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On models of irreducible q-representations of the Lie algebra $\mathcal{G}(0, 1)$

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Abstract

In this paper, the irreducible q-representations of $\mathcal{G}(0, 1)$ are discussed. We construct one and two variable models of irreducible q-representations of $\mathcal{G}(0, 1)$ in terms of q-derivative operator, and derive identities based on it.

Keywords: Lie algebra $\mathcal{G}(0, 1)$; Basic hypergeometric series

1. Introduction

The idea of irreducible q-representations of a Lie algebra was first introduced by Manocha [5]. The models of the special complex Lie algebra $\mathrm{sl}(2, \mathbb{C})$ were constructed and using the techniques of fractional q-calculus, special function identities were derived involving q-hypergeometric functions. Later, in Sahai [7], the q-Euler integral transformation was utilized to obtain q-difference dilation operator models of irreducible q-representations of $\mathrm{sl}(2, \mathbb{C})$. In this paper, we extend this idea to the Lie algebra $\mathcal{G}(0, 1)$. Precisely, we prove a classification theorem for irreducible q-representations of the Lie algebra $\mathcal{G}(0, 1)$ and give one and two variable models of this Lie algebra in terms of q-derivative operators. Section-wise treatment is as follows.

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In Section 2, we list various results from the theory of q-special functions, needed for our discussion. The fractional q-derivative of order λ, Δ_x^λ, is defined. Next we introduce an operator $\mathcal{D}_q$ defined as $\mathcal{D}_q f(x) = \Delta_x^{\beta-\gamma} x^{\beta-1} f(x)$ and then make use of generalized q-Leibniz rule to obtain operator expressions for $\mathcal{D}_q(x\Delta_x)\mathcal{D}_q^{-1}$ and $\mathcal{D}_q(x)\mathcal{D}_q^{-1}$.

In Section 3, we discuss the irreducible q-representations of the Lie algebra $\mathcal{G}(0,1)$ and prove a classification theorem. Based on the theorem, we construct canonical models of irreducible q-representations of $\mathcal{G}(0,1)$ in one and $(m+1)$-variables. In Section 4, we obtain identities based on the one variable model. In Section 5, we construct two variable models of representations $R_q(\alpha,\mu)$ and $\uparrow_q(\mu)$ of $\mathcal{G}(0,1)$ in terms of q-derivative and q-dilation operators in which ${}_1\phi_0$ functions appear as the basis functions. These models are then transformed, with the help of a theorem, to the new models of irreducible q-representations of $\mathcal{G}(0,1)$ in terms of q-derivative and inverse q-derivative operators of fractional order with basis functions involving the q-hypergeometric functions ${}_2\phi_1$. Finally, in Section 6, these models are exploited for identities.

2. Preliminaries

The generalized basic or q-hypergeometric series ${}_r\phi_s$ is defined as [3]

$$ {}_r\phi_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}; q, x\right) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n\cdots(a_r;q)_n}{(b_1;q)_n\cdots(b_s;q)_n(q;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} x^n, \tag{1} $$

where q-shifted factorial $(a;q)_n$ is defined by

$$ (a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \qquad (a;q)_\infty = \prod_{r=0}^{\infty}\left(1-q^r a\right). \tag{2} $$

In other words,

$$ (a;q)_n = \begin{cases} (1-a)(1-aq)\cdots(1-aq^{n-1}), & n = 1,2,\dots, \\ 1, & n = 0, \\ [(1-aq^{-1})(1-aq^{-2})\cdots(1-aq^{-n})]^{-1}, & n = -1,-2,\dots. \end{cases} \tag{3} $$

The series ${}_r\phi_s$ terminates if one of the numerator parameter is of the form q^{-m}, $m = 0,1,2,\dots,$ and $q \neq 0$. When $0 < |q| < 1$, the series ${}_r\phi_s$ converges absolutely for all x if $r \leqslant s$; and for $|x| < 1$ if $r = s+1$. If $|q| > 1$ and $|x| < \frac{|b_1\cdots b_s|}{|a_1\cdots a_r|}$, then also ${}_r\phi_s$ converges absolutely. It diverges for $x \neq 0$ if $0 < |q| < 1$ and $r > s+1$, and if $|q| > 1$ and $|x| > \frac{|b_1\cdots b_s|}{|a_1\cdots a_r|}$, unless it terminates.

The q-analogue of the binomial function is

$$ {}_1\phi_0\left(\begin{matrix} a \\ - \end{matrix}; q, x\right) = \frac{(ax;q)_\infty}{(x;q)_\infty}, \quad |q| < 1,\ |x| < 1. \tag{4} $$

The q-analogues of the exponential functions are

$$ e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n} = \frac{1}{(x;q)_\infty}, \quad |x| < 1, \tag{5} $$

and

$$ E_q(x) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} x^n}{(q;q)_n} = (-x;q)_\infty. \tag{6} $$

$E_q(x)$ converges for all x.

We also make use of the function

$$\Gamma_q(\alpha) = \frac{e_q(q^\alpha)}{e_q(q)}(1-q)^{1-\alpha} \tag{7}$$

defined for $\alpha \neq 0, -1, -2, \ldots$.

This is a q-analogue of the gamma function and satisfies the functional equation

$$\Gamma_q(\alpha+1) = [\alpha]_q \Gamma_q(\alpha), \tag{8}$$

where $[\alpha]_q = \frac{1-q^\alpha}{1-q}$ is the q-analogue of α [3,4].

We need the q-analogue of the general double hypergeometric series in the form [3]

$$\begin{aligned} &\Phi^{A:B:C}_{D:E:F}\left(\begin{matrix} a_A : b_B; c_C \\ d_D : e_E; f_F \end{matrix}; q; x, y\right) \\ &\quad = \sum_{m,n=0}^{\infty} \frac{(a_A;q)_{m+n}(b_B;q)_m(c_C;q)_n}{(d_D;q)_{m+n}(q, e_E;q)_m(q, f_F;q)_n} \\ &\qquad \times \left[(-1)^{m+n} q^{\binom{m+n}{2}}\right]^{D-A} \left[(-1)^m q^{\binom{m}{2}}\right]^{1+E-B} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+F-C} x^m y^n, \end{aligned} \tag{9}$$

where $q \neq 0$ when $\min(D-A, 1+E-B, 1+F-C) < 0$. The series (9) converges absolutely for $|x|, |y| < 1$ when $\min(D-A, 1+E-B, 1+F-C) \geqslant 0$ and $|q| < 1$.

The q-analogue of the Lauricella function [2] is defined by

$$\begin{aligned} &\Phi_D\left(\begin{matrix} a; b_1, \ldots, b_m \\ c \end{matrix}; q; x_1, \ldots, x_m\right) \\ &\quad = \sum_{n_1,\ldots,n_m=0}^{\infty} \frac{(a;q)_{n_1+\cdots+n_m}(b_1;q)_{n_1}\cdots(b_m;q)_{n_m}}{(c;q)_{n_1+\cdots+n_m}} \frac{x_1^{n_1}}{(q;q)_{n_1}} \cdots \frac{x_m^{n_m}}{(q;q)_{n_m}}. \end{aligned} \tag{10}$$

The q-derivative operator is defined by

$$\Delta_x\big(f(x)\big) = \frac{f(x) - f(qx)}{(1-q)x} = \frac{(1-T_x)}{(1-q)x} f(x), \tag{11}$$

where the q-dilation operator T_x is given by $T_x[f(x)] = f(qx)$. From (11) it follows that

$$\Delta_x^n\big(x^p\big) = \frac{\Gamma_q(p+1)}{\Gamma_q(p-n+1)} x^{p-n}. \tag{12}$$

The above derivative formula can be extended to a fractional q-derivative operator of order λ as

$$\Delta_x^\lambda\big(x^\mu\big) = \frac{\Gamma_q(\mu+1)}{\Gamma_q(\mu-\lambda+1)} x^{\mu-\lambda}, \quad \mu \neq -1, -2, \ldots. \tag{13}$$

The generalized q-Leibniz formula for q-fractional derivative of product of two functions in terms of q-derivatives of each [5], is established as

$$\Delta_x^\lambda\big[f(x)g(x)\big] = \sum_{r=0}^{\infty} \begin{bmatrix} \lambda \\ r \end{bmatrix}_q q^{-r(\lambda-r)} \Delta_x^{\lambda-r} f\big(xq^r\big) \Delta_x^r g(x), \tag{14}$$

where the q-binomial coefficient is defined by

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_q = \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\beta+1)\Gamma_q(\alpha-\beta+1)}, \quad \alpha, \beta \in \mathbb{C},\ |q| < 1. \tag{15}$$

To construct new models of irreducible q-representations of $\mathcal{G}(0,1)$, we introduce the operators $\mathcal{D}_q$ and $\mathcal{D}_q^{-1}$ defined as

$$\mathcal{D}_q f(x) = \Delta_x^{\beta-\gamma}\big[x^{\beta-1} f(x)\big], \tag{16}$$

$$\mathcal{D}_q^{-1} f(x) = x^{1-\beta}\Delta_x^{\gamma-\beta}\big[f(x)\big]. \tag{17}$$

Indeed, in general

$$\mathcal{D}_q\mathcal{D}_q^{-1}\big[f(x)\big] = f(x) = \mathcal{D}_q^{-1}\mathcal{D}_q\big[f(x)\big]. \tag{18}$$

Using (14), we obtain the following:

$$\mathcal{D}_q(x\Delta_x)\mathcal{D}_q^{-1} = q^{1-\gamma}x\Delta_x + [1-\gamma]_q, \tag{19}$$

$$\mathcal{D}_q(x)\mathcal{D}_q^{-1} = xq^{\beta-\gamma} + [\beta-\gamma]_q\Delta_x^{-1}, \tag{20}$$

where Δ_x^{-1} is q-integral in disguise. As we shall see later, Eqs. (19) and (20) will be instrumental in obtaining new models of $\mathcal{G}(0,1)$.

3. The Lie algebra $\mathcal{G}(0,1)$ and its q-representations

For any pair of complex numbers (a,b) the 4-dimensional complex Lie algebra $\mathcal{G}(a,b)$ with basis $\mathcal{J}^+$, $\mathcal{J}^-$, $\mathcal{J}^0$ and $\mathcal{E}$ is defined by

$$\begin{aligned} &\big[\mathcal{J}^+,\mathcal{J}^-\big] = 2a^2\mathcal{J}^0 - b\mathcal{E},\\ &\big[\mathcal{J}^0,\mathcal{J}^+\big] = \mathcal{J}^+, \qquad \big[\mathcal{J}^0,\mathcal{J}^-\big] = -\mathcal{J}^-,\\ &\big[\mathcal{J}^+,\mathcal{E}\big] = \big[\mathcal{J}^-,\mathcal{E}\big] = \big[\mathcal{J}^0,\mathcal{E}\big] = \mathcal{O}, \end{aligned} \tag{21}$$

where [.,.] is the commutator bracket and $\mathcal{O}$ is the additive identity element.

For special choices of the parameters a, b, $\mathcal{G}(a,b)$ essentially coincides with one of the Lie algebras sl(2), $\mathcal{G}(0,1)$ and $L(T_3)$. Indeed, it can be shown that [6, Lemma 2.1]

$$\mathcal{G}(a,b) \cong \begin{cases} \mathcal{G}(1,0) \cong \mathrm{sl}(2)\oplus(\mathcal{E}), & \text{if } a \neq 0,\\ \mathcal{G}(0,1), & \text{if } a = 0,\ b \neq 0,\\ \mathcal{G}(0,0) \cong L(T_3)\oplus(\mathcal{E}), & \text{if } a = b = 0, \end{cases}$$

where $(\mathcal{E})$ is the 1-dimensional Lie algebra generated by $\mathcal{E}$. The q-representations of sl(2) have been extensively studied in [5,7,8,10]. Further, the p,q-representations of the Lie algebra $\mathcal{G}(1,0)$ were studied in [9]. We now extend this idea to the Lie algebra $\mathcal{G}(0,1)$.

The Lie algebra $\mathcal{G}(0,1)$ is the space of 4×4 matrices of the form

$$\alpha = \begin{pmatrix} 0 & x_2 & x_4 & x_3\\ 0 & x_3 & x_1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad x_1, x_2, x_3, x_4 \in \mathbb{C}, \tag{22}$$

with the Lie product $[\alpha,\beta] = \alpha\beta - \beta\alpha$.

The matrices

$$\mathcal{J}^+ = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{J}^- = \begin{pmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{J}^0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{E} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{23}$$

with commutation relations

$$[\mathcal{J}^0, \mathcal{J}^+] = \mathcal{J}^+, \qquad [\mathcal{J}^0, \mathcal{J}^-] = -\mathcal{J}^-, \qquad [\mathcal{J}^+, \mathcal{J}^-] = -\mathcal{E}, \tag{24}$$

and

$$[\mathcal{E}, \mathcal{J}^+] = [\mathcal{E}, \mathcal{J}^-] = [\mathcal{E}, \mathcal{J}^0] = \mathcal{O}, \tag{25}$$

where $\mathcal{O}$ is the 4×4 zero matrix, constitute a basis for $\mathcal{G}(0,1)$.

Let V_q be a complex vector space consisting of q-special functions with a basis $\{\phi_\lambda\colon \lambda \in S\}$ such that the functions $\{f_\lambda = \lim_{q\to 1} \phi_\lambda\colon \lambda \in S\}$ form a basis for a vector space, say, V. Let $A(V_q)$ be the associative algebra of all linear operators on V_q over the complex field.

A q-representation of $\mathcal{G}(0,1)$ on V_q is a mapping $\rho_q : \mathcal{G}(0,1) \to A(V_q)$ satisfying the following conditions:

(1) $\rho_q(ax + by) = a\rho_q(x) + b\rho_q(y)$.
(2) There exists a Lie algebra representation ρ of $\mathcal{G}(0,1)$ on V such that $\lim_{q\to 1} \rho_q(x)\phi_\lambda = \rho(x) f_\lambda$, for all $x, y \in \mathcal{G}(0,1)$ and $a, b \in \mathbb{C}$.
(3) If we denote

$$J_q^+ = \rho_q(\mathcal{J}^+), \qquad J_q^- = \rho_q(\mathcal{J}^-), \qquad J_q^0 = \rho_q(\mathcal{J}^0), \qquad E_q = \rho_q(\mathcal{E}), \tag{26}$$

then

$$\begin{aligned} &J_q^0 J_q^+ - q J_q^+ J_q^0 = J_q^+, \\ &q J_q^0 J_q^- - J_q^- J_q^0 = -J_q^-, \\ &q J_q^+ J_q^- - J_q^- J_q^+ = -q^{u-1} E_q, \\ &[E_q, J_q^+] = [E_q, J_q^-] = [E_q, J_q^0] = 0. \end{aligned} \tag{27}$$

A q-representation ρ_q of $\mathcal{G}(0,1)$ is said to be irreducible if there is no proper subspace W_q of V_q which is invariant under ρ_q.

If we define the operator C_q on V_q by

$$C_q = q J_q^+ J_q^- - q^u J_q^0 E_q, \tag{28}$$

it is easy to check that

$$\begin{aligned} &q J_q^+ C_q = C_q J_q^+, \\ &J_q^- C_q = q C_q J_q^-, \\ &J_q^0 C_q = C_q J_q^0, \\ &E_q C_q = C_q E_q. \end{aligned} \tag{29}$$

Obviously, as $q \to 1$, ρ_q reduces to a Lie algebra representation ρ of $\mathcal{G}(0,1)$ on V.

Let $S_q = \{[\lambda - u]_q\colon \lambda \in S\}$ be the spectrum of J_q^0, and let the q-representation ρ_q satisfy the conditions:

(i) ρ_q is irreducible,

(ii) each eigenvalue of J_q^0 has multiplicity equal to one. (30)

Conditions (30) guarantee that S_q, and for that matter S, is countable and that there exists a basis for V_q consisting of eigenvectors f_λ of J_q^0.

Following the analysis as in [6, p. 40], we note:

(i) Let $\lambda \in S$, then the equation $(J_q^0 J_q^+ - q J_q^+ J_q^0) f_\lambda = J_q^+ f_\lambda$ implies either $J_q^+ f_\lambda = \xi_{\lambda+1} f_{\lambda+1}$, where $\xi_{\lambda+1}$ is a nonzero constant and $\lambda + 1 \in S$, or $J_q^+ f_\lambda = 0$. Similarly, the equation $(q J_q^0 J_q^- - J_q^- J_q^0) f_\lambda = -J_q^- f_\lambda$ implies either $J_q^- f_\lambda = \eta_\lambda f_{\lambda-1}$, where η_λ is a nonzero constant and $\lambda - 1 \in S$, or $J_q^- f_\lambda = 0$. The equation $[E_q, J_q^0] f_\lambda = 0$ implies $E_q f_\lambda = \mu_\lambda f_\lambda$ for some constants μ_λ and $[C_q, J_q^0] f_\lambda = 0$ implies $C_q f_\lambda = a_\lambda f_\lambda$ for some constants a_λ.

(ii) S is connected in the sense: $S = \{\lambda + n\colon\ n$ is an integer such that $n_1 < n < n_2\}$, where n_1 and n_2 are integers. We do not exclude the possibility that $n_1 = -\infty$ or $n_2 = \infty$.

(iii) If $\lambda, \lambda + 1 \in S$, then $\xi_{\lambda+1}, \eta_{\lambda+1} \neq 0$, since otherwise the irreducibility of ρ_q would be violated.

(iv) Suppose that $\lambda, \lambda + 1 \in S$. Then the equation $[E_q, J_q^+] f_\lambda = 0$ gives $\xi_{\lambda+1}(\mu_{\lambda+1} - \mu_\lambda) = 0$. Therefore $\mu_{\lambda+1} = \mu_\lambda$. Hence $\mu_\lambda = \mu$, a constant for all $\lambda \in S$. Further, $q J_q^+ C_q f_\lambda = C_q J_q^+ f_\lambda$ implies $\xi_{\lambda+1}(a_{\lambda+1} - q a_\lambda) = 0$. Hence $a_{\lambda+1} = q a_\lambda$.

(v) The equation $C_q f_\lambda = a_\lambda f_\lambda$, that is,

$$\left(q J_q^+ J_q^- - q^u J_q^0 E_q\right) f_\lambda = a_\lambda f_\lambda$$

leads to the following relation:

$$q \xi_\lambda \eta_\lambda = a_\lambda + \mu q^u [\lambda - u]_q, \tag{31}$$

defined for all $\lambda \in S$, where $\eta_\lambda = 0$ if $\lambda - 1 \notin S$. Using

$$q J_q^+ J_q^- = J_q^- J_q^+ - q^{u-1} E_q, \tag{32}$$

we have

$$q \xi_{\lambda+1} \eta_{\lambda+1} = a_{\lambda+1} + \mu q^u [\lambda + 1 - u]_q \tag{33}$$

defined for all $\lambda \in S$ where $\xi_{\lambda+1} = 0$ if $\lambda + 1 \notin S$.

(vi) The representation ρ_q of $\mathcal{G}(0,1)$ is uniquely determined by $a_\lambda, a_{\lambda+1} = q a_\lambda$, and the spectrum S_q of J_q^0. The nonzero constants ξ_λ and η_λ are not unique, and may be chosen arbitrarily, subject only to condition (31).

Denote by $R_q(\alpha, u, \mu)$ the q-representations of $\mathcal{G}(0,1)$ defined for all $\alpha, u, \mu \in \mathbb{C}$ such that $\mu \neq 0$, $0 \leqslant \operatorname{Re}\alpha < 1$, α is not an integer and $S = \{\alpha + n\colon\ n = 0, \pm 1, \pm 2, \ldots\}$; and by $\uparrow_q (u, \mu)$, the q-representations defined for all $u, \mu \in \mathbb{C}$ such that $\mu \neq 0$ and $S = \{0, 1, 2, \ldots\}$.

We now have the following theorem:

Theorem 1. *Every q-representation ρ_q of $\mathcal{G}(0,1)$ satisfying conditions* (30) *and* (31) *is isomorphic to either a q-representation $R_q(\alpha, u, \mu)$ or a q-representation $\uparrow_q (u, \mu)$. For these cases, there is a basis of V_q consisting of vectors f_λ defined for each $\lambda \in S$ such that*

$$\begin{aligned}
&J_q^0 f_\lambda = [\lambda - u]_q f_\lambda,\\
&J_q^+ f_\lambda = \mu q^{u-1} f_{\lambda+1},\\
&J_q^- f_\lambda = [\lambda]_q f_{\lambda-1},\\
&E_q f_\lambda = \mu f_\lambda,\\
&C_q f_\lambda = \mu [u]_q q^\lambda f_\lambda.
\end{aligned}\tag{34}$$

(*On the right-hand side of these equations, we assume that* $f_\lambda = 0$ *if* $\lambda \notin S$.)

Proof. Set $a_\lambda = \mu q^\lambda [u]_q$ in (31). This gives $\xi_\lambda \eta_\lambda = \mu q^{u-1}[\lambda]_q$. Choose $\xi_\lambda = \mu q^{u-1}$ and $\eta_\lambda = [\lambda]_q$. Two cases arise:

Case 1. If $\lambda = 0$ is not in S then $\eta_\lambda \neq 0$ for all λ. Choose a complex number α such that $0 \leqslant \operatorname{Re}\alpha < 1$ and α is not an integer. Then the spectrum is $S = \{\alpha + n\colon n = 0, \pm 1, \pm 2, \ldots\}$.
Case 2. If $\lambda = 0$ is in S then since $\eta_0 = 0$, we get the spectrum as $S = \{0, 1, 2, \ldots\}$. □

3.1. Models of irreducible q-representations

A model in one variable for each of $R_q(\alpha, u, \mu)$ and $\uparrow_q(u, \mu)$ is

$$\begin{aligned}
&J_q^0 = (1-q)^{-1}\big(1 - q^{-u} T_t\big),\\
&J_q^+ = \mu q^{u-1} t,\\
&J_q^- = (1-q)^{-1} t^{-1} (1 - T_t),\\
&E_q = \mu I,\\
&f_\lambda(t) = t^\lambda,
\end{aligned}\tag{35}$$

where, for $R_q(\alpha, u, \mu)$, $\lambda \in S = \{\alpha + n\colon \alpha \in C - \{0\}, 0 \leqslant \operatorname{Re}\alpha < 1, n = 0, \pm 1, \pm 2, \ldots\}$, while for $\uparrow_q(u, \mu)$, $\lambda \in S = \{0, 1, 2, \ldots\}$. The model (35) satisfies the commutation relations (27) and (29) as well as (34).

However, for obtaining identities involving q-hypergeometric functions we shall consider models $R_q(\alpha, u, \mu)$ and $\uparrow_q(u, \mu)$ corresponding to $u = 0$, in which case they will be denoted by $R_q(\alpha, \mu)$ and $\uparrow_q(\mu)$, respectively.

3.1.1. One-variable models of $R_q(\alpha, \mu)$ and $\uparrow_q(\mu)$

A model in one variable for each of $R_q(\alpha, \mu)$ and $\uparrow_q(\mu)$ is

$$\begin{aligned}
&J_q^0 = (1-q)^{-1}(1 - T_t),\\
&J_q^+ = \mu q^{-1} t,\\
&J_q^- = (1-q)^{-1} t^{-1} (1 - T_t),\\
&E_q = \mu I,\\
&f_\lambda(t) = t^\lambda,
\end{aligned}\tag{36}$$

where, for $R_q(\alpha, \mu)$, $\lambda \in S = \{\alpha + n\colon \alpha \in C - \{0\}, 0 \leqslant \operatorname{Re}\alpha < 1, n = 0, \pm 1, \pm 2, \ldots\}$; while for $\uparrow_q(\mu)$, $\lambda \in S = \{0, 1, 2, \ldots\}$.

3.1.2. $(m+1)$-variable models of $R_q(\alpha,\mu)$ and $\uparrow_q(\mu)$

A model in $(m+1)$-variable for each of $R_q(\alpha,\mu)$ and $\uparrow_q(\mu)$ is

$$\begin{aligned}
&J_q^0 = (1-q)^{-1}(1-T_t),\\
&J_q^+ = \mu q^{-1} t(1-x_1T_t),\\
&J_q^- = (1-q)^{-1}t^{-1}\prod_{i=1}^m T_{x_i}^{-1}\left(\prod_{i=1}^m T_{x_i} - T_t\right),\\
&E_q = \mu I,\\
&f_\lambda(x_1,\dots,x_m,t) = \Phi_D\left(\begin{matrix} q^{-\lambda}; q,\dots,q \\ q\end{matrix}; q; q^\lambda x_1,\dots,q^\lambda x_m\right)t^\lambda,
\end{aligned}\tag{37}$$

where Φ_D is the q-Lauricella function defined by (10).

For $R_q(\alpha,\mu)$, $\lambda\in S=\{\alpha+n\colon\ \alpha\in C-\{0\},\ 0\leqslant \operatorname{Re}\alpha<1,\ n=0,\pm1,\pm2,\dots\}$; while for $\uparrow_q(\mu)$, $\lambda\in S=\{0,1,2,\dots\}$.

It can be verified that the models (36) and (37) obey the following:

$$\begin{aligned}
&J_q^0 f_\lambda = [\lambda]_q f_\lambda,\\
&J_q^+ f_\lambda = \mu q^{-1} f_{\lambda+1},\\
&J_q^- f_\lambda = [\lambda]_q f_{\lambda-1},\\
&E_q f_\lambda = \mu f_\lambda,\\
&C_q f_\lambda = 0, \quad \text{where } C_q = qJ_q^+J_q^- - J_q^0E_q,
\end{aligned}\tag{38}$$

as well as

$$\begin{aligned}
&J_q^0J_q^+ - qJ_q^+J_q^0 = J_q^+,\\
&qJ_q^0J_q^- - J_q^-J_q^0 = -J_q^-,\\
&qJ_q^+J_q^- - J_q^-J_q^+ = -q^{-1}E_q,\\
&[E_q,J_q^+] = [E_q,J_q^-] = [E_q,J_q^0] = 0
\end{aligned}\tag{39}$$

and (29).

4. Identities based on one variable model

Considering that $v(t) = {}_1\phi_1\left(\begin{smallmatrix}a\\c\end{smallmatrix}; q, t\right)$ is a solution of

$$\big[t(1-aT_t) - (1-T_t)\big(1-cq^{-1}T_t\big)\big]v(t) = 0, \tag{40}$$

we have that $u(t) = {}_1\phi_1\left(\begin{smallmatrix}a\\c\end{smallmatrix}; q, t\right)t^\alpha$, $q^\alpha = a$, is a simultaneous solution of

$$\big(qJ_q^+J_q^- - J_q^0E_q\big)u(t) = 0 \tag{41}$$

and

$$\left[\frac{q}{1-q}J_q^+J_q^0 - E_q\big(\big[a^{-1}\big]_q + a^{-1}J_q^0\big)\big(\big[a^{-1}q^{-1}c\big]_q + a^{-1}q^{-1}cJ_q^0\big)\right]u(t) = 0. \tag{42}$$

Note that $t^{-\alpha}u$ is analytic at $t=0$.

Now, as in [1],

$$\big[e_q\big(sJ_q^+\big)u\big](t)=\frac{1}{(\frac{s\mu t}{q};q)_\infty}\,{}_1\phi_1\left(\begin{matrix}a\\c\end{matrix};q,t\right)t^\alpha \tag{43}$$

and

$$\big[E_q\big(sJ_q^-\big)u\big](t)=\frac{(\frac{s}{(1-q)t};q)_\infty}{(\frac{as}{(1-q)t};q)_\infty}\,{}_2\phi_1\left(\begin{matrix}a,\ \frac{as}{(1-q)t}\\c\end{matrix};q,t\right)t^\alpha. \tag{44}$$

Using Weisner's expansion [11], we get

$$\big[E_q\big(sJ_q^-\big)u\big](t)=\sum_{n=-\infty}^{\infty}j_n t^{\alpha+n} \tag{45}$$

which leads to the following identity, after some rescaling:

$$\begin{aligned}&\frac{(\frac{\omega}{t};q)_\infty}{(\frac{a\omega}{t};q)_\infty}\,{}_2\phi_1\left(\begin{matrix}a,\ \frac{a\omega}{t}\\c\end{matrix};q,t\right)\\&\quad=\sum_{n=-\infty}^{\infty}\frac{\Gamma_q(\gamma)\Gamma_q(\alpha+n)}{\Gamma_q(\alpha)\Gamma_q(\gamma+n)\Gamma_q(n+1)}\,{}_2\phi_2\left(\begin{matrix}aq^n,aq^{n+1}\\cq^n,q^{n+1}\end{matrix};q,\omega\right)t^n.\end{aligned} \tag{46}$$

5. Two variable models

We have the following two variable model for the Lie algebra $\mathcal{G}(0,1)$:

$$\begin{aligned}&J_q^0=t\Delta_t=(1-q)^{-1}(1-T_t),\\&J_q^+=\mu q^{-1}t(1-xT_t),\\&J_q^-=t^{-1}T_x^{-1}(t\Delta_t-x\Delta_x)=(1-q)^{-1}t^{-1}T_x^{-1}(T_x-T_t),\\&E_q=\mu I,\end{aligned} \tag{47}$$

$$f_\lambda(x,t)=\frac{(x;q)_\infty}{(q^\lambda x;q)_\infty}t^\lambda={}_1\phi_0\left(\begin{matrix}q^{-\lambda}\\-\end{matrix};q,q^\lambda x\right)t^\lambda. \tag{48}$$

For $R_q(\alpha,\mu)$, $\lambda\in S=\{\alpha+n\colon\ \alpha\in C-\{0\},\ 0\leqslant\operatorname{Re}\alpha<1,\ n=0,\pm1,\pm2,\dots\}$, while for $\uparrow_q(\mu)$, $\lambda\in S=\{0,1,2,\dots\}$.

Now we use the operators $\mathcal{D}_q$ and $\mathcal{D}_q^{-1}$ defined in (16) and (17) and the formulae (19) and (20) to obtain new models of $R_q(\alpha,\mu)$ and $\uparrow_q(\mu)$ in terms of q-inverse difference operators with basis functions in terms of q-hypergeometric functions ${}_2\phi_1$. To accomplish this, we present the following theorem.

Theorem 2. *Let ρ_q be a q-representation of $\mathcal{G}(0,1)$ in terms of $\{J_q^+,J_q^-,J_q^0,E_q\}$ with basis functions $\{f_\lambda\colon\ \lambda\in S\}$. Then ρ_q is also a q-representation of $\mathcal{G}(0,1)$ in terms of $\{K_q^+,K_q^-,K_q^0,E_q\}$ with basis functions $\{h_\lambda\colon\ \lambda\in S\}$ where*

$$\begin{aligned}&K_q^+=\mathcal{D}_qJ_q^+\mathcal{D}_q^{-1},\qquad K_q^-=\mathcal{D}_qJ_q^-\mathcal{D}_q^{-1},\qquad K_q^0=\mathcal{D}_qJ_q^0\mathcal{D}_q^{-1},\\&E_q=\mathcal{D}_qE_q\mathcal{D}_q^{-1},\qquad h_\lambda=\mathcal{D}_qf_\lambda.\end{aligned}$$

Proof. The proof follows from the fact that

$$J_q^+ f_\lambda(x,t) = \mu q^{-1} f_{\lambda+1}(x,t)$$

can be rewritten as

$$\begin{aligned}
&J_q^+\mathcal{D}_q^{-1}\big(\mathcal{D}_q f_\lambda(x,t)\big) = \mu q^{-1}\mathcal{D}_q^{-1}\mathcal{D}_q f_{\lambda+1}(x,t),\\
&\big(\mathcal{D}_q J_q^+\mathcal{D}_q^{-1}\big)h_\lambda(x,t) = \mu q^{-1} h_{\lambda+1}(x,t),\\
&K_q^+ h_\lambda(x,t) = \mu q^{-1} h_{\lambda+1}(x,t), \quad \text{etc.} \qquad \square
\end{aligned}$$

We apply Theorem 2 to models (47) and (48) to induce new models of $R_q(\alpha,\mu)$ and $\uparrow_q(\mu)$ as under:

Models of $R_q(\alpha,\mu)$:

$$\begin{aligned}
&K_q^0 = t\Delta_t,\\
&K_q^+ = \mu q^{-1} t\big(1 - xq^{\beta-\gamma}T_t - [\beta-\gamma]_q \Delta_x^{-1} T_t\big),\\
&K_q^- = (1-q)^{-1} t^{-1} T_x^{-1}\big(T_x - cq^{-1}T_t\big),\\
&E_q = \mu I, \qquad (49)\\
&h_\lambda(x,t) = \frac{\Gamma_q(\beta)}{\Gamma_q(\gamma)} x^{\gamma-1}\, {}_2\phi_1\left(\begin{matrix} q^{-\lambda}, b \\ c\end{matrix}; q, q^\lambda x\right) t^\lambda, \qquad (50)
\end{aligned}$$

where $\lambda \in S = \{\alpha + n\colon \alpha \in C - \{0\}, 0 \leqslant \operatorname{Re}\alpha < 1, n = 0, \pm 1, \pm 2, \ldots\}$.

A model of $\mathcal{G}(0,1)$ for $\uparrow_q(\mu)$ is same as above with $\lambda \in S = \{0,1,2,\ldots\}$.

It can be verified that the K-model given above satisfies

$$\begin{aligned}
&K_q^0K_q^+ - qK_q^+K_q^0 = K_q^+,\\
&qK_q^0K_q^- - K_q^-K_q^0 = -K_q^-,\\
&qK_q^+K_q^- - K_q^-K_q^+ = -q^{-1}E_q,\\
&\big[E_q, K_q^+\big] = \big[E_q, K_q^-\big] = \big[E_q, K_q^0\big] = 0 \qquad (51)
\end{aligned}$$

as well as

$$\begin{aligned}
&K_q^0 h_\lambda = [\lambda]_q h_\lambda,\\
&K_q^+ h_\lambda = \mu q^{-1} h_{\lambda+1},\\
&K_q^- h_\lambda = [\lambda]_q h_{\lambda-1},\\
&E_q h_\lambda = \mu h_\lambda,\\
&C_q' h_\lambda = 0, \quad \text{where } C_q' = qK_q^+K_q^- - K_q^0E_q. \qquad (52)
\end{aligned}$$

Also,

$$\begin{aligned}
&qK_q^+C_q' = C_q'K_q^+,\\
&K_q^-C_q' = qC_q'K_q^-,\\
&K_q^0C_q' = C_q'K_q^0,\\
&E_qC_q' = C_q'E_q. \qquad (53)
\end{aligned}$$

6. Identities based on two variable model

6.1. Model of $R_q(\alpha, \mu)$

We have shown in (52) that

$$h_\lambda(x,t) = \frac{\Gamma_q(\beta)}{\Gamma_q(\gamma)} x^{\gamma-1}\, {}_2\phi_1\left(\begin{matrix} q^{-\lambda}, b \\ c \end{matrix}; q, q^\lambda x\right) t^\lambda \tag{54}$$

is a solution of $C'_q h_\lambda(x,t) = 0$ where $C'_q = qK_q^+ K_q^- - K_q^0 E_q$.

It therefore follows that

$$u(x,t) = \frac{\Gamma_q(\beta)}{\Gamma_q(\gamma)} x^{\gamma-1} \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q,c';q)_n}\, {}_2\phi_1\left(\begin{matrix} a^{-1}q^{-n}, b \\ c \end{matrix}; q, aq^n x\right) t^{\alpha+n} \tag{55}$$

also satisfies $C'_q u(x,t) = 0$. In view of the fact that $K_q^- C'_q = qC'_q K_q^-$, we have

$$C'_q\big[E_q\big(sK_q^-\big)u\big](x,t) = 0, \tag{56}$$

where

$$\begin{aligned}
&\big[E_q\big(sK_q^-\big)u\big](x,t) \\
&\quad = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} s^n}{(q;q)_n} K_q^{-^n} u(x,t) \\
&\quad = \frac{\Gamma_q(\beta)}{\Gamma_q(\gamma)} x^{\gamma-1} \frac{(\frac{\omega}{t};q)_\infty}{(\frac{a\omega}{t};q)_\infty} \sum_{n=0}^{\infty} \frac{(a, \frac{a\omega}{t};q)_n}{(q,c';q)_n}\, {}_2\phi_2\left(\begin{matrix} a^{-1}q^{-n}, b \\ \frac{a^{-1}q^{1-n}t}{\omega}, c \end{matrix}; q, \frac{qxt}{\omega}\right) t^{\alpha+n}.
\end{aligned} \tag{57}$$

Using Weisner's expansion, we get

$$\big[E_q\big(sK_q^-\big)u\big](x,t) = \sum_{n=-\infty}^{\infty} a_n h_\lambda(x,t), \tag{58}$$

which leads to the following identity:

$$\begin{aligned}
&\frac{(\frac{\omega}{t};q)_\infty}{(\frac{a\omega}{t};q)_\infty} \sum_{n=0}^{\infty} \frac{(a, \frac{a\omega}{t};q)_n t^n}{(q,c';q)_n}\, {}_2\phi_2\left(\begin{matrix} a^{-1}q^{-n}, b \\ \frac{a^{-1}q^{1-n}t}{\omega}, c \end{matrix}; q, \frac{qxt}{\omega}\right) \\
&\quad = \sum_{n=-\infty}^{\infty} \frac{\Gamma_q(\gamma)\Gamma_q(\alpha+n)}{\Gamma_q(\alpha)\Gamma_q(\gamma+n)\Gamma_q(n+1)}\, {}_2\phi_2\left(\begin{matrix} aq^n, aq^{n+1} \\ c'q^n, q^{n+1} \end{matrix}; q, \omega\right) \\
&\qquad \times {}_2\phi_1\left(\begin{matrix} a^{-1}q^{-n}, b \\ c \end{matrix}; q, aq^n x\right) t^n.
\end{aligned} \tag{59}$$

6.2. Model of $\uparrow_q(\mu)$

As shown in (52),

$$h_n(x,t) = \frac{\Gamma_q(\beta)}{\Gamma_q(\gamma)} x^{\gamma-1}\, {}_2\phi_1\left(\begin{matrix} q^{-n}, b \\ c \end{matrix}; q, q^n x\right) t^n \tag{60}$$

satisfies $C_q' h_n(x,t)=0$. This in turn gives that

$$u(x,t)=\frac{\Gamma_q(\beta)}{\Gamma_q(\gamma)}x^{\gamma-1}\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q,c';q)_n}\,{}_2\phi_1\left(\begin{matrix}q^{-n},b\\ c\end{matrix};q,q^n x\right)t^n \tag{61}$$

satisfies $C_q' u(x,t)=0$. In view of the fact that $K_q^- C_q' = qC_q' K_q^-$, we have

$$C_q'\bigl[E_q\bigl(sK_q^-\bigr)u\bigr](x,t)=0, \tag{62}$$

where

$$\begin{aligned}\bigl[E_q\bigl(sK_q^-\bigr)u\bigr](x,t)&=\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}s^n}{(q;q)_n}{K_q^-}^n u(x,t)\\ &=\frac{\Gamma_q(\beta)}{\Gamma_q(\gamma)}x^{\gamma-1}\Phi_{1:1;0}^{1:1;1}\left(\begin{matrix}a:b;\frac{-s}{(1-q)t}\\ c':c;-\end{matrix};q;xt,t\right).\end{aligned} \tag{63}$$

Using the expansion

$$\bigl[E_q\bigl(sK_q^-\bigr)u\bigr](x,t)=\sum_{n=0}^{\infty}c_n h_n(x,t), \tag{64}$$

we get the following identity:

$$\begin{aligned}&\Phi_{1:1;0}^{1:1;1}\left(\begin{matrix}a:b;\frac{\omega}{t}\\ c':c;-\end{matrix};q;xt,t\right)\\ &\quad=\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q,c';q)_n}\,{}_2\phi_1\left(\begin{matrix}q^{-n},b\\ c\end{matrix};q,q^n x\right)\,{}_1\phi_1\left(\begin{matrix}aq^n\\ c'q^n\end{matrix};q,\omega\right)t^n.\end{aligned} \tag{65}$$

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