Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 326 (2007) 521–532
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

On Taylor and other joint spectra for commuting n-tuples of operators

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Received 10 November 2004
Available online 17 April 2006
Submitted by R. Curto

Abstract

Let **R** and **S** be commuting n-tuples of operators. We will give some spectral relations between **RS** and **SR** that extend the case of single operators. We connect the Taylor spectrum, the Fredholm spectrum and some other joint spectra of **RS** and **SR**. Applications to Aluthge transforms of commuting n-tuples are also provided.

Keywords: n-Tuples; Taylor spectrum; Joint spectra; Aluthge transforms

1. Introduction

Let X be a Banach space and let $\mathcal{L}(X)$ denote the Banach algebra of all bounded linear operators on X. For $T \in \mathcal{L}(X)$, let $\sigma(T)$, $\ker(T)$ and $\operatorname{Im}(T)$ denote the spectrum of T, the null space and the range space of T, respectively.

An operator $\mathbf{T} = (T_1, \ldots, T_n) \in \mathcal{L}(X)^n$ is called a commuting n-tuple if $T_iT_j = T_jT_i$ for every $1 \leqslant i, j \leqslant n$. Spectral properties of commuting n-tuples received important attention during last decades. Systematic investigations have been carried out to extend (and to apply) known results for single operators to commuting n-tuples.

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.03.008

It is well known that for any single operators R and S, the following equality holds:

$$\sigma(RS) \setminus \{0\} = \sigma(SR) \setminus \{0\}. \tag{1}$$

It has also been shown that RS and SR share, except perhaps at zero, all their spectral and local spectral properties, see [2–4] for example.

Equation (1) has been extended to criss-cross commuting n-tuples. That is operators $\mathbf{R}$ and $\mathbf{S}$ that satisfy $R_i S_j R_k = R_k S_j R_i$ and $S_i R_j S_k = S_k R_j S_i$ for every $1 \leqslant i, j, k \leqslant n$. The spectrum in this context is taken to be the Taylor spectrum [8,17].

We study in this paper common spectral properties of the n-tuples $\mathbf{RS}$ and $\mathbf{SR}$ when $\mathbf{R}$ and $\mathbf{S}$ are commuting n-tuples satisfying some commutation relations. Here $\mathbf{R} = (R_1, \dots, R_n)$ and $\mathbf{S} = (S_1, \dots, S_n)$, and we write $\mathbf{RS} = (R_1 S_1, \dots, R_n S_n)$.

We mention at this stage that the commutativity condition under scope here can be stated even if $\mathbf{R}$ and $\mathbf{S}$ do not have the same length or are elements in an arbitrary Banach algebra.

We introduce in Section 2 basic definitions of Taylor spectrum, Fredholm spectrum and related properties of the associated Koszul complex.

In Sections 3 and 4, we show that if $\mathbf{R} = (R_1, \dots, R_n)$ and $\mathbf{S} = (S_1, \dots, S_n)$ are commuting n-tuples such that

$$R_i S_j = S_j R_i \quad \text{for every } i \neq j, \tag{2}$$

then $\mathbf{RS}$ and $\mathbf{SR}$ are commuting n-tuples. Moreover, for $\lambda = (\lambda_1, \dots, \lambda_n)$ such that $\prod_i \lambda_i \neq 0$, we have that $\lambda - \mathbf{RS}$ is Taylor (respectively Taylor–Fredholm, Weyl) invertible if and only if $\lambda - \mathbf{SR}$ is.

This yields the following extension of Eq. (1) for commuting n-tuples:

$$\Sigma(\mathbf{RS}) \setminus [0] = \Sigma(\mathbf{SR}) \setminus [0], \tag{3}$$

where Σ stands for the Taylor spectrum, the Taylor–Fredholm spectrum or the Taylor–Weyl spectrum of the commuting n-tuples $\mathbf{SR}$ and $\mathbf{RS}$, respectively and where

$$[0] = \left\{ \lambda \in \mathbb{C}^n \colon \prod_{i=1}^{n} \lambda_i = 0 \right\}.$$

We notice in passing that the commutativity condition considered here does not imply the criss-cross commutativity and vice versa. In fact, if $\mathbf{R}$ or $\mathbf{S}$ has the identity operator as one of its coordinates, then criss-cross commutativity implies $\mathbf{RS} = \mathbf{SR}$, while the commutativity condition (2) does not take in account identity coordinates. On the other hand, the n-tuples $\mathbf{R} = (R, \dots, R)$ and $\mathbf{S} = (S, \dots, S)$ are always criss-cross commuting, while the commutativity condition (2) holds only in the trivial case $\mathbf{RS} = \mathbf{SR}$. On the other hand, if $\mathbf{R} = (S_1^*, \dots, S_n^*)$ then commutativity condition (2) is just double commutativity while criss-cross commutativity will require R_i to be quasi-normal.

Note also, as shown by Example 1 (Section 3), that (1) is not valid generally in our context.

It is shown in Sections 3 and 4 that Eq. (3) is valid for Taylor and Fredholm spectra. Section 5 is devoted to split spectra, as a consequence we derive that (3) is valid for the joint approximate point spectrum, defect spectrum and other joint spectra.

The last section is focused on applications, we retrieve some results from [6,7,13] given for Aluthge transforms of doubly commuting n-tuples. Concluding remarks, concerning possible extensions end this paper.

2. Basic definitions

Let $\mathbf{e}=\{e_1,e_2,\ldots,e_n\}$ be indeterminates and define $\Lambda_n[\mathbf{e}]$ to be the *exterior algebra* on the generators $e_1,e_2,\ldots,e_n$. That is the linear space over the complex plane $\mathbb{C}$ endowed with an anti-commutative exterior product $e_i\wedge e_j=-e_j\wedge e_i$ $(1\leqslant i,j\leqslant n)$. For $F=\{i_1,\ldots,i_p\}\subset\{1,\ldots,n\}$ with $i_1<\cdots<i_p$, we write $e_F=e_{i_1}\wedge\cdots\wedge e_{i_p}$. The exterior algebra over $\mathbb{C}$ is then given by

$$\Lambda_n[\mathbf{e}]=\left\{\sum_F\alpha_F e_F\colon\ e_F=e_{i_1}\wedge\cdots\wedge e_{i_p}\text{ and }\alpha_F\in\mathbb{C}\right\}.$$

We let here $e_\emptyset$ to be the identity element for the exterior product. If we denote $\Lambda_n^k[\mathbf{e}]=\{\sum_{|F|=k}\alpha_F e_F\colon\ \alpha_F\in\mathbb{C}\}$, where $|F|$ is the *cardinal* of F, then clearly $\dim\Lambda_n^k[\mathbf{e}]=C_n^k$ for every $k\leqslant n$, $\Lambda_n^k[\mathbf{e}]\wedge\Lambda_n^l[\mathbf{e}]=\Lambda_n^{k+l}[\mathbf{e}]$ and $\Lambda_n[\mathbf{e}]=\bigoplus_{k=0}^n\Lambda_n^k$.

Given a Banach space X, the exterior algebra over X is defined to be

$$\Lambda_n[\mathbf{e},X]=\left\{\sum_F x_F e_F\colon\ e_F=e_{i_1}\wedge\cdots\wedge e_{i_p}\text{ and }x_F\in X\right\}.$$

The subspaces $\Lambda_n^p[\mathbf{e},X]=\{\sum_{|F|=p}x_Fe_F\colon\ x_F\in X\}$, for $p\leqslant n$ are given in a similar way. Naturally $\Lambda_n^0[\mathbf{e},X]$, $\Lambda_n^1[\mathbf{e},X]$ and $\Lambda_n^n[\mathbf{e},X]$ can be identified with X, X^n and X, respectively.

Since no confusion is possible we will omit any reference to the indeterminate $e_1,e_2,\ldots,e_n$ and for short we will write $\Lambda_n^k[X]$ and $\Lambda_n[X]$ for $\Lambda_n^k[\mathbf{e},X]$ and $\Lambda_n[\mathbf{e},X]$, respectively.

If $T\in\mathcal{L}(X)$, we keep the same symbol T to denote the operator defined on $\Lambda_n[X]$ by

$$T\left(\sum_F x_Fe_F\right)=\sum_F Tx_Fe_F.$$

For each $i\in\{1,2,\ldots,n\}$, let $E_i:\Lambda_n[X]\to\Lambda_n[X]$ be the left multiplication operator by e_i: $E_i(e_F)=e_i\wedge e_F$. It is usually called the *creation operator*. With any commuting n-tuple $\mathbf{T}=(T_1,\ldots,T_n)$ we associate the linear mapping defined over $\Lambda_n[X]$ by

$$\delta_{\mathbf{T}}=\sum_{i=1}^n T_i\otimes E_i:\sum_F x_Fe_F\to\sum_F\sum_{i=1}^n T_ix_Fe_i\wedge e_F.$$

Clearly,

$$\delta_{\mathbf{T}}^2(x_Fe_F)=\sum_{i,j=1}^n T_jT_ix_Fe_j\wedge e_i\wedge e_F=\sum_{i<j}T_iT_jx(e_i\wedge e_j+e_j\wedge e_i)\wedge e_F=0.$$

It follows that $\delta_{\mathbf{T}}^2=0$ and hence that $\operatorname{Im}(\delta_T)\subset\ker(\delta_T)$. Since $\delta_{\mathbf{T}}$ takes $\Lambda_n^{k-1}[X]$ to $\Lambda_n^k[X]$, the last statement is equivalent to: $\operatorname{Im}(\delta_{T\,|\Lambda_n^{k-1}[X]})\subset\ker(\delta_{T\,|\Lambda_n^k[X]})$ for every $k\leqslant n$.

Set $\delta_{\mathbf{T}}^k:=\delta_{\mathbf{T}|\Lambda_n^k[X]}$. We construct a co-chain complex $K(\mathbf{T})$, called the *Koszul complex* associated with $\mathbf{T}$ on X as follows:

$$K(\mathbf{T}):\mathbf{0}\xrightarrow{\delta_{\mathbf{T}}^{-1}}\Lambda_n^0[X]\xrightarrow{\delta_{\mathbf{T}}^0}\Lambda_n^1[X]\xrightarrow{\delta_{\mathbf{T}}^1}\cdots\xrightarrow{\delta_{\mathbf{T}}^{n-1}}\Lambda_n^n[X]\xrightarrow{\delta_{\mathbf{T}}^n}\mathbf{0}.$$

The operator $\mathbf{T}$ is said to be non-singular, or Taylor invertible, if $\ker\delta_{\mathbf{T}}^k=\operatorname{Im}\delta_{\mathbf{T}}^{k-1}$ for $k=0,\ldots,n$, equivalently $\ker\delta_{\mathbf{T}}=\operatorname{Im}\delta_{\mathbf{T}}$. The associated Koszul complex is said to be exact in this case. The Taylor spectrum of $\mathbf{T}$ on X^n is then the set

$$\sigma_T(\mathbf{T})=\left\{\lambda\in\mathbb{C}^{\mathbf{n}}\colon\ \mathbf{K}(\mathbf{T}-\lambda)\text{ is not exact}\right\}.$$

The Taylor spectrum coincide with the usual spectrum in the case of single operators. We refer to [19–22] for a detailed study of this spectrum.

The n-tuple $\mathbf{T}$ is said to be Fredholm, if $\operatorname{Im}\delta_{\mathbf{T}}^{k-1}$ is closed and $\ker\delta_{\mathbf{T}}^{k}/\operatorname{Im}\delta_{\mathbf{T}}^{k-1}$ is finite dimensional for every $k = 0, \ldots, n$, equivalently $\operatorname{Im}\delta_{\mathbf{T}}$ is closed and $\ker\delta_{\mathbf{T}}/\operatorname{Im}\delta_{\mathbf{T}}$ is finite dimensional. At this point, we mention that the condition $\operatorname{Im}\delta_{\mathbf{T}}^{k-1}$ is closed, derived from $\dim\ker\delta_{\mathbf{T}}^{k}/\operatorname{Im}\delta_{\mathbf{T}}^{k-1} < \infty$ as observed in [22].

The Fredholm (called also the essential) spectrum is then

$$\sigma_{Te}(\mathbf{T}) := \{\lambda \in \mathbb{C}^n\colon\ \mathbf{T} - \lambda \text{ is not Fredholm}\}.$$

Given a Fredholm n-tuple $\mathbf{T}$, the index of $\mathbf{T}$ is defined by the *Euler characteristic number* of the associated Koszul complex. Namely

$$\operatorname{ind}(\mathbf{T}) = \sum_{k=0}^{n} (-1)^k \dim H^k(\mathbf{T}),$$

where $H^k(\mathbf{T}) = \ker\delta_{\mathbf{T}}^{k}/\operatorname{Im}\delta_{\mathbf{T}}^{k-1}$, $k = 1, \ldots, n$, are the associated cohomology groups. A Fredholm operator is said to be Taylor–Weyl if $\operatorname{ind}(\mathbf{T}) = 0$ [12], and the Taylor–Weyl spectrum is

$$\sigma_{Tw}(\mathbf{T}) := \{\lambda \in \mathbb{C}^n\colon\ \mathbf{T} - \lambda \text{ is not Taylor–Weyl}\}.$$

3. Taylor spectrum of RS and SR

In this section, we investigate the link of the switching operation and the stability of the Taylor spectrum and other joint spectra of **RS** and **SR**. In the sequel, for $i = 1, \ldots, n$, we set

$$\mathbf{SR}_{\{i\}} = (S_1R_1, S_2R_2, \ldots, R_iS_i, \ldots, S_nR_n).$$

(We put in the ith coordinate R_iS_i instead of S_iR_i.) Then, we have

Theorem 1. *Let* $\mathbf{R}$ *and* $\mathbf{S}$ *be commuting* n*-tuples such that* $R_kS_j = S_jR_k$ *for* $k \neq j$*. Then*

(1) **SR** *and* **RS** *are commuting* n*-tuples.*
(2) *For every* $i \in \{1, \ldots, n\}$*, and* $\lambda = (\lambda_1, \ldots, \lambda_n)$ *such that* $\lambda_i \neq 0$*, we have* $\lambda - \mathbf{SR}$ *is Taylor invertible if and only if* $\lambda - \mathbf{SR}_{\{i\}}$ *is.*

The following lemma is crucial in our proof and is of independent interest.

Lemma 1. *Let* $\mathbf{T} = (T_1, \ldots, T_n) \in \mathcal{L}(X)^n$ *be a commuting* n*-tuple, then*

$$T_j = E_j^*\delta_{\mathbf{T}} + \delta_{\mathbf{T}}E_j^*$$

for every $j = 1, \ldots, n$*. In particular,*

$$T_j(\ker(\delta_{\mathbf{T}})) \subset \operatorname{Im}(\delta_{\mathbf{T}}).$$

Proof. Recall from [22] that

$$E_j^*E_i + E_jE_i^* = \delta_{ij}I \quad \text{(Kronecker delta)}.$$

Since $\delta_{\mathbf{T}} = \sum_j T_j \otimes E_j$, we have

$$E_j^*\delta_{\mathbf{T}} + \delta_{\mathbf{T}}E_j^* = E_j^*\sum_i T_i \otimes E_i + \sum_i T_i \otimes E_iE_j^* = \sum_i T_i \otimes \delta_{ij}I = T_j. \qquad \square$$

Proof of Theorem 1. We recall that $\mathbf{T} = (T_1, \dots, T_n)$ is Taylor invertible if and only if for every $x \in \ker \delta_{\mathbf{T}}$ there exists $y \in \Lambda_n[X]$ such that

$$x = \delta_{\mathbf{T}}(y).$$

Suppose now that $\mathbf{T} = (\lambda_1 - S_1 R_1, \dots, \lambda_i - S_i R_i, \dots, \lambda_n - S_n R_n)$ is Taylor invertible and let $\mathbf{T}'_i = (\lambda_1 - S_1 R_1, \dots, \lambda_i - R_i S_i, \dots, \lambda_n - S_n R_n)$. Since $S_k R_j = R_j S_k$ for every $k \neq j$, we obtain the following intertwining relations:

$$\begin{cases} S_i \delta_{\mathbf{T'_i}} = \delta_{\mathbf{T}} S_i, \\ \delta_{\mathbf{T'_i}} R_i = R_i \delta_{\mathbf{T}}. \end{cases} \tag{4}$$

If $x \in \ker \delta_{\mathbf{T}'_i}$, then from Eq. (4), we have $S_i x \in \ker \delta_{\mathbf{T}}$ and because $\delta_{\mathbf{T}}$ is Taylor invertible, we get $S_i x = \delta_{\mathbf{T}} y$ for some $y \in \Lambda[X]$. We write the identity $\lambda_i x = (\lambda_i - R_i S_i)x + R_i S_i x$, then we use Eq. (4) and Lemma 1 to conclude that

$$\lambda_i x = (\lambda_i - R_i S_i)x + \delta_{\mathbf{T}'_i} R_i y \in \mathrm{Im}(\delta_{\mathbf{T}'_i})$$

and since $\lambda_i \neq 0$, we get $x \in \mathrm{Im}(\delta_{\mathbf{T}'_i})$. Consequently $\mathbf{T}'_i$ is Taylor invertible.

The reverse implication is obtained by symmetry. □

Let $\mathcal{I}$ be a subset of $\{1, \dots, n\}$, we associate with $\mathcal{I}$ the "*partially switched*" operator $\mathbf{SR}_{\mathcal{I}}$ of $\mathbf{SR}$ defined as follows:

$$\mathbf{SR}_{\mathcal{I}} := (Q_1, \dots, Q_n),$$

where $Q_i = R_i S_i$ if $i \in \mathcal{I}$, and $Q_i = S_i R_i$ otherwise. Clearly $\mathbf{SR}_{\emptyset} = \mathbf{SR}$ while $\mathbf{SR}_{\{1,\dots,n\}} = \mathbf{RS}$. The symmetric formula $(\mathbf{SR}_{\mathcal{I}})_{\mathcal{I}} = \mathbf{SR}$ enables us to show in all our proofs only one implication each time we have to show an equivalence statement.

The following corollary is a more general version of Theorem 1 and is obtained by a finite induction argument.

Corollary 1. *Under the assumptions of Theorem* 1, *for any* $\mathcal{I}, \mathcal{J} \subset \{1, \dots, n\}$ *and* λ *such that* $\prod_{i \in \mathcal{I} \cup \mathcal{J}} \lambda_i \neq 0$, *we have*

$$\lambda - \mathbf{SR}_{\mathcal{I}} \text{ is Taylor invertible} \quad \text{if and only if} \quad \lambda - \mathbf{SR}_{\mathcal{J}} \text{ is.}$$

Denote $[0]^{\mathcal{K}} = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n\colon \prod_{i \in \mathcal{K}} \lambda_i = 0\}$; here $\mathcal{K} \subset \{1, \dots, n\}$. For convenience, we put $[0] =: [0]^{\{1,\dots,n\}}$ and $[0]^{\emptyset} = \emptyset$.

Using the observations above, we get the following result that extends Eq. (1) for commuting n-tuples.

Theorem 2. *Under the assumptions of Theorem* 1, *for all* $\mathcal{I}, \mathcal{J} \subset \{1, \dots, n\}$, *we have*

$$\sigma_T(\mathbf{SR}_{\mathcal{I}}) \setminus [0]^{\mathcal{I} \cup \mathcal{J}} = \sigma_T(\mathbf{SR}_{\mathcal{J}}) \setminus [0]^{\mathcal{I} \cup \mathcal{J}}.$$

In particular,

$$\sigma_T(\mathbf{RS}) \setminus [0] = \sigma_T(\mathbf{SR}) \setminus [0].$$

The set $[0]^{\mathcal{K}}$ cannot be replaced in general by $\{(0, \dots, 0)\}$ as shown by the following example.

Example 1. Let U be a non-invertible isometry and consider the commuting 2-tuples $\mathbf{S} = (U, I)$ and $\mathbf{R} = (U^*, I)$. Then $\mathbf{S}$ and $\mathbf{R}$ satisfy the assumptions of Theorem 1. Moreover, $\mathbf{RS} = (I, I)$ and $\mathbf{SR} = (P, I)$, where $I - P$ is the orthogonal projection on $\ker U^*$. We have $\sigma_T(\mathbf{RS}) = \{(1, 1)\}$ and $\sigma_T(\mathbf{SR}) = \{(1, 1), (0, 1)\}$.

It is clear that

$$\sigma_T(\mathbf{RS}) \setminus [0]^{\{1\}} = \sigma_T(\mathbf{SR}) \setminus [0]^{\{1\}}$$

and

$$\sigma_T(\mathbf{RS}) \setminus \{(0, 0)\} \neq \sigma_T(\mathbf{SR}) \setminus \{(0, 0)\}.$$

We also have

Corollary 2. *Under the assumptions of Theorem* 1*, we get*

(1) *If* $0 \notin \sigma(R_i S_i) = \sigma(S_i R_i)$ *for every* $i \in \mathcal{I} \subset \{1, \ldots, n\}$*, then*
$$\sigma_T(\mathbf{RS}) \setminus [0]^{\mathcal{I}^c} = \sigma_T(\mathbf{SR}) \setminus [0]^{\mathcal{I}^c}.$$
The set $\mathcal{I}^c$ *is the complement of* $\mathcal{I}$ *in* $\{1, \ldots, n\}$.

(2) *If* $0 \notin \sigma(R_i S_i) = \sigma(S_i R_i)$ *for every* $i = 1, \ldots, n$*, then* $\sigma_T(\mathbf{RS}) = \sigma_T(\mathbf{SR})$.

Proof. For $i = 1, \ldots, n$, let $P_i : \mathbb{C}^n \to \mathbb{C}$ denote the orthogonal projection on the ith coordinate. Then by the spectral mapping theorem, we have $P_i \sigma_T(\mathbf{T}) = \sigma(T_i)$. Theorem 2 completes the proof. $\square$

4. Fredholm operators and essential spectrum

We devote this section to Fredholm invertibility of operators $\mathbf{RS}$ and $\mathbf{SR}$. We keep the same notations of the previous section.

In the line of Theorem 1, we have the following result.

Theorem 3. *Let* $\mathbf{R}$ *and* $\mathbf{S}$ *be commuting* n*-tuples such that* $R_k S_j = S_j R_k$ *for* $k \neq j$*. Let* $i \in \{1, \ldots, n\}$ *and let* $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ *be such that* $\lambda_i \neq 0$*. Then,*

$$\lambda - \mathbf{SR}_{\{i\}} \text{ is Fredholm} \quad \text{if and only if} \quad \lambda - \mathbf{SR} \text{ is Fredholm.}$$

Proof. Suppose that $\mathbf{T} = \lambda - \mathbf{SR}$ is Fredholm, which means that $\operatorname{Im} \delta_{\mathbf{T}}^{k-1}$ is closed and that $H^k(\mathbf{T}) =: \ker \delta_{\mathbf{T}}^k / \operatorname{Im} \delta_{\mathbf{T}}^{k-1}$ is finite dimensional for every $k = 0, \ldots, n$. We will show that $\mathbf{T}'_i = \lambda - \mathbf{SR}_{\{i\}}$ is also Fredholm.

For $k = 1, \ldots, n$, consider the linear transformation

$$\begin{aligned} s_i : \ker \delta_{\mathbf{T}'_i}^k &\longrightarrow H^k(\mathbf{T}) \\ x &\longmapsto \pi(S_i x), \end{aligned}$$

where π is the canonical surjection onto $H^k(\mathbf{T})$. Then s_i is a well-defined linear continuous transformation. Moreover, if $s_i(x) = \pi(0)$ for some $x \in \ker \delta_{\mathbf{T}'_i}^k$, then $S_i x \in \operatorname{Im} \delta_{\mathbf{T}}^{k-1}$. Applying R_i we get $R_i S_i x \in R_i \operatorname{Im} \delta_{\mathbf{T}}^{k-1} \subset \operatorname{Im} \delta_{\mathbf{T}'_i}^{k-1}$. Now, using Lemma 1, we obtain

$$\lambda_i x = (\lambda_i - R_i S_i)x + R_i S_i x \in \operatorname{Im} \delta_{\mathbf{T}'_i}^{k-1}$$

and since $\lambda_i \neq 0$, we get $x \in \operatorname{Im} \delta_{\mathbf{T}'_i}^{k-1}$.

Thus $\ker s_i \subset \operatorname{Im} \delta_{\mathbf{T}'_i}^{k-1}$. The reverse inclusion being trivial we have $\ker s_i = \operatorname{Im} \delta_{\mathbf{T}'_i}^{k-1}$. It follows that $\operatorname{Im} \delta_{\mathbf{T}'_i}^{k-1}$ is closed. Moreover, the following diagram

$$\begin{tikzcd} \ker \delta_{\mathbf{T}'_i}^{k} \arrow[r, "s_i"] \arrow[d, "\pi'"'] & H^k(\mathbf{T}) \\ \ker \delta_{\mathbf{T}'_i}^{k} / \ker s_i \arrow[ur, "\tilde{s}_i"'] & \end{tikzcd}$$

is commutative; and hence $\tilde{s}_i : H^k(\mathbf{T}'_i) = \ker \delta_{\mathbf{T}'_i}^{k} / \ker s_i \to H^k(\mathbf{T})$ is a linear injection.

We deduce that $\dim H^k(\mathbf{T}'_i) \leqslant \dim H^k(\mathbf{T})$ and consequently $\mathbf{T}'_{\mathbf{i}}$ is Fredholm.

By symmetry again, we obtain the required equivalence. □

The following corollary is immediate from the proof given above.

Corollary 3.

(1) $\operatorname{Im} \delta_{\mathbf{T}'_i}^{k-1}$ *is closed if and only if* $\operatorname{Im} \delta_{\mathbf{T}}^{k-1}$ *is.*
(2) *If* $\operatorname{Im} \delta_{\mathbf{T}'_i}^{k-1}$ *is closed, then* $\dim H^k(\mathbf{T}'_i) = \dim H^k(\mathbf{T})$.
(3) *In particular, if* $\mathbf{T}$ *is Fredholm, then*

$$\operatorname{ind} \mathbf{T} = \operatorname{ind} \mathbf{T}'_i.$$

Arguing as in the above section, we obtain

Theorem 4. *Under the assumptions of Theorem* 1*, for* $\lambda = (\lambda_1, \dots, \lambda_n)$ *such that* $\prod_{i \in \mathcal{I} \cup \mathcal{J}} \lambda_i \neq 0$*, we have*

(1) $\lambda - \mathbf{SR}_{\mathcal{I}}$ *is Fredholm if and only if* $\lambda - \mathbf{SR}_{\mathcal{J}}$ *is.*
(2) *If* $\lambda - \mathbf{SR}_{\mathcal{I}}$ *is Fredholm, then* $\operatorname{ind}(\lambda - \mathbf{SR}_{\mathcal{I}}) = \operatorname{ind}(\lambda - \mathbf{SR}_{\mathcal{J}})$.
(3) *In particular, for every* λ *such that* $\prod_i \lambda_i \neq 0$*, we have* $\lambda - \mathbf{SR}$ *is Fredholm if and only if* $\lambda - \mathbf{RS}$ *is Fredholm, and in this case,*

$$\operatorname{ind}(\lambda - \mathbf{SR}) = \operatorname{ind}(\lambda - \mathbf{RS}).$$

The next corollary follows from Theorem 4.

Corollary 4. *Under the assumptions of Theorem* 1*, we have*

$$\sigma_{Te}(\mathbf{SR}_{\mathcal{I}}) \setminus [0]^{\mathcal{I} \cup \mathcal{J}} = \sigma_{Te}(\mathbf{SR}_{\mathcal{J}}) \setminus [0]^{\mathcal{I} \cup \mathcal{J}}.$$

In particular, we have

$$\sigma_{Te}(\mathbf{SR}) \setminus [0] = \sigma_{Te}(\mathbf{RS}) \setminus [0].$$

Outlining the proof of Corollary 2, we get the following result.

Corollary 5. *Under the assumptions of Theorem* 1*, we have*

(1) *If* $0 \notin \sigma_e(R_i S_i) = \sigma_e(S_i R_i)$ *for every* $i \in \mathcal{I} \subset \{1, \ldots, n\}$*, then*

$$\sigma_{Te}(\mathbf{RS}) \setminus [0]^{\mathcal{I}^c} = \sigma_{Te}(\mathbf{SR}) \setminus [0]^{\mathcal{I}^c},$$

where $\mathcal{I}^c$ *is the complement of* $\mathcal{I}$ *in* $\{1, \ldots, n\}$.

(2) *If* $0 \notin \sigma_e(R_i S_i) = \sigma_e(S_i R_i)$ *for every* $i = 1, \ldots, n$*, then* $\sigma_{Te}(\mathbf{RS}) = \sigma_{Te}(\mathbf{SR})$.

Corollary 6. *Under the assumptions of Theorem* 1*,*

$$\sigma_{Tw}(\mathbf{SR}_{\mathcal{I}}) \setminus [0]^{\mathcal{I} \cup \mathcal{J}} = \sigma_{Tw}(\mathbf{SR}_{\mathcal{J}}) \setminus [0]^{\mathcal{I} \cup \mathcal{J}},$$

and hence

$$\sigma_{Tw}(\mathbf{RS}) \setminus [0] = \sigma_{Tw}(\mathbf{SR}) \setminus [0].$$

5. Taylor split spectrum

Recall the following definitions:

Let $\sigma^{\delta,k}(\mathbf{T})$ be the complement of all complex numbers λ for which

$$H^p(\mathbf{T} - \lambda) = O \quad \forall p,\ n - k \leqslant p \leqslant n,$$

and $\sigma^{\pi,k}(\mathbf{T})$ be the complement of all complex numbers λ for which

$$H^p(\mathbf{T} - \lambda) = O \quad \forall p,\ 0 \leqslant p \leqslant k,$$

and $H^{k+1}(\mathbf{T} - \lambda)$ is separated in its natural quotient topology.

$\sigma_e^{\delta,k}(\mathbf{T})$ is the complement of all complex numbers λ for which

$$\dim H^p(\mathbf{T} - \lambda) < \infty \quad \forall p,\ n - k \leqslant p \leqslant n,$$

and $\sigma_e^{\pi,k}(\mathbf{T})$ is the complement of all complex numbers λ for which

$$\dim H^p(\mathbf{T} - \lambda) < \infty \quad \forall p,\ 0 \leqslant p \leqslant k,$$

and $H^{k+1}(\mathbf{T} - \lambda)$ is separated in its natural quotient topology.

$\sigma^{\delta,k}(\mathbf{T})$ and $\sigma^{\pi,k}(\mathbf{T})$ are called the defect and the approximate point spectra of degree k for $\mathbf{T}$.

The defect and the approximate point spectrum of $\mathbf{T}$ are defined as

$$\sigma_\delta(\mathbf{T}) = \sigma^{\delta,0}(\mathbf{T}) \quad \text{and} \quad \sigma_\pi(\mathbf{T}) = \sigma^{\pi,0}(\mathbf{T}) \quad \big(\text{denoted also by } \sigma_{\mathrm{ap}}(\mathbf{T})\big).$$

We give now a version of Theorem 2 for the above spectra.

Theorem 5. *Under the assumptions of Theorem* 1*, if* $\mathcal{I}, \mathcal{J} \subset \{1, \ldots, n\}$*, we have*

(1) $\sigma^{\pi,k}(\mathbf{SR}_{\mathcal{I}}) \setminus [0]^{\mathcal{I} \cup \mathcal{J}} = \sigma^{\pi,k}(\mathbf{SR}_{\mathcal{J}}) \setminus [0]^{\mathcal{I} \cup \mathcal{J}}$,

(2) $\sigma^{\delta,k}(\mathbf{SR}_{\mathcal{I}}) \setminus [0]^{\mathcal{I} \cup \mathcal{J}} = \sigma^{\delta}(\mathbf{SR}_{\mathcal{J}}) \setminus [0]^{\mathcal{I} \cup \mathcal{J}}$,

(3) $\sigma_e^{\pi,k}(\mathbf{SR}_{\mathcal{I}}) \setminus [0]^{\mathcal{I} \cup \mathcal{J}} = \sigma_e^{\pi,k}(\mathbf{SR}_{\mathcal{J}}) \setminus [0]^{\mathcal{I} \cup \mathcal{J}}$,

(4) $\sigma_e^{\delta,k}(\mathbf{SR}_{\mathcal{I}}) \setminus [0]^{\mathcal{I} \cup \mathcal{J}} = \sigma_e^{\delta}(\mathbf{SR}_{\mathcal{J}}) \setminus [0]^{\mathcal{I} \cup \mathcal{J}}$.

It is easy to see that the last results are still valid for $\sigma_p(.)$ the joint point spectrum where

$$\sigma_p(\mathbf{T}) = \{\lambda \in \mathbb{C}^n,\ \exists x \in X \setminus \{0\}\colon\ (T_i - \lambda_i)x = 0 \text{ for } i = 1, \ldots, n\}.$$

Recall that an operator L on a Banach space X has a generalized inverse if there exists an operator M on X such that $LML = L$.

Definition 1. Let $\mathbf{T} = (T_1, \ldots, T_n)$ be an n-tuple of commuting operators on a Banach space X. We say that $\mathbf{T}$ is *split regular* if it is Taylor regular and the mapping $\delta_{\mathbf{T}} : \Lambda[e, X] \to \Lambda[e, X]$ has a generalized inverse. The *split spectrum* $\sigma_S(\mathbf{T})$ is the set of all $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ such that the n-tuple $\mathbf{T} = (T_1 - \lambda_1, \ldots, T_n - \lambda_n)$ is not split regular.

The following result characterizes the split regular n-tuples of operators and could be found in [18].

Proposition 1. *Let $\mathbf{T} = (T_1, \ldots, T_n)$ be an n-tuple of mutually commuting operators on a Banach space X. The following conditions are equivalent*:

(1) $\mathbf{T}$ *is split regular*;
(2) $\mathbf{T}$ *is Taylor regular and* $\ker \delta_{\mathbf{T}}^k$ *is a complemented subspace of* $\Lambda[e, X]^k$ *for* $k = 0, \ldots, n-1$;
(3) *there exist operators* $W_1, W_2 : \Lambda[e, X] \to \Lambda[e, X]$ *such that* $W_1\delta_{\mathbf{T}} + \delta_{\mathbf{T}} W_2 = I_{\Lambda[e,X]}$;
(4) *there exists an operator* $V : \Lambda[e, X] \to \Lambda[e, X]$ *such that* $V\delta_{\mathbf{T}} + \delta_{\mathbf{T}} V = I$, $V^2 = 0$ *and* $V : \Lambda[e, X]^p \subset \Lambda[e, X]^{p-1}$ *for* $p = 0, \ldots, n$. *Equivalently, there are operators* $V_p : \Lambda[e, X]^{p+1} \to \Lambda[e, X]^p$ *such that* $V_{p-1}V_p = 0$ *and* $V_p\delta_{\mathbf{T}}^p + \delta_{\mathbf{T}}^{p-1} V_{p-1} = I_{\Lambda[e,X]^p}$ *for every* p (*for* $p = 0$ *and* $p = n$ *this reduces to* $V_0\delta_{\mathbf{T}}^0 = I_{\Lambda[e,X]^0}$ *and* $\delta_{\mathbf{T}}^{n-1} V_{n-1} = I_{\Lambda[e,X]^n}$, *respectively*).

It is then clear that we have also similar results for split spectrum.

Remark 1. We have:

(1) The Taylor split spectrum is definable in Banach algebras as easily as for operators on Banach spaces and also coincides with both the Taylor spectrum and the Taylor split spectrum of the induced system of left multiplication operators, and also of the induced system of right multiplication operators on the Banach algebra.
(2) The left and the right spectrum are the parts of the split spectrum analogous to the approximate point and defect spectrum as parts of the Taylor spectrum.
(3) There are also split versions of Taylor–Fredholm and Taylor–Weyl spectrum, related or not to the Taylor split spectrum on the Calkin algebra.

6. Applications and concluding remarks

Switching properties are related to many problems in operator theory. Examples of such applications are given in [4,5] where this concept has been used to extend some results on Aluthge transform. Also, it has been useful to obtain some stability spectral properties for operator matrices and extensions.

In this section, we apply the same concept to give some results in the line of [7–10,13].

6.1. Aluthge and Duggal transforms

Let $T \in \mathcal{L}(H)$ be a bounded operator on some Hilbert space H and $U|T|$ be its polar decomposition, where $|T| = (T^*T)^{1/2}$ and U is the appropriate partial isometry. The Generalized Aluthge transform of T is given by $\tilde{T}(t) = |T|^{1-t}U|T|^t$ where $0 < t < 1$.

Set $S = |T|^t$ and $R = U|T|^{1-t}$. Then clearly $RS = T$ and $SR = \tilde{T}(t)$. In particular $\tilde{T}$ and T almost have the same spectral properties. For further details, we refer to [1,14–16] in the case $t = \frac{1}{2}$.

The "n-tuples version" of the last results has been given first in [9,10,13], for n-tuples of p-hyponormal operators and lately in [6] for doubly commuting n-tuples of operators. It is worthy to note that we may retrieve a part of the last results by applying the previous section to the doubly commuting n-tuples $\mathbf{T} = (T_1, \ldots, T_n)$ and its Aluthge transform $\tilde{\mathbf{T}}(t) = (\tilde{T}_1(t_1), \ldots, \tilde{T}_n(t_n))$ for all $t = (t_1, \ldots, t_n) \in]0, 1[^n$.

It is not hard to see that the following result from [11] given for pairs remains true for n-tuples.

Lemma 2. [11, Theorem 2] *Let $\mathbf{T} = (T_1, T_2)$ be a doubly commuting pair. If $U_1|T_1|$ and $U_2|T_2|$ are their respective polar decompositions, then U_1, U_1^* and $|T_1|$ commute with U_2, U_2^* and $|T_2|$.*

It has been shown in [14,15] for a single operator T, that

$$\Sigma(\tilde{T}) = \Sigma(T) \tag{5}$$

for $\Sigma \in \{\sigma, \sigma_p, \sigma_{\mathrm{ap}}, \sigma_e\}$. Recall that an n-tuple $\mathbf{T} = (T_1, \ldots, T_n)$ is doubly commuting provided that $\mathbf{T}$ and $\mathbf{T}^*$ satisfy (2). The latter equality is extended for doubly commuting n-tuples as follows.

Theorem 6. [6] *Let $\mathbf{T} = (T_1, \ldots, T_n)$ be a doubly commuting n-tuple and for $\mathbf{t} = (t_1, \ldots, t_n) \in]0, 1[^n$, let $\tilde{\mathbf{T}}(t) = (\tilde{T}_1(t), \ldots, \tilde{T}_n(t))$ be its Aluthge transform. Then*

(1)
$$\sigma_T\big(\tilde{\mathbf{T}}(\mathbf{t})\big) = \sigma_T(\mathbf{T}), \quad \sigma_{Te}\big(\tilde{\mathbf{T}}(\mathbf{t})\big) = \sigma_{Te}(\mathbf{T});$$
$$\sigma_p\big(\tilde{\mathbf{T}}(\mathbf{t})\big) = \sigma_p(\mathbf{T}), \quad \sigma_{\mathrm{ap}}\big(\tilde{\mathbf{T}}(\mathbf{t})\big) = \sigma_{\mathrm{ap}}(\mathbf{T}).$$

(2)
$$\sigma_T\big(\tilde{\mathbf{T}}(\mathbf{t})^*\big) \setminus [0] = \sigma_T(\mathbf{T}^*) \setminus [0], \quad \sigma_{Te}\big(\tilde{\mathbf{T}}(\mathbf{t})^*\big) \setminus [0] = \sigma_{Te}(\mathbf{T}^*) \setminus [0];$$
$$\sigma_p\big(\tilde{\mathbf{T}}(\mathbf{t})^*\big) \setminus [0] = \sigma_p(\mathbf{T}^*) \setminus [0], \quad \sigma_{\mathrm{ap}}\big(\tilde{\mathbf{T}}(\mathbf{t})^*\big) \setminus [0] = \sigma_{\mathrm{ap}}(\mathbf{T}^*) \setminus [0].$$

Moreover, if U_i is unitary for every i, then we have

(3)
$$\sigma_T\big(\tilde{\mathbf{T}}(\mathbf{t})^*\big) = \sigma_T(\mathbf{T}^*), \quad \sigma_{Te}\big(\tilde{\mathbf{T}}(\mathbf{t})^*\big) = \sigma_{Te}(\mathbf{T}^*);$$
$$\sigma_p\big(\tilde{\mathbf{T}}(\mathbf{t})^*\big) = \sigma_p(\mathbf{T}^*), \quad \sigma_{\mathrm{ap}}\big(\tilde{\mathbf{T}}(\mathbf{t})^*\big) = \sigma_{\mathrm{ap}}(\mathbf{T}^*).$$

(1) We set $S_i = |T_i|^{t_i}$ and $R_i = U_i|T_i|^{1-t_i}$, then $R_iS_i = T_i$ and $S_iR_i = \tilde{T}_i(t_i)$.

Of course, **R** and **S** satisfy our commutativity conditions (cf., the lemma above). We conclude by appealing the previous results that $\Sigma(\tilde{\mathbf{T}}(\mathbf{t})) \setminus [0] = \Sigma(\tilde{\mathbf{T}}) \setminus [0]$, where $\Sigma \in \{\sigma, \sigma_e, \sigma_p, \sigma_{\mathrm{ap}}\}$.

The equalities in (1) seem to be due the fact that we have furthermore the double commutativity and even more.

(2) To obtain our requirement, it suffices to set $S_i = |T_i|^{t_i}$ and $R_i = |T_i|^{1-t_i}U^*$.

The Duggal transplant of $T = U|T|$ is defined to be $\hat{T} = |T|U$.

For a doubly commuting n-tuple $\mathbf{T} = (T_1, \ldots, T_n)$, we can also consider Duggal transplant of T which is $\hat{\mathbf{T}} = (\hat{T}_1, \ldots, \hat{T}_n)$.

We apply again the results of the previous section to a doubly commuting n-tuple $\mathbf{T} = (T_1, \ldots, T_n)$ and its Duggal transplant.

Theorem 7. *Let $\mathbf{T} = (T_1, \ldots, T_n)$ be a doubly commuting n-tuple and let $\hat{\mathbf{T}} = (\hat{T}_1, \ldots, \hat{T}_n)$ be its Duggal transplant. Then*

$$\Sigma(\hat{\mathbf{T}}) \setminus [0] = \Sigma(\mathbf{T}) \setminus [0], \qquad \Sigma(\hat{\mathbf{T}}^*) \setminus [0] = \Sigma(\mathbf{T}^*) \setminus [0],$$

where $\Sigma \in \{\sigma, \sigma_e, \sigma_p, \sigma_{\mathrm{ap}}, \ldots\}$.

It is easy to see that those results may be also given for an n-tuple with mixed coordinates formed by either Duggal transplants or Aluthge transforms. Namely, let us define, for two disjoint subsets $\mathcal{I}$ and $\mathcal{J}$ of $\{1, \ldots, n\}$, the n-tuples $\mathbf{T}_{\mathcal{I},\mathcal{J}} := (Q_1, \ldots, Q_n)$ where $Q_i = \tilde{T}_i$ if $i \in \mathcal{I}$, $Q_j = \hat{T}_j$ if $j \in \mathcal{J}$, and $Q_k = T_k$, if k is neither in $\mathcal{I}$ nor in $\mathcal{J}$. Then, we have the following result:

Theorem 8. *Let $\mathbf{T} = (T_1, \ldots, T_n)$ be a doubly commuting n-tuple and $\mathcal{I}$ and $\mathcal{J}$ be two disjoint subsets of $\{1, \ldots, n\}$. Then*

$$\Sigma(\hat{\mathbf{T}}_{\mathcal{I},\mathcal{J}}) \setminus [0]^{\mathcal{I}\cup\mathcal{J}} = \Sigma(\mathbf{T}) \setminus [0]^{\mathcal{I}\cup\mathcal{J}}, \qquad \Sigma(\hat{\mathbf{T}}^*_{\mathcal{I}\cup\mathcal{J}}) \setminus [0]^{\mathcal{I}\cup\mathcal{J}} = \Sigma(\mathbf{T}^*) \setminus [0]^{\mathcal{I}\cup\mathcal{J}},$$

where $\Sigma \in \{\sigma, \sigma_e, \sigma_p, \sigma_{\mathrm{ap}}, \ldots\}$.

6.2. Concluding remarks

(1) The iterated Aluthge transforms are defined by the recursive formula $T^{(n)} = \tilde{T}^{(n-1)}$ with $T^{(1)} = T$. Common spectral properties of $\tilde{T}^n$, $n \geqslant 1$, have been investigated in [16]. The main results obtained for the Aluthge transforms of commuting n-tuples are extendible to their iterated Aluthge transforms. This is carried out by applying inductively the previous section.

(2) The value $n = 1$ permits to retrieve the case of single operators. Thus some parts of [3,4, 14–16] are recaptured. We mention that the proofs given in this paper do not depend heavily on the ones given for single operators.

Acknowledgments

We thank the referee for his suggestions and for pointing out to us the given version of Lemma 1.

This paper was written while the second author was visiting the University of Lille 1. He is grateful to the UFR of mathematics of this university.

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