JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 4, 65-71 (1962)

On Transcendental Equations Related to Differential-Difference Equations

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I. INTRODUCTION

In a recent report, W. L. Miranker [3] discussed existence, uniqueness, and stability theorems for systems of nonlinear differential-difference equations of the form

$$\sum_{j=0}^{n_1} A_j y'(t - t_j') + \sum_{j=0}^{n_2} B_j y(t - t_j) = f(y, t). \tag{1}$$

Here the A_j and B_j are real constant $N \times N$ matrices, the tj' and tj are real constants, and f and y are real N-dimensional column vectors. The function f is regarded as given, and y is the unknown. Associated with this equation is the *characteristic matrix*

$$H(s) = \sum_{j=0}^{n_1} A_j s e^{-t_j' s} + \sum_{j=0}^{n_2} B_j e^{-t_j s}. \tag{2}$$

Miranker showed that if the roots of the characteristic equation

$$\det H(s) = 0 \tag{3}$$

all satisfy

$$\operatorname{Re}(s) \leqq -c_1 < 0, \tag{4}$$

then under suitable conditions on f the zero solution of the equation in (1) is asymptotically stable as $t \to +\infty$.

Miranker also remarked that a necessary condition for (4), at least for a single scalar equation and commensurable lags t_j' and t_j, is that

$$\inf_{j=0,\ldots,n_1} t_j' \leqq \inf_{j=0,\ldots,n_2} t_j, \tag{5}$$

in view of certain theorems of Pontryagin [4], and he raised the question of the precise relation between the conditions in (4) and (5). The present note is intended as a clarification of this relation.

We first remark that (5) is certainly not sufficient to insure (4). In fact, consider the single scalar equation

$$a_0 y'(t) + a_1 y'(t-\omega) + b_0 y(t) = 0, \qquad a_0 a_1 \neq 0. \tag{6}$$

For this equation the characteristic function is

$$h(s) = a_0 s + a_1 s e^{-\omega s} + b_0 = e^{-\omega s}\left[s e^{\omega s}\left(a_0 + \frac{b_0}{s}\right) + a_1 s\right].$$

The problem of determining the asymptotic location of the zeros of such a transcendental function is one which has been solved. The procedure is outlined in an expository paper of Langer [2], and in more detail in a forthcoming book of Bellman and Cooke [1]. For the characteristic function of the equation in (6), the zeros of large modulus are asymptotically equal to those of the comparison function

$$a_0 s e^{\omega s} + a_1 s = s(a_0 e^{\omega s} + a_1).$$

Clearly these zeros are given by $s = 0$ and

$$s = \omega^{-1} \log(-a_1/a_0), \qquad a_1 a_0 \neq 0. \tag{7}$$

The equation in (6) has the form in (1) with $t_0' = 0$, $t_1' = \omega$, $t_0 = 0$, and thus the inequality in (5) is satisfied. However, the zeros of the comparison function are given by

$$s = \omega^{-1}\{\log|-a_1/a_0| + i[\arg(-a_1/a_0) + 2k\pi]\}, \quad (k = 0, \pm 1, \cdots) \tag{8}$$

and consequently if $|a_1/a_0| > 1$, infinitely many zeros of $h(s)$ satisfy $\operatorname{Re}(s) > 0$.

In refs. [1] and [2] already cited, it is shown that the roots of large modulus of the equation in (3) are grouped in a finite number of *chains*. Some of these chains, which we call ***retarded***, have the property that $\operatorname{Re}(s) \to -\infty$ as $|s| \to \infty$ along the chain. Others, called ***advanced***, have the property that $\operatorname{Re}(s) \to +\infty$ as $|s| \to \infty$ along the chain. *Neutral* chains have the property that $\operatorname{Re}(s)$ is bounded as $|s| \to \infty$ along the chain. Roots in a retarded chain or in an advanced chain lie asymptotically along a curve with an equation of the form $|s^m e^s| =$ constant. Such a curve is, for $|s|$ large, similar to an ordinary exponential curve in appearance. Roots in a neutral chain may lie asymptotically along a vertical line in the s-plane, as for the equation in (6) above, or may exhibit a more complicated behavior.

A conjecture which quickly occurs to one is that the condition in (5), while not sufficient to insure that (4) holds, might be sufficient to insure that all root chains of (3) are of neutral or retarded type, or equivalently that there is a constant c, possibly positive, such that all roots lie in the half-plane $\mathrm{Re}\,(s) \leqq c$. We shall show that even this weaker assertion is not quite correct, but that a somewhat restricted form of the assertion *is* correct.

II. The Distribution Diagram

In order to discuss the conjecture in the last paragraph, we shall need to explain the procedure by which the various root chains of the equation $\det H(s) = 0$ can be located. We can assume that the numbers t_j' and t_j are so arranged that

$$t_0' < t_1' < \cdots < t_{n_1}' \qquad \text{and} \qquad t_0 < t_1 < \cdots < t_{n_2} \tag{9}$$

and we can assume that none of the matrices A_j, B_j is the zero matrix.

Let us first suppose that $t_{n_1}' \geqq t_{n_2}$. We then let

$$G(s) = \exp\,(t_{n_1}'s)\,H(s) = \sum_{j=0}^{n_1} A_j s e^{\alpha_j s} + \sum_{j=0}^{n_2} B_j e^{\beta_j s} \tag{10}$$

where

$$\alpha_j = t_{n_1}' - t_j', \qquad \beta_j = t_{n_1}' - t_j. \tag{11}$$

Thus from (9) we see that

$$\begin{aligned} &0 = \alpha_{n_1} < \alpha_{n_1-1} < \cdots < \alpha_1 < \alpha_0 \\ &0 \leqq \beta_{n_2} < \beta_{n_2-1} < \cdots < \beta_1 < \beta_0. \end{aligned} \tag{12}$$

The roots of $\det H(s) = 0$ are the same as the roots of $\det G(s) = 0$. If we now write down the matrix $G(s)$, and calculate its determinant, we find that it has the form

$$g(s) = \det G(s) = \sum_{i=0}^{m} p_i(s) e^{\gamma_i s} \tag{13}$$

where the numbers γ_i are linear combinations, with nonnegative integral coefficients, of the α_j and β_j, and where each $p_i(s)$ is a polynomial in s of degree at most equal to N, the dimension of the system in (1). We can suppose the γ_i so ordered that

$$0 = \gamma_m < \gamma_{m-1} < \cdots < \gamma_0, \tag{14}$$

and we can write

$$p_i(s) = \sum_{k=0}^{N} p_{ik} s^k. \tag{15}$$

We now locate the root chains in the following way. For each i, $(0 \leqq i \leqq m)$ let k_i denote the largest value of k in (15) for which $p_{ik} \neq 0$. Thus

$$p_i(s) = p_{ik_i} s^{k_i} (1 + o(1)) \text{ as } |s| \to \infty. \tag{16}$$

We now plot the points P_i: (γ_i, k_i), $0 \leqq i \leqq m$ on a rectangular coordinate system. If for some i all p_{ik} are zero, that is, if $p_i(s) = 0$, no point is plotted. We now draw the unique polygonal line L which

(i) joins P_0 with P_m,
(ii) has vertices only at points of the set P_i,
(iii) is convex upward (or straight), and
(iv) is such that no points P_i lie above it.

This graph will be called the distribution diagram associated with $g(s)$. It can be shown, cf. [1], that there is a root chain of $f(s) = 0$ associated with each segment of L. If the slope of the segment is positive, the roots form a retarded chain; if the slope is negative, the roots form an advanced chain; and if the slope is zero, the roots form a neutral chain.

On the other hand, suppose that $t'_{n_1} < t_{n_2}$. We then let

$$G(s) = \exp(t_{n_2} s) H(s) = \sum_{j=0}^{n_1} A_j s e^{\alpha_j s} + \sum_{j=0}^{n_2} B_j e^{\beta_j s} \tag{17}$$

where

$$\alpha_j = t_{n_2} - t'_j, \qquad \beta_j = t_{n_2} - t_j. \tag{18}$$

Thus

$$0 < \alpha_{n_1} < \cdots < \alpha_1 < \alpha_0$$

$$0 = \beta_{n_2} < \cdots < \beta_1 < \beta_0. \tag{19}$$

Once again the relations in (13), (14), (15), and (16) are valid. The distribution diagram is formed in the same way as before.

III. A Counterexample

We shall now show by an example that *the condition* (5) *is not even sufficient to insure that all root chains are of neutral or retarded type.* Let the vector y have components y_1 and y_2 and consider the system

$$\begin{aligned} &y_1'(t) + y_2'(t) + y_1'(t-1) + y_1(t) = 0 \\ &y_1'(t) + y_2'(t) + 2y_1'(t-1) + y_2'(t-1) + 2y_1(t) + y_2(t) = 0. \end{aligned} \tag{20}$$

This has the form in (1) with $n_1 = 1$, $n_2 = 0$, $N = 2$, and with $t_0' = 0$, $t_1' = 1$, $t_0 = 0$, and

$$A_0 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

The inequality in (5) is clearly satisfied. However, a simple calculation yields

$$g(s) = \det \{e^s H(s)\} = (e^s + s)^2.$$

The points on the indicator diagram are (0,2), (1,1), and (2,0), and the roots form an advanced chain.

IV. A Necessary and Sufficient Condition

If we impose the further condition that $\det A_0 \neq 0$, the condition in (5) does, however, turn out to be sufficient in order that all roots of $g(s) = 0$ lie in a left half-plane. The extra condition $\det A_0 \neq 0$ is actually a rather natural one to impose, as we shall now explain. In the first place, the condition $\inf t_j' \leqq \inf t_j$ guarantees that the largest argument appearing in (1) occurs as the argument of one of the y' terms. This would suggest that the rate of change of y at any time could be determined as a function of the rates of change and values of y at previous times—that is, that the solution depends only on past time. This is the situation which one would expect to encounter in most applications, and to give rise to the simplest mathematics. However, if $\det A_0 = 0$, then the system in (1) cannot always be solved for y' at the largest argument, and the solution may in a sense depend on future time. Thus it is physically reasonable to require that $\det A_0$ be nonzero. Accordingly the following theorem seems to be useful.

Theorem. *Consider the system in (1) and suppose the lags t_j' and t_j are arranged as in (9). Assume that* $\det A_0 \neq 0$. *Then the condition in (5), $t_0' \leqq t_0$, is sufficient to insure the existence of a constant c such that all roots of (3) lie in the half-plane* $\operatorname{Re}(s) \leqq c$. *Conversely, if $t_0' > t_0$, and if in addition* $\det B_0 \neq 0$, *then there is a sequence of roots whose real parts approach $+\infty$.*

To prove this theorem, we expand the determinant $g(s)$ of $G(s)$, and examine the distribution diagram. The determinant $g(s)$ is

$$g(s) = \left| \sum_{j=0}^{n_1} a_{\mu\nu}^j s e^{\alpha_j s} + \sum_{j=0}^{n_2} b_{\mu\nu}^j e^{\beta_j s} \right|_{\mu,\nu=1,2,\dots,N} \tag{21}$$

where $A_j = (a_{\mu\nu}^j)$, $B_j = (b_{\mu\nu}^j)$.

Suppose first that $t_0' < t_0$, so that $\alpha_0 > \beta_0$. Since α_0 is the largest α and β_0 is the largest β, we see that in the equation in (13) we have $\gamma_0 = N\alpha_0$. The term in the expansion of $g(s)$ containing $e^{N\alpha_0 s}$ is clearly

$$| a^0_{\mu\nu} s e^{\alpha_0 s} |_{\mu,\nu} = s^N e^{N\alpha_0 s} \det A_0.$$

Since $\det A_0 \neq 0$, the distribution diagram contains the point $(N\alpha_0, N)$, which is the farthest point to the right on the diagram. Since no other point on the diagram can have ordinate greater than N, every segment of the polygonal line L must have zero or positive slope. Consequently all root chains are neutral or retarded.[1]

Next suppose that $t_0' = t_0$. Then $\alpha_0 = \beta_0$. Now $\gamma_0 = N\alpha_0 = N\beta_0 = k\alpha_0 + l\beta_0$ for $k + l = N$. The term of $g(s)$ containing $e^{\gamma_0 s}$ is

$$| a^0_{\mu\nu} s e^{\alpha_0 s} + b^0_{\mu\nu} e^{\beta_0 s} |_{\mu,\nu} = e^{\gamma_0 s}(s^N \det A_0 + \cdots + \det B_0).$$

Once again the distribution diagram contains the point $(N\alpha_0, N)$, and all root chains are neutral or retarded.

It remains to prove the existence of an advanced root chain in case $t_0' > t_0$ and $\det A_0 \neq 0$, $\det B_0 \neq 0$. Now we have $\alpha_0 < \beta_0$, and $\gamma_0 = N\beta_0$. The term in $g(s)$ containing $e^{\gamma_0 s}$ is

$$| a^0_{\mu\nu} e^{\beta_0 s} | = e^{N\beta_0 s} \det B_0.$$

Since $\det B_0 \neq 0$, the distribution diagram contains the point $(N\beta_0, 0)$. On the other hand, let us look for terms in the expansion of $g(s)$ which contain $e^{N\alpha_0 s} s^N$. Since the power of s is N, no factor of such a term can involve $e^{\beta_j s}$, that is, all factors must be of the form $a^j_{\mu\nu}\ s e^{\alpha_j s}$. Since $\alpha_0 > \alpha_1 > \cdots > \alpha_{n_1}$, the value of j in each factor must be zero. Thus the required terms are given by

$$| a^0_{\mu\nu} s e^{\alpha_0 s} | = s^N e^{N\alpha_0 s} \det A_0.$$

Consequently the distribution diagram still contains the point $(N\alpha_0, N)$. Since the diagram contains $(N\alpha_0, N)$ and $(N\beta_0, 0)$, and $N\alpha_0 < N\beta_0$, there must be a segment of negative slope, and therefore an advanced root chain. This completes the proof of the theorem.

We remark that the condition $t_0' \leqq t_0$ is not by itself a necessary condition in order that all roots satisfy $\operatorname{Re}(s) \leqq c$. That is, all roots may lie in a left

[1] In certain cases it is possible that all points of the distribution diagram will have equal abscissas, or in other words that only one exponential in (13) has nonzero coefficient. Then there are but a finite number of characteristic roots, the roots of the nonvanishing polynomial in (13), and certainly there is a constant c for which all roots lie in the half-plane $\operatorname{Re}(s) \leqq c$.

half-plane and yet we may have $t_0' > t_0$. In such a case, of course, det B_0 must be zero. To construct an example of such behavior, we consider a system with $n_1 = 1$, $n_2 = 0$, and $N = 2$. Then $g(s)$ has the form

$$g(s) = | a_{\mu\nu}^0 s e^{\alpha_0 s} + a_{\mu\nu}^1 s e^{\alpha_1 s} + b_{\mu\nu}^0 e^{\beta_0 s} |_{\mu,\nu=1,2}.$$

When expanded, $g(s)$ contains exponentials $e^{\gamma_i s}$ where γ_i takes the values $2\alpha_0$, $2\alpha_1$, $2\beta_0$, $\alpha_0 + \alpha_1$, $\alpha_0 + \beta_0$, $\alpha_1 + \beta_0$. Since det $A_0 \neq 0$, one of the terms is s^2 (det A_0) exp $(2\alpha_0 s)$. By choosing values properly, it is possible to make the coefficient of $e^{\gamma_i s}$ zero for every $\gamma_i > 2\alpha_0$. Thus the point $(2\alpha_0, 2)$ will be farthest to the right on the distribution diagram, every segment of the diagram will have nonnegative slope, and no advanced chains will be possible. For example, consider the system

$$\begin{aligned} &y_1'(t-1) + y_1'(t-2) + y_1(t) + y_2(t) = 0, \\ &2y_1'(t-1) + y_2'(t-1) + 2y_1'(t-2) + y_2'(t-2) + y_1(t) + y_2(t) = 0. \end{aligned} \tag{22}$$

This has the form in (1) with $t_0' = 1$, $t_1' = 2$, $t_0 = 0$, $\alpha_0 = 1$, $\alpha_1 = 0$, $\beta_0 = 2$, and

$$A_0 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \quad A_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \quad B_0 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Evidently det $A_0 = 1$, det $B_0 = 0$, and $t_0' > t_0$. We find that

$$g(s) = \begin{vmatrix} e^{2s} + se^s + s & e^{2s} \\ e^{2s} + 2se^s + 2s & e^{2s} + se^s + s \end{vmatrix} = (e^{2s} + 2e^s + 1)s^2.$$

The roots all lie in a neutral chain along the imaginary axis.

Additional results on the distribution of roots of the equation in (3) can be found in the book [1] already cited.

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