sets* we mean a Θ-computable mapping $\lambda\xi K_\xi$ with the property that for each Θ-finite set K there is ξ such that $K = K_\xi$. Such an enumeration always exists since every Θ-finite set is W_ε^σ for some ε and σ. An enumeration can of course be chosen with somewhat care, e.g. we may require $K_\xi \subseteq L^\xi$.

Definition 2.1. Let A and B be sets, f a function and $\lambda\xi K_\xi$ a fixed enumeration of Θ-finite sets.

(i) f is *weakly Θ-computable in* B (denoted $f \leq_w B$) if there is a Θ-s.c. set W such that for all $\boldsymbol{x}$, y,

$$f(\boldsymbol{x}) \to y \Leftrightarrow \exists \xi, \eta(\langle \boldsymbol{x}, y, \xi, \eta\rangle \in W \quad \& \quad K_\xi \subseteq B \quad \& \quad K_\eta \cap B = \emptyset)$$

A is weakly Θ-computable in B ($A \leq_w B$) in case $c_A \leq_w B$.

(ii) A is *Θ-computable in* B (denoted $A \leq B$) if there is a Θ-s.c. set W such that for all γ, δ

$$K_\gamma \subseteq A \quad \& \quad K_\delta \cap A = \emptyset \Leftrightarrow \exists \xi, \eta(\langle \gamma, \delta, \xi, \eta\rangle \in W \quad \& \quad K_\xi \subseteq B \quad \& \quad K_\eta \cap B = \emptyset)$$

The definitions are independent of the particular enumeration of Θ-finite sets. We define the upper semi-lattice of degrees in the usual way using the transitive reducibility $\leq$. $A \equiv B$ denotes $A \leq B$ & $B \leq A$. The join of deg(A) and deg(B), deg(A)$\vee$deg(B), is deg($A \oplus B$) where

$$A \oplus B = \{\langle x, 0\rangle : x \in A\} \cup \{\langle x, 1\rangle : x \in B\}$$

The notions of *weakly Θ-s.c. in* and *Θ-s.c. in* are easily abstracted from (i) and (ii) of Definition 2.1. Thus A is Θ-s.c. in B if there is a Θ-s.c. set W such that for each γ

$$K_\gamma \subseteq A \Leftrightarrow \exists \xi, \eta(\langle \gamma, \xi, \eta\rangle \in W \quad \& \quad K_\xi \subseteq B \quad \& \quad K_\eta \cap B = \emptyset)$$

The sets weakly Θ-s.c. in B are enumerated by putting

$$W_\varepsilon^B = \{x : \exists \xi, \eta(\langle x, \xi, \eta\rangle \in W_\varepsilon \quad \& \quad K_\xi \subseteq B \quad \& \quad K_\eta \cap B = \emptyset)\}$$

It follows immediately from the definitions that a set is (weakly) Θ-computable in B iff both it and its complement are (weakly) Θ-s.c. in B, and that a set is weakly Θ-s.c. in B iff it is the domain of a function weakly Θ-computable in B.

To define a reducibility notion corresponding to definability is technically somewhat more complicated. From an infinite theory Θ and a set $B \subseteq U$ we construct a new theory $\Theta[B]$ and say that $f \leq_d B$ if f is $\Theta[B]$-computable. In addition to the obvious requirement that $\Theta[B]$ should have the usual closure and enumeration properties, i e that $\Theta[B]$ should be a precomputation theory, we want B to be $\Theta[B]$-computable, $\Theta \preccurlyeq \Theta[B]$ ($\preccurlyeq$ is the relation between precomputation theories given in [6]), and quantification over initial segments of $\preccurlyeq$ to be $\Theta[B]$-computable The latter means that the functional E should be $\Theta[B]$-computable where

$$E^{\preccurlyeq}(f, z) \to \begin{cases} 0 & \text{if} \quad \exists x < z(f(x) \to 0), \\ 1 & \text{if} \quad \forall x < z(f(x) \to 1) \end{cases}$$

Furthermore $\Theta[B]$ should have a computable selection operator in order for the $\Theta[B]$-s c relations and $\Theta[B]$-finite sets to behave properly

The theory $\Theta[B]$ will be the least fixed point of an inductive operator Γ defined by clauses 0–VIII Clause 0 introduces the characteristic function of B and clause I makes $\Theta \preccurlyeq \Theta[B]$ using Axiom 2 for Θ Clauses II–VI correspond to clauses IX′–XIII′ in [6] Finally, clauses VII and VIII introduce the functional $E^{\preccurlyeq}$ and a selection operator respectively Having already opted for multi-valued theories we make the selection operator take all its possible values

The β-th iteration of Γ is defined as $\Theta^{\beta}[B] = \Gamma(\Theta^{<\beta}[B])$ where $\Theta^{<\beta}[B] = \bigcup\{\Theta^{\gamma}[B] \ \gamma < \beta\}$ Thus the least fixed point of Γ is $\Theta[\mathrm{B}] = \bigcup_{\beta} \Theta^{\beta}[\mathrm{B}]$

There is no need to give the detailed construction We only note that all clauses have the following important property A tuple $(\varepsilon, \boldsymbol{x}, z)$ is added to $\Theta^{\beta}[B]$ only if $\varepsilon, \boldsymbol{x}, z$ and $\langle \varepsilon, \boldsymbol{x}, z \rangle$ are elements of L^{β} For $(\varepsilon, \boldsymbol{x}, z) \in \Theta^{\beta}[B]$, set $|\varepsilon, \boldsymbol{x}, z|_{\Theta[B]}$ = least ordinal β such that $(\varepsilon \ \boldsymbol{x}, z) \in \Theta^{\beta}[\beta]$ Using this notion of length of computations, $\Theta[B]$ is a computation theory in the sense of Moschovakis One can show that $\Theta[B]$ is either an infinite theory or a Spector theory (defined in [3] and [6]) depending on whether U is $\Theta[B]$-infinite or $\Theta[B]$-finite

In [9] Sacks defines α-recursion relative to a set $B \subseteq \alpha$ to be Σ_1-recursion on α relative to the structure $\langle \mathrm{L}(\alpha, B), \in, B \rangle$, where $\mathrm{L}(\alpha, B)$ is the result of relativizing $\mathrm{L}(\alpha)$ to B by adding $x \in B$ to the atomic formulas We regard the theory $\Theta[B]$ as the relativization of an infinite theory Θ to a set B Suppose Θ is a formulation of α-recursion theory Then one can show that $\Theta[B]$ is an infinite theory if and only if $\langle \mathrm{L}(\alpha, B), \in, B \rangle$ is admissible, in which case the notions of $\Theta[B]$-finite and $\Theta[B]$-s c agree with α-B-finite and α-B-r e

Definition 2.2. (i) $f \leqslant_d B$ if f is $\Theta[B]$-computable
(ii) $A \leqslant_d B$ if c_A is $\Theta[B]$-computable

Lemma 2.3. *$A \leqslant_d B$ if and only if $\Theta[A] \preccurlyeq \Theta[B]$*

Corollary. *$\leqslant_d$ is transitive*

The proof of the lemma is standard The required mapping p is defined by cases using the second recursion theorem for $\Theta[B]$ The "if" direction of

$(\varepsilon, \boldsymbol{x}, z) \in \Theta[A] \Leftrightarrow (p(\varepsilon, n), \boldsymbol{x}, z) \in \Theta[B]$ is shown by induction on $|p(\varepsilon, n), \boldsymbol{x}, z|_{\Theta[B]}$, while the "only if" direction is shown by induction on $|\varepsilon, \boldsymbol{x}, z|_{\Theta[A]}$

Using the corollary we define d-degrees by

$$\text{d-deg}(A) = \{B \;\; A \leq_d B \;\&\; B \leq_d A\}$$

The d-degrees form an upper semi-lattice in the usual way

Lemma 2.4. $f \leq_w B \Rightarrow f \leq_d B$

Proof. Let $\lambda\xi K_\xi$ be an enumeration (in Θ) of Θ-finite sets It follows from the $\Theta[B]$-computability of $E^<$ and $\Theta \leq \Theta[B]$ that $K_\xi \subseteq B$ and $K_\eta \cap B = \emptyset$ are $\Theta[B]$-computable relations Suppose $f \leq_w B$ using W, i e ,

$$f(\boldsymbol{x}) \to y \Leftrightarrow \exists \xi, \eta(\langle \boldsymbol{x}, y, \xi, \eta \rangle \in W \;\&\; K_\xi \subseteq B \;\&\; K_\eta \cap B = \emptyset)$$

Recalling that ν takes all its possible values in $\Theta[B]$ we have

$$f(\boldsymbol{x}) = (\nu y[\langle \boldsymbol{x}, (y)_1, (y)_2, (y)_3 \rangle \in W \;\&\; K_{(y)_2} \subseteq B \;\&\; K_{(y)_3} \cap B = \emptyset])_1$$

From the lemma we conclude that $A \leq B \Rightarrow A \leq_w B \Rightarrow A \leq_d B$ None of the implications can be reversed since Driscoll [2] has shown that $\leq_w$ need not be transitive even on Θ-s c sets

We now introduce the analogues of two notions due to Sacks [8] Recalling the definition of W_e^B let

$$^{\sigma}W_e^B = \{x \;\; \exists \xi, \eta(\langle x, \xi, \eta \rangle \in W_e^\sigma \;\&\; K_\xi \subseteq B \;\&\; K_\eta \cap B = \emptyset)\}$$

***Definition* 2.5.** (i) A set B is *regular* if $B \cap K$ is Θ-finite whenever K is Θ-finite

(ii) A set B is *hyperregular* if whenever $K \subseteq W_e^B$ and K is Θ-finite then there is σ such that $K \subseteq {}^{\sigma}W_e^B$

Hyperregularity has the following equivalent formulation in terms of functions B is hyperregular if and only if whenever $f \leq_w B$, $K \subseteq \operatorname{dom} f$ and K is Θ-finite, then $\exists z(\forall x \in K)(\exists y < z)(f(x) \to y)$

Every Θ-computable set is hyperregular (Lemma 1 4) and every hyperregular Θ-s c set is regular (proved in [15]) A useful characterization of the regular Θ-s c sets is the following Suppose $\lambda\sigma W^\sigma$ is a $(\leq)$-enumeration of a set W Let $V^\sigma = W^\sigma - \bigcup\{W^\tau \;\; \tau < \sigma\}$ For obvious reasons we say that $\lambda\sigma V^\sigma$ is a *disjoint* $(\leq)$-*enumeration* of W Then W is regular if and only if

$$(\forall \beta < |\leq|)(\exists \sigma)(\forall \tau > \sigma)(V^\tau \cap L^\beta = \emptyset)$$

The problem of non-regularity can be avoided in the usual way when studying Θ-s c degrees for adequate theories

Theorem 2.6. *Suppose Θ is an adequate theory Then for every Θ-s c set B there is* *a regular Θ-s c set D such that $B \equiv D$ D may be chosen such that $\forall x\ (\forall y \sim x)(x \in D \Rightarrow y \in D)$*

The theorem is due to Sacks [8] for α-recursion theory. Its proof (which we omit) in our more general setting is modelled on Sacks' original proof in [8] A proof of a weaker, but for our purposes sufficient, version of Theorem 2 6 can be found in [15]

Now we set out to show that for any sets A and B, $A \leqslant B \Leftrightarrow A \leqslant_{\mathrm{d}} B$ if B is regular and hyperregular Given disjoint sets B_1 and B_2 we obtain a theory $\Theta[B_1, B_2]$ by altering clause 0 in the definition of $\Theta[B]$ as follows

0′ If $\langle 0,0\rangle, x, 0, \langle\langle 0,0\rangle, x, 0\rangle \in \mathrm{L}^{\beta}$ & $x \in B_1$,

then $(\langle 0,0\rangle, x, 0) \in \Theta^{\beta}[B_1, B_2]$

If $\langle 0,0\rangle, x, 1, \langle\langle 0,0\rangle, x, 1\rangle \in \mathrm{L}^{\beta}$ & $x \in B_2$,

then $(\langle 0,0\rangle, x, 1) \in \Theta^{\beta}[B_1\ B_2]$

Thus $\Theta[B] = \Theta[B, U\text{-}B]$ For each σ, ξ, η and m define

$$ {}^{m}H^{\sigma}_{\xi\,\eta} = \{\langle \varepsilon, \boldsymbol{x}, y\rangle\ \ (\varepsilon, \boldsymbol{x}, y) \in \Theta^{\rho\ (\sigma)}[K_{\xi}, K_{\eta}], \mathrm{lh}(\boldsymbol{x}) = m\} $$

Lemma 2.7. *${}^{m}H^{\sigma}_{\xi\,\eta}$ is Θ-finite uniformly in m, σ, ξ, η*

Proof. ${}^{m}H^{\sigma}_{\xi\,\eta}$ can be defined by induction on σ with respect to $\leqslant$ considering all cases in the definition of $\Theta[K_{\xi}, K_{\eta}]$

By an easy induction on σ we have

Lemma 2.8. *If $\langle \varepsilon, \boldsymbol{x}, y\rangle \in {}^{m}H^{\sigma}_{\xi\,\eta}$ & $K_{\xi} \subseteq B$ & $K_{\eta} \cap B = \emptyset$, then $(\varepsilon, \boldsymbol{x}, y) \in \Theta^{\rho\ (\sigma)}[B]$*

Theorem 2.9. *Let B be a regular set Then* (i)–(iii) *below are equivalent*

(i) *B is hyperregular,*

(ii) *$\Theta[B]$ is an infinite theory,*

(iii) $\forall f (f \leqslant_{\mathrm{w}} B \Leftrightarrow f \leqslant_{\mathrm{d}} B)$

Proof. (i)$\Rightarrow$(ii) To show $\Theta[B]$ is an infinite theory it suffices to show $\Theta^{|\leqslant|}[B] = \Theta^{<|\leqslant|}[B]$ Since $|\leqslant|$ is a limit ordinal we need only consider the case of universal quantification whose inductive clause is

If $\langle 7,0\rangle, x, \langle\langle 7,0\rangle, \varepsilon, x, 1\rangle \in \mathrm{L}^{\beta}$ & $(\forall y < x)[(\varepsilon, y, 1) \in \Theta^{<\beta}[B]]$,

then $(\langle 7,0\rangle, \varepsilon, x, 1) \in \Theta^{\beta}[B]$

So suppose $(\varepsilon, y, 1) \in \Theta^{<|\leqslant|}[B]$ for each $y < x$ It follows from the regularity of B that for each $y < x$ there are σ, ξ, η such that $\langle \varepsilon, y, 1\rangle \in {}^{1}H^{\sigma}_{\xi\,\eta}$ where $K_{\xi} \subseteq B$ and

$K_\eta \cap B = 0$. Letting

$$W^\sigma = \{\langle y, \xi, \eta\rangle \in L^\sigma : \langle \varepsilon, y, 1\rangle \in {}^1H^\sigma_{\xi,\eta}\},$$

this can be reformulated as $L^x \subseteq W^B$ where $\lambda\sigma W^\sigma$ is a $(\leqslant)$-enumeration of W. By the hyperregularity of B, $L^x \subseteq {}^\tau W^B$ for some τ. But then $\langle \varepsilon, y, 1\rangle \in \Theta^{\rho^{\leqslant}(\tau)}[B]$ for each $y < x$ by Lemma 2.8, and hence $(\langle 7, 0\rangle, \varepsilon, x, 1) \in \Theta^{<|\leqslant|}[B]$.

(ii)$\Rightarrow$(iii). Suppose f is $\Theta[B]$-computable with a $\Theta[B]$-index ε. Then by (ii), Lemma 2.8 and the regularity of B,

$$\begin{aligned} f(\boldsymbol{x}) \to y &\Leftrightarrow \exists \beta < |\leqslant|((\varepsilon, \boldsymbol{x}, y) \in \Theta^\beta[B]) \\ &\Leftrightarrow \exists \sigma, \xi, \eta(\langle \varepsilon, \boldsymbol{x}, y\rangle \in {}^m H^\sigma_{\xi,\eta} \quad \& \quad K_\xi \subseteq B \quad \& \quad K_\eta \cap B = \emptyset) \end{aligned}$$

It follows that $f \leqslant_w B$.

(iii)$\Rightarrow$(i). Assume (iii). Then every $\Theta[B]$-s.c. set has a $(\leqslant)$-enumeration in $\Theta[B]$. For suppose V is $\Theta[B]$-s.c. Then $V = W^B$ for some Θ-s.c. W by (iii). Put

$$V^\sigma = \{x < \sigma : \exists \xi, \eta < \sigma(\langle x, \xi, \eta\rangle \in W^\sigma \;\&\; K_\xi \subseteq B \;\&\; K_\eta \cap B = \emptyset)\}$$

Then $\lambda\delta V^\sigma$ is a $(\leqslant)$-enumeration in $\Theta[B]$ of V. It follows that U is $\Theta[B]$-infinite (and in fact that $\Theta[B]$ is an infinite theory). Suppose a Θ-finite set $K \subseteq W^B_e$. Let $f(x) = \mu\sigma[x \in {}^\sigma W^B_e]$. Then for each $x \in K$, $L^{f(x)}$ is $\Theta[B]$-finite uniformly in x. Thus $M = \bigcup\{L^{f(x)} : x \in K\}$ is $\Theta[B]$-finite and hence bounded by some σ. Then $K \subseteq {}^\sigma W^B_e$, so B is hyperregular.

Note that regularity was not needed in going from (iii) via (ii) to (i). The regular hyperregular sets can be characterized as those sets B for which every $\Theta[B]$-finite set is Θ-finite. Of course, whenever (iii) holds for B it follows that $A \leqslant B \Leftrightarrow A \leqslant_d B$. Just let

$$f(\gamma, \delta) = 0 \Leftrightarrow K_\gamma \subseteq A \quad \& \quad K_\delta \cap A = \emptyset$$

Before defining the jump of a set we introduce yet another notion of reducibility.

Definition 2.10. A set A is *many-one reducible* to a set B, $A \leqslant_m B$, if there is a Θ-computable mapping $\lambda x H_x$ whose values are (canonical Θ-indices for) non-empty Θ-finite sets such that

(i) $x \in A \Leftrightarrow H_x \subseteq B$

(ii) $x \notin A \Leftrightarrow H_x \cap B = \emptyset$

Note that $A \leqslant_m B \Rightarrow A \leqslant B$ and $A \leqslant_m B \;\&\; B \leqslant_w C \Rightarrow A \leqslant_w C$.

Following Shore [12] and Simpson [13] we want the jump of a set B to be a $\leqslant_m$ complete set B' weakly Θ-s.c. in B, i.e. whenever A is weakly Θ-s.c. in B, then

$A \leqslant_m B'$ Letting $\lambda \varepsilon W_\varepsilon$ be a (not necessarily the) standard $(\leqslant)$-parametrization of Θ-s c sets we make the following definition.

Definition 2.11 The jump of a set B is the set

$$B' = \{\varepsilon . \exists \xi, \eta(\langle \xi, \eta \rangle \in W_\varepsilon \;\&\; K_\xi \subseteq B \;\&\; K_\eta \cap B = \emptyset\}$$

Our only requirement on the $(\leqslant)$-parametrization used in the definition is that (iii) in Proposition 2 12 below must hold This is certainly the case for a $(\leqslant)$-parametrization obtained from the standard one as in Lemma 1 7

Proposition 2.12. (i) $B \leqslant_m B'$ *but not* $B' \leqslant_w B$ (*so* $B < B'$)

(ii) $B \leqslant D \Leftrightarrow B' \leqslant_m D'$

(iii) D *is weakly* Θ-s c *in* $B \Leftrightarrow D \leqslant_m B'$

(iv) B' *is weakly* Θ-s c *in* B

Thus the jump is well defined and increasing on degrees However, it may not be increasing on d-degrees as is readily seen by considering a non-hyperregular d-degree This is not surprising since $\leqslant_d$ in general is a much stronger reducibility notion than $\leqslant$ The proper notion of "semi-computable in B" for $\leqslant_d$ is $\Theta[B]$-s c Thus we want the jump (in this connection called d-jump) of a set B to be a complete $\Theta[B]$-s c set

Definition 2.13. The d-jump of a set B is the set

$$B^d = \{\langle \varepsilon, x \rangle \; \{\varepsilon\}_{\Theta[B]}(x) \downarrow \}$$

It is easily verified that the analogue for the d-jump of Proposition 2 12 holds Of course, in case B is regular and hyperregular, then $B' \equiv_m B^d$

3. Σ_2-functions

It is clear that in case the domain of an infinite computation theory is not computably wellordered, one cannot consider a unique requirement at a given stage of a priority construction There is thus a need to consider a Θ-finite block of requirements at each stage The obvious way to block requirements is in terms of the levels of the given prewellorder letting each level make up one block This method suffices for Θ-finite injury arguments where elements in at most one set of requirements can be injured more than a fixed finite number of times In particular, a weak positive solution to Post's problem was obtained in [15] for every adequate theory using this method

In proving the splitting theorem for an admissible ordinal α, Shore [11] developed a technique of blocking requirements into $\sigma 2cf(\alpha)$ many α-finite sets S G Simpson [14] was the first to note that this technique could also be used to prove a version of the Friedberg Muchnik theorem for thin admissible sets This led us to develop Shore's blocking technique for adequate theories Θ

A set A is said to be Σ_0 and Π_0 if it is Θ-computable A is Σ_{n+1} if $x \in A \Leftrightarrow \exists y(\langle x, y\rangle \in B)$ where B is Π_n, and A is Π_{n+1} if its complement is Σ_{n+1} A function f is Σ_n if its graph $G_f = \{\langle \boldsymbol{x}, y\rangle\ f(\boldsymbol{x}) \to y\}$ is Σ_n

Let $\mathscr{S}$ be the class of functions on U satisfying

$$f(x_1, \ldots, x_i, \ldots, x_n) \to z \quad \& \quad f(x_1, \ldots, x_i', \ldots, x_n) \to z' \quad \& \quad x_i \sim x_i' \Rightarrow z \sim z'$$

Functions in $\mathscr{S}$ will be identified in the obvious way with partial single-valued functions on $|\leqslant|$ Thus by a function in $\mathscr{S} \cap \Sigma_n$ we will interchangeably mean a Σ_n-function in $\mathscr{S}$ or a function on $|\leqslant|$ induced by a Σ_n-function in $\mathscr{S}$ It is shown in [15] that $|\leqslant|$ is admissible and that every $|\leqslant|$-recursive function is in $\mathscr{S} \cap \Sigma_1$

Let $f'(\alpha, \gamma)$ be a partial single-valued function on $|\leqslant|$ Then $\lim_\alpha f'(\alpha, \gamma) = \delta$ iff $\exists \beta (\forall \alpha \geqslant \beta)(f'(\alpha, \gamma) = \delta)$ For $f, f' \in \mathscr{S}$ we say that $\lim_\sigma f'(\sigma, x) \simeq f(x)$ if this is the case for the induced functions on $|\leqslant|$, where $\simeq$ has its usual meaning

Lemma 3.1. *Let Θ be an adequate theory Suppose $f \in \mathscr{S} \cap \Sigma_2$ is total (on $|\leqslant|$) Then there is a total Θ-computable function $f' \in \mathscr{S}$ such that* $\lim_\sigma f'(\sigma, x) \simeq f(x)$

Proof. Since G_f is Σ_2 it follows that $f \leqslant_w A$, say using W, where A is Θ-s c and by Theorem 2 6) regular Let $\lambda\sigma A^\sigma$ and $\lambda\sigma W^\sigma$ be $(\leqslant)$-enumerations of A and W respectively Let N_x^σ be the Θ-finite set of minimal $\eta < \sigma$ such that

$$(\exists y < \sigma)(\exists x' \sim x)(\langle x', y, \eta\rangle \in W^\sigma \ \& \ K_\eta \cap A^\sigma = \emptyset)$$

Define

$$f'(\sigma, x) = \begin{cases} \mu y[(\exists \eta \in N_x^\sigma)(\exists x' \sim x)(\langle x', y, \eta\rangle \in W^\sigma) & \text{if } N_x^\sigma \neq \emptyset \\ \sigma & \text{else} \end{cases}$$

Then f' is total and in $\mathscr{S} \cap \Sigma_1$

Suppose $f(\alpha) = \beta$ (on $|\leqslant|$) Choose x, y such that $\rho\ (x) = \alpha$, $\rho_{<}(y) = \beta$ and $f(x) \to y$, and choose η such that $\langle x, y, \eta\rangle \in W \ \& \ K_\eta \cap A = \emptyset$ By the regularity of A we can choose σ sufficiently large so that $y < \sigma$, $\langle x, y, \eta\rangle \in W^\sigma$ and $(U - A) \cap \mathrm{L}^\eta = (U - A^\sigma) \cap \mathrm{L}^\eta$ Suppose $\tau \geqslant \sigma$ Then $N_x^\tau \neq \emptyset$ since η is a candidate Let $\xi \in N_x^\tau$ There is $x' \sim x$ and y' such that $\langle x', y', \xi\rangle \in W^\tau \ \& \ K_\xi \cap A^\tau = \emptyset$ Since $\xi \leqslant \eta$ and (we may assume our enumeration of Θ-finite sets to satisfy) $K_\xi \subseteq \mathrm{L}^\xi$, $K_\xi \cap A = \emptyset$ But then $\langle x', y', \xi\rangle$ is a correct computation of f, i e $f(x') \to y'$ Since $f \in \mathscr{S}$ and $x' \sim x$, we must have $y' \sim y$ Thus $\lim_\sigma f'(\sigma, \alpha) = \beta$

Definition 3.2. The Σ_2-cof(α) is the least ordinal β for which there is a function $f \in \mathscr{S} \cap \Sigma_2$ with domain β and range unbounded in α

Lemma 3.3. *Let Θ be an adequate theory Then* Σ_2-cof $(|\leqslant|) = \Sigma_2$-cof $(|\leqslant|^*)$

Proof. Let $k \in \mathscr{S}$ be a total Θ-computable function with range in $|{\leqslant}|^*$ such that $\{\beta : k(\beta) < \alpha\}$ is bounded for each $\alpha < |{\leqslant}|^*$. Such a k can be defined from a $(\leqslant)$-enumeration of a Θ-s.c. non-Θ-computable set $W \subseteq L^{|\leqslant|^*}$. Suppose $f \in \mathscr{L} \cap \Sigma_2$ with domain β is unbounded in $|{\leqslant}|$. Then $g(\alpha) = k(f(\alpha))$ is an $\mathscr{L} \cap \Sigma_2$ function unbounded in $|{\leqslant}|^*$. Thus Σ_2-cof$(|{\leqslant}|^*) \leqslant \Sigma_2$-cof$(|{\leqslant}|)$.

For the converse inequality suppose $f \in \mathscr{L} \cap \Sigma_2$ with domain β is unbounded in $|{\leqslant}|^*$. Let $g(x) = \mu\sigma[\forall\tau \geqslant \sigma(f(x) < k(\tau))]$. Then $g \in \mathscr{L}$ and g is unbounded in $|{\leqslant}|$. It follows from Lemma 3.1 and some easily shown closure properties of Σ_n and Π_n sets that g is Σ_2.

By a $(\leqslant)$-*sequence of* Θ-*s.c. sets* we mean a Θ-computable mapping r such that $x \sim y \Rightarrow W_{r(x)} = W_{r(y)}$.

Lemma 3.4. *Suppose* $\alpha < \Sigma_2$-cof$(|{\leqslant}|)$ *and* $\langle I_x : x < \alpha\rangle$ *is a* $(\leqslant)$-*sequence of* Θ-s.c. *sets such that for each* $x < \alpha$, I_x *is* Θ-*finite. Then* $\bigcup\{I_x : x < \alpha\}$ *is* Θ-*finite.*

Proof. Let α be least for which such a sequence exists whose union is not Θ-finite. Let $W = \bigcup\{I_x : x < \alpha\}$ and let $\lambda\sigma W^\sigma$ be a $(\leqslant)$-enumeration of W. Define $g(x) = \mu\sigma[I_x \subseteq W^\sigma]$. Then $g \in \mathscr{L} \cap \Sigma_2$ and g is defined on L^α. But $g(L^\alpha)$ is unbounded in U since W is not Θ-finite, i.e. Σ_2-cof$(|{\leqslant}|) \leqslant \alpha$.

Assume for the remaining part of this section that Θ is an adequate theory. We are going to divide the projectum $L^{|\leqslant|^*}$ into Σ_2-cof$(|{\leqslant}|)$ many Θ-finite blocks M_α, each bounded strictly below $|{\leqslant}|^*$. Clearly Σ_2-cof$(|{\leqslant}|) \leqslant |{\leqslant}|^*$. Suppose first that Σ_2-cof$(|{\leqslant}|) = |{\leqslant}|^*$. In this case we let $M_\alpha = M_\alpha^\sigma = \{x : x \sim \alpha\}$ for each $\alpha < |{\leqslant}|^*$. Then each M_α is Θ-finite uniformly in α.

Now suppose Σ_2-cof$(|{\leqslant}|) < |{\leqslant}|^*$. We are going to define Θ-finite approximations M_α^σ to our blocks M_α uniformly from σ and α. Furthermore

$$(\forall\alpha < \Sigma_2\text{-cof}(|{\leqslant}|))(\exists\sigma)(\forall\tau \geqslant \sigma)(\forall\beta < \alpha)(M_\beta^\tau = M_\beta),$$

i.e. our approximation will be "tame".

Let $g : \Sigma_2$-cof$(|{\leqslant}|) \to |{\leqslant}|^*$ be a $\mathscr{S} \cap \Sigma_2$ function unbounded in $|{\leqslant}|^*$, and let $g' \in \mathscr{S}$ be Θ-computable such that $\lim_\sigma g'(\sigma, \alpha) = g(\alpha)$ and ran $g' \subseteq L^{|\leqslant|^*}$. These functions exist by Lemma 3.1 and Lemma 3.3. Define

$$h(\sigma, \alpha) = \mu\gamma[(\forall\beta < \alpha)(g'(\sigma, \beta) < \gamma)]$$

and put $M_\alpha^\sigma = \{\varepsilon : h(\sigma, \alpha) \leqslant \varepsilon < h(\sigma, \alpha + 1)\}$. Note that a canonical Θ-index for M_α^σ is obtained uniformly from α and σ and that each M_α^σ is bounded strictly below $|{\leqslant}|^*$. To show $\lambda\alpha\sigma M_\alpha^\sigma$ is tame, let

$$I_\beta = \{\sigma : (\exists\tau > \sigma)(g'(\tau, \beta) \neq g'(\sigma, \beta))\}.$$

Fix $\alpha < \Sigma_2$-cof$(|{\leqslant}|)$. Then $\langle I_\beta : \beta < \alpha + 1\rangle$ is a $(\leqslant)$-sequence of Θ-s.c. sets such that each I_β is Θ-finite. Applying Lemma 3.4 we obtain

$$\exists\sigma(\forall\beta\leqslant\alpha)(\forall\tau\geqslant\sigma)(g'(\tau,\beta)\sim g'(\sigma,\beta)),$$

i.e. $$\exists\sigma(\forall\beta<\alpha)(\forall\tau\geqslant\sigma)(M_\beta^\tau=M_\beta^\sigma).$$

Let $M_\beta=M_\beta^\sigma$ for sufficiently large σ. It remains to show

$$\bigcup\{M_\beta : \beta<\Sigma_2\text{-cof}\,(|\leqslant|)\}=\mathrm{L}^{|\leqslant|^*}.$$

Fix $\varepsilon<|\leqslant|^*$ and choose least α for which $\varepsilon<h(\sigma,\alpha)$ where σ is fixed and sufficiently large. Such α exists since g is unbounded in $|\leqslant|^*$. By the definition of h there is $\beta<\alpha$ such that $\varepsilon\leqslant g'(\sigma,\beta)$. But then $\varepsilon<h(\sigma,\beta+1)$, so by the choice of α, $\alpha=\beta+1$ and $h(\sigma,\beta)\leqslant\varepsilon$.

4. The splitting theorem

For parts (i) and (ii) of our main theorem we need assume Θ has a reasonable pairing function. By this we mean that for each $\alpha<|\leqslant|^*$ there is $\beta<|\leqslant|^*$ such that $\mathrm{L}^\alpha\times\mathrm{L}^\alpha=\{\langle x,y\rangle : x,y\in\mathrm{L}^\alpha\}\subseteq\mathrm{L}^\beta$. Surely any adequate Θ that comes to mind has a reasonable pairing function.

Theorem 4.1. *Suppose Θ is an adequate theory with a reasonable pairing function. Let C be a regular Θ-*s.c. *set and let D be a Θ-*s.c. *non-Θ-computable set. Then there are Θ-*s.c. *sets A and B such that $C=A\cup B$, $A\cap B=\emptyset$, $A\leqslant C$, $B\leqslant C$ and*

(i) *$\Theta[A]$ and $\Theta[B]$ are adequate theories (so in particular A and B are hyperregular),*

(ii) *$A'\equiv B'\equiv 0'$,*

(iii) *$D\not\leqslant_w A$ and $D\not\leqslant_w B$.*

Before proving Theorem 4.1 we state some of its usual corollaries. First we need the following lemma.

Lemma 4.2. *If A and B are disjoint regular Θ-*s.c. *sets, then* $\deg(A\cup B)=\deg(A)\vee\deg(B)$ *and* d-deg$(A\cup B)=$ d-deg$(A)\vee$ d-deg(B).

Proof. Clearly $A\cup B\leqslant A\oplus B$. For the converse we note that

$$U-A=(U-A\cup B)\cup B.$$

Using the regularity of B we have

$$K_\gamma\cap A=\emptyset\Leftrightarrow\exists\eta(K_\eta\subseteq K_\gamma\ \&\ K_\gamma-K_\eta\subseteq B\ \&\ K_\eta\cap(A\cup B)=\emptyset),$$

i.e. $A\leqslant A\cup B$. The proof for d-degrees does not use regularity.

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ vary over Θ-s.c. degrees (Θ-s.c. d-degrees) and let $\mathbf{a}'$ denote the jump (the d-jump) of $\mathbf{a}$.

Corollary 4.3. (i) $(\forall \mathbf{c} > \mathbf{0})(\exists \mathbf{a}, \mathbf{b})(\mathbf{c} = \mathbf{a} \vee \mathbf{b} \ \& \ \mathbf{a} < \mathbf{c} \ \& \ \mathbf{b} < \mathbf{c} \ \& \ \mathbf{a} \mid \mathbf{b})$,

(ii) $(\forall \mathbf{d})(\mathbf{0} < \mathbf{d} < \mathbf{0}' \Rightarrow \exists \mathbf{a}(\mathbf{d} \mid \mathbf{a} \ \& \ \mathbf{a}' = \mathbf{0}'))$.

The proofs are entirely similar to the ones found in [10] and [11], using the main theorem and Lemma 4.2.

Corollary 4.4. (i) $\exists \mathbf{a}, \mathbf{b}(\mathbf{0} < \mathbf{a} < \mathbf{b} \ \& \ \mathbf{a}' = \mathbf{b}')$,

(ii) $\exists \mathbf{a}, \mathbf{b}(\mathbf{0} < \mathbf{a} < \mathbf{b} \ \& \ \mathbf{a}' < \mathbf{b}')$,

(iii) $\exists \mathbf{a}, \mathbf{b}(\mathbf{a} \mid \mathbf{b} \ \& \ \mathbf{a}' = \mathbf{b}' = (\mathbf{a} \vee \mathbf{b})')$

(iv) $\exists \mathbf{a}, \mathbf{b}(\mathbf{a} \mid \mathbf{b} \ \& \ \mathbf{a}' = \mathbf{b}' = \mathbf{a} \vee \mathbf{b})$

We now proceed with the proof of Theorem 4.1. Our description of the construction will be in terms of A only whenever the description in terms of B is analogous. In case $\lambda\sigma H^\sigma$ is a $(\leqslant)$-enumeration of Θ-finite sets we use the notation $H^{<\sigma} = \bigcup\{H^\tau : \tau < \sigma\}$. By Theorem 2.6 we may assume D to be regular and satisfy $\forall x(\forall y \sim x)(x \in D \Rightarrow y \in D)$. Let $\lambda\sigma D^\sigma$ be a $(\leqslant)$-enumeration of D and let $\lambda\sigma C^\sigma$ be a *disjoint* $(\leqslant)$-enumeration of C. We are going to define $(\leqslant)$-enumerations $\lambda\sigma A^\sigma$ and $\lambda\sigma B^\sigma$ of A and B inductively on the prewellorder $\leqslant$. If $\sigma \sim \tau$, then the set constructions at stage σ and stage τ will be identical though the indices used may differ. At stage σ, C^σ will be added to precisely one of $A^{<\sigma}$ and $B^{<\sigma}$. Thus A and B will be Θ-s.c., $C = A \cup B$ and $A \cap B = \emptyset$. Furthermore $A \leqslant C$ and $B \leqslant C$. For let $q(\xi) = \mu\sigma[(K_\xi - C^{<\sigma}) \cap C = \emptyset]$. Then $q \leqslant_w C$ and q is total by the regularity of C. Clearly $K_\xi \cap A = \emptyset \Leftrightarrow K_\xi \cap A^{q(\xi)} = \emptyset$, so $A \leqslant C$.

In order to satisfy (i) and (ii) of the theorem, some care is needed in choosing a $(\leqslant)$-parametrization $\lambda\varepsilon\sigma W^\sigma_\varepsilon$ of Θ-s.c. sets, besides requiring $\{\varepsilon : W_\varepsilon \neq \emptyset\} \subseteq \mathrm{L}^{|\leqslant|^*}$. First of all we want $\lambda\varepsilon\sigma W^\sigma_\varepsilon$ to be repetitive in the following sense. For each $\alpha, \varepsilon < |\leqslant|^*$ there is $\delta < |\leqslant|^*$ and σ such that $\alpha < \delta$ and $\forall\tau > \sigma(W^\tau_\varepsilon = W^\tau_\delta)$. Then we want Definition 2.11 of the jump to make sense for our choice of $(\leqslant)$-parametrization. Let $\lambda\varepsilon\sigma V^\sigma_\varepsilon$ be a $(\leqslant)$-parametrization obtained as in Lemma 1.7 from the standard one, such that $\{\varepsilon : V_\varepsilon \neq \emptyset\} \subseteq \mathrm{L}^{|\leqslant|^*}$. Let $W^\sigma_\varepsilon = V^\sigma_{(\varepsilon)_1}$. Then $\lambda\varepsilon\sigma W^\sigma_\varepsilon$ has the required properties.

To make $\Theta[A]$ and $\Theta[B]$ into adequate theories, the construction is split into two cases.

Definition 4.5. Suppose $\beta < |\leqslant|$. Then

$$\operatorname{cof}(\beta) = \mu\alpha[\exists\Theta\text{-computable } q : \mathrm{L}^\beta \to \mathrm{L}^\alpha \text{ such that}$$
$$\forall\varepsilon \in \mathrm{L}^\alpha \exists\gamma < \beta(q^{-1}(\varepsilon) \subseteq \mathrm{L}^\gamma)]$$

Remark. Since $\beta<|\leqslant|$, $q^{-1}(\varepsilon)$ may be considered a Θ-finite set with an index obtained uniformly from ε. Note that $\operatorname{dom} q = L^{\beta}$.

If $|\leqslant|^{+}=|\leqslant|$ or $|\leqslant|^{+}<|\leqslant|$ and $\operatorname{cof}(|\leqslant|^{+})<|\leqslant|^{+}$, then attempts are made to preserve computations $x \in W_\varepsilon^A$ for $x<\varepsilon$. In case $|\leqslant|^{+}<|\leqslant|$ and $\operatorname{cof}(|\leqslant|^{+})=|\leqslant|^{+}$, additional attempts are made to preserve computations on initial segments of $L^{|\leqslant|^{+}}$.

Assume we have shown $A\leqslant_{w}0'$. A' is weakly Θ-s.c. in A by Proposition 2.12. By the hyperregularity of A, A' is in fact Θ-s.c. in A and hence Θ-s.c. in $0'$. Let $A^{\backprime}$ denote the jump of A using the standard $(\leqslant)$-parametrization $\lambda\varepsilon V_\varepsilon$. Then

$$\begin{aligned} K_\delta\cap A^{\backprime}=\emptyset &\Leftrightarrow \neg(\exists\eta\in \cup\{V_\varepsilon : \varepsilon\in K_\delta\})(K_\eta\cap A=\emptyset)\\ &\Leftrightarrow f(\delta)\in A^{\backprime} \end{aligned}$$

where f is a Θ-computable mapping giving a standard index for $\bigcup\{V_\varepsilon : \varepsilon\in K_\delta\}$. Thus $(U-A^{\backprime})$ is weakly Θ-s.c. in $0'$ iff $(U-A^{\backprime})$ is Θ-s.c. in $0'$. Both A' and $A^{\backprime}$ satisfy Proposition 2.12 (iii) and (iv), so $A'\equiv_m A^{\backprime}$. Thus $A^{\backprime}\leqslant_w 0'$ since $A'\leqslant_w 0'$, and hence $(U-A^{\backprime})$ is Θ-s.c. in $0'$. But then (again using $A'\equiv_m A^{\backprime}$) $(U-A')$ is Θ-s.c. in $0'$. Since both A' and its complement are Θ-s.c. in $0'$, $A'\leqslant 0'$. Thus it suffices to make $A'\leqslant_w 0'$ in order to satisfy (ii).

To make $A'\leqslant_w 0'$, attempts are made to preserve computations showing $\varepsilon\in A'$ by creating a requirement for such a computation. Then one can effectively from $0'$ look through the list of requirements to determine whether or not $\varepsilon\in A'$.

Finally, to insure that for no ε, $(U-D)=W_\varepsilon^A$, we use the usual approach of trying to preserve computations $x\in W^A$ for minimal x not in D. In case $(U-D)=W_\varepsilon^A$ for some ε we would eventually preserve a correct computation for each $x\in W_\varepsilon^A$, i.e. W_ε^A would be Θ-s.c. Thus computations $x\in W_\varepsilon^A$ will eventually stop being preserved. However we need have Θ-finite blocks of requirements to settle down by some stage of the construction. Towards this end we use Shore's technique of letting each block play the role of a single requirement in trying to preserve a computation $x\in W_\varepsilon^A$ for $x\notin D$ and some ε in the block considered. Furthermore, to avoid the problem of never finishing creating requirements with arguments from a fixed level of $\leqslant$, we utilize the fact that D was chosen to have the property $\forall x(\forall y\sim x)(x\in D\Rightarrow y\in D)$. Thus there is a need to create a requirement preserving a computation $x\in W_\varepsilon^A$ only if no other computation $y\in W_\varepsilon^A$ for $y\sim x$ is being preserved.

Let M_α^σ and M_α for $\alpha<\Sigma_2\text{-cof}(|\leqslant|)$ be the Θ-finite blocks described in Section 3. We will create sets $R_{A,i}$ $(R_{B,i})$ of requirements for $i<3$. $R_{A,0}$ will insure that $\Theta[A]$ is adequate, $R_{A,1}$ that $A'\leqslant_w 0'$, and $R_{A,2}$ that $D\not\leqslant_w A$. S_A denotes the set of A-requirements (i.e. requirements in $\bigcup\{R_{A,i} : i<3\}$) injured during the construction. $R_{A,i}^\sigma$ and S_A^σ denote the Θ-finite parts of $R_{A,i}$ and S_A obtained by stage σ. Each requirement will be of the form $\langle\varepsilon, x, F\rangle$ where F is (a canonical Θ-index for) a Θ-finite set. Such a requirement in $R_{A,i}$ is called an $\varepsilon-A$ requirement or an $\alpha - A$ requirement (at σ) in case $\varepsilon \in M_\alpha$ ($\varepsilon \in M^\sigma_\alpha$) It is said to have argument x In case $F \cap A^\sigma = \emptyset$ it is said to be *active* at σ, else it is *inactive* $\varepsilon \in M^\sigma_\alpha$ is an *inactive* $\alpha - A$ *reduction procedure at* σ in case there is an active $\varepsilon - A$ requirement in $R^\sigma_{A,2}$ preserving a computation $x \in W^A_\varepsilon$ for some $x \in D^\sigma$, i e there is

$$\langle \varepsilon, x, F\rangle \in R^{<\sigma}_{A,2} - S^{<\sigma}_A \quad \text{s.t} \quad \exists \eta < \sigma(\langle x, \eta\rangle \in W^\sigma_\varepsilon \;\&\; K_\eta \subseteq F \;\&\; x \in D^\sigma)$$

If no such requirement exists, then ε is an *active* $\alpha - A$ *reduction procedure at* σ

Let r $|{\leqslant}| \to \Sigma_2\text{-cof}\,(|{\leqslant}|)$ be a Θ-computable function such that

$$(\forall \alpha < \Sigma_2\text{-cof}\,(|{\leqslant}|))(\forall \beta)(\exists \gamma > \beta)(r(\gamma) = \alpha),$$

where α, β and γ vary over $|{\leqslant}|$ The function r indicates which part of the construction to concern ourselves with at a given stage

The construction at stage σ Suppose $r(\sigma) = \alpha$ We describe only the construction of A-requirements, the construction of B-requirements being analogous

First we construct requirements making $\Theta[A]$ adequate The construction is split into two cases

Case I $|{\leqslant}|^* = |{\leqslant}|$ or $\text{cof}\,(|{\leqslant}|^*) < |{\leqslant}|^* < |{\leqslant}|$ Let

$$K^\sigma = \{\langle \varepsilon, x\rangle \in M^\sigma_\alpha \times (\bigcup\{M^\sigma_\beta \;\; \beta \leqslant \alpha\}) \;\; (\exists \eta < \sigma)(\langle x, \eta\rangle \in W^\sigma_\varepsilon$$
$$\& \; K_\eta \cap A^{<\sigma} = \emptyset) \;\&\; (\forall w \in R^{<\sigma}_{A,0} - S^{<\sigma}_A)((w)_1 \neq \varepsilon \vee (w)_2 \neq x)\}$$

Thus $\langle \varepsilon, x\rangle \in K^\sigma$ only if there is a computation $x \in W^{A^\sigma}_\varepsilon$ which is not already being preserved by an active requirement A requirement for each $\langle \varepsilon, x\rangle \in K^\sigma$ preserving such a computation will be created. Letting

$$F^\sigma_{\varepsilon,x} = \bigcup\{K_\eta \;\; \langle x, \eta\rangle \in W^\sigma_\varepsilon \;\&\; K_\eta \cap A^{<\sigma} = \emptyset\}$$

we put

$$R^\sigma_{A,0} = R^{<\sigma}_{A,0} \cup \{\langle \varepsilon, x, F^\sigma_{\varepsilon,x}\rangle \;\; \langle \varepsilon, x\rangle \in K^\sigma\}$$

Case II $\text{Cof}(|{\leqslant}|^*) = |{\leqslant}|^* < |{\leqslant}|$ Let

$$\begin{aligned} K^\sigma = \{\langle \varepsilon, x\rangle \in M^\sigma_\alpha \times L^{|\leqslant|^*} \;\; & (\exists \eta < \sigma)(\langle x, \eta\rangle \in W^\sigma_\varepsilon \;\&\; K_\eta \cap A^{<\sigma} = \emptyset) \\ & \& \;\; (\forall w \in R^{<\sigma}_{A,0} - S^{<\sigma}_A)((w)_1 \neq \varepsilon \vee (w)_2 \neq x) \\ & \& \;\; [(\forall y < x)(\exists w \in R^{<\sigma}_{A,0} - S^{<\sigma}_A)((w)_1 = \varepsilon \\ & \& \;\; (w)_2 = y) \vee x \in \bigcup\{M^\sigma_\beta \;\; \beta \leqslant \alpha\}]\} \end{aligned}$$

To show that A is hyperregular in this case, we need preserve computations on initial segments of $L^{|\leqslant|^*}$ In addition, in order to show $\Theta[A]$ is adequate, we need preserve computations $x \in W^A_\varepsilon$ for $x \in \bigcup\{M_\beta \;\; \beta \leqslant \alpha\}$. $F^\sigma_{\varepsilon,x}$ and $R^\sigma_{A,0}$ are defined as in the previous case

Next we construct requirements making $A' \leqslant_w 0'$ Let

$$I^{\sigma} = \{\varepsilon \in M^{\sigma}_{\alpha} \;\; (\exists \eta \in W^{\sigma}_{\varepsilon})(K_{\eta} \cap A^{<\sigma} = \emptyset)$$
$$\& \;\; (\forall w \in R^{<\sigma}_{A\,1} - S^{<\sigma}_{A})((w)_1 \neq \varepsilon)\}$$

Letting $G^{\sigma}_{\varepsilon} = \bigcup\{K_{\eta} \;\; \eta \in W^{\sigma}_{\varepsilon} \;\&\; K_{\eta} \cap A^{<\sigma} = \emptyset\}$ we put

$$R^{\sigma}_{A\,1} = R^{<\sigma}_{A\,1} \bigcup \{\langle \varepsilon, 0, G^{\sigma}_{\varepsilon}\rangle \;\; \varepsilon \in I^{\sigma}\}$$

Finally we construct requirements making $D \not\leqslant_w A$ Let H^{σ} be the Θ-finite set of minimal x such that for each $x' \sim x$, $x' \notin D^{\sigma}$ and $\neg(\exists \langle \varepsilon, x', F\rangle \in R^{<\sigma}_{A\,2} - S^{<\sigma}_{A})$ ("ε is an active $\alpha - A$ reduction procedure at σ") Next let

$$N^{\sigma} = \{\langle \varepsilon, x\rangle \in M^{\sigma}_{\alpha} \times H^{\sigma} \;\; \text{"}\varepsilon \text{ is an active } \alpha - A \text{ reduction procedure at } \sigma\text{"}$$
$$\& \; (\exists \eta < \sigma)(\langle x, \eta\rangle \in W^{\sigma}_{\varepsilon} \;\&\; K_{\eta} \cap A^{<\sigma} = \emptyset)\}$$

Letting $F^{\sigma}_{\varepsilon} = \bigcup\{K_{\eta} \;\; (\exists x \in H^{\sigma})(\langle x, \eta\rangle \in W^{\sigma}_{\varepsilon} \;\&\; K_{\eta} \cap A^{<\sigma} = \emptyset)\}$ we put

$$R^{\sigma}_{A\,2} = R^{<\sigma}_{A\,2} \cup \{\langle \varepsilon, x, F^{\sigma}_{\varepsilon}\rangle \;\; \langle \varepsilon, x\rangle \in N^{\sigma}\}$$

To establish our priorities let

$$J^{\sigma}_{A} = \{\langle \varepsilon, x, F\rangle \in R^{\sigma}_{A} - S^{<\sigma}_{A} \;\; F \cap C^{\sigma} \neq \emptyset\} \quad \text{where} \quad R^{\sigma}_{A} = \bigcup\{R^{\sigma}_{A\,i} \;,\; i < 3\}$$

J^{σ}_{A} is the set of active A-requirements which would be injured in case C^{σ} were added to A Using the notation $(H)_i = \{(w)_i \;\; w \in H\}$, define $f_A(\sigma) = \mu\beta[(J^{\sigma}_{A})_1 \cap M^{\sigma}_{\beta} \neq \emptyset]$ in case such β exists and let $f_A(\sigma) = |\leqslant|$ otherwise It is clear from the definition of the blocks M^{σ}_{β} (considering the split in that definition) that f_A and f_B may be viewed as Θ-computable functions If $f_A(\sigma) \leqslant f_B(\sigma)$, let $B^{\sigma} = B^{<\sigma} \cup C^{\sigma}$ and $A^{\sigma} = A^{<\sigma}$ If $f_B(\sigma) < f_A(\sigma)$, let $A^{\sigma} = A^{<\sigma} \cup C^{\sigma}$ and $B^{\sigma} = B^{<\sigma}$

To complete the construction, let $S^{\sigma}_{A} = \{\langle \varepsilon, x, F\rangle \in R^{\sigma}_{A} \;\; F \cap A^{\sigma} \neq \emptyset\}$

Lemma 4.6. *For each $\alpha < \Sigma_2$-cof (|≤|, the set of $\alpha - A$ and $\alpha - B$ requirements is Θ-finite*

Proof. The proof is by induction on α Fix $\alpha < \Sigma_2$-cof (|≤|) and assume the set of $\beta - A$ and $\beta - B$ requirements is Θ-finite for each $\beta < \alpha$ By the tameness of our blocking there is a stage σ_0 by which all blocks M^{σ}_{β} for $\beta \leqslant \alpha$ have settled down Let

$$I_{\beta} = \{\sigma > \sigma_0 \;\; (\exists w \in R^{\sigma}_{A} \cup R^{\sigma}_{B} - R^{<\sigma}_{A} \cup R^{<\sigma}_{B})((w)_1 \in M^{\sigma}_{\beta})\}$$

Then I_{β} is Θ-finite for each $\beta < \alpha$ by our induction hypothesis so $\bigcup\{I_{\beta} \;\; \beta < \alpha\}$ is Θ-finite by Lemma 3 4 Thus, using the regularity of C, we can assert the existence of $\sigma_1 \geqslant \sigma_0$ such that all β-requirements for $\beta < \alpha$ have been created by σ_1 and no such β-requirement will meet C^{τ} for $\tau \geqslant \sigma_1$ It follows that $f_A(\tau) \geqslant \alpha$

and $f_B(\tau) \geqslant \alpha$ for $\tau \gtrsim \sigma_1$ and hence, by our priorities, no $\alpha - A$ requirement will be injured beyond σ_1

Now we show the existence of $\sigma_2 \gtrsim \sigma_1$ beyond which no $\alpha - A$ requirement in $R_{A\,1}$ is created Let

$$T_1 = \{\varepsilon \in M_\alpha \ (\exists \sigma \gtrsim \sigma_1)\exists w \in R^\sigma_{A\,1} - R^{<\sigma}_{A\,1})((w)_1 = \varepsilon)\}$$

T_1 is Θ-s c and hence, by the adequacy of Θ, Θ-finite After σ_1 only permanent $\alpha - A$ requirements are created As is readily seen from the definition of I^σ, at most one permanent ε-requirement is created for each $\varepsilon \in M_\alpha$ Thus the existence of σ_2 follows from T_1 being Θ-finite

Next we show the existence of $\sigma_3 \gtrsim \sigma_2$ beyond which no $\alpha - A$ requirement in $R_{A\,0}$ is created We need consider two cases

Case A $|\leqslant|^* = |\leqslant|$ or $\mathrm{cof}(|\leqslant|^+) < |\leqslant|^* < |\leqslant|$ The set

$$\{\langle\varepsilon, x\rangle \in M_\alpha \times \bigcup\{M_\beta \ \beta \leqslant \alpha\} \ (\exists \sigma \gtrsim \sigma_2)(\exists w \in R^\sigma_{A\,0} - R^{<\sigma}_{A\,0}) \\ ((w)_1 = \varepsilon \ \& \ (w)_2 = x)\}$$

is Θ-finite by adequacy and the assumption on the pairing function The existence of σ_3 then follows as above

Case B $\mathrm{Cof}(|\leqslant|^+) = |\leqslant|^* < |\leqslant|$ Let

$$T_0 = \{\varepsilon \in M_\alpha \ (\forall x < |\leqslant|^*)(\exists \sigma)(\exists w \in R^\sigma_{A\,0} - S^{\sigma}_{A'}) \\ ((w)_1 = \varepsilon \ \& \ (w)_2 = x)\}$$

T_0 is the set of $\varepsilon \in M_\alpha$ for which there is a permanent ε-requirement with argument x for each $\in L^{|\leqslant|^*}$ T_0 is Θ-finite by adequacy and hence there is $\sigma'_2 \gtrsim \sigma_2$ by which stage all such requirements are created

Suppose there is $\gamma < |\leqslant|^*$ such that if an $\alpha - A$ requirement in $R_{A\,0}$ is created beyond σ'_2 then its argument is less than γ Then the existence of $\sigma_3 \gtrsim \sigma'_2$ follows just as in the former case.

Suppose no such γ exists For each $x \in L^{|\leqslant|^*}$ let

$$q(x) = \nu\varepsilon[(\exists \sigma \gtrsim \sigma'_2)(\exists w \in R^\sigma_{A\,0} - R^{<\sigma}_{A\,0}) \\ ((w)_1 = \varepsilon \ \& \ (w)_2 \gtrsim x \ \& \ \varepsilon \in M_\alpha)]$$

Then $q \ L^{|\leqslant|^*} \rightarrow M_\alpha$ is total. Fix $\varepsilon \in M_\alpha$ If there is a permanent ε-requirement with argument x for each $x \in L^{|\leqslant|^*}$, then $q^{-1}(\varepsilon) = \emptyset$ by our choice of σ' Else there is $x < |\leqslant|^*$ such that there is no permanent ε-requirement with argument x If $x \in \bigcup\{M_\beta \ \beta \leqslant \alpha\}$, then $q^{-1}(\varepsilon) \subseteq \bigcup\{M_\beta \ \beta \leqslant \alpha\}$, else $q^{-1}(\varepsilon) \subseteq L^{\rho_{\leqslant}(x)+1}$ In either case $q^{-1}(\varepsilon)$ is bounded strictly below $|\leqslant|^*$ But then $\mathrm{cof}(|\leqslant|^+) < |\leqslant|^*$, contradicting our case hypothesis

Finally we show the existence of $\sigma \gtrsim \sigma_3$ beyond which no $\alpha - A$ requirement in $R_{A\,2}$ is created First note that an $\alpha - A$ reduction procedure inactive at some $\tau \gtrsim \sigma_1$ will remain inactive forever, since no $\alpha - A$ requirement is injured beyond

σ_1 The set of $\alpha - A$ reduction procedures which become inactive beyond σ_1 is Θ-s c and hence Θ-finite Thus there is $\sigma_4 \geqslant \sigma_1$ beyond which no $\alpha - A$ reduction procedure is made inactive

Suppose $\sigma_4 \leqslant \sigma < \tau$ and $l(\sigma) = l(\tau) = \alpha$ From the choice of σ_4 it is easily seen that $H^\sigma \leqslant H^\tau$ (i e $x \in H^\sigma$ & $y \in H^\tau \Rightarrow x \leqslant y$) Moreover, if an $\alpha - A$ requirement is created at σ, then $H^\sigma < H^\tau$ It follows that either the set of $\alpha - A$ requirements is Θ-finite or for each $x \notin D$ there is a permanent $\alpha - A$ requirement $\langle \varepsilon, x', F\rangle$ where $x' \sim x$ and ε is a reduction procedure active beyond σ_4 If the latter were the case D would be Θ-computable contrary to our hypothesis For then

$$x \notin D \Leftrightarrow (\exists \tau \geqslant \sigma_4)(\exists x' \sim x)(\exists \langle \varepsilon, x', F\rangle \in R^{\tau}_{A,2} - S^{\tau}_{A})$$

("ε is an active $\alpha - A$ reduction procedure at τ)

This completes the proof that the set of $\alpha - A$ requirements is Θ-finite Using the regularity of C choose $\sigma_5 \geqslant \sigma_1$ sufficiently large for all $\alpha - A$ requirements to have been created and such that no C^τ will meet an $\alpha - A$ requirement for $\tau \geqslant \sigma_5$ No $\alpha - B$ requirement is injured beyond σ_5 since $f_X(\tau) > \alpha$ whenever $\tau \geqslant \sigma_5$ To show that the set of $\alpha - B$ requirements is Θ-finite we can thus repeat the above argument with B in place of A starting with σ_5 in place of σ_1

Lemma 4.7. *A and B are hyperregular*

Proof. The proof splits into three cases

Case 1 $|\leqslant|^r = |\leqslant|$ Suppose $H \subseteq W^A_\varepsilon$ where H is Θ-finite We need to show the existence of τ such that $H \subseteq {}^{\tau}W^A_\varepsilon$ Recall that our $(\leqslant)$-parametrization of Θ-s c sets was chosen to be repetitive Choose β_0 such that $H \subseteq \bigcup\{M_\gamma \ \gamma < \beta_0\}$ and choose $\alpha \geqslant \beta_0$ for which there is $\delta \in M_\alpha$ such that $W_\varepsilon = W_\delta$ Let σ be sufficiently large for all $\alpha - A$ requirements to have settled down Then for each $x \in H$ there is a permanent δ-requirement with argument x in $R^\sigma_{A,0}$ For if this was not the case for some $x \in H$, choose η such that $\langle x, \eta\rangle \in W_\delta$ and $K_\eta \cap A = \emptyset$ Let $\tau > \sigma$ be such that $l(\tau) = \alpha$ and $\langle x, \eta\rangle \in W^\tau_\delta$ Then $\langle \delta, x\rangle \in K^\tau$ so a δ-requirement with argument x would be put into $R^\tau_{A,0}$ contradicting the choice of σ Let $x \in H$ and choose $\langle \delta, x, F\rangle \in R^\sigma_{A,0} - S^\sigma_A$ Then there is η such that $\langle x, \eta\rangle \in W^\sigma_\delta$ and $K_\eta \subseteq F$ But $\langle \delta, x, F\rangle$ is a permanent requirement so $F \cap A = \emptyset$, i e $x \in {}^{\sigma}W^A_\delta$ Thus $H \subseteq {}^{\sigma}W^A_\delta$ Choose τ such that $W^\sigma_\delta \subseteq W^\tau_\varepsilon$ Then $H \subseteq {}^{\tau}W^A_\varepsilon$

Before proceeding to the remaining cases we note that by easy manipulations using a projection function one can show the following If $|\leqslant|^r < |\leqslant|$, then a set A is hyperregular iff for every ε, $L^{|\leqslant|^r} \subseteq W^A_\varepsilon \Rightarrow \exists\sigma(L^{|\leqslant|^r} \subseteq {}^{\sigma}W^A_\varepsilon)$

Case 2 $\mathrm{Cof}(|\leqslant|^r) = |\leqslant|^r < |\leqslant|$ Suppose $L^{|\leqslant|^r} \subseteq W^A_\varepsilon$ and let $\varepsilon \in M_\alpha$ Choose σ sufficiently large for all $\alpha - A$ requirements to have settled down Recall from the construction that in this case we attempted to preserve computations on initial segments of $L^{|\leqslant|^r}$ Thus using an argument similar to the one above there is for each $x \in L^{|\leqslant|^*}$ a permanent ε-requirement with argument x in R^σ_{A0} preserving a correct computation $x \in W^A_\varepsilon$. Thus $L^{|\leqslant|^*} \subseteq {}^\sigma W^A_\varepsilon$.

Case 3. $\mathrm{Cof}(|\leqslant|^*) < |\leqslant|^* < |\leqslant|$. Let $\mathrm{cof}(|\leqslant|^*) = \gamma$ and let $q: L^{|\leqslant|^*} \to L^\gamma$ be as in Definition 4.5. Recalling the remark following that definition we view $q^{-1}(x)$ as a set Θ-finite uniformly in x. Define the Θ-computable mapping $\lambda\varepsilon\sigma V^\sigma_\varepsilon$ by

$$V^\sigma_\varepsilon = V_0 \cup \{\langle x, \eta\rangle \in L^\gamma \times L^\sigma : (\forall y \in q^{-1}(x))(\exists \xi < \sigma)(\langle y, \xi\rangle \in W^\sigma_\varepsilon \ \&\ K_\xi \subseteq K_\eta)\}$$

where

$$V_0 = \{\langle x, \eta_0\rangle : x \in L^\gamma \ \&\ q^{-1}(x) = \emptyset\} \quad \text{and} \quad K_{\eta_0} = \emptyset$$

Claim: $L^{|\leqslant|^*} \subseteq W^A_\varepsilon \Leftrightarrow L^\gamma \subseteq V^A_\varepsilon$

To prove the claim, assume $L^{|\leqslant|^*} \subseteq W^A_\varepsilon$ and let $x \in L\gamma$. If $q^{-1}(x) = \emptyset$, then $x \in V^A_\varepsilon$. Suppose $q^{-1}(x) \neq \emptyset$. Then $q^{-1}(x)$ is bounded strictly below $|\leqslant|^*$. Let α, δ and σ_0 be such that $q^{-1}(x) \subseteq \bigcup\{M_\beta : \beta \leqslant \alpha\}, \delta \in M_\alpha$ and $\forall \tau \geqslant \sigma_0 (W^\tau_\varepsilon = W^\tau_\delta)$. Choose $\sigma \geqslant \sigma_0$ sufficiently large for all $\alpha - A$ requirements to have settled down. Then as in the first case there is a permanent δ-requirement in R^σ_{A0} with argument y for each $y \in q^{-1}(x)$. Let

$$K_\eta = \bigcup\{F : \langle\delta, y, F\rangle \in R^\sigma_{A0} - S^\sigma_A \ \&\ y \in q^{-1}(x)\}$$

Then $\langle x, \eta\rangle \in V^\tau_\varepsilon$ for $\tau \geqslant \sigma$ and $\tau > \eta$. Furthermore $K_\eta \cap A = \emptyset$ since only permanent requirements were used to obtain K_η. It follows that $x \in V^A_\varepsilon$.

Conversely assume $L^\gamma \subseteq V^A_\varepsilon$ and let $y \in L^{|\leqslant|^*}$. Choose x, η and σ such that $y \in q^{-1}(x)$, $\langle x, \eta\rangle \in V^\sigma_\varepsilon$ and $K_\eta \cap A = \emptyset$. Then there is ξ such that $\langle y, \xi\rangle \in W^\sigma_\varepsilon$ and $K_\xi \subseteq K_\eta$. Thus $y \in {}^\sigma W^A_\varepsilon$.

Suppose $L^{|\leqslant|^*} \subseteq W^A_\varepsilon$. By the claim, $L^\gamma \subseteq V^A_\varepsilon$. Choose α and δ such that $L^\gamma \subseteq \bigcup\{M_\beta : \beta \leqslant \alpha\}, \delta \in M_\alpha$ and $V_\varepsilon = W_\delta$. By the usual argument there is σ such that $L^\gamma \subseteq {}^\sigma W^A_\delta$. Let τ be such that $W^\sigma_\delta \subseteq V^\tau_\varepsilon$. Then $L^\gamma \subseteq {}^\tau V^A_\varepsilon$ so by the last half of the proof of the claim, $L^{|\leqslant|^*} \subseteq {}^\tau W^A_\varepsilon$.

Lemma 4.8. $\Theta[A]$ *and* $\Theta[B]$ *are adequate theories*.

Proof. $\Theta[A]$ is an infinite theory by Theorem 2.9 since A is hyperregular and regular. Clearly $|\leqslant|^*_\Theta \geqslant |\leqslant|^*_{\Theta[A]}$. We show $|\leqslant|^+_{\Theta[A]} \geqslant |\leqslant|^*_\Theta$. Let $V \subseteq L^\beta$ be a $\Theta[A]$-s.c. set where $\beta < |\leqslant|^*_\Theta$. Then, again using Theorem 2.9, V is weakly Θ-s.c. in A. Let α and δ be such that $V \subseteq \bigcup\{M_\beta : \beta \leqslant \alpha\}$, $\delta \in M_\alpha$ and $V = W^A_\delta$. A permanent δ-requirement with argument x is put into R_{A0} for each $x \in V$. Let σ be sufficiently large for all α-requirements to have settled down. Then

$$x \in V \Leftrightarrow (\exists w \in R^\sigma_{A0} - S^\sigma_A)((w)_1 = \delta \ \&\ (w)_2 = x),$$

so V is Θ-finite and hence $\Theta[B]$-finite.

Lemma 4.9. $A' \equiv B' \equiv 0'$

Proof. As already remarked, it suffices to show $A' \leq_w 0'$ Let

$$q(\varepsilon) = \mu\sigma[\forall\tau \gtrsim \sigma)(\forall w \in (R^{\tau}_{A} - R^{<\tau}_{A}) \cup (S^{\tau}_{A} - S^{<\tau}_{A}))((w)_1 > \varepsilon)]$$

q is defined on all of $L^{\leq|*}$ by Lemma 4 6 Furthermore $q \leq_w 0'$ since q is a Σ_2-function Clearly $\varepsilon \in A' \Leftrightarrow \varepsilon \in (R^{q(\varepsilon)}_{A\,1} - S^{q(\varepsilon)}_{A})_1$ and hence $A' \leq_w 0'$

Lemma 4.10. *$D \not\leq_w A$ and $D \not\leq_w B$*

Proof. Suppose $(U - D) = W_\varepsilon^{A}$ Choose α and σ_0 such that $\varepsilon \in M_\alpha$, all $\alpha - A$ requirements have settled down by stage σ_0 and no $\delta \in M_\alpha$ becomes an inactive $\alpha - A$ reduction procedure beyond σ_0 Note that ε is an active $\alpha - A$ reduction procedure at σ_0, for else an erroneous computation would be preserved Choose a minimal $x \notin D$ such that there is no $x' \sim x$ for which $\langle \delta, x', F\rangle \in R^{\sigma_0}_{A,2} - S^{\sigma_0}_{A}$ where δ is an active $\alpha - A$ reduction procedure at σ_0 By the regularity of D there is $\sigma_1 \geq \sigma_0$ such that $L^x \cap D = L^x \cap D^{\sigma_1}$ Let $\tau \geq \sigma_1$ be such that $x \in {}^{\tau}W_\varepsilon^{A}$ and $\iota(\tau) = \alpha$ Then $H^{\tau} = \{x' \; x' \sim x\}$ and $\langle \varepsilon, x\rangle \in N^{\tau}$ It follows that an ε-requirement with argument x will be created at τ, contradicting the fact that $\tau \gtrsim \sigma_0$

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