Weighted Sobolev embedding with unbounded and decaying radial potentials

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Abstract
We prove embedding results of weighted $W^{1,p}(\mathbb{R}^N)$ spaces of radially symmetric functions. The results then are used to obtain ground and bound state solutions of quasilinear equations with unbounded or decaying radial potentials.

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1. Introduction

The motivation of the paper is concerned with nonlinear elliptic equations
\begin{equation}
-\Delta_p u + V(x)|u|^{p-2}u = f(x, u) \tag{1.1}
\end{equation}
where $\Delta_p u = \text{div}(\vert \nabla u \vert^{p-2}\nabla u)$ is the \(p\)-Laplacian operator, and $1 < p < N$. We will focus on potentials whose spatial dependence may be unbounded, decaying and vanishing. We will con-
centrate on the radially symmetric potentials. When $p = 2$, these type equations have been studied recently (e.g., [1–14,21–23,25,27,28,30,32]), and in [15] a quasilinear problem in bounded domains was considered with Hardy type potentials. The variational framework for (1.1) requires weighted Sobolev type embedding and in [29] we have studied the case for $p = 2$. There seems to be little work on the case $p \neq 2$ (see [20,24]), to the best of our knowledge.

In this paper, we consider the general case for $p$ by further extending and developing the ideas and techniques in our recent work [29] in which some embedding results was established for the case $p = 2$. For some cases, our results in the current paper are new even for the case of $p = 2$.

These embedding results can be used to study the existence of ground states of (1.1) where the dependence on $x$ is radial and can tend to infinity or zero somewhere. More precisely we shall focus on the following model equation

$$
\begin{aligned}
-\Delta_p u + V(|x|)u^{p-1} &= Q(|x|)u^{q-1}, \quad u > 0, \text{ in } \mathbb{R}^N, \\
|u(x)| &\to 0 \text{ as } |x| \to \infty.
\end{aligned}
$$

Throughout the paper we assume $V$ and $Q$ are continuous, nonnegative functions in $(0, \infty)$.

We make the following assumptions.

(V) There exist real numbers $a$ and $a_0$ such that $\liminf_{r \to \infty} \frac{V(r)}{r^a} > 0,$ and $\liminf_{r \to 0} \frac{V(r)}{r^{a_0}} > 0.$

(Q) There exist real numbers $b$ and $b_0$ such that $\limsup_{r \to \infty} \frac{Q(r)}{r^b} < \infty,$ and $\limsup_{r \to 0} \frac{Q(r)}{r^{b_0}} < \infty; \ Q(r) > 0.$

Let $C_0^\infty(\mathbb{R}^N)$ denote the collection of smooth functions with compact support and

$$
C_0^{\infty,r}(\mathbb{R}^N) = \{ u \in C_0^{\infty}(\mathbb{R}^N) \mid u \text{ is radial} \}.
$$

Let $D_1^{1,p}(\mathbb{R}^N)$ be the completion of $C_0^{\infty,r}(\mathbb{R}^N)$ under

$$
\|u\|_p = \int_{\mathbb{R}^N} |\nabla u|^p \, dx.
$$

Define for $p \geq 1$ and $q \geq 1$

$$
L_p(\mathbb{R}^N; V) = \left\{ u : \mathbb{R}^N \to \mathbb{R} \mid u \text{ is measurable, } \int_{\mathbb{R}^N} V(|x|) |u|^p \, dx < \infty \right\},
$$

$$
L_q(\mathbb{R}^N; Q) = \left\{ u : \mathbb{R}^N \to \mathbb{R} \mid u \text{ is measurable, } \int_{\mathbb{R}^N} Q(|x|)|u|^q \, dx < \infty \right\}.
$$

Then define $W_1^{1,p}(\mathbb{R}^N; V) = D_1^{1,p}(\mathbb{R}^N) \cap L_p(\mathbb{R}^N; V)$, which is a Banach space under

$$
\|u\|_V^p = \int_{\mathbb{R}^N} |\nabla u|^p + V(|x|)|u|^p \, dx.
$$
Define
\[ L^\infty(\mathbb{R}^N; Q) = \{ u : \mathbb{R}^N \to \mathbb{R} \mid u \text{ is measurable, } \|u\|_{L^\infty(\mathbb{R}^N; Q)} < \infty \}, \]
where
\[ \|u\|_{L^\infty(\mathbb{R}^N; Q)} = \text{ess sup}_{x \in \mathbb{R}^N} |Q(|x|)u(x)|. \]

To state our first type embedding results, we define the following relations between \( p \) and \( a, b \) or \( a_0, b_0 \):

\[
q_* = \begin{cases} 
\frac{p^2(N-1+b)-ap}{p(N-1)+a(p-1)}, & b \geq a > -p, \\
\frac{p(N+b)}{N-p}, & b \geq -p, -p \geq a, \\
\frac{p(N+b_0)}{N-p}, & b_0 \geq -p, a_0 \geq -p,
\end{cases}
\]

\[
q^* = \begin{cases} 
\frac{p(N+b_0)-a_0p}{p(N-1)+a_0(p-1)}, & -p \geq a_0 > -\frac{N-1}{p-1} p, b_0 \geq a_0, \\
\infty, & a_0 \leq -\frac{N-1}{p-1} p, b_0 \geq a_0,
\end{cases}
\]

\[
q^{**} = \frac{p^2(N-1+b_0)-a_0p}{p(N-1)+a_0(p-1)}, \quad b_0 \leq a_0 < -\frac{N-1}{p-1} p.
\]

Now we have the following embedding theorems.

**Theorem 1.** Let \( 1 < p < N \). Assume (V) and (Q). Then we have the embedding
\[ W_{r,1}^{1,p}(\mathbb{R}^N; V) \hookrightarrow L^q(\mathbb{R}^N; Q) \]
for \( q_* \leq q \leq q^* \) when \( q^* < \infty \) and for \( q_* \leq q < \infty \) when \( q^* = \infty \).

Furthermore, the embedding is compact for \( q_* < q < q^* \). And if \( b < \max\{a, -p\} \) and \( b_0 > \min\{-p, a_0\} \), the embedding is also compact for \( q = p \).

**Theorem 2.** Let \( 1 < p < N \). Assume (V) and (Q) with \( b_0 \leq a_0 < -\frac{N-1}{p-1} p \). Then we have the embedding
\[ W_{r,1}^{1,p}(\mathbb{R}^N; V) \hookrightarrow L^q(\mathbb{R}^N; Q) \quad \text{for } \infty > q \geq \max\{q_*, q^{**}\}. \]

Furthermore, the embedding is compact for \( \infty > q > \max\{q_*, q^{**}\} \).

To state our second type embedding result, we define
\[
\gamma_0 = \min \left\{ \frac{N-p}{p}, \frac{p(N-1)+a_0(p-1)}{p^2} \right\},
\]
\[
\gamma_\infty = \max \left\{ \frac{N-p}{p}, \frac{p(N-1)+a(p-1)}{p^2} \right\}.
\]
Theorem 3. Let $1 < p < N$. Assume (V) and (Q) with $b_0 \geq \gamma_0$ and $b \leq \gamma_\infty$. Then we have the embedding

$$W^{1,p}_r(\mathbb{R}^N; V) \hookrightarrow L^\infty(\mathbb{R}^N; Q).$$

Furthermore, the embedding is compact if $b_0 > \gamma_0$ and $b < \gamma_\infty$.

These results are extensions of our earlier work in [29] where $p = 2$ was considered. We point out that Theorems 2 and 3 are completely new results even for the case $p = 2$. The parameters range for $q^*$ to be infinity is enlarged compared with the result in [29]. As applications, we consider the existence of ground state solutions of (1.2).

Theorem 4. Let $1 < p < N$. Assume (V) and (Q) with the corresponding $q_*$ and $q^*$ (respectively, $q_*$ and $q^{**}$) defined such that $q_* < q < q^*$ (respectively, $q > \max\{q_*, q^{**}\}$). Then Eq. (1.2) has a ground state solution $u \in W^{1,p}_r(\mathbb{R}^N; V)$, namely $u$ satisfies

$$\frac{\int_{\mathbb{R}^N} |\nabla u|^p + V(|x|)u^p \, dx}{\left(\int_{\mathbb{R}^N} Q(|x|)u^q \, dx\right)^{\frac{p}{q}}} = \inf_{v \in W^{1,p}_r(\mathbb{R}^N; V), v \neq 0} \frac{\int_{\mathbb{R}^N} |\nabla v|^p + V(|x|)|v|^p \, dx}{\left(\int_{\mathbb{R}^N} Q(|x|)|v|^q \, dx\right)^{\frac{p}{q}}}.$$

If $b < \max\{a, -p\}$ and $b_0 > \min\{-p, a_0\}$, then Eq. (1.2) also has a ground state solution $u \in W^{1,p}_r(\mathbb{R}^N; V)$ for $q = p$ (which corresponds to a first eigenfunction of the first eigenvalue for the p-Laplacian).

We close up with some more remarks. This paper is a continuation of our earlier work [29] which in turn was motivated by some recent works in dealing with nonlinear Schrödinger type equations having unbounded or decaying potentials (e.g., [1–6,8–14,21–23,27,28,30,32]). The motivations is twofold. First of all, through these works a general principle has been revealed that with radially symmetric potentials the existence of ground state solutions holds for a large class of nonlinearity and a wider range of growth and decay of potentials than for general potentials. On the other hand, for equations like (1.2) with radial potentials and $p = 2$, some partial results were available for the existence of ground state solutions which was established by a variety of different methods (e.g., [5,11,12,14,17,18,21,22,27,30–32]). Our results in [29] unify and generalize the existing results but also establish a unified framework. In the current paper, we not only further develop the methods in [29] to deal with the $p$-Laplacian case but also improve the results in [29] for the case $p = 2$.

The main results are proved in Section 3 while some preliminaries are given in Section 2 some of which may be of independent interests.

2. Preliminary lemmas

In this section we give a few preliminary results some of which would be of their own interests. These then will be used to prove the main results in next section. We use $C_i$ to denote various constants independent of the functions, and for any set $A \subset \mathbb{R}^N$, $A^c$ denotes the complement of $A$. Denote by $B_r$ the ball in $\mathbb{R}^N$ centered at 0 with radius $r$. 
Lemma 1. Assume that \( 1 < p < N \). Then there exists \( \hat{C} = \hat{C}(N, p) > 0 \) such that for all \( u \in D^{1,p}_r(\mathbb{R}^N) \),
\[
|u(x)| \leq \hat{C}|x|^{-\frac{N-p}{p}}\|\nabla u\|_{L^p(\mathbb{R}^N)}.
\] (2.1)

Proof. Up to a standard density argument, we only consider \( u \in C^\infty_{0,r}(\mathbb{R}^N) \). Denote by \( \omega_N \) the volume of the unit sphere in \( \mathbb{R}^N \). We have
\[
-u(r) = u(\infty) - u(r) = \int_r^\infty u'(s) \, ds.
\]

Thus
\[
|u(r)| \leq \int_r^\infty |u'(s)| \, ds = \int_r^\infty |u'(s)| \left( s^{-\frac{N-1}{p}} s^{-\frac{N-1}{p}} \right) \, ds \leq \left( \int_r^\infty |u'(s)|^{\frac{p}{p}} s^{N-1} \, ds \right)^{\frac{1}{p}} \left( \int_r^\infty s^{-\frac{N-1}{p}} \, ds \right)^{\frac{p-1}{p}} \omega_N^{\frac{1}{p}} \left( \int_{\mathbb{R}^N} |\nabla u|^p \, dx \right)^{\frac{1}{p}} \left( \int_r^\infty s^{-\frac{N-1}{p}} \, ds \right)^{\frac{p-1}{p}}.
\]

Since
\[
\int_r^\infty s^{-\frac{N-1}{p}} \, ds = \frac{p-1}{N-p} r^{\frac{p-N}{p-1}},
\]
we see that (2.1) holds with
\[
\hat{C} := \hat{C}(N, p) = \omega_N^{\frac{1}{p}} \left( \frac{p-1}{N-p} \right)^{\frac{p-1}{p}}.
\]

Lemma 1 extends a result in [21] which was for \( p = 2 \) and known as Ni’s inequality. Use Lemma 1, we can prove

Lemma 2. Let \( 1 < p < N \). Assume \( p \leq q < \infty \) and write \( q = \frac{p(N+c)}{N-p} \) for some \( -p \leq c < \infty \). Then there exists \( \tilde{C} > 0 \) such that for all \( u \in D^{1,p}_r(\mathbb{R}^N) \)
\[
\left( \int_{\mathbb{R}^N} |x|^c |u|^q \, dx \right)^{\frac{p}{q}} \leq \tilde{C} \int_{\mathbb{R}^N} |\nabla u|^p \, dx.
\] (2.2)
Proof. By a standard density argument, it suffices to consider the case that \( u \in C^\infty_{0, r}(\mathbb{R}^N) \). For such a \( u \), by using Lemma 1, we obtain

\[
\int_{\mathbb{R}^N} |x|^c |u|^q \, dx = \omega_N \int_0^\infty r^{N-1+c} |u(r)|^q \, dr
\]

\[
= - \frac{q \omega_N}{N+c} \int_0^\infty r^{N+c} |u(r)|^{q-2} u(r) u'(r) \, dr
\]

\[
\leq \frac{p \omega_N}{N-p} \int_0^\infty r^{N+c} |u(r)|^{q-1} |u'(r)| \, dr
\]

\[
= \frac{p \omega_N}{N-p} \int_0^\infty r^{N+c - \frac{N-1}{p}} |u(r)|^{q-1} |u'(r)| r^{\frac{N-1}{p}} \, dr
\]

\[
\leq \frac{p \omega_N}{N-p} \left( \int_0^\infty |u'(r)|^{p} r^{N-1} \, dr \right) \left( \int_0^\infty r^{(N+c - \frac{N-1}{p})} |u(r)|^{\frac{q(p-1)}{p}} \, dr \right)^{\frac{p-1}{p}}
\]

\[
= \frac{p \omega_N}{N-p} \left( \int_0^\infty |u'(r)|^{p} r^{N-1+c} |u(r)|^{q} \, dr \right) \left( \int_0^\infty r^N |u(r)|^{q} \, dr \right)^{\frac{p-1}{p}}
\]

where the constant \( \hat{C} \) was given in Lemma 1. It follows that

\[
\left( \int_{\mathbb{R}^N} |x|^c |u|^q \, dx \right)^{\frac{p}{q}} \leq \hat{C} \int_{\mathbb{R}^N} |\nabla u|^p \, dx
\]

where

\[
\hat{C} = \omega_N^{-\frac{q-p}{q}} \left( \frac{p-1}{N-p} \right)^{\frac{(p-1)(q-p)}{q}} \left( \frac{p}{N-p} \right)^{\frac{p^2}{q}}
\]

depends only on \( p, q \) and \( N \). The proof is finished. \( \square \)

Lemma 2 with \( p = 2 \) was due to [16], and was reproved by somewhat different arguments in [19,26]. It should be pointed out that when \( p = q \), i.e., \( c = -p \), the inequality (2.2) holds for all \( u \in D^{1,p}(\mathbb{R}^N) \). This is the Hardy inequality.
Lemma 3. Let $1 < p < N$. For all $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$

$$
\int_{\mathbb{R}^N} |\nabla u|^p \, dx \geq \left( \frac{N-p}{p} \right)^p \int_{\mathbb{R}^N} |u|^p \, dx.
$$

(2.3)

Now we give some lemmas which involve the conditions on $V$ and $Q$. First we note that under (V) the functional

$$
\|u\|_V = \left( \int_{\mathbb{R}^N} |\nabla u|^p + V(|x|)|u|^p \, dx \right)^{1/p}
$$

defines a norm so $W^{1,p}_r(\mathbb{R}^N; V)$ is well defined. In fact we only need that $\|u\|_V = 0$ implies $u = 0$. If $\int_{\mathbb{R}^N} |\nabla u|^p \, dx = 0$, $u$ is a constant. As $\lim \inf_{|x| \to \infty} V(|x|)|x|^{-\alpha} > 0$, one has that $u = 0$.

Lemma 4. Assume (V) with $a > -\frac{N-1}{p-1}$. Then there exists $C > 0$ such that for all $u \in W^{1,p}_r(\mathbb{R}^N; V)$,

$$
|u(x)| \leq C|x|^{-\frac{p(N-1)+a(p-1)}{p^2}} \|u\|_V, \quad |x| \gg 1.
$$

(2.4)

Proof. It follows from (V) that there exists $R > 0$ such that for some constant $C_0 > 0$,

$$
V(|x|) \geq C_0 |x|^a, \quad |x| \geq R.
$$

For $u \in W^{1,p}_r(\mathbb{R}^N; V)$, when $\alpha > -(N-1)$, one has

$$
\frac{d}{dr} (r^{\alpha+N-1}|u|^p) = p \cdot r^{\alpha+N-1}|u|^{p-2}u \frac{du}{dr} + (\alpha + N - 1)|u|r^{\alpha+N-2}
\geq p \cdot r^{\alpha+N-1}|u|^{p-2}u \frac{du}{dr}.
$$

Take $\alpha = \frac{a(p-1)}{p}$, then for $r > R$,

$$
|u|^p r^{\alpha+N-1} \leq \int_r^\infty \left| |u|^{p-1}s^{a+N-1}|u'(s)| \right| ds
\leq p \int_r^\infty \left| |u'(s)|s^{\frac{N-1}{p}}s^{\frac{p-1}{p}(N-1)}|u|^{p-1} \right| ds
\leq p \left( \int_r^\infty |u'(s)| s^{N-1} \, ds \right)^{\frac{1}{p}} \left( \int_r^\infty s^{\frac{p-1}{p}} |u|^{p(N-1)} \, ds \right)^{\frac{p-1}{p}}
$$
\[ \leq p^{\frac{1}{p}} \left( \int_{\mathbb{R}^N \setminus B_r(0)} |\nabla u|^p \, dx \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^N \setminus B_r(0)} |x|^a |u|^p \, dx \right)^{\frac{p-1}{p}} \]

\[ \leq p^{\frac{1}{p}} \omega^{-1} C_0^{-\frac{p-1}{p}} \int_{\mathbb{R}^N} |\nabla u|^p + V(|x|)|u|^p \, dx. \]

It follows that

\[ |u(x)| \leq C|x|^{-\frac{p(N-1)+a(p-1)}{p^2}} \|u\|_V, \quad |x| > R. \quad \Box \]

**Lemma 5.** Assume (V). Then there exist \( r_0 > 0 \) and \( \tilde{C}_0 > 0 \) such that for all \( u \in W^{1,p}_r(\mathbb{R}^N; V) \cap D^{1,p}_0(B_r(0)) \),

\[ |u(x)| \leq \tilde{C}_0|x|^{-\frac{p(N-1)+a(p-1)}{p^2}} \|u\|_V, \quad 0 < |x| \leq r_0. \tag{2.5} \]

**Proof.** It follows from (V) that there exists \( r_0 > 0 \) such that for some constant \( C_0 > 0 \),

\[ V(|x|) \geq C_0|x|^{a_0}, \quad 0 < |x| \leq r_0. \]

For \( u \in W^{1,p}_r(\mathbb{R}^N; V) \cap D^{1,p}_0(B_r(0)) \), one has

\[ \frac{d}{dr} (r^{\beta+N-1} |u|^p) = p \cdot r^{\beta+N-1} |u|^{p-2} u \frac{du}{dr} + (\beta + N - 1) |u|^p r^{\beta+N-2}. \]

Thus for \( 0 < r \leq r_0 \),

\[ r^{\beta+N-1} |u|^p \leq p \int_r^{r_0} |u|^{p-1} s^{\beta+N-1} |u'(s)| \, ds - (\beta + N - 1) \int_r^{r_0} |u|^p s^{\beta+N-2} \, ds. \]

When \( \beta \geq a_0 + 1 \),

\[ \int_r^{r_0} |u|^p s^{\beta+N-2} \, ds = \int_r^{r_0} |u|^p s^{a_0+N-1}s^{\beta-a_0-1} \, ds \]

\[ \leq \omega^{-1}_N r^{\beta-a_0-1}_0 \int_{B_{r_0}(0) \setminus B_r(0)} |x|^{a_0} |u|^p \, dx \]

\[ \leq \omega^{-1}_N C_0^{-1} r^{\beta-a_0-1}_0 \|u\|_V^p. \]

Now we take \( \beta = \frac{p-1}{p} a_0 \), then
\[
\int_r^{r_0} |u|^{p-1} s^{\beta+N-1} |u'(s)| \, ds \\
= \int_r^{r_0} |u'(s)| s^{\frac{N-1}{p}} s^{\beta+N-1} |u|^{p-1} \, ds \\
\leq \left( \int_r^{r_0} |u'(s)|^{p} s^{N-1} \, ds \right)^{\frac{1}{p}} \left( \int_r^{r_0} s^{\frac{p-1}{p}} |u|^{p} s^{N-1} \, ds \right)^{\frac{p-1}{p}} \\
\leq \omega_N^{-1} \left( \int_{B_{r_0}(0) \setminus B_r(0)} |\nabla u|^p \, dx \right)^{\frac{1}{p}} \left( \int_{B_{r_0}(0) \setminus B_r(0)} |x|^{a_0} |u|^p \, dx \right)^{\frac{p-1}{p}} \\
\leq \omega_N^{-1} C_0 \frac{p-1}{p} \int_{\mathbb{R}^N} |\nabla u|^p + V(|x|)|u|^p \, dx \\
= \omega_N^{-1} C_0 \frac{p-1}{p} \| u \|_V^p.
\]

Since

\[
\beta + N - 1 \geq 0 \iff a_0 \geq -\frac{p}{p-1} (N-1),
\]

it follows that \( \beta + N - 1 \leq 0 \) implies \( \beta - a_0 - 1 \geq \frac{N-p}{p-1} \). Then from above arguments we have

\[
|u(x)| \leq \tilde{C}_0 r^{p(N-1)+a_0(p-1)} \| u \|_V, \quad 0 < |x| \leq r_0,
\]

where the constant \( \tilde{C}_0 = \tilde{C}_0(a_0, r_0, N, p) \).

When \( p = 2 \), (2.4) and (2.5) are related to the Strauss Radial Lemma in [31] in which no weights effects were considered. In [27,29], (2.4) and (2.5) were proved for the case \( p = 2, a \geq -2(N-1) \) and \( a_0 \geq -2(N-1) \). The case of \( a_0 < -\frac{N-1}{p-1} \) is new, even for the case \( p = 2 \). We do not know whether the constants involved in the last few lemmas are sharp except in Lemma 3 the constant is sharp.

**Lemma 6.** Let \( 1 < p < N, 1 \leq q \leq \infty \). Then for any \( 0 < r < R < \infty \) with \( R \gg 1 \), the following embedding is compact

\[
W^1,q_r(B_R \setminus B_r; V) \hookrightarrow L^q(B_R \setminus B_r; Q).
\]

**Proof.** The proof is very much similar to the case of \( p = 2 \) which is given in [29], and is omitted. \( \square \)
3. The proofs of main theorems

In this section we give the proof of the main results in the paper.

Proof of Theorem 1. First we prove that the embedding is continuous. It suffices to show

\[ S_r(V, Q) := \inf_{u \in W^{1,p}_r(\mathbb{R}^N; V)} \frac{\int_{\mathbb{R}^N} |\nabla u|^p + V(|x|)|u|^p \, dx}{\left( \int_{\mathbb{R}^N} Q(|x|)|u|^q \, dx \right)^{\frac{p}{q}}} > 0. \] (3.1)

Assume for the contrary that

\[ S_r(V, Q) = 0. \]

Then there exists \( \{u_n\} \subset W^{1,p}_r(\mathbb{R}^N; V) \) such that

\[ \int_{\mathbb{R}^N} |\nabla u_n|^p + V(|x|)|u_n|^p \, dx = o(1) \quad \text{as } n \to \infty, \] (3.2)

\[ \int_{\mathbb{R}^N} Q(|x|)|u_n|^q \, dx = 1, \quad \text{for all } n \in \mathbb{N}. \] (3.3)

We will get a contradiction by showing

\[ \lim_{n \to \infty} \int_{\mathbb{R}^N} Q(|x|)|u_n|^q \, dx = 0. \] (3.4)

By (V) and (Q), there exist \( R_0 > r_0 > 0 \), for some \( C_0 > 0 \),

\[ \begin{cases} Q(|x|) \leq C_0|x|^b, & V(|x|) \geq C_0|x|^a, \quad \text{for } |x| \geq R_0, \\ Q(|x|) \leq C_0|x|^b_0, & V(|x|) \geq C_0|x|^a_0, \quad \text{for } 0 < |x| \leq r_0. \end{cases} \]

For \( R > R_0 \) and \( 0 < r < r_0 \), we estimate the integrals \( \int_{B_r} Q(|x|)|u_n|^q \, dx \) and \( \int_{B_R} Q(|x|)|u_n|^q \, dx \) in different cases according to the definitions of \( q^* \) and \( q_s \).

We first estimate \( \int_{B_r} Q(|x|)|u_n|^q \, dx \).
Case 1.1: $a_0 \geq -p$, $b_0 \geq -p$. Writing $q = \frac{p(N+c)}{N-p}$, by $q \leq q^*$, $\alpha_1 := b_0 - c \geq 0$. Hence by Lemma 2 and (3.2),

$$
\int_{B_r} Q(|x|)|u_n|^q \, dx \leq C_0 \int_{B_r} |x|^{b_0} |u_n|^q \, dx
\leq C_0 r^{b_0-c} \int_{B_r} |x|^c |u_n|^q \, dx
\leq C_1 r^{b_0-c} \left( \int_{\mathbb{R}^N} |\nabla u_n|^p \, dx \right)^{\frac{q}{p}}
= r^{b_0-c} \cdot o(1) \quad \text{as } n \to \infty.
$$

(3.5)1

Case 1.2: $-p \geq a_0 > -\frac{N-1}{p-1} p$, $b_0 \geq a_0$. By $q \leq q^*$, it holds

$$
\alpha_2 := b_0 - a_0 - (q - p) \frac{p(N-1) + a_0(p-1)}{p^2} \geq 0.
$$

We choose a cut-off function $\phi$ such that $\phi(x) = 1$ for $0 \leq |x| \leq \frac{r_0}{2}$, and $\phi(x) = 0$ for $|x| > r_0$. Then by Lemma 5, for $r < \frac{r_0}{2}$,

$$
\int_{B_r} Q(|x|)|u_n|^q \, dx \leq C_0 \int_{B_r} |x|^{b_0} |\phi u_n|^q \, dx
= C_0 \int_{B_r} |x|^{b_0-a_0} |\phi u_n|^{q-p} |x|^{a_0} |\phi u_n|^p \, dx
\leq C_2 \|\phi u_n\|_{V}^{q-p} \int_{B_r} |x|^{b_0-a_0-(q-p) \frac{p(N-1) + a_0(p-1)}{p^2}} V(|x|)|u_n|^p \, dx
\leq C_3 r^{b_0-a_0-(q-p) \frac{p(N-1) + a_0(p-1)}{p^2}} \|\phi u_n\|^q_{V}
= r^{b_0-a_0-(q-p) \frac{p(N-1) + a_0(p-1)}{p^2}} \cdot o(1) \quad \text{as } n \to \infty.
$$

(3.5)2

Case 1.3: $a_0 \leq -\frac{N-1}{p-1} p$, $b_0 \geq a_0$. In this case $q^* = \infty$. For any $\infty > q > p$, it always holds

$$
\alpha_3 := b_0 - a_0 - (q - p) \frac{p(N-1) + a_0(p-1)}{p^2} \geq 0.
$$
With the same function \( \phi \) given in Case 1.2, we have by Lemma 5, for \( r < \frac{r_0}{2} \),

\[
\int_{B_r} Q(|x|)|u_n|^q \, dx \leq C_0 \int_{B_r} |x|^{b_0} |\phi u_n|^q \, dx
\]

\[
= C_0 \int_{B_r} |x|^{b_0-a_0} |\phi u_n|^{q-p} |x|^{a_0} |\phi u_n|^p \, dx
\]

\[
\leq C_4 ||\phi u_n||_V^{q-p} \int_{B_r} |x|^{b_0-a_0-(q-p)\frac{p(N-1)+a_0(p-1)}{p^2}} V(|x|)|u_n|^p \, dx
\]

\[
\leq C_4 r^{b_0-a_0-(q-p)\frac{p(N-1)+a_0(p-1)}{p^2}} \|u_n\|_V^q \cdot o(1) \quad \text{as } n \to \infty. \tag{3.5}_3
\]

**Case 1.4:** \( q = p \). In this case it requires \( b_0 \geq \min\{ -p, a_0 \} \). When \( a_0 \geq -p \), \( \alpha_4 := b_0 + p \geq 0 \). We still take the same function \( \phi \) as in Case 1.2, then by Lemma 3, for \( r < \frac{r_0}{2} \),

\[
\int_{B_r} Q(|x|)|u_n|^p \, dx \leq C_0 \int_{B_r} |x|^{b_0+p} |x|^{-p} |\phi u_n|^p \, dx
\]

\[
\leq C_0 r^{b_0+p} \int_{B_r} |x|^{-p} |\phi u_n|^p \, dx
\]

\[
\leq C_5 r^{b_0+p} \int_{\mathbb{R}^N} |\nabla u_n|^p \, dx
\]

\[
= r^{b_0+p} \cdot o(1) \quad \text{as } n \to \infty. \tag{3.5}_4
\]

When \( a_0 \leq -p \), \( \alpha_4' := b_0 - a_0 \geq 0 \). Hence by Lemma 3, we have

\[
\int_{B_r} Q(|x|)|u_n|^p \, dx \leq C_0 \int_{B_r} |x|^{b_0-a_0} |x|^{a_0} |u_n|^p \, dx
\]

\[
\leq r^{b_0-a_0} \int_{B_r} V(|x|)|u_n|^p \, dx
\]

\[
= r^{b_0-a_0} \cdot o(1) \quad \text{as } n \to \infty. \tag{3.5}'_4
\]

Next we estimate \( \int_{B_r} Q(|x|)|u_n|^q \, dx \).

**Case 2.1:** \( b \geq a > -p \). It follows from \( q \geq q_* \) that

\[
\beta_1 := b - a - (q-p)\frac{p(N-1)+a(p-1)}{p^2} \leq 0.
\]
Thus by Lemma 4 and (3.2), for \( R > R_0 \)

\[
\int_{B_R} Q(|x|)|u_n|^q \, dx \leq C_0 \int_{B_R} |x|^{b}|u_n|^q \, dx
\]

\[
= C_0 \int_{B_R} |x|^{b-a}|u_n|^{q-p}|x|^a|u_n|^p \, dx
\]

\[
\leq C_6 \|u_n\|_V^{q-p} \int_{B_R} |x|^{b-a-(q-p)} \frac{p(N-1)+a(p-1)}{p^2} V(|x|)|u_n|^p \, dx
\]

\[
\leq C_7 R^{b-a-(q-p)} \frac{p(N-1)+a(p-1)}{p^2} \|u_n\|_V^q
\]

\[
= R^{b-a-(q-p)} \frac{p(N-1)+a(p-1)}{p^2} \cdot o(1) \quad \text{as } n \to \infty. \quad (3.6)_1
\]

**Case 2.2:** \( b \geq -p, a \leq -p \). Writing \( q = \frac{p(N+c)}{N-p} \), then \( q \geq q_* \) implies \( \beta_2 := c - b \geq 0 \). Hence by Lemma 2, for \( R > R_0 \), we have

\[
\int_{B_R} Q(|x|)|u_n|^q \, dx \leq C_0 \int_{B_R} |x|^{b-c}|x|^c|u_n|^q \, dx
\]

\[
\leq C_8 R^{b-c} \left( \int_{\mathbb{R}^N} |\nabla u_n|^p \, dx \right)^q
\]

\[
= R^{b-c} \cdot o(1) \quad \text{as } n \to \infty. \quad (3.6)_2
\]

**Case 2.3:** \( b \leq \max\{a, -p\}, q > p = q_* \). For \( R > R_0 \), when \( a > -p, b \leq a \), it always holds

\[
\beta_3 := b - a -(q - p) \frac{p(N-1)+a(p-1)}{p^2} < 0,
\]

so we get similar to (3.6)_1 that

\[
\int_{B_R} Q(|x|)|u_n|^q \, dx \leq C_7 R^{b-a-(q-p)} \frac{p(N-1)+a(p-1)}{p^2} \|u_n\|_V^q
\]

\[
= R^{b-a-(q-p)} \frac{p(N-1)+a(p-1)}{p^2} \cdot o(1) \quad \text{as } n \to \infty, \quad (3.6)_3
\]

and when \( a \leq -p, b \leq -p \), then by writing \( q = \frac{p(N+c)}{N-p} \), it always holds \( \beta_3' := b - c \leq 0 \), we have similar to (3.6)_2 that

\[
\int_{B_R} Q(|x|)|u_n|^q \, dx \leq C_8 R^{b-c} \|u_n\|_V^q = R^{b-c} \cdot o(1) \quad \text{as } n \to \infty. \quad (3.6)'_3
\]
Case 2.4: \( b \leq \max\{a, -p\} \), \( q = p = q_* \). For \( R > R_0 \), when \( a > -p \), we have \( \beta_4 := b - a \leq 0 \), thus

\[
\int_{B_R^c} Q(|x|)|u_n|^p \, dx \leq C_0 \int_{B_R^c} |x|^{b-a}|x|^a|u_n|^p \, dx
\]

\[
\leq R^{b-a} \int_{B_R^c} V(|x|)|u_n|^p \, dx
\]

\[
= R^{b-a} \cdot o(1) \quad \text{as } n \to \infty, \quad (3.6)_4
\]

and when \( a \leq -p \), we have \( \beta'_4 := b + p \leq 0 \), then by Lemma 3,

\[
\int_{B_R^c} Q(|x|)|u_n|^p \, dx \leq C_0 \int_{B_R^c} |x|^{b+p}|x|^{-p}|u_n|^p \, dx
\]

\[
\leq C_8 R^{b+p} \int_{\mathbb{R}^N} |\nabla u_n|^p \, dx
\]

\[
= R^{b+p} \cdot o(1) \quad \text{as } n \to \infty. \quad (3.6)'_4
\]

Now we write

\[
\int_{\mathbb{R}^N} Q(|x|)|u_n|^q \, dx = \int_{B_r} Q(|x|)|u_n|^q \, dx + \int_{B_r^c} Q(|x|)|u_n|^q \, dx + \int_{B_R \setminus B_r} Q(|x|)|u_n|^q \, dx.
\]

When \( q^* \) is finite and \( q_* \leq q \leq q^* \), it follows from (3.5) \((i = 1, 2)\), (3.6) \((j = 1, 2, 3)\) and Lemma 6 that (3.4) holds. When \( q^* \) is infinite and \( q_* \leq q < \infty \), by (3.5) \((j = 1, 2, 3)\) and Lemma 6 we get (3.4). Therefore the embedding is continuous in each cases.

When \( b_0 \geq \min\{a_0, -p\} \), \( b \leq \max\{a, -p\} \), the embedding for \( q = p \) follows from (3.5) \((j = 1, 2, 3)\) or (3.6) \((j = 1, 2, 3)\), (3.6)' \((j = 1, 2, 3)\), (3.6)' \((j = 1, 2, 3)\)' and Lemma 6.

Now we show the embedding obtained above is compact. Let \( \{u_n\} \subset W^{1,p}_r(\mathbb{R}^N ; V) \) be such that

\[
\|u_n\|_V \leq C. \quad (3.7)
\]

Without loss of generality, we may assume

\[
u_n \rightharpoonup 0 \quad \text{in } W^{1,p}_r(\mathbb{R}^N ; V) \quad \text{as } n \to \infty. \quad (3.8)
\]

To show the compactness, we only need to show that

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} Q(|x|)|u_n|^q \, dx = 0. \quad (3.9)
\]
As \( q_* < q < q^* \), the exponents \( \alpha_i \) of \( r \) in the estimates (3.5)_i \( (i = 1, 2, 3) \) are strictly positive, and the exponents \( \beta_j \) of \( R \) in the estimates (3.6)_j \( (j = 1, 2, 3) \) are strictly negative, we get by similar arguments as above the following estimates

\[
\int_{B_r} Q(|x|) |u_n|^p \, dx \leq Cr^{\alpha_i} \|u_n\|^q_V, \quad i = 1, 2, 3, \tag{3.10}
\]

\[
\int_{B^c_R} Q(|x|) |u_n|^p \, dx \leq CR^{\beta_j} \|u_n\|^q_V, \quad j = 1, 2, 3. \tag{3.11}
\]

By (3.7), (3.10), (3.11) and Lemma 6, we get (3.9). Hence the embedding is compact in each case.

For \( b < \max\{a, -p\} \) and \( b_0 > \min\{a_0, -p\} \), the compactness of \( W^{1,p}_r(\mathbb{R}^N; V) \hookrightarrow L^p(\mathbb{R}^N; Q) \) follows from similar arguments as (3.5) or (3.5)', (3.6) or (3.6)', and (3.7).

The proof of Theorem 1 is finished.

**Proof of Theorem 2.** To show the embedding, we still show (3.1) holds. Assume it is not true, then there is a sequence \( \{u_n\} \subset W^{1,p}_r(\mathbb{R}^N; V) \) satisfying (3.2) and (3.3). We will get a contradiction by showing (3.4) holds. As before we need to estimate the integrals

\[
\int_{B_r} Q(|x|) |u_n|^q \, dx \text{ for } 0 < r \leq r_0, \quad \text{and } \int_{B^c_R} Q(|x|) |u_n|^q \, dx \text{ for } R > R_0.
\]

As \( q \geq q_* \), the estimates of \( \int_{B^c_R} Q(|x|) |u_n|^q \, dx \) are same as that in the proof of Theorem 1.

It follows from \( q \geq q^{**} \) that

\[
\gamma := b_0 - a_0 - (q - p) \frac{p(N-1) + a_0(p-1)}{p^2} \geq 0. \tag{3.12}
\]

With the same function \( \phi \) given above, we have by Lemma 5, for \( r < \frac{r_0}{2} \),

\[
\int_{B_r} Q(|x|) |u_n|^q \, dx \leq C_4 r^b_0 - a_0 - (q - p) \frac{p(N-1) + a_0(p-1)}{p^2} \|u_n\|^q_V
\]

\[
= r^b_0 - a_0 - (q - p) \frac{p(N-1) + a_0(p-1)}{p^2} \cdot o(1) \quad \text{as } n \to \infty. \tag{3.13}
\]

Now when \( q \geq \max\{q_*, q^{**}\} \), we can get (3.4) by (3.13), (3.6)_j \( (j = 1, 2, 3) \) and Lemma 6.

When \( q > \max\{q_*, q^{**}\} \), we see \( \gamma > 0 \), the compactness of the embedding is proved as before.

The proof is complete. □

**Proof of Theorem 3.** To show the embedding

\[
W^{1,p}_r(\mathbb{R}^N; V) \hookrightarrow L^\infty(\mathbb{R}^N; Q),
\]

we need to show that

\[
S_\infty(V, Q) := \inf_{u \in W^{1,p}_r(\mathbb{R}^N; V)} \frac{\|u\|_V}{\|u\|_{L^\infty(\mathbb{R}^N; Q)}} > 0. \tag{3.14}
\]
Assume for the contrary that
\[ S_\infty(V, Q) = 0. \]

Then there exists \( \{u_n\} \subset W^{1,p}_r(\mathbb{R}^N; V) \) such that
\[
\int_{\mathbb{R}^N} |\nabla u_n|^p + V(|x|)|u_n|^p \, dx = o(1) \quad \text{as } n \to \infty, \tag{3.15}
\]
\[
\|u_n\|_{L^\infty(\mathbb{R}^N; Q)} \equiv 1, \quad \text{for all } n \in \mathbb{N}. \tag{3.16}
\]

By (V) and (Q), there exist \( R_0 > r_0 > 0 \), for some \( C_0 > 0 \),
\[
\begin{align*}
Q(|x|) &\leq C_0|x|^b_0, \quad \text{for } |x| \geq R_0, \\
Q(|x|) &\leq C_0|x|^b_0, \quad \text{for } 0 < |x| \leq r_0.
\end{align*}
\]

For \( R > R_0 \) and \( 0 < r < r_0 \), we estimate
\[
\sup_{x \in B_r(0) \setminus \{0\}} |Q(|x|)u_n(x)|, \quad \sup_{x \in B_r^c} |Q(|x|)u_n(x)|. \]

When \( a_0 \geq -p, \gamma_0 = \frac{N-p}{p} \), by Lemma 1, we have
\[
\sup_{x \in B_r(0) \setminus \{0\}} |Q(|x|)u_n(x)| \leq C_0 \sup_{x \in B_r(0) \setminus \{0\}} |x|^{b_0} \|u_n(x)\|
\leq C_1 \sup_{x \in B_r(0) \setminus \{0\}} |x|^{b_0 - \frac{N-p}{p}} \|u_n\|_V
\leq C_1 r^{b_0 - \frac{N-p}{p}} \|u_n\|_V \tag{3.17}
\]
provided \( b_0 \geq \gamma_0 \). When \( a_0 \leq -p, \gamma_0 = \frac{p(N-1)+q_0(p-1)}{p^2} \), it follows from Lemma 5 that
\[
\sup_{x \in B_r(0) \setminus \{0\}} |Q(|x|)u_n(x)| \leq C_0 \sup_{x \in B_r(0) \setminus \{0\}} |x|^{b_0} \|u_n(x)\|
\leq C_1 \sup_{x \in B_r(0) \setminus \{0\}} |x|^{b_0 - \frac{p(N-1)+q_0(p-1)}{p^2}} \|u_n\|_V
\leq C_1 r^{b_0 - \frac{p(N-1)+q_0(p-1)}{p^2}} \|u_n\|_V \tag{3.18}
\]
provided \( b_0 \geq \gamma_0 \).
On the other hand, when \( a \leq -p \), \( \gamma_\infty = \frac{N-p}{p} \), by Lemma 1, we have

\[
\sup_{x \in \mathbb{B}^c_R} |Q(|x|)u_n(x)| \leq C_0 \sup_{x \in \mathbb{B}^c_R} |x|^b |u_n(x)| \\
\leq C_2 \sup_{x \in \mathbb{B}^c_R} |x|^b \frac{N-p}{p} \|u_n\|_V \\
\leq C_2 R^{b-N-p} \frac{N-p}{p} \|u_n\|_V
\]

(3.19)

provided \( b \leq \gamma_\infty \). When \( a \geq -p \), \( \gamma_\infty = \frac{p(N-1)+a(p-1)}{p^2} \), by Lemma 4, we have

\[
\sup_{x \in \mathbb{B}^c_R} |Q(|x|)u_n(x)| \leq C_0 \sup_{x \in \mathbb{B}^c_R} |x|^b |u_n(x)| \\
\leq C_2 \sup_{x \in \mathbb{B}^c_R} |x|^b \frac{p(N-1)+a(p-1)}{p^2} \|u_n\|_V \\
\leq C_2 R^{b-p(N-1)+a(p-1)} \frac{p}{p^2} \|u_n\|_V
\]

(3.20)

provided \( b \leq \gamma_\infty \). Now as \( b_0 \geq \gamma_0 \) and \( b \leq \gamma_\infty \), by above estimates, Lemma 6 and (3.15), we get

\[\|u_n\|_{L^\infty(\mathbb{R}^N; \mathbb{Q})} \to 0 \quad \text{as} \quad n \to \infty,\]

a contradiction with (3.16). Thus the embedding is continuous.

Now we show the embedding is compact when \( b_0 > \gamma_0 \) and \( b < \gamma_\infty \). Let \( \{u_n\} \subset W^{1,p}_{r}(\mathbb{R}^N; V) \) be such that

\[u_n \to 0 \quad \text{in} \quad W^{1,p}_{r}(\mathbb{R}^N; V) \quad \text{as} \quad n \to \infty,\]

it suffices to show that

\[\lim_{n \to \infty} \|u_n\|_{L^\infty(\mathbb{R}^N; \mathbb{Q})} = 0.\]

This follows from the above estimates and Lemma 6 since the exponents of \( r \) in (3.17) and (3.18) are strictly positive and the exponents of \( R \) in (3.19) and (3.20) are strictly negative. The proof is complete. \( \square \)

**Proof of Theorem 4.** The existence of a ground state solution follows from the compact embedding immediately. \( \square \)

We finish the paper with a few more remarks. First, more general equations than (1.2) can be considered by making use of the embedding theorems established in this paper. For simplicity, let us state a result for the following equation

\[
\begin{aligned}
-\Delta_p u + V(|x|)|u|^{p-2}u &= Q(|x|)f(u) \quad \text{in} \quad \mathbb{R}^N, \\
|u(x)| &\to 0 \quad \text{as} \quad |x| \to \infty.
\end{aligned}
\]

(3.21)
We assume the conditions on $V$ and $Q$ so that the corresponding $q_*$ and $q^*$ are defined as in Section 1 such that $q_* < q^*$ (respectively, $q_*$ and $q^{**}$ are defined). We assume $f \in C(\mathbb{R}, \mathbb{R})$, $f(0) = 0$; there exists $C > 0$, $q_* < q_1 \leq q_2 < q^*$ (respectively, $\max\{q_*, q^{**}\} < q_1 \leq q_2 < \infty$) such that

$$|f(u)| \leq C(|u|^{q_1-1} + |u|^{q_2-1});$$

and there exists $\mu > p$ such that

$$0 < \mu F(u) \leq uf(u), \quad \forall u \in \mathbb{R}.$$

Here $F(u) = \int_0^u f(s) \, ds$. Then we have

**Theorem 5.** Under the above conditions, Eq. (3.21) has a positive solution. If in addition, $f$ is odd in $u$, (3.21) has infinitely many solutions in $W^{1,p}_r(\mathbb{R}^N; V)$.

This would follow from some standard techniques and we omit the proof.

We remark that the result still holds for $V$ slightly negative and $Q$ vanishing somewhere. Under condition (V), the potential $V$ is positive near 0 and near infinity. Then we see $V$ can be negative somewhere as long as the embedding in Lemma 6 holds. The rest of the arguments are without changes.

Another remark we would like to make is that instead of the whole space $\mathbb{R}^N$ we may consider bounded ball domains or exterior spherical domains; for the former we only need conditions near the origin and for the latter we need only conditions near infinity. Thus the conditions are weaker for the embedding in these special cases. We leave the precise statements to the interested readers.

Finally we remark that our methods should be useful in dealing with more general type weighted Sobolev inequalities. We will come back to this in another project.

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