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A COVERING THEOREM FOR HYPERBOLIC 3-MANIFOLDS AND ITS APPLICATIONS

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1. INTRODUCTION

IN THIS paper we will prove a theorem which describes how geometrically infinite ends of topologically tame hyperbolic 3-manifolds cover. In particular, we prove that if $\hat{E}$ is a geometrically infinite end of a topologically tame hyperbolic 3-manifold $\hat{N}$ which covers (by a local isometry) another hyperbolic 3-manifold N, then either the covering projection is finite-to-one on some neighborhood of $\hat{E}$ or N has a finite cover which fibers over the circle. This theorem generalizes a theorem of Thurston [34] proved for geometrically tame hyperbolic 3-manifolds whose (relative) compact cores have incompressible boundary. The key technical result in this paper, called the filling theorem, is that there exists a neighborhood of any geometrically infinite end such that passing through any point is a convenient simplicial hyperbolic surface in the appropriate homotopy class. As a consequence of the filling theorem we see that for any hyperbolic 3-manifold there exists an upper bound to the injectivity radius at any point in the convex core.

Our main application of the covering theorem is a characterization of exactly which covers of an infinite volume, topologically tame hyperbolic 3-manifold $N = \mathbf{H}^3/\Gamma$ are geometrically finite. Associated to any geometrically infinite end of N there is a geometrically infinite peripheral subgroup of Γ. We will prove that a finitely generated subgroup $\hat{\Gamma}$ of Γ is geometrically finite if and only if it does not contain a finite index subgroup of a geometrically infinite peripheral subgroup. We remark that using the original results of Bonahon [5] and Thurston [34] it is possible to understand covers of N whose (relative) compact cores have incompressible boundary. We will also discuss briefly the applications of the covering theorem to understanding algebraic and geometric limits of sequences of Kleinian groups.

A discussion of these results in the simpler setting when $\hat{N}$ has no cusps as well as a discussion of their relationship with various conjectures is given in [9].

2. STATEMENT OF RESULTS

Let $N = \mathbf{H}^3/\Gamma$ be a complete hyperbolic 3-manifold. The *convex core* $C(N)$ of N is defined to be the smallest convex submanifold such that the inclusion map is a homotopy equivalence. (Explicitly, $C(N)$ is the quotient by Γ of the convex hull $CH(L_\Gamma)$ of Γ's limit set.) Recall that the injectivity radius $inj_N(x)$ of N at a point x is defined to be half the length of the shortest homotopically non-trivial loop through x. We define $N_{thin(\varepsilon)}$ to be the set of points in N with injectivity radius less than or equal to ε and $N_{thick(\varepsilon)}$ to be the set of points in

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N with injectivity radius greater than or equal to ε. There exists a constant $\mathcal{M}_3$ (called the Margulis constant) such that if $\varepsilon < \mathcal{M}_3$ then every component $N_{thin(\varepsilon)}$ is either

(a) a torus cusp, i.e. a horoball in $\mathbf{H}^3$ modulo a parabolic action of $\mathbf{Z} \oplus \mathbf{Z}$,
(b) a rank one cusp, i.e. a horoball in $\mathbf{H}^3$ modulo a parabolic action of $\mathbf{Z}$, or
(c) a solid torus neighborhood of a geodesic

(see [3] or [27]). We let N_ε^0 denote the complement in N of all the cuspidal parts of $N_{thin(\varepsilon)}$. When considering N_ε^0 we will always assume that $\varepsilon < \mathcal{M}_3$.

If N has a finitely generated fundamental group, there exists a compact submanifold M, called the *relative compact core* such that the inclusion map of M into N is a homotopy equivalence and M intersects every rank one component of ∂N_ε^0 in an annulus and every rank two component of ∂N_ε^0 in the full torus (see [23] or [22]). The ends of N_ε^0 are in one-to-one correspondence with the components of $N_\varepsilon^0 - M$ (see [5]). An end E of N_ε^0 is said to be *geometrically finite* if it has a neighborhood U such that $U \cap C(N) = \emptyset$. Otherwise, E is said to be *geometrically infinite*. N will be said to be *topologically tame* if it is homeomorphic to the interior of a compact 3-manifold. Our main theorem will be:

THE COVERING THEOREM. *Let $\hat{N}$ be a topologically tame hyperbolic 3-manifold which covers another hyperbolic 3-manifold N by a local isometry $p: \hat{N} \to N$. If $\hat{E}$ is a geometrically infinite end of $\hat{N}_\varepsilon^0$ then either*

(a) *$\hat{E}$ has a neighborhood $\hat{U}$ such that p is finite-to-one on $\hat{U}$, or*

(b) *N has finite volume and has a finite cover N' which fibers over the circle such that if N_S denotes the cover of N' associated to the fiber subgroup then $\hat{N}$ is finitely covered by N_S. Moreover, if $\hat{N} \neq N_S$, then $\hat{N}$ is homeomorphic to the interior of a twisted I-bundle which is doubly covered by N_S.*

The covering theorem is a generalization of a similar theorem of Thurston's. Thurston's theorem applied to the case where N is geometrically tame and the fundamental group of every component of $\partial M - \partial N_{thin(\varepsilon)}$ injects into $\pi_1(N)$. (An exposition of Thurston's original proof is contained in [28]. His proof makes use of the various compactness theorems for pleated surfaces, an expository treatment of which is given in [10].) The fundamental group of every component of $\partial M - \partial N_{thin(\varepsilon)}$ injecting into $\pi_1(N)$ is equivalent to Condition B which we now state.

Condition B. A hyperbolic 3-manifold $N = \mathbf{H}^3/\Gamma$ is said to satisfy Bonahon's condition (B) if for every non-trivial free decomposition $A * B$ of Γ, there exists a parabolic element of Γ which is not conjugate into either A or B.

Combining Thurston's original theorem with Bonahon's work [5] one obtains a version of the covering theorem which holds whenever N satisfies Condition B.

The key technical tool in the proof of the covering theorem is the filling theorem. The filling theorem should be regarded as a generalization of Theorem 9.5.13 in [34]. We have chosen to make use of simplicial hyperbolic surfaces, rather than pleated surfaces, as it makes the exposition quicker and more elementary, especially in the case when Γ does not satisfy Bonahon's condition (B). In the statement S denotes the interior of the compact surface $\bar{S}$. (See Sections 3 and 4 for the definitions of parabolic extension and convenient simplicial hyperbolic surface.)

THE FILLING THEOREM. *Let N be a topologically tame hyperbolic 3-manifold and E a geometrically infinite end of N_ε^0. Then E has a neighborhood U homeomorphic to $\bar{S} \times [0, \infty)$ whose* *parabolic extension* U_P *is homeomorphic to* $S \times [0, \infty)$ *such that every point in some subneighborhood* $\hat{U} = \bar{S} \times [k, \infty) \subset U$ *is in the image of some convenient simplicial hyperbolic surface* $f_x: S \to U_P$ *such that* f_x *is (properly) homotopic (within* U_P*) to* $S \times \{0\}$*. Moreover, given any* $A > 0$ *we may choose subneighborhood* $\hat{U}^A$ *of* $\hat{U}$ *such that if* $x \in \hat{U}_P^A$ *and* γ *is any compressible curve or accidental parabolic on S, then* $f_x(\gamma)$ *has length at least A.*

One immediate consequence of the filling theorem is:

COROLLARY A. *If N is a topologically tame hyperbolic* 3*-manifold, then there exists K such that* $inj_N(x) \leq K$ *for all points* $x \in C(N)$.

We will also explain, in Section 6, how one may use the technique of proof of the filling theorem to prove Thurston's result that if Γ is any Kleinian group which admits a type-preserving isomorphism to a cofinite area Fuchsian group, then there exists an upper bound, depending only on the type of the Fuchsian group, on the injectivity radius of any point in the convex core of $N = \mathbf{H}^3/\Gamma$.

We may now state our topological characterization of which covers of an infinite volume, topologically tame hyperbolic 3-manifold are geometrically finite.

COROLLARY B. *Let* $N = \mathbf{H}^3/\Gamma$ *be an infinite volume topologically tame hyperbolic* 3*-manifold. Then if* $\hat{\Gamma}$ *is a finitely generated subgroup of* Γ *either*

(a) $\hat{N} = \mathbf{H}^3/\hat{\Gamma}$ *is geometrically finite, or*

(b) $\hat{N}_\varepsilon^0$ *has a geometrically infinite end* $\hat{E}$ *such that* $p: \hat{N} \to N$ *is finite-to-one on some neighborhood* $\hat{U}$ *of* $\hat{E}$.

We can give a more group-theoretic formulation of this characterization. Let M be a relative compact core for $N = \mathbf{H}^3/\Gamma$. Let S be a component of $\partial M - \partial N_\varepsilon^0$ and H be the subgroup of Γ determined by the inclusion of $\pi_1(S)$ into Γ. We will call H a *geometrically infinite peripheral subgroup* if the end of N_ε^0 determined by S is geometrically infinite. Notice that there are only finitely many (conjugacy classes of) geometrically infinite peripheral subgroups of Γ.

COROLLARY C. *Let* $N = \mathbf{H}^3/\Gamma$ *be an infinite volume topologically tame hyperbolic* 3*-manifold. Then if* $\hat{\Gamma}$ *is a finitely generated subgroup of* Γ *either*

(a) $\hat{N} = \mathbf{H}^3/\hat{\Gamma}$ *is geometrically finite, or*

(b) $\hat{\Gamma}$ *contains a (conjugate of a) finite index subgroup of a geometrically infinite peripheral subgroup.*

One specific application of our characterization is:

COROLLARY D. *Let* $N = \mathbf{H}^3/\Gamma$ *be an infinite volume, topologically tame hyperbolic* 3*-manifold. Then if* $\hat{\Gamma}$ *is a finitely generated, infinite index, free subgroup of* Γ *without parabolics it is geometrically finite and hence it is a Schottky group.*

For example, any finitely generated infinite index subgroup of a purely hyperbolic surface group is a Schottky group. Scott and Swarup [31] have previously established, using a result of Cannon and Thurston [12], that all infinite index finitely generated subgroups of the fiber subgroup of a Kleinian group whose associated 3-manifold fibers over the circle are geometrically finite.

For covers of finite-volume hyperbolic 3-manifolds the situation is not as satisfactory. The basic obstruction is that covers of finite volume hyperbolic 3-manifolds with finitely generated fundamental groups are not known to be topologically tame. However, we may generalize results of Culler and Shalen [14] to obtain a partial result. Recall that a subgroup $\hat{\Gamma}$ is said to be a *virtual fiber subgroup* of Γ if $\mathbf{H}^3/\Gamma$ has a finite cover which fibers over the circle and $\hat{\Gamma}$ is the subgroup associated to the fiber.

COROLLARY E. *Let $N = \mathbf{H}^3/\Gamma$ be a finite-volume hyperbolic 3-manifold. Let $\hat{\Gamma}$ be a finitely generated subgroup of Γ such that there exists an epimorphism $H:\Gamma \to \mathbf{Z}$ such that $\hat{\Gamma}$ lies entirely in the kernel. Then $\hat{\Gamma}$ is geometrically finite if and only if it does not contain a virtual fiber subgroup with index at most two.*

Thurston's original application of his covering theorem was to understand when algebraically convergent sequences of Kleinian groups also converged geometrically. In a final section we briefly discuss this application and prove a related theorem (Theorem 9.2) which is an easy consequence of the covering theorem.

The basic motivation for the covering theorem is that every geometrically infinite end of a topologically tame hyperbolic 3-manifold $\hat{N}$ is homeomorphic to $S \times [0, \infty)$ and every point in some neighborhood of the end is in the image of a simplicial hyperbolic surface whose intrinsic geometry has curvature ≤ -1. If the covering is infinite-to-one on this end then some point in N is in the image of infinitely many simplicial hyperbolic surfaces in $\hat{N}$. Since there is a limit to how congested the geometry can get locally we see that there really must be only finitely many of these surfaces up to local homotopy. We may then construct a map of a 3-manifold Q which fibers over the circle into N such that $i_*:\pi_1(Q) \to \pi_1(N)$ is injective, which allows us to conclude that possibility (b) occurs.

In Section 3 we bring together some definitions and in Section 4 we discuss basic properties of simplicial hyperbolic surfaces. In Section 5 we explain some basic methods for constructing continuous families of simplicial hyperbolic surfaces. In Section 6 we use the techniques developed in Section 5 to prove the filling theorem. In Section 7 we bring together the proof of the covering theorem and in Section 8 we discuss the application to understanding which subgroups are geometrically finite. In Section 9 we discuss geometric and algebraic limits.

3. DEFINITIONS

We begin by defining what we will mean by a triangulation of a punctured surface (following [15–17]). Let $\hat{F}$ be a closed surface with a finite number of distinguished points $\{p_1, \ldots, p_n\}$ (called *punctures*) and let F denote $\hat{F} - \{p_1, \ldots, p_n\}$. ($F$ is called a *punctured surface*.) Let V denote a finite collection of points in F and let $\hat{V}$ denote $V \cup \{p_1, \ldots, p_n\}$. A *curve system* $(\alpha_1, \ldots, \alpha_m\}$ on $(\hat{F}, \hat{V})$ is a disjoint (except at endpoints) collection of embedded arcs in $\hat{F}$ with endpoints in $\hat{V}$ no two of which are ambient isotopic (rel $\hat{V}$) and none of which is homotopic to a point (rel $\hat{V}$). A *triangulation* of (F, V) is the restriction to F of a maximal curve system for $(\hat{F}, \hat{V})$. The points in V are called the *vertices* of the triangulation. Two triangulations are equivalent if they are ambiently isotopic (rel $\hat{V}$). These triangulations are not triangulations in the traditional sense as a single edge may be two edges of a face and a single vertex may be 1, 2, or all 3 of the vertices of a given face. For

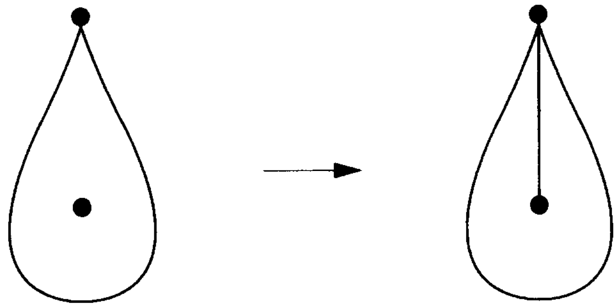

Fig. 1. A punctured monogon and its completion.

example, see Fig. 1, a triangle may be obtained by completing a punctured monogon by appending an arc running from the puncture to the vertex.

Given a triangulation of (F, V) one may perform an *elementary move* to obtain a new triangulation (see Fig. 2). Given a triangulation T with one vertex, one considers a quadrilateral in T bounded by e_1, e_2, e_3, and e_4 and with diagonal e_5. If the other diagonal of the quadrilateral is labelled e_6, we obtain a new triangulation by replacing e_5 by e_6. Harer [15, 16] proved that any two triangulations of (F, V) are related by a sequence of such elementary moves. This will be a key technical result used in Section 6. We will assume throughout this paper that each edge of any triangulation we use has at least one vertex, i.e. that no edge has both endpoints at punctures.

Let F be a punctured surface. We say that a proper map $g: F \to N$ of F into a hyperbolic 3-manifold N *weakly preserves parabolicity* if $g_*: \pi_1(F) \to \pi_1(N)$ has the property that $g_*(\gamma)$ is parabolic if γ is parabolic (in a finite area hyperbolic structure on F). Let $f: F \to N$ be a proper map which weakly preserves parabolicity. We say that f is a *simplicial prehyperbolic surface* if there exists a triangulation T such that f maps each edge of T to a geodesic arc and each face of T to a non-degenerate, totally geodesic triangle in N. The map f induces a piecewise Riemannian metric on F which has constant negative curvature except (perhaps) at the vertices. f is said to be an *simplicial hyperbolic surface* if the total angle about each interior vertex of T is $\geq 2\pi$. We will call the induced piecewise Riemannian metric τ on F a *simplicial hyperbolic structure* and it will occasionally be useful to consider $f: (S, \tau) \to N$ to be a pathwise isometry. The key property of simplicial hyperbolic structures is that they curvature ≤ -1 at every point, where we regard vertices with angles more than 2π as having concentrated negative curvature. For example, it follows from the Gauss–Bonnet theorem for hyperbolic triangles that the area of (S, τ) is at most $2\pi|\chi(S)|$. In particular, if we define $ang(v)$ to be the total angle about a vertex v, then (see [7])

$$area(S, \tau) = 2\pi(\chi(S)) - \sum_{v \in T} (ang(v) - 2\pi).$$

Let E be an end of N_ε^0 which has a neighborhood U homeomorphic to $\bar{F} \times [0, \infty)$. We now define the *parabolic extension* U_P of U. As always, we encourage the reader to assume that the manifold does not contain any cusps when first reading the paper and then return later to the technical difficulties caused by the presence of cusps. If $\bar{F}$ is closed then $U_P = U$. If $\bar{F}$ is not closed, we append to U a portion of each thin part which lies adjacent to U.

Let $(\partial\bar{F})_i$ be a component of the boundary of $\bar{F}$. $(\partial\bar{F})_i \times [0, \infty)$ lies on the boundary of a component of P_i of $N_{thin(\varepsilon)}$ which is a rank one cusp. This component P_i has a natural parametrization as $S^1 \times (-\infty, \infty) \times [0, \infty)$ which is induced by normalizing so that the

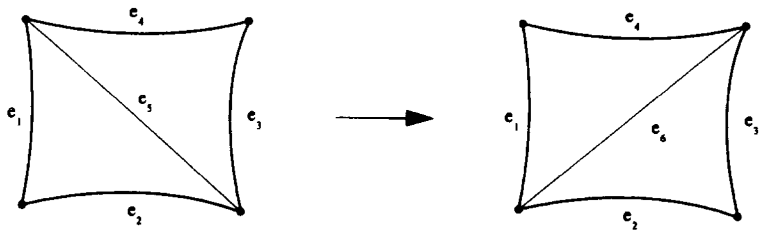

Fig. 2. The elementary move on a triangulation.

horoball is centered at ∞ (in the (z, t) coordinates for the upper half-space model) and is determined by the equation $t \geq c$ for some constant c (depending only on ε) and the parabolic action of $\mathbf{Z}$ is generated by $z \mapsto z + 2\pi$. We give it coordinates (θ_i, r_i, s_i) in which the Riemannian metric is of the form

$$e^{-2s_i}(G^2\, d\theta_i^2 + dr_i^2) + ds_i^2$$

where G is a constant depending only on ε. In these coordinates, $\theta_i = \operatorname{Re} z (\operatorname{mod} 2\pi)$, $r_i = H(\operatorname{Im} z)$ and $s_i = \log(t) + J$ for some constants H and J depending only on ε.

In this normalization, $(\partial \bar{F})_i \times [0, \infty)$ is a subset C_i of $S^1 \times (-\infty, \infty) \times \{0\}$. Let

$$B(C_i) = \{(\theta_i, r_i, s_i) \mid (\theta_i, r_i, 0) \in C_i \text{ and } s_i \geq 0\}$$

denote the portion of P_i lying "above" C_i in this parametrization. To form U_P we append $\bigcup_{P_i} B(\partial U \cap P_i)$ to U, where the union is taken over all rank-one cusps adjacent to U. U_P is homeomorphic to $F \times [0, \infty)$ where F is homeomorphic to the interior of $\bar{F}$. See Fig. 3 for a picture of how this might look. In general if A is any subset of U we may define its parabolic extension A_P to be $A \bigcup_{P_i} B(A \cap \partial P_i)$.

We will call a neighborhood U of E *well-cusped* if whenever P_i is a rank one cusp adjacent to U, then ∂U intersects P_i in a set of the form $\{(\theta_i, r_i, s_i) \mid s_i = 0 \text{ and } r_i \geq R(U, i)\}$ for some $R(U,i)$ or $\{(\theta_i, r_i, s_i) \mid s_i = 0 \text{ and } r_i \leq R(U, i)\}$ for some $R(U, i)$. Alternatively, we could say that ∂U_P intersects ∂P_i in a horocycle.

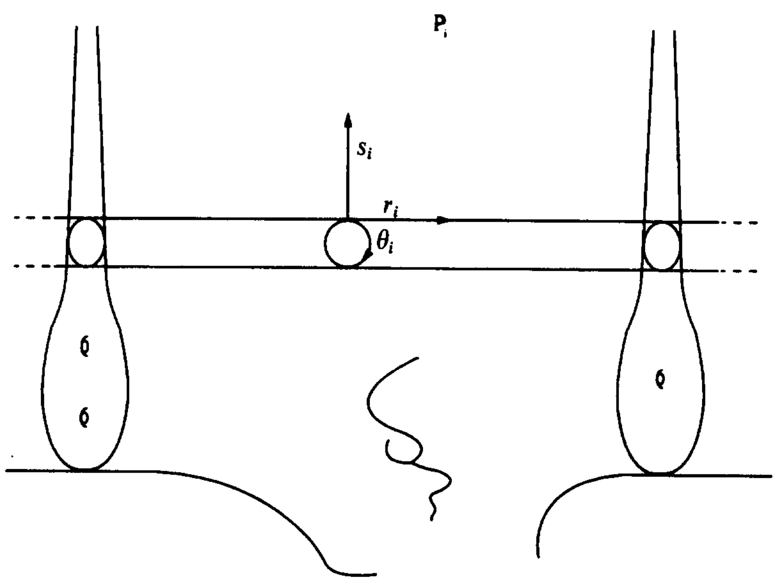

Fig. 3.

We say that an end E of N_ε^0 is *simply degenerate* if there exists a neighborhood U of E homeomorphic to $\bar{S}\times[0,\infty)$ whose parabolic extension U_P is homeomorphic to $S\times[0,\infty)$ (where S is the interior of $\bar{S}$) and there exists a sequence of simplicial hyperbolic surfaces $\{f_n: S\to U_P\}$, such that $f_n(S)$ is (properly) homotopic to $S\times\{0\}$ within U_P and $f_n(S)$ leaves every compact subset of N. Such a sequence of simplicial hyperbolic surfaces is said to *exit E*. A hyperbolic 3-manifold N with finitely generated fundamental group is said to be *geometrically tame* if all the ends of N_ε^0 are either simply degenerate or geometrically finite. Notice that if a hyperbolic 3-manifold is geometrically tame it is also topologically tame. It is conjectured that all hyperbolic 3-manifolds with finitely generated fundamental group are both topologically and geometrically tame. The two main existence theorems we will use are:

THEOREM 3.1 (Bonahon [5]). *If $N=\mathbf{H}^3/\Gamma$ is a hyperbolic 3-manifold and Γ satisfies Condition B, then N is geometrically tame.*

THEOREM 3.2. (Canary [8]). *If N is a topologically tame hyperbolic 3-manifold, then N is geometrically tame.*

Let $f: S\to N$ be a simplicial hyperbolic surface. We will say that a homotopically non-trivial curve γ in S is *compressible* if $f(\gamma)$ is homotopically trivial in N. We will say that a homotopically non-trivial curve γ in S is an *accidental parabolic* if it is not homotopic to a cusp of S but $f(\gamma)$ is homotopic into a cusp of N.

It is often helpful to be able to assume that one's simplicial hyperbolic surfaces contain no compressible curves or accidental parabolics of length less than some predetermined constant K. By working in a sufficiently small neighborhood of the end, this can always be accomplished.

LEMMA 3.3 (Lemma 7.1 in [8]). *Let N be a topologically tame hyperbolic 3-manifold and let E be a simply degenerate end of N_ε^0. Let U be a neighborhood of E which is homeomorphic to $\bar{S}\times[0,\infty)$. Then given any $K>0$ there exists a well-cusped subneighborhood U^K of U which is homeomorphic to $\bar{S}\times[0,\infty)$ Such that if $f: S\to U_P^K$ is a simplicial hyperbolic surface (properly) homotopic to ∂U_P (within U_P) and γ is any compressible curve or accidental parabolic on S, then $f(\gamma)$ has length at least K and $f(\gamma)$ is not contained entirely within any component of $N_{thin(\varepsilon)}$.*

4. USEFUL AND CONVENIENT SIMPLICIAL HYPERBOLIC SURFACES

In the bulk of the paper we will restrict ourselves to simplicial hyperbolic surfaces with certain technical properties which will make them easier to work with. In this section, we will introduce two special classes of simplicial hyperbolic surfaces and investigate some of their basic properties.

We will define a simplicial hyperbolic surface $p: S\to N$ (with associated triangulations T) to be *useful* if T has only one vertex v (aside from the ideal vertices at the cusps), and that one of the edges e (called the *distinguished* edge) of T is mapped to a closed geodesic. The full statement of the main theorem in [8] also includes the following technically useful fact.

PROPOSITION 4.1. *Let E be a simply degenerate end of a topologically tame hyperbolic 3-manifold; then there exists a sequence of useful simplicial hyperbolic surfaces $\{p_i: S\to N\}$ exiting E.*

Bonahon [5] observes that if $f: S \to N$ is a simplicial pre-hyperbolic surface and v is any vertex of the associated triangulation T such that there exists an edge e, such that $f(e \cup v)$ is a closed geodesic, then the total angle about that vertex is at least 2π. We will now generalize this observation to obtain a criterion which guarantees that a simplicial pre-hyperbolic surface is indeed a simplicial hyperbolic surface. To a vertex v and an edge e with endpoint v, we associate a point in $T^1_{f(v)}(\mathbf{H}^3)$ which is the unit tangent vector in the direction of $f(e)$ (pointing away from $f(v)$). Let $S_f(v)$ denote the set of points in $T^1_{f(v)}(\mathbf{H}^3)$ associated to edges with endpoints at v. If an edge has both endpoints at v it will produce two points in $S_f(v)$. We will say that a vertex v is *NLSC* with respect to f, if there is no open hemisphere in $T^1_{f(v)}(\mathbf{H}^3)$ which contain $S_f(v)$. (NLSC stands for not locally strictly convex. An alternative definition is that, working in the universal cover, there exists no open half-space H containing $\tilde{f}(\tilde{v})$ such that all the edges emanating from $\tilde{f}(\tilde{v})$ lie in H.) Notice that the vertex of a useful simplicial hyperbolic surface satisfies NLSC.

LEMMA 4.2. *If $f: S \to N$ is a simplicial pre-hyperbolic surface such that all vertices v of the associated triangulation T are NLSC with respect to f, then f is a simplicial hyperbolic surface.*

Proof. Recall that any path in the unit 2-sphere which is not contained in any open hemisphere has length at least 2π. Therefore, the total angle about any vertex satisfying NSLC must be at least 2π. □

We will make use of the following application of Lemma 4.2.

LEMMA 4.3. *Let $f: S \to N$ be a simplicial pre-hyperbolic surface with associated triangulation T. Let $\tilde{f}: \tilde{S} \to \mathbf{H}^3$ be a lift of f. Let v be a vertex of T and $\tilde{v}$ be one of its lifts. If $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$, and $\tilde{e}_4$ are edges of $\tilde{T}$ which have $\tilde{v}$ as one endpoint, then the total angle about $\tilde{v}$ in $\tilde{f}(\tilde{S})$ is at least 2π if $\tilde{v}$ lies in the tetrahedron spanned by the other endpoints of $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$, and $\tilde{e}_4$. Thus, the total angle about v in $f(S)$ is at least 2π.*

Another helpful property of simplicial hyperbolic surfaces whose vertices satisfy NLSC is that one can bound the horizontal (r_i) extent of their intersection with a rank one cusp by the horizontal extent of their intersection with the boundary of the cusp.

LEMMA 4.4. *Let P_i be a rank one cusp of N with coordinates $(\theta_i, r_i, s_i) \in S^1 \times (-\infty, \infty) \times [0, \infty)$ and $f: S \to N$ be a simplicial hyperbolic surface whose vertices satisfy NLSC. If $x \in f^{-1}(P_i)$, let $r_i(x)$ denote the r_i coordinate of $f(x)$. Then,*

$$\max_{x \in f^{-1}(P_i)} r_i(x) = \max_{x \in f^{-1}(\partial P_i)} r_i(x)$$

and

$$\min_{x \in f^{-1}(P_i)} r_i(x) = \min_{x \in f^{-1}(\partial P_i)} r_i(x).$$

Proof. We first of all observe that it is sufficient to consider the values of r_i on the edges of the associated triangulation T. If neither of the vertices of an edge e of T map into P_i, then the edge attains both its maximum and minimum values of r_i on the boundary of its intersection with P_i (which may be easily seen by looking at the universal cover).

If e is an edge which has one or both of its vertices mapped into P_i, then the maximum and minimum of r_i over each segment of $f(e) \cap P_i$ are obtained either on the boundary of the cusp or at a vertex contained within the cusp. However, condition NLSC guarantees that emanating from each vertex which is mapped into P_i is an edge which has a non-decreasing value of r_i as one heads away from v and does not have as its other vertex a puncture which is mapped to P_i. Similarly, condition NLSC guarantees that emanating from each vertex which is mapped into P_i is an edge which has a non-increasing value of r_i as one heads away from v and does not have as its other vertex a puncture which is mapped to P_i. Therefore, the maximum or minimum of r_i over all of $f^{-1}(P_i)$ cannot occur at a vertex in the interior of $f^{-1}(P)$ and hence must occur at the boundary. □

We may define the injectivity radius $inj_{(S,\tau)}(x)$ of a point x in a simplicial hyperbolic structure τ on a surface S to be half the length of the shortest homotopically non-trivial loop through x. However, there does not exist a uniform Margulis constant for simplicial hyperbolic structures. We may see this by gluing together equilateral hyperbolic triangles with very short sides to produce a simplicial hyperbolic structure on a surface S with arbitrarily small diameter and cone angle arbitrarily close to $2\pi(1 + \chi(S))$. However, we will be able to establish a usable analogue of the thick-thin decomposition for simplicial hyperbolic surfaces satisfying the conclusion of Lemma 3.3.

We will call a simplicial hyperbolic surface $f:(S,\tau)\to N$ *convenient* if its associated triangulation has at most two vertices, each of these vertices has property NLSC, every accidental parabolic and compressible curve has length at least $\mathcal{M}_3$ and no accidental parabolic or compressible curve is contained entirely within a non-compact component of $N_{thin(\varepsilon)}$ if $\varepsilon < \mathcal{M}_3$. In this case, we will let $(S,\tau)^0_\varepsilon$ denote (S,τ) with the non-compact components of $(S,\tau)_{thin(\varepsilon)}$ removed. The following lemma is a strengthened version of Bonahon's bounded diameter lemma [5].

LEMMA 4.5 (Strengthened Bounded Diameter Lemma). *Let $f:(S,\tau)\to N$ be a convenient simplicial hyperbolic surface. Let $\varepsilon < \mathcal{M}_3$. Then the following hold:*

(i) *If $x, y \in S$, then x and y may be joined by a path R such that $f(R)\cap N_{thick(\varepsilon)}$ has length at most C (where C depends only on ε and the topological type of S).*

(ii) *If P is a non-compact component of $N_{thin(\varepsilon)}$, then every component V of $(S,\tau)_{thin(\varepsilon)}$ such that $f(V)\cap P \neq \emptyset$ is non-compact.*

(iii) *If V is any non-compact component of $(S,\tau)_{thin(\varepsilon)}$ then every homotopically non-trivial curve in V is homotopic to the (unique) puncture contained in V. Moreover, V is homeomorphic to a half-open annulus which encloses a puncture of S.*

Proof. We first observe that our hypotheses imply that $f((S,\tau)_{thin(\varepsilon)})$ is contained in $N_{thin(\varepsilon)}$. Let R be the shortest path in (S,τ) joining x to y. Then we may observe that if $R_\varepsilon = R\cap(S,\tau)_{thick(\varepsilon)}$ is non-empty, then the $\varepsilon/2$-neighborhood about R_ε is embedded in (S,τ) and has area at least $(\varepsilon/2)l(R_\varepsilon)$, where $l(R_\varepsilon)$ denotes the length of R_ε. Thus, since (S,τ) has area at most $2\pi|\chi(S)|$, we see that there is a uniform bound on the length of R_ε. Statement (i) then follows from the fact that $f(R - R_\varepsilon) \subset N_{thin(\varepsilon)}$.

Let V be a component of $(S,\tau)_{thin(\varepsilon)}$ such that $f(V)$ intersects a non-compact component P of $N_{thin(\varepsilon)}$. We first observe that this implies that $f(V)\subset P$, since $f((S,\tau)_{thin(\varepsilon)})\subset N_{thin(\varepsilon)}$. Passing through every point $x\in V$ there exists a homotopically non-trivial simple closed curve R_x of length $<\varepsilon$. The fact that no accidental parabolics have length $<\mathcal{M}_3$ guarantees that R_x is homotopic into a cusp (i.e. puncture) of (S,τ). Let $\pi:(\tilde S,\tilde\tau)\to(S,\tau)$ be the universal covering of S with the induced simplicial hyperbolic structure. Let γ be a covering transformation associated to the puncture which R_x is homotopic to. Since $d_{\tilde\tau}(x,\gamma(x))$ is a convex function on $\tilde S$ (see [35] for example) and γ is a parabolic isometry, we see that

$V(\varepsilon, \gamma) = \{x \in \tilde{S} \mid d(x, \gamma(x)) \leq \varepsilon\}$ is convex, connected and non-compact. We may observe that $\pi_1(V) \cong \mathbf{Z}$ is generated by γ since otherwise there would be a non-peripheral curve on S whose image lay entirely within P. Statement (ii) then follows from the fact that $\pi(V(\varepsilon, \gamma)) = V$.

Let V be a non-compact component of $(S, \tau)_{thin(\varepsilon)}$. Since f is proper and convenient, $f(V)$ lies entirely within a non-compact component P of $N_{thin(\varepsilon)}$. Statement (iii) then follows immediately from the argument given above by observing that $V = \pi(V(\varepsilon, \gamma))$ and $\pi_1 = (V) = \mathbf{Z}$. □

We will also make use of the following application of Lemma 4.4.

LEMMA 4.6. *Let U be a neighborhood of a simply degenerate end of N_ε^0. Let $f: (S, \tau) \to U$ be a convenient simplicial hyperbolic surface. If A is a subset of U_P and $f((S, \tau)_\varepsilon^0) \subset A$, then $f(S)$ is contained in the parabolic extension $(A \cap N_\varepsilon^0)_P$* of $A \cap N_\varepsilon^0$.

5. CONTINUOUS FAMILIES OF SIMPLICIAL HYPERBOLIC SURFACES

In this section we will explain how to construct continuous families of simplicial hyperbolic surfaces by interpolating between pairs of simplicial hyperbolic surfaces which differ only in some basic way. In the next section we will put these basic steps together to produce families of simplicial hyperbolic surfaces "filling up" an entire simply degenerate end.

We now explain one basic construction of a simplicial pre-hyperbolic surface which we will use throughout the section. We begin with a proper map $h: S \to N$ which weakly preserves parabolicity and a triangulation T such that every edge e of T with both endpoints at the same vertex is mapped to a homotopically non-trivial curve in N by h and no two vertices of T are mapped to the same point. First homotope h, keeping $h(V)$ fixed, to a map $h_1: S \to N$ such that if e is a compact edge of T then $h_1(e)$ is the unique geodesic arc joining its endpoints to itself in the homotopy class of $h(e)$. We then (properly) homotope h_1, keeping the image of each compact edge fixed, to a map h_2 such that each infinite edge e of T is taken to the infinite geodesic with the same endpoint which is properly homotopic to $h(e)$. This can be visualized best in the universal cover, where a lift of the infinite geodesic has an endpoint at the fixed point of a parabolic element of Γ. We then homotope h_2, fixing $h_2(T)$, to a map $F(h, T): S \to N$ such that each face of T is taken to the totally geodesic triangle spanned by the image of its edges. This procedure always results in a simplicial pre-hyperbolic surface.

Given a simplicial pre-hyperbolic surface $f: (S, \tau) \to N$ and a path $\gamma: [0, 1] \to N$ such that there exists a vertex v of T such that $f(v) = \gamma(0)$, we may construct a map $H: S \times [0, 1] \to N$ such that $H(\cdot, t) = H_t$ is a simplicial pre-hyperbolic surface (for all $t \in [0, 1]$), $H_t(v) = \gamma(t)$ and $H_0 = f$. We will call H the continuous family of simplicial pre-hyperbolic surfaces obtained by *dragging f along γ*. We first choose $\delta > inj_{(S,\tau)}(x)$ such that no other vertex of T lies within δ of v. We then construct $R(\cdot, t): S \times [0, 1] \to N$ such that $R(\cdot, t)$ agrees with f on the complement of $B_{\delta/2}(v)$, the ball of radius $\delta/2$ about v in (S, τ), $R(\cdot, t)(v) = \gamma(t)$, and $R(\cdot, t)(B_{\delta/2}(v)) \subset f(B_{\delta/2}) \cup \gamma([0, t])$. Let $H(\cdot, t) = F(R(\cdot, t), T)$. Notice that the family of simplicial pre-hyperbolic surfaces obtained is independent of the choice of R, since if we had chosen R differently (but with the same restrictions) the homotopy classes of the images of the edges (under $R(\cdot, t)$) would be independent of the choices made.

If $h: S \to N$ is a useful simplicial hyperbolic surface with associated triangulation T and distinguished edge $\hat{e}$ then we may drag h along the geodesic $h(\hat{e} \cup v)$ to produce a continuous family of actual simplicial hyperbolic surfaces. In general, this produces a non-compact 1-parameter family $H: S \times R \to N$ of distinct simplicial hyperbolic surfaces. Notice that if $H_1: S \to N$ is the result of dragging $h(v)$ one full twist about $h(\hat{e} \cup v)^*$, then the homotopy class (but not the free homotopy class) of $H_1(e \cup v)$ for other edges e of T is altered. In fact, this will produce all useful simplicial hyperbolic surfaces freely homotopic to h which are associated to the triangulation T and have distinguished edge $\hat{e}$.

LEMMA 5.1. *Let $f: S \to N$ and $g: S \to N$ be two (properly) homotopic useful simplicial hyperbolic surfaces associated to the same triangulation T and with the same distinguished edge $\hat{e}$. Then there exists a homotopy $H: S \times [0, 1] \to N$ such that $H(\cdot, t)$ is a useful simplicial hyperbolic surface with triangulation T and distinguished edge $\hat{e}$ and $H(\cdot, 0) = f$ and $H(\cdot, 1) = g$.*

Proof. Let $H': S \times [0, 1] \to N$ be a (proper) homotopy between f and g. Since N is atoroidal we may assume that $H'_t(\hat{e} \cup v) = f(\bar{e} \cup v)$ for all $t \in [0, 1]$. Let $H_t = F(H'_t, T)$. H_t is the desired homotopy. □

This allows us to interpolate between useful simplicial hyperbolic surfaces with the same associated triangulation and distinguished edge. We now explain how to interpolate between useful simplicial hyperbolic surfaces with the same associated triangulations, but different distinguished edges.

LEMMA 5.2. *Let h be a useful simplicial hyperbolic surface with associated triangulation T and distinguished edge $\hat{e}$. Let $\bar{e}$ be another edge of T such that $h(\bar{e} \cup v)$ has a closed geodesic representative $h(\bar{e} \cup v)^*$. Then there exists a continuous family $J: S \times [0, 2] \to N$ of simplicial hyperbolic surfaces with only one vertex (which satisfies NLSC) joining h to a useful simplicial hyperbolic surface $\bar{h}$ with associated triangulation T and distinguished edge $\bar{e}$.*

Proof. Let $W: S^1 \times [0, 1] \to N$ be a homotopy between $h(\bar{e} \cup v)$ and its geodesic representative $h(\bar{e} \cup v)^*$. Let α_1 and α_2 be components of the pre-images of $\hat{e} \cup v$ and $\bar{e} \cup v$ to $\tilde{S}$ which intersect at the point $\tilde{v}$ which is a lift of v. Lift both h and W to maps $\tilde{h}: \tilde{S} \to \mathbf{H}^3$ and $\tilde{W}: \mathbf{R} \times [0, 1] \to \mathbf{H}^3$ such that $\tilde{W}(\mathbf{R} \times \{0\}) = \tilde{h}(\alpha_2)$. Let $\gamma_1 = \tilde{h}(\alpha_1)$ and let $\gamma_2 = \tilde{W}(\mathbf{R} \times \{1\})$. Let R be the unique common perpendicular joining γ_1 to γ_2. Let $x = R \cap \gamma_1$ and let P be a geodesic arc in γ_1 joining $\tilde{h}(\tilde{v})$ to x. We may drag h along (the projection to N of) P to produce a continuous family of useful simplicial hyperbolic surfaces $G: S \times [0, 1] \to N$. We may then drag G_1 along (the projection to N of) R to construct a family of simplicial pre-hyperbolic surfaces $H: S \times [1, 2] \to N$ such that $H_2(v)$ lies on the closed geodesic in the homotopy class of $h(\bar{e} \cup v)$. Thus, H_2 is a useful simplicial hyperbolic surface with associated triangulation T and distinguished edge $\bar{e}$.

It remains to check that any intermediate simplicial pre-hyperbolic surface H_t ($1 \le t \le 2$) in this family is actually a simplicial hyperbolic surface. Let $\tilde{H}: \mathbf{H}^2 \times [1, 2] \to \mathbf{H}^3$ be a lift of H such that $\tilde{H}_1 = \tilde{G}_1$ (where $\tilde{G}_1$ arises as a portion of a lift of G such that $\tilde{G}_0 = \tilde{h}$). Let g_1 be the hyperbolic isometry having axis γ_1 and g_2 be the hyperbolic isometry having axis γ_2. Then there are four edges of $\tilde{H}_t(\tilde{T})$ with endpoints at $\tilde{H}_t(\tilde{v})$ whose other endpoints are $g_1(\tilde{H}_t(\tilde{v}))$, $g_1^{-1}(\tilde{H}_t(\tilde{v}))$, $g_2(\tilde{H}_t(\tilde{v}))$, and $g_2^{-1}(\tilde{H}_t(\tilde{v}))$. We now simply observe that $(\tilde{H}_t(\tilde{v}))$ lies in the tetrahedron spanned by these endpoints and apply Lemma 4.3 to conclude that H_t is a simplicial hyperbolic surface. (Notice that the geodesic arc joining $g_1(\tilde{H}_t(\tilde{v}))$ to $g_1^{-1}(\tilde{H}_t(\tilde{v}))$ and the geodesic arc joining $g_2(\tilde{H}_t(\tilde{v}))$ to $g_2^{-1}(\tilde{H}_t(\tilde{v}))$ intersect the infinite geodesic containing R on opposite sides of $(\tilde{H}_t(\tilde{v}))$.)

Let J be formed by concatenating G and H. □

We now want to be able to alter the triangulation by an elementary move.

LEMMA 5.3. *Let h be a useful simplicial hyperbolic surface with associated triangulation T and distinguished edge $\hat{e}$. Let e_1, e_2, e_3, and e_4 bound a quadrilateral in T with diagonal $e_5 \neq \hat{e}$. Let e_6 be the other diagonal of this quadrilateral and T' the triangulation obtained by replacing e_5 with e_6. Assume that $h(e_6 \cup v)$ is homotopically non-trivial. Then we may construct a continuous family of simplicial hyperbolic surfaces with at most two vertices (each having property NLSC) joining h to a useful simplicial hyperbolic surface h' with associated triangulation T' and distinguished edge $\hat{e}$.*

Proof. Let $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4, \tilde{e}_5$, and $\tilde{e}_6$ be the lifts of e_1, e_2, e_3, e_4, e_5, and e_6 to $\tilde{S}$ such that $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$, and $\tilde{e}_4$ form a quadrilateral with diagonals $\tilde{e}_5$ and $\tilde{e}_6$. Let $\tilde{h}: \tilde{S} \to \mathbf{H}^3$ be a lift of h. Let $\tilde{h}(\tilde{e}_6)^*$ denote the geodesic arc joining the endpoints of $\tilde{h}(\tilde{e}_6)$. Notice that $\tilde{h}(\tilde{e}_1)$, $\tilde{h}(\tilde{e}_2)$, $\tilde{h}(\tilde{e}_3)$, $\tilde{h}(\tilde{e}_4)$, $\tilde{h}(\tilde{e}_5)$ and $\tilde{h}(\tilde{e}_6)^*$ form a tetrahedron X in $\mathbf{H}^3$. Let v' denote the intersection of e_5 and e_6 and let $\tilde{v}' = \tilde{e}_5 \cap \tilde{e}_6$. Let R be a geodesic in X joining $\tilde{h}(\tilde{v}')$ to $\tilde{h}(\tilde{e}_6)^*$. We obtain a new triangulation $\bar{T}$ of S by adding e_6 to T (which now has an additional vertex at $v' = e_5 \cap e_6$).

We now drag h along (the projection to N of) R to obtain a continuous family $H: S \times [0, 1] \to N$ of simplicial pre-hyperbolic surfaces with associated triangulation $\bar{T}$. In fact, Lemma 4.3 assures us that H_t is a simplicial hyperbolic surface for all t, since $H_t(v)$ still lies in the image of an edge which is mapped to a closed geodesic and $\tilde{H}_t(\tilde{v}')$ lies in the tetrahedron spanned by the endpoints of X. (These same observations guarantee that each of the vertices has property NLSC.) Notice that H_1 maps e_6 to the geodesic arc $h(e_6)^*$, and we may remove the vertex v' and the edge e_5. Viewed in this manner, H_1 is a useful simplicial hyperbolic surface with associated triangulation T'. □

However, in the presence of accidental parabolics or compressible curves, extra complications may occur. For example, we may attempt to change our distinguished edge to an edge e' such that $e' \cup v$ is an accidental parabolic. In this case we may construct a continuous family of simplicial hyperbolic surfaces in which the length of curve $H_t(e' \cup v)$ converges to 0.

LEMMA 5.4. *Let $h: S \to N$ be a useful simplicial hyperbolic surface with associated triangulation T and distinguished edge $\hat{e}$. Let e' be an edge of T such that $(e' \cup v)$ is an accidental parabolic. Then there exists a continuous family $H: S \times [0, \infty) \to N$ of simplicial hyperbolic surfaces with only one vertex (which satisfies NLSC) such that the length of $H_t(e' \cup v)$ decreases and converges to* 0.

Proof. Let $W: S^1 \times (\mathbf{R}_+ \cup \{0\}) \to N$ be a proper homotopy of $h(e' \cup v)$ into a cusp of N. Let α be a component of the pre-image of $e_6 \cup v$ in $\tilde{S}$ and let $\tilde{v}$ be a lift of v lying on α. Let $\tilde{h}: \tilde{S} \to \mathbf{H}^3$ and $\tilde{W}: \mathbf{R} \times (\mathbf{R}_+ \cup \{0\}) \to \mathbf{H}^3$ be lifts of h and W such that $\tilde{h}(\alpha) = \tilde{W}(\mathbf{R} \times \{0\})$. Let $t \in R$ be such that $\tilde{h}(\tilde{v}) = \tilde{W}(t, 0)$. Let R be a geodesic ray properly homotopic (rel endpoint) to $\tilde{W}(\{t\} \times (\mathbf{R} \cup \{0\}))$. Notice that R has an endpoint at infinity which is a parabolic fixed point. We may then drag h along (the projection to N of) R to produce a continuous family $H: S \times [0, \infty) \to N$ of simplicial pre-hyperbolic surfaces such that the length of $H_t(e' \cup v)$ decreases and converges to 0. We now simply check that the vertex of T satisfies NLSC for all H_t and then apply Lemma 4.2 to see that H is actually a continuous family of simplicial hyperbolic surfaces. □

If S is not incompressible another awkward possibility may occur. When one makes an elementary move, one may attempt to add a new edge e_6 such that $e_6 \cup v$ is compressible. We handle this in a manner similar to Lemma 5.4.

LEMMA 5.5. *Let $h: S \to N$ be a useful hyperbolic surface with associated triangulation T and distinguished edge $\hat{e}$. Let e_1, e_2, e_3, and e_4 form a quadrilateral with diagonal $e_5 \neq \hat{e}$ such that the "other diagonal" e_6 determines a compressible curve $e_6 \cup v$. Then there exists a continuous family $H: S \times [0, r) \to N$ of simplicial hyperbolic surfaces with at most two vertices (each satisfying NLSC) such that the length of $H_t(e_6)$ decreases and converges to 0.*

Proof. We now adapt the notation in the proof of Lemma 5.3. In this case the quadrilateral spanned by $\tilde{h}(\tilde{e}_1)$, $\tilde{h}(\tilde{e}_2)$, $\tilde{h}(\tilde{e}_3)$, and $\tilde{h}(\tilde{e}_4)$ must actually be a triangle spanned by $\tilde{h}(\tilde{e}_1) = \tilde{h}(\tilde{e}_3)$, $\tilde{h}(\tilde{e}_2) = \tilde{h}(\tilde{e}_4)$, and $\tilde{h}(\tilde{e}_5)$. Let R be a geodesic in this triangle joining $\tilde{h}(\tilde{v}')$ to the opposite vertex. If we drag h along R we produce a continuous family $H: S \times [0, r) \to N$ of simplicial hyperbolic surfaces with at most two vertices (each satisfying NLSC) with associated triangulation $\bar{T}$ such that the length of $H_t(e_6 \cup v)$ decreases and converges to 0. $\square$

6. A PROOF OF THE FILLING THEOREM

In this section we will prove the key technical result in the paper. We show that every point in some neighborhood of a simply degenerate end lies in the image of a convenient simplicial hyperbolic surface. The presence of a convenient simplicial hyperbolic surface implies, roughly, that the geometry of the hyperbolic 3-manifold at that point is at least as complicated as that of a hyperbolic surface. For example, if the surface is incompressible, the injectivity radius in the hyperbolic 3-manifold is less than the injectivity radius of the surface at the point and one can find an upper bound for the injectivity radius of a surface depending only on its topological type. Moreover, if we know that all compressible curves on the surface are sufficiently long then we can still obtain an upper bound on the injectivity radius of the 3-manifold purely in terms of the topological type of the surface. (This simple observation will essentially yield a proof of Corollary A.)

THE FILLING THEOREM. *Let N be a hyperbolic 3-manifold and E a simply degenerate end of N_ε^0. Then E has a neighborhood U homeomorphic to $\bar{S} \times [0, \infty)$ whose parabolic extension U_P is homeomorphic to $S \times [0, \infty)$ such that every point in some subneighborhood $\hat{U} = \bar{S} \times [k, \infty) \subset U$ is in the image of some convenient simplicial hyperbolic surface $f_x: S \to U_P$ such that f_x is (properly) homotopic (within U_P) to $S \times \{0\}$. Moreover, given any $A > 0$ we may choose subneighborhood $\hat{U}^A$ of $\hat{U}$ such that if $x \in \hat{U}_P^A$ and γ is any compressible curve or accidental parabolic on S, then $f_x(\gamma)$ has length at least A.*

Proof. Since we already know, by Proposition 4.1, that there exists a sequence of useful simplicial hyperbolic surfaces exiting the end, it would suffice to prove that we can construct a continuous family of convenient simplicial hyperbolic surfaces interpolating between any two useful simplicial hyperbolic surfaces. If S is incompressible and has no accidental parabolics this is always possible; however in general we will not be able to do this. Given two useful simplicial hyperbolic surfaces in U_P we will construct a family beginning at the first which either terminates at the second surface or terminates by meeting the boundary of U_P.

We will say that a simplicial hyperbolic surface $f: S \to U_P$ is *U-nice* if it is properly homotopic to ∂U_P within U_P. Lemma 3.3 allows us to choose a well-cusped neighborhood U of E which is homeomorphic to $\bar{S} \times [0, \infty)$ such that if $f: (S, \tau) \to N$ is a U-nice simplicial hyperbolic surface then every compressible curve and every accidental parabolic has length at least $\mathscr{M}_3$ and no compressible curve or accidental parabolic has image contained entirely within a cusp of N.

LEMMA 6.1. *Let N be a topologically tame hyperbolic 3-manifold. Let F be a geometrically infinite end of N_ε^0 with a neighborhood V which is homeomorphic to $\bar{R} \times [0, \infty)$. Then there exists a compact subset $K(V)$ of V, which may be taken to be of the form $\bar{R} \times [0, k]$, such that every V-nice convenient simplicial hyperbolic surface which intersects ∂V_P lies within $K(V)_P$.*

Proof. Let C be chosen as in the strengthened bounded diameter lemma. Let D be the set of points in V_P which can be reached by a path B originating in $\partial_1 V_P = \partial V_P \cap N_\varepsilon^0$ such that the length of the portion of B which is not contained in any compact component of $N_{thin(\varepsilon)}$ is at most C. Since $\partial_1 V_P$ is compact, we see that D is compact. Thus the strengthened bounded diameter lemma guarantees that if $f:(R, \tau) \to V_P$ is a convenient simplicial hyperbolic surface then $f((R, \tau)_\varepsilon^0) \subset D$. Since D is compact we may find k such that if $K(V) = \bar{R} \times [0, k]$ then $D_P \subset K(V)_P$. Lemma 4.6 then guarantees that $f(R) \subset K(V)_P$. $\square$

Let $g: S \to U_P$ and $h: S \to U_P$ be two U-nice useful simplicial hyperbolic surfaces with associated triangulations T_g and T_h and distinguished edges e_g and e_h. We assume that each puncture of S is the endpoint of only one edge of T_g and only one edge of T_h. If S is not a four-times-punctured sphere, we may construct a finite sequence $(T_0, e_0), \ldots, (T_m, e_m)$ such that T_i is a triangulation of S, e_i is an edge of T_i, $e_i \cup v$ is non-peripheral, and $T_0 = T_g, e_0 = e_g, T_m = T_h$ and $e_m = e_h$.† Moreover, (T_i, e_i) is obtained from (T_{i-1}, e_{i-1}) by either making an elementary move on T_{i-1} which does not affect e_{i-1} to obtain T_i or by keeping T_{i-1} fixed and changing e_{i-1} to e_i. We will call a pair (T_i, e_i) *workable* if $e_i \cup v$ is not an accidental parabolic and $g(e \cup v)$ is homotopically non-trivial whenever e is a compact edge of T_i.

Now suppose that (T_i, e_i) is workable and we have produced a useful simplicial hyperbolic surface g_i with associated triangulation T_i and distinguished edge e_i and that (T_{i+1}, e_{i+1}) is workable. Then we may apply Lemma 5.2 or Lemma 5.3 to obtain a continuous family of simplicial hyperbolic surfaces (whose associated triangulations have at most two vertices, each of which satisfies NLSC) joining g_i to g_{i+1} where g_{i+1} is a useful simplicial hyperbolic surface with associated triangulation T_{i+1} and distinguished edge e_{i+1}.

We now notice that whenever a U-nice simplicial hyperbolic surface has an associated triangulation with at most two vertices, each of which satisfies NLSC, then it is convenient.

If every pair (T_i, e_i) is workable then we may produce a continuous family of simplicial hyperbolic surfaces joining g to g_m where g_m is a useful simplicial hyperbolic surface with associated triangulation T_h and distinguished edge e_h. Moreover, each of the simplicial hyperbolic surfaces in this family has an associated triangulation with at most two vertices (each of which satisfies NLSC). We may then use Lemma 5.1 to produce a continuous family of useful simplicial hyperbolic surfaces joining g_m to h. If each of the simplicial hyperbolic surfaces in the family joining g to h has an image contained entirely within U_P then we have produced a continuous family of U-nice convenient simplicial hyperbolic surfaces joining g to h. If not, we may produce a continuous family of U-nice convenient simplicial hyperbolic surfaces originating with g and terminating in a convenient simplicial hyperbolic surface which intersects ∂U_P. In the second case, Lemma 6.1 guarantees that the terminal simplicial hyperbolic surface has an image contained entirely within $K(U)_P$.

Now suppose that (T_i, e_i) is workable and we have produced a useful simplicial hyperbolic surface g_i with associated triangulation T_i and distinguished edge e_i and that (T_{i+1}, e_{i+1}) is not workable. We may then use Lemma 5.4 or Lemma 5.5 to construct

† In a note added in proof we will complete the proof in the case where S is a four-times-punctured sphere.

a continuous family of simplicial hyperbolic surfaces (whose associated triangulations have at most two vertices, each of which satisfies NLSC) in which the length of either a compressible curve or an accidental parabolic converges to 0. Recalling that on any U-nice simplicial hyperbolic surface compressible curves and accidental parabolics must have length at least $\mathcal{M}_3$, we see that the continuous family must eventually intersect ∂U_P. Thus, if some pair (T_{i+1}, e_{i+1}) is not workable then we may produce a continuous family of U-nice convenient simplicial hyperbolic surfaces originating with g and terminating in a convenient simplicial hyperbolic surface which intersects ∂U_P.

Let M be a manifold with boundary and R a subset of M. We will say that a point $x \in M$ is *enclosed* by R if every properly embedded (infinite) half-ray originating at x intersects R.

If there exists a continuous family of U-nice convenient simplicial hyperbolic surfaces joining g to h, then every point in N_ε^0 enclosed by $(g(S) \cup h(S)) \cap N_\varepsilon^0$ has a U-nice convenient simplicial hyperbolic surface passing through it. Similarly, if there exists a continuous family of U-nice convenient simplicial hyperbolic surfaces originating with g and terminating in a convenient simplicial hyperbolic surface which intersects ∂U_P, then every point in N_ε^0 enclosed by $(\partial K(U)_P \cup g(S)) \cap N_\varepsilon^0$ has a U-nice convenient simplicial hyperbolic surface passing through it.

Recall that we have an infinite sequence $\{f_i : S \to U_P\}$ of U-nice useful simplicial hyperbolic surfaces exiting E. We may assume that each puncture of S is the endpoint of only one edge of the triangulation associated to f_i (for each i). Let X be the subset of U which is enclosed by $(K(U)_P \cup f_1(S)) \cap N_\varepsilon^0$. Notice that X is compact and every point in $U - X$ has a U-nice convenient simplicial hyperbolic surface passing through it. We now simply choose a subneighborhood $\hat{U}$ of U of the form $\bar{S} \times [R, \infty)$ (for some $R \geq 0$) which is contained within $U - X$.

Lemma 3.3 guarantees that we can find a subneighborhood U^A of $\hat{U}$ which is homeomorphic to $S \times [0, \infty)$ such that if $f: (S, \tau) \to N$ is a U-nice simplicial hyperbolic surface with image in U_P^A, then every compressible curve and every accidental parabolic has length at least A. So we may take $\hat{U}^A$ to be any subneighborhood of U^A which does not intersect $K(U^A)$. □

The work of Curt McMullen [24] suggests that typically, even in the absence of parabolics, there is no lower bound on the injectivity radius of a geometrically infinite, topologically tame hyperbolic 3-manifold (see also [6]). However, one easy consequence of the filling theorem is the existence of an upper bound for the injectivity radius of the convex core. Notice that if N has a geometrically finite end then there is no upper bound on the injectivity radius over all of N.

COROLLARY A. *If N is a topologically tame hyperbolic 3-manifold, then there exists K such that $inj_N(x) \leq K$ for all points $x \in C(N)$.*

Proof. Let $E_1, \ldots, E_k$ be the ends of $C(N) \cap N_\varepsilon^0$. Let E_i be homeomorphic to $S_i \times [0, \infty)$ for all i. For each i choose A_i such that for any simplicial hyperbolic structure on S_i every point on S_i has a homotopically non-trivial curve through it of length $\leq 2A_i$. (We may do this since every embedded ball of radius r in a simplicial hyperbolic structure has area greater than a ball of size r in hyperbolic space, and every simplicial hyperbolic structure on S_i has area at most $2\pi|\chi(S_i)|$.) The filling theorem guarantees that there exists a neighborhood U^i which is homeomorphic to $\bar{S}_i \times [0, \infty)$ of E_i and a subneighborhood $\hat{U}_i$ such that if x is any point in $\hat{U}_i$ then it is in the image of a U-nice convenient simplicial hyperbolic surface $f_x : S_i \to U_P^i$ such that every compressible curve has length at least $2A_i + 1$. There exists a homotopically non-trivial curve β through (any point in) $f^{-1}(x)$ of length at most

$2A_i$ which is therefore not compressible. Thus, $inj_N(x) \le A_i$. Let W denote $C(N) \cap N_\varepsilon^0 - \bigcup_i \hat{U}_i$. Since W is compact there exists A_W such that $inj_N(x) \le A_W$ for all $x \in W$. Therefore, one may take $K = \max\{A_1, \ldots, A_k, A_W, \varepsilon\}$. □

It would be nice to obtain a uniform upper bound on the injectivity radius of all points in the convex core which depended only on the topology of N. Thurston has obtained such a result in the case that N is homotopy equivalent to a surface. We will sketch how one may use the technique of proof of the filling theorem to prove such a result.

THEOREM 6.2. (Thurston [34, 21] and Bonahon [5]). *Let Θ be a cofinite area, torsion-free Fuchsian group. Then there exists a constant $K > 0$ such that if Γ is a Kleinian group such that there exists a type-preserving isomorphism between Θ and Γ, then $inj_N(x) \le K$ for all $x \in C(N)$ where $N = \mathbf{H}^3/\Gamma$.*

Proof. Let $S = \mathbf{H}^3/\Theta$. The assumptions, together with the work of Bonahon [5], guarantee that N is isomorphic to $S \times \mathbf{R}$ and that $C(N)$ is isomorphic to either $S \times \mathbf{R}$, $S \times [0, \infty)$ or $S \times [0, 1]$. We will only explain the proof in the case that N is homeomorphic to $S \times [0, \infty)$ since it contains all the essential details used in the proof of both of the other cases.

Assume that $C(N)$ is homeomorphic to $S \times [0, \infty)$. Let $f: S \to N$ be the pleated surface whose image is the boundary of $C(N)$. Lemmas 6.1 and 6.2 of [11] guarantee that there exists a useful simplicial hyperbolic surface $g: S \to N$ and a homotopy $H: S \times I \to N$ such that $H(x, 0) = f(x)$, $H(x, 1) = g(x)$ and $H(x, \cdot): I \to N$ is a path of length less than A for all $x \in S$ (where A is a universal constant depending only on the topological type of S). Moreover, there exists a universal constant B, again depending only on the topological type of S, such that every point in S and hence every point in $g(S)$ has injectivity radius at most B.

Let $y \in C(N)$. If $y \in H(S \times I)$, then there exists a point $z \in g(S)$ such that $d(y, z) \le A$. Hence, since $inj_N(z) \le B$, we see that $inj_N(y) \le A + B$. If y does not lie in $H(S \times I)$, then the work of Bonahon [5] guarantees that there exists a useful simplicial hyperbolic surface $h: S \to N$ such that y lies homologically between $g(S)$ and $h(S)$. One may then repeatedly apply Lemmas 5.2 and 5.3 to obtain a continuous family of simplicial hyperbolic surfaces $J: S \times I \to N$ joining g to h. (Notice that given our assumptions there are no accidental parabolics or compressible curves.) Hence, there exists $t \in I$ such that y lies in the image of the simplicial hyperbolic surface $H(\cdot, t)$. Thus, $inj_N(y) \le B$. Hence $inj_N(y) \le A + B$ for all points $y \in C(N)$. The proof in the other cases is similar. □

However if N is homeomorphic to a handlebody there is no uniform upper bound on the injectivity radius of points in the convex core. One may see this by examining the sequence of Schottky groups generated by two hyperbolic isometries with fixed translation distance whose axes are farther and farther apart. However, Curt McMullen (see [4]) has proposed the following conjecture.

CONJECTURE. *Given a compact 3-manifold M there exists a constant K such that if N is a hyperbolic 3-manifold homotopy equivalent to M, then the convex core of N does not contain an embedded ball of radius K.*

A positive solution to the above conjecture could be useful in the understanding of quasiconformal deformation theory of Kleinian groups and in the spectral theory of hyperbolic 3-manifolds.

Remarks: (1) Another, perhaps conceptually simpler, way to prove the filling theorem is to first construct a Riemannian 3-manifold with pinched negative curvature and hyperbolic cusps whose compact core is acylindrical which has an end isometric to E. (In [8] we construct such a Riemannian 3-manifold with freely indecomposable fundamental group. The arguments in [7] may be modified to produce such a Riemannian 3-manifold with acylindrical compact core.) One may then construct a continuous family of convenient simplicial ruled surfaces joining any two useful simplicial ruled surfaces. (Thus we avoid some of the unpleasantries in the construction which are caused by accidental parabolics and compressible curves.) If one is sufficiently far out the end, then these convenient simplicial ruled surfaces are associated to convenient simplicial hyperbolic surfaces in E.

(2) The filling theorem is the analogue for simplicial hyperbolic surfaces of the key technical lemma in the proof of Theorem 9.5.13 in Thurston's lecture notes [34]. In Section 9.5 Thurston shows how to obtain a continuous family of "hyperbolic" surfaces interpolating between any two (incompressible) pleated surfaces, pleated along essentially complete laminations. This allows him to find a continuous family of "hyperbolic" surfaces filling up a simply degenerate end. We make use of simplicial hyperbolic surfaces as they are easier to manipulate and our proof of the filling theorem is correspondingly simpler than Thurston's proof of Theorem 9.5.13. One may also prove an analogue of the filling theorem using harmonic maps (see [26]). In both cases the technology to prove such theorems is only fully developed when N satisfies Bonahon's condition (B). However, it seems likely that in both cases the techniques could be developed to produce such generalizations. For pleated surfaces, the work of Otal [29] seems likely to lead to such a proof.

Notice that the ability to interpolate between any two pleated surfaces makes the proof of Theorem 6.2 nearly immediate, and yields a better constant.

(3) Theorem 6.2 remains true if the isomorphism only weakly preserves parabolicity; however there are a few additional technical details in the proof.

7. A PROOF OF THE COVERING THEOREM

THE COVERING THEOREM. *Let $\hat{N}$ be a topologically tame hyperbolic 3-manifold which covers ($p: \hat{N} \to N$) another hyperbolic 3-manifold N by a local isometry. If $\hat{E}$ is a geometrically infinite end of $\widehat{N_\varepsilon^0}$*, then either

(a) *$\hat{E}$ has a neighborhood $\hat{U}$ such that p is finite-to-one on $\hat{U}$*, or

(b) *N has finite volume and has a finite cover N' which fibers over the circle such that if N_S denotes the cover of N' associated to the fiber subgroup then $\hat{N}$ if finitely covered by N_S. Moreover, if $\hat{N} \neq N_S$, then $\hat{N}$ is homeomorphic to the interior of a twisted I-bundle which is doubly covered by N_S.*

Proof. First choose a neighborhood U of $\hat{E}$ as in the filling theorem such that through every point in some subneighborhood $\hat{U}$ there passes a convenient simplicial hyperbolic surface $\hat{f}: (S, \tau) \to U_P$ (properly homotopic to ∂U_P). Suppose that $p|_{\hat{U}}$ is not finite-to-one. Then there exists $x \in N$ such that $p^{-1}(x) \cap \hat{U}$ contains infinitely many points. We may assume, without loss of generality, that $inj_N(x) \geq \mathcal{M}_3$. Label the points in $p^{-1}(x) \cap \hat{U}$ by $\{x_1, \ldots, x_n, \ldots\}$. We see immediately that $inj_{\hat{N}}(x_i) \geq \mathcal{M}_3$ for all i.

For the remainder of the proof we will assume a fixed identification of Γ with $\pi_1(N, x)$ and $\hat{\Gamma}$ with $\pi_1(\hat{N}, x_1)$.

Let $\hat{f}_i = \hat{f}_{x_i}: (S, \tau_i) \to \hat{N}$ be the convenient simplicial hyperbolic surfaces (properly homotopic to ∂U_P) given by the filling theorem. These descend to simplicial hyperbolic surfaces $f_i = p \circ \hat{f}_i: (S, \tau_i) \to N$ passing through x, such that every accidental parabolic and compressible curve on $f_i(S)$ has length $\geq \mathcal{M}_3$. Let C be as in the proof of the strengthened bounded diameter lemma. Let K be the set of points in N which can be joined to x by paths B such that the length of the portion of B which is not contained in any compact component of $N_{thin(\varepsilon)}$ is less than C. K is a compact set and $f_i((S, \tau_i)^0_\varepsilon) \subset K$. Therefore there exists $\delta > 0$ such that every point in $(S, \tau_i)^0_\varepsilon$ has injectivity radius greater than δ for all i.

We will call a system of generators $\{g_1, \dots, g_n\}$ for $\pi_1(S, *)$ *minimal* if they have curve representatives $\{\alpha_1, \dots, \alpha_n\}$ which are disjoint (except at $*$) and such that $S - \{\alpha_1, \dots, \alpha_n\}$ is the union of a single connected polygon and a finite number of punctured monogons. We will say that a minimal generating system $\{g_1, \dots, g_n\}$ for $\pi_1(S, x)$ and a minimal generating system $\{g'_1, \dots, g'_n\}$ for $\pi_1(S, x')$ are *topologically equivalent* if there exist a homeomorphism $F:(S, x) \to (S, x')$ such that $F_*(g_i) = (g'_i)$ for all i. Notice that every system of minimal generators for $\pi_1(S)$ has the same number of elements. Moreover, there are only finitely many topological equivalence classes of minimal generators for $\pi_1(S)$.

It will be useful to notice that if there is a lower bound on the injectivity radius of a convenient simplicial hyperbolic surface then one can find a minimal generating system all of whose curves have bounded length. Lemma 7.1 holds in much more generality than is claimed below.

LEMMA 7.1. *Let $\hat{f}:(S, \tau) \to \hat{N}$ be a convenient simplicial hyperbolic surface such that every point in $(S, \tau)^0_\varepsilon$ has injectivity radius $\geq \delta$. Then if $* \in (S, \tau)^0_\varepsilon$ there exists a minimal set of generators $\{g_1, \dots, g_n\}$ for $\pi_1(S, *)$ all of whom have curves representative with length $\leq C$, where C depends only on δ and the topological type of S.*

Proof. Let $\varepsilon = \min\{\delta, \mathcal{M}_3/2\}$. Let $S_0 = (S, \tau)^0_\varepsilon$. Lemma 4.5 guarantees that $S - S_0$ consists of finitely many punctured monogons. Let $\{\beta_1, \dots, \beta_p\}$ denote the components of the boundary of $(S, \tau)^0_\varepsilon$. We may construct a pair of pants decomposition of S_0 such that the length of each curve is bounded by a constant depending only on ε and the topological type of S (as in Lemma 3 of Section II.3.3 in [1]). If S_0 has positive genus, let β_{p+1} be a non-separating curve in this pair of pants decomposition. Let S_1 be the metric completion of $S_0 - \beta_{p+1}$. We may similarly find a non-separating curve β_{p+2} on S_1. We continue this process until S_q has genus 0.

Since every point in S_q has injectivity radius at least δ and each boundary component has length bounded by a constant depending only on ε and the topological type of S, we can easily conclude that there exists a bound, depending only on δ and the topological type of S, on the diameter of S_q. Hence there exists a disjoint collection of simple arcs $\{\mu_1, \dots, \mu_{p+q}\}$ of bounded length such that μ_i joins $*$ to β_i. Let α_i be a simple closed curve passing through $*$ formed by slightly perturbing $\mu_i \beta_i \bar{\mu}_i$. We may also assume that the α_i are mutually disjoint, except at $*$.

Let T_0 denote the completion of $S - \{\alpha_1, \dots, \alpha_{p+q}\}$. If T_0 has only one boundary component then we are finished and $\{\alpha_1, \dots, \alpha_{p+q}\}$ is the desired minimal generating system of bounded length. If not, we first notice again that T_0 has diameter bounded by a constant depending only on ε and the topological type of S. Let $*^0_1$ and $*^0_2$ be points on different components of ∂T_0 which both correspond to $*$. Then there exists an arc μ_{p+q+1} joining $*^0_1$ and $*^0_2$ of bounded length. Let α_{p+q+1} be the image of this arc on S. Let T_1 be the completion of $T_1 - \mu_{p+q+1}$. If T_1 has only one boundary component then $\{\alpha_1, \dots, \alpha_{p+q+1}\}$ is the desired minimal generating system of uniformly bounded length. If not, we continue the process until we have found our minimal generating system. (Notice that the number of steps taken by this process depends only on the topological type of S.) $\square$

Let y_i be a point in $f_i^{-1}(x_i)$ and let $\{g^i_1, \dots, g^i_n\}$ be a minimal generating system for $\pi_1(S, y_i)$ with disjoint representatives $\{\alpha^i_1, \dots, \alpha^i_n\}$ having length (in τ_i) at most C. We may pass to a subsequence such that for all i and j there exists an isomorphism $F_{ij}: \pi_1(S, y_i) \to \pi_1(S, y_j)$ given by $F_{ij}(g_k^i) = g_k^j$ for all $1 \leq k \leq n$. In particular, $g \in \pi_1(S, y_i)$ is peripheral or trivial if and only if $F_{ij}(g)$ is peripheral or trivial.

LEMMA 7.2. *Given A, there exists I such that if $i \geq I$, then every compressible curve and accidental parabolic on (S, τ_i) has length at least A.*

Proof. Let $\hat{U}^A$ be the neighborhood of $\hat{E}$ given by the filling theorem. Recall that if $x \in U^A$, then $f_x: (S, \tau) \to U_P$ is a convenient simplicial hyperbolic surface which is properly homotopic to ∂U_P and every accidental parabolic and compressible curve has length at least A. Since the sequence $\{x_i\}$ leaves every compact subset of U, there exists I such that if $i \geq I$, then $x_i \in \hat{U}^A$. $\square$

Choose I so that if $i \geq I$ then every accidental parabolic and compressible curve on (S, τ_i) has length at least $C + 1$. Then the elements $\{(f_i)_*(g_1^i), \ldots, (f_i)_*(g_n^i)\}$ generate $f_i^*(\pi_1(S, y_i)) \subset \pi_1(N, x)$ and have representatives $\{f_i(\alpha_1), \ldots, f_i(\alpha_n)\}$ all of length at most C and all of which are homotopically non-trivial. There are only finitely many homotopy classes of curves passing through x of length at most C. Thus there are only finitely many possibilities for the set $\{(f_i)_*(g_1^i), \ldots, (f_i)_*(g_n^i)\}$.

Suppose that S has compressible curves or accidental parabolics. Then each group generated by $\{(f_i)_*(g_1^i), \ldots, (f_i)_*(g_n^i)\}$ has compressible words or accidental parabolics, i.e. words in $\{(f_i)_*(g_1^i), \ldots, (f_i)_*(g_n^i)\}$ which are either trivial or associated to parabolic elements of Γ such that the associated words in $\{g_1^i, \ldots, g_n^i\}$ are not either trivial or associated to peripheral elements of $\pi_1(S)$. As only finitely many such different sets arise, there exists R such that each set has a compressible word or accidental parabolic of word length at most R. Thus (S, τ_i) has an accidental parabolic or compressible curve of length at most CR for all i. However, this contradicts Lemma 7.2.

We may now assume that S contains no accidental parabolics or compressible curves. We are thus in the case of Thurston's original theorem and we could simply refer to his result. However, we will give a proof in our framework, both for consistency's sake and because there is not a complete, self-contained proof of Thurston's original theorem in print.

Since there are only finitely any possibilities for $\{(f_i)_*(g_1^i), \ldots, (f_i)_*(g_n^i)\}$, we may pass to a subsequence such that $(f_i)_*(g_k^i) = (f_j)_*(g_k^j)$ for all i, j and k. Let $G = (f_i)_*(\pi_1(S, y_i)) \subset \pi_1(N, x)$. Then we get a sequence of isomorphisms $(f_i)_*: \pi_1(S, y_i) \to G$. Let $T_{i(i+1)}$ be an arc in $\hat{N}$ joining x_i to x_{i+1}. Construct T_{ij} by concatenating $T_{k(k+1)}$ for all $i \leq k \leq j-1$. Let h_{ij} be the homotopy class of $p(T_{ij})$ in $\pi_1(N, x)$. For all i and j we get an automorphism $H_{ij}: G \to G$ given by taking g to $h_{ij} g h_{ij}^{-1}$. Since T_{ij} is not closed we see that h_{ij} does not lie in G. Moreover, H_{ij} is not an inner automorphism, since otherwise there exists $\beta \in G$, such that if $g \in G$ then βh_{ij} commutes with g. If g is any hyperbolic element of G this implies that βh_{ij} and g have the same axis. However, G contains hyperbolic elements with distinct axes, so we have obtained a contradiction.

We now quote a result from Canary *et al.* [10] which played a role in Thurston's original proof.

LEMMA 7.3. *Let T be a finite area hyperbolic surface. Let $g: T \to T$ be a homeomorphism which induces a non-trivial automorphism $g_*: \pi_1(T) \to \pi_1(T)$. If there exists n such that g_*^n is an inner automorphism, then $g_*: H_1(T; \mathbf{Z}_3) \to H_1(T; \mathbf{Z}_3)$ is non-trivial.*

We then observe that if $i < j < k$, then $H_{ik} = H_{jk} \circ H_{ij}$. Therefore we can find j and k such that $(H_{jk})_{\#}$ is trivial and thus that no finite power of H_{jk} is an inner automorphism. Let $\phi : S \to S$ be a homeomorphism of S such that $\phi_* = H_{jk}$. Let M_ϕ be the mapping torus of ϕ and let Γ' be the subgroup of Γ generated by G and h_{jk}. $\pi_1(M_\phi)$ is generated by $\pi_1(S)$ and an element α with the property that $\alpha \circ g \circ \alpha^{-1} = \phi_*(\alpha)$. We then define a homomorphism $F_* : \pi_1(M_\phi) \cong \pi_1(S) *_z \to \Gamma' = \langle G, h_{jk} \rangle$ by taking $\pi_1(S)$ to G (using the identification given by $(f_j)_*$) and $\mathbf{Z} = \langle \alpha \rangle$ to the subgroup generated by h_{jk}. Suppose that F_* is not an isomorphism. Then there exists an element of the form $\gamma\alpha^n$ in the kernel of F_* where $\gamma \in \pi_1(S)$. But this implies that $F_*(\alpha)^n \in \pi_1(S)$, which implies that H_{jk} has a finite power which is an inner automorphism. This contradiction establishes that F_* is an isomorphism and thus is associated to a homotopy equivalence F. We now only need to show that F is homotopic to a homeomorphism to complete the proof.

If S is a closed surface then M_ϕ is closed. Let $N' = \mathbf{H}^3/\Gamma'$. Since N' is a 3-manifold without boundary and is homotopy equivalent to a closed 3-manifold, we see, by simply looking at the homology, that N' is itself closed. Waldhausen's theorem (see [36] or [18]) then applies directly to show that F is homotopic to a homeomorphism. Since N' is closed we see immediately that N' is a finite cover of N.

If S is not compact, each end of M_ϕ has a fibered neighborhood which is homeomorphic to $T^2 \times (0, \infty)$. We obtain M' by removing such a neighborhood of each of the ends of M_ϕ. We may regard F as a homotopy equivalence from the compact manifold M' to $(N')^0_\varepsilon$. Every boundary component T of M' is an incompressible torus. Thus $F_*(\pi_1(T))$ is a free-abelian subgroup of $\pi_1(N')$ which implies that $F(T)$ is homotopic into a rank-two cusp of N and hence into a boundary component of $(N')^0_\varepsilon$. Moreover, since F_* is an isomorphism and the subgroups associated to different boundary components of M' are not conjugate, we see that F may be taken to be a degree 1 homeomorphism on the boundary. Again a homology argument (as in Theorem 5.2.18 in [10]) guarantees that $(N')^0_\varepsilon$ is compact and that each boundary component of $(N')^0_\varepsilon$ is in the image of F. (One may also double across the image of the boundary of M_ϕ and across $F(\partial M_\phi)$ in N' and apply the original argument.) Then we may apply Waldhausen's theorem again to show that F is homotopic to a homeomorphism. Since N' has finite volume we see that it is a finite cover of N.

It now only remains to show that the fiber subgroup G of Γ' has finite index in $\hat{\Gamma}$. We see immediately that

$$G \subset (\hat{\Gamma} \cap \Gamma') \subset \Gamma.$$

Suppose that there exists an element $g \in (\hat{\Gamma} \cap \Gamma') - G$. Every element in Γ' has the form $\gamma(h_{jk})^n$ where $\gamma \in G$. So $g = \gamma(h_{jk})^n$ where $n \neq 0$, which implies that $(h_{jk})^n \in \hat{\Gamma}$. In this case, $\hat{\Gamma} \cap \Gamma'$ would be a finite index subgroup of Γ', which would imply that $\hat{N}$ has finite volume, giving a contradiction. Therefore, $\hat{\Gamma} \cap \Gamma' = G$ which implies, since Γ' has finite index in Γ, that G has finite index in $\hat{\Gamma}$. Therefore, by Theorem 10.5 in [18], we see that G has index at most two in $\hat{\Gamma}$ and that $\hat{\Gamma}$ is itself a surface group. Furthermore, if $\hat{\Gamma} \neq G$, then $\hat{N}$ is homeomorphic to the interior of a twisted I-bundle over a surface. □

Remark. Thurston's original proof of his covering theorem does not use the full power of the filling theorem. We chose to develop the extra structure in the filling theorem because we believe it clarifies the proof of the covering theorem as well as being a powerful tool in its own right. Corollary A is a first application of the extra power of the filling theorem.

8. SUBGROUPS OF TOPOLOGICALLY TAME KLEINIAN GROUPS

Our covering theorem allows us to understand exactly which covers of a topologically tame, infinite volume, hyperbolic 3-manifold N are geometrically finite. The key observation is that all the covers of N are themselves topologically tame.

PROPOSITION 8.1 (Proposition 3.2 in [8]). *If N is an infinite volume, topologically tame hyperbolic 3-manifold and $\hat{N}$ is a cover of N with finitely generated fundamental group, then $\hat{N}$ is also topologically tame.*

The following characterization is an almost immediate consequence of our covering theorem.

COROLLARY B. *Let $N = \mathbf{H}^3/\Gamma$ be an infinite volume topologically tame hyperbolic 3-manifold. Then if $\hat{\Gamma}$ is a finitely generated subgroup of Γ either*

(a) *$\hat{\Gamma}$ is geometrically finite, or*

(b) *$\widehat{N_\varepsilon^0}$ has a geometrically infinite end $\hat{E}$ such that $p: \hat{N} \to N$ is finite-to-one on some neighborhood $\hat{U}$ of $\hat{E}$.*

Proof. Let $\hat{N} = \mathbf{H}^3/\hat{\Gamma}$. Suppose $\hat{N}$ is not geometrically finite. Then, by Proposition 8.1, $\widehat{N_\varepsilon^0}$ is topologically tame and has a geometrically infinite end $\hat{E}$. Since N does not have finite volume, the covering theorem implies that $p: \hat{N} \to N$ is finite-to-one on some neighborhood $\hat{U}$ of $\hat{E}$. □

Corollary C gives a more group-theoretic characterization of which subgroups are geometrically finite. Notice that there are finitely many conjugacy classes of geometrically infinite peripheral subgroups of N, one conjugacy class for each simply degenerate end of N_ε^0.

COROLLARY C. *Let $N = \mathbf{H}^3/\Gamma$ be an infinite volume, topologically tame hyperbolic 3-manifold. Then if $\hat{\Gamma}$ is a finitely generated subgroup of Γ either*

(a) *$\hat{\Gamma}$ is geometrically finite, or*

(b) *$\hat{\Gamma}$ contains a (conjugate of a) finite index subgroup of a geometrically infinite peripheral subgroup.*

Proof. Suppose that $\hat{\Gamma}$ is not geometrically finite. Then there exists a geometrically infinite end $\hat{E}$ of $\widehat{N_\varepsilon^0}$. The covering theorem guarantees that p is finite-to-one on some neighborhood of $\hat{E}$. Let $\hat{U}$ be a neighborhood of $\hat{E}$ which is homeomorphic to $\bar{S} \times [0, \infty)$. Let A be chosen such that in every simplicial hyperbolic structure τ on S there exists a curve of length $\leq A$ through every point on S. We may assume, by the filling theorem, that every point in $\hat{U}$ has a convenient simplicial hyperbolic surface passing through it such that every compressible curve or accidental parabolic has length at least $A + 1$. Therefore, if $x \in \hat{U}$, there exists a homotopically non-trivial curve through x of length $\leq A$ which is not homotopic into a cusp. Thus, the same holds for $p(x)$ which implies that there exists ε' such that $p(x) \subset N_{\varepsilon'}^0$. Therefore, $p(\hat{U}) \subset N_{\varepsilon'}^0$. So we may assume, by passing to a subneighborhood of $\hat{U}$ if necessary, that there exists an end E of $N_{\varepsilon'}^0$ with a neighborhood U homeomorphic to $\bar{T} \times [0, \infty)$ (for some compact surface $\bar{T}$) such that $p(\hat{U}) \subset U$. Moreover, $p(\hat{U}_P) \subset U_P$. Since $\hat{U}$ contains a sequence of closed geodesics exiting every compact subset, so does U. Therefore E is simply degenerate. Now since p is a local isometry, the boundary of $p(\hat{U}_P \cap \widehat{N^0_{\varepsilon'}})$ is contained in $p(\partial(\hat{U}_P \cap \widehat{N^0_{\varepsilon'}}))$. Notice that $\partial_1 \hat{U} = \partial \hat{U}_P \cap \widehat{N^0_{\varepsilon'}}$ is compact and $p(\partial(\hat{U}_P \cap \widehat{N^0_{\varepsilon'}}) - \partial_1 U_P)) \subset N - N^0_\varepsilon$. Therefore, since $p|_{\hat{U}}$ is proper, we can find a neighborhood U' of E such that $\hat{U}_P$ maps onto U'.

Let $\hat{i}_*(\pi_1(\hat{U}))$ denote the image of $\pi_1(\hat{U})$ in $\pi_1(\hat{N})$ and $i_*(\pi_1(U))$ denote the image of $\pi_1(U)$ in $\pi_1(N)$. (In this discussion one should assume that the basepoint $*_N$ has been chosen in U' and that $*_{\hat{N}} \in \hat{U} \cap p^{-1}(*_N)$.) So we see that $p_*(\hat{i}_*(\pi_1(\hat{U})))$ is contained in $i_*(\pi_1(U))$ and that $i_*(\pi_1(U))$ is a geometrically infinite peripheral subgroup. It only remains to show that $p_*(\hat{i}_*(\pi_1(\hat{U})))$ has finite index in $i_*(\pi_1(U))$. First notice that if γ is any element of $i_*(\pi_1(U))$ it has some representative which lies entirely within U' and does not intersect $p(\partial \hat{U}_P)$. Therefore, since p is a covering map which is finite-to-one from $\hat{U}_P$ onto U', it has some closed pre-image in $\hat{U}_P$. Group-theoretically this means that there exists $n_\gamma \neq 0$ such that $\gamma^{n_\gamma} \in p_*(\hat{i}_*(\pi_1(\hat{U})))$. The proof is then finished by the following group-theoretic fact whose hyperbolic proof was provided by Jim Anderson.

PROPOSITION 8.2 (Anderson). *Let M be a compact 3-manifold with a non-toroidal boundary component such that the interior of M admits a hyperbolic structure and let G be a finitely generated subgroup of $\pi_1(M)$. If given any element $g \in \pi_1(M)$, there exists $n_g \neq 0$ such that $g^{n_g} \in G$, then G has finite index in $\pi_1(M)$.*

Proof. By Thurston's geometrization theorem (see [27]) we may assume that there exists a geometrically finite hyperbolic 3-manifold $N = \mathbf{H}^3/\Gamma$ homeomorphic to the interior of M, whose only cusps are torus cusps. Since the proposition clearly holds for $\mathbf{Z}$ and $\mathbf{Z} \oplus \mathbf{Z}$ we may assume that Γ is non-elementary. Let Γ_1 be the subgroup of Γ corresponding to G. Let x be a fixed point of a hyperbolic element γ in Γ. Since $\gamma^{n_\gamma} \in \Gamma_1$ for some $n_\gamma > 0$ we see that $x \in L_{\Gamma_1}$. Since fixed points of hyperbolic elements are dense in the limit sets of non-elementary Kleinian groups, we see that $L_{\Gamma_1} = L_\Gamma$. Since M has a non-toroidal boundary component, $D_\Gamma = S^2_\infty - L_\Gamma$ is non-empty. Thus D_{Γ_1}/Γ_1 covers D_Γ/Γ. But Ahlfors' finiteness theorem [2] implies that D_{Γ_1}/Γ_1 has finite area, so that the cover must be finite-to-one. Therefore, since Γ and Γ_1 both act effectively as groups of isometries of D_Γ, Γ_1 is a finite index subgroup of Γ; hence the proposition follows. This also concludes the proof of Corollary C. □

One especially nice case is when one has a finitely generated free purely hyperbolic subgroup. In that case the subgroup must be geometrically finite (unless it has finite index). A Kleinian group Γ is called a *Schottky group* if and only if (the ε-neighborhood of) the convex core is a handlebody. Geometrically finite, free, purely hyperbolic groups are Schottky groups as the ε-neighborhood of their convex core is an irreducible compact 3-manifold with a free fundamental group and hence a handlebody (see [18]). See Maskit's book [25] for a discussion of Schottky groups.

COROLLARY D. *Let $N = \mathbf{H}^3/\Gamma$ be a topologically tame hyperbolic 3-manifold with at least one end. Then if $\hat{\Gamma}$ is a finitely generated, infinite index, free subgroup of Γ without parabolics it is geometrically finite and hence it is a Schottky group.*

Proof. Since $\pi_1(\hat{N})$ is a finitely generated free group and $\hat{N}$ is topologically tame, $\hat{N}$ is homeomorphic to the interior of a handlebody. Since $\hat{\Gamma}$ contains no parabolics, $\widehat{N^0_\varepsilon} = \hat{N}$ has only one end $\hat{E}$. Since $p : \hat{N} \to N$ is infinite-to-one, the covering must be infinite-to-one on $\hat{E}$. Since N has infinite volume, the covering theorem tells us that $\hat{E}$ and hence $\hat{N}$ is geometrically finite. □

REMARK. The above argument also shows that if Γ is topologically tame and $\hat{\Gamma}$ is any finitely generated, infinite index, purely hyperbolic subgroup whose compact core has only one end, then $\hat{\Gamma}$ is geometrically finite.

We now briefly discuss the situation for covers of finite-volume hyperbolic 3-manifolds. Here the situation is hampered by the fact that covers with finitely generated fundamental group are not known to be topologically tame. (It is a conjecture of Simon [32] that they all are topologically tame.) However, we can still produce a characterization of which topologically tame covers are geometrically finite.

Let $N = \mathbf{H}^3/\Gamma$ be a finite volume hyperbolic 3-manifold. A subgroup $\hat{\Gamma}$ of Γ is called a *virtual fiber subgroup* if there exists a finite cover N' of N which fibers over the circle and $\hat{\Gamma}$ is the subgroup of $\pi_1(N')$ associated to the fiber (see [33] for a discussion of virtual fiber subgroups). A subgroup $\hat{\Gamma}$ of Γ is called topologically tame if $\hat{N} = \mathbf{H}^3/\hat{\Gamma}$ is topologically tame. The following corollary is an immediate consequence of the covering theorem.

COROLLARY 8.3. *Let $N = \mathbf{H}^3/\Gamma$ be a finite volume hyperbolic 3-manifold. Let $\hat{\Gamma}$ be a topologically tame subgroup of Γ. Then $\hat{\Gamma}$ is geometrically finite if and only if it does not contain a virtual fiber subgroup with index at most two.*

The easiest situation in which one can apply Corollary 8.3 is when the subgroup satisfies Bonahon's condition (B). The most commonly used application of this type is that any cover of a closed hyperbolic 3-manifold which is homotopy equivalent to a closed surface is either quasi-Fuchsian or a virtual fiber (see for example [13]). This result can be obtained directly from the works of Bonahon and Thurston.

There are however some other instances where one knows that a subgroup is topologically tame. One may generalize the arguments in the proof of Theorem 10.2 in [14] to prove that all covers of finite-volume hyperbolic 3-manifold whose associated subgroups lie in the kernel of a surjection to $\mathbf{Z}$ are topologically tame. We will mimic their arguments and point out the occasional differences which are caused by the presence of parabolics.

PROPOSITION 8.4. *Let $N = \mathbf{H}^3/\Gamma$ be a finite-volume hyperbolic 3-manifold. Let $\hat{\Gamma}$ be a finitely generated subgroup of Γ. If there exists an epimorphism $H:\Gamma \to \mathbf{Z}$ such that $\hat{\Gamma}$ lies entirely in the kernel, then $\hat{\Gamma}$ is topologically tame. In particular, if the abelianization of $\hat{\Gamma}$ has infinite index in $H_1(N)$, then $\hat{\Gamma}$ is topologically tame.*

Proof. Let $M = N_\varepsilon^0$. Then M is a compact 3-manifold with torus boundary. It follows from standard arguments in 3-manifold topology (see [18]) that H is induced by a map $f: M \to S^1$ such that $f^{-1}(0)$ is a (possibly disconnected) properly embedded incompressible surface. We may extend the map $f: M \to S^1$ to $\bar{f}: N \to S^1$ simply by taking any point x in a torus cusp of N to the point $f(y)$ where y is the point in ∂N_ε^0 nearest to x. Notice that $\bar{f}^{-1}(0)$ is a properly embedded incompressible surface in N.

Let Γ_1 be the kernel of H, $N_1 = \mathbf{H}^3/\Gamma_1$, and $p_1: N_1 \to N$ be the covering map. $\bar{f}$ lifts to a map $\bar{f}_1: N_1 \to \mathbf{R}$. Let X be a finite 2-complex such that $\pi_1(X)$ is isomorphic to $\hat{\Gamma}$. Let $\phi: X \to M$ be constructed such that $\phi_*(\pi_1(X)) = \hat{\Gamma}$. Then ϕ lifts to a map $\tilde{\phi}: X \to N_1$. There exists an integer D such that $\tilde{\phi}(X) \subset \bar{f}_1^{-1}([-D, D])$. Let $K_1 = \tilde{f}^{-1}([-D, D]) \cap p_1^{-1}(M)$ and let $\Gamma_2 = \pi_1(K_1) \subset \Gamma_1$. ∂K_1 lies in $p_1^{-1}(f^{-1}(0)) \cup p_1^{-1}(\partial M)$. Each component of $\partial K_1 \cap p_1^{-1}(f^{-1}(0))$ is incompressible and each component of $\partial K_1 \cap p_1^{-1}(\partial M)$ is parabolic. Therefore, Γ_2 satisfies Bonahon's condition (B). Hence, by Bonahon's theorem [5],

$N_2 = \mathbf{H}^3/\Gamma_2$ is topologically tame. Therefore, since $\hat{\Gamma} \subset \Gamma_2$, Proposition 8.1 implies that $\hat{\Gamma}$ is topologically tame. □

One may combine Corollary 8.3 and Proposition 8.4 to prove Corollary E.

COROLLARY E. *Let $N = \mathbf{H}^3/\Gamma$ be a finite-volume hyperbolic 3-manifold. Let $\hat{\Gamma}$ be a finitely generated subgroup of Γ such that there exists an epimorphism $H: \Gamma \to \mathbf{Z}$ such that $\hat{\Gamma}$ lies entirely in the kernel. Then $\hat{\Gamma}$ is geometrically finite if and only if it does not contain a virtual fiber subgroup of index at most two.*

9. ALGEBRAIC AND GEOMETRIC LIMITS

We will say that a sequence $\{\rho_i: \pi_1(M) \to Isom_+(\mathbf{H}^3)\}$ of discrete faithful representations *converges algebraically* to a representation $\rho_\infty: \pi_1(M) \to Isom_+(\mathbf{H}^3)$ if for all $g \in \pi_1(M)$, $\rho_i(g)$ converges to $\rho_\infty(g)$. Jorgensen's inequality [19] then implies that ρ_∞ is itself a discrete faithful representation. We say that a sequence $\{\Gamma_i\}$ of discrete subgroups of $Isom_+(\mathbf{H}^3)$ *converges geometrically* to a discrete subgroup Γ it if converges in the Chabauty topology on closed subsets of $Isom_+(\mathbf{H}^3)$ (see [10] or [20]). Explicitly, Γ_i converges Γ if and only if

(i) if there exists a subsequence Γ_{n_j} of Γ_n and elements $\gamma_{n_j} \in \Gamma_{n_j}$ such that γ_{n_j} converges to γ, then $\gamma \in \Gamma$, and

(ii) if $\gamma \in \Gamma$, then there exists $\gamma_n \in \Gamma_n$ (for all n) such that γ_n converges to γ.

We recall [10] that this is equivalent to $\mathbf{H}^3/\Gamma_i$ converging to $\mathbf{H}^3/\Gamma$ in the sense of Gromov. We will say that a sequence $\{\rho_i: \pi_1(M) \to Isom_+(\mathbf{H}^3)\}$ of discrete faithful representations *converges strongly* to a representation $\rho_\infty: \pi_1(M) \to Isom_+(\mathbf{H}^3)$ if it converges algebraically and $\rho_i(\pi_1(M))$ converges geometrically to $\rho_\infty(\pi_1(M))$.

Thurston's original application of his covering theorem for geometrically tame hyperbolic 3-manifolds satisfying Bonahon's condition (B) was to prove that sequences which converged algebraically to representations without new parabolics also converged strongly. We state the full version of his theorem which contains additional information (see also [28]). This theorem played a role in Thurston's proof of his geometrization theorem (see [27]). In Thurston's original result he had to assume that N_i was geometrically tame for all i and he was able to conclude that N_∞ was geometrically tame. With the advent of Bonahon's result [5] such an assumption is no longer necessary.

THEOREM 9.1 (Thurston [34]). *Let $\{\rho_i\}$ be a sequence in $AH(M)$ which converges algebraically to $\rho_\infty \in AH(M)$. Let $N_i = \mathbf{H}^3/\rho_i(\pi_1(M))$ and $N_\infty = \mathbf{H}^3/\rho_\infty(\pi_1(M))$. Let (M, P) be a fixed pared manifold. Further assume that for all i there exists a homeomorphism of pairs*

$$\phi_i: (int(M) \cup P, P) \to ((N_i)^0_\varepsilon, \partial(N_i)^0_\varepsilon)$$

which is homotopic to the homotopy equivalence induced by ρ_i and that N_i satisfies Condition B. If $\rho_\infty(g)$ is parabolic if and only if $\rho_i(g)$ is parabolic for all i, then N_∞ is homeomorphic to N_i (for all i) and $\{\rho_i\}$ converges strongly to ρ_∞.

We will not attempt to give a full proof of Theorem 9.1 as we would have to introduce the theory of pleated surfaces to do so. However, the following theorem is an almost immediate consequence of our covering theorem.

THEOREM 9.2. *Let $\{\rho_i: \pi_1(M) \to Isom_+(\mathbf{H}^3)\}$ be a sequence of discrete faithful representations converging algebraically to $\rho_\infty: \pi_1(M) \to Isom_+(\mathbf{H}^3)$, a representation with image Γ. If the limit set L_Γ is all of S^2 and $N = \mathbf{H}^3/\Gamma$ is topologically tame, then $\{\rho_i\}$ converges strongly to ρ_∞.*

Proof. Theorem 3.2 implies that if N is topologically tame and $L_\Gamma = S^2$ then all ends of N_ε^0 are simply degenerate. Let $\Gamma_i = \rho_i(\pi_1(M))$, and let $\{\Gamma_k\}$ be a convergent subsequence of $\{\Gamma_i\}$ with geometric limit Γ'. Then Γ' is discrete and $\Gamma \subset \Gamma'$ (see [20]). In particular, they prove that given $x_0 \in \mathbf{H}^3$, there exists ε_0 such that $d(x_0, \gamma_i(x_0)) \geq \varepsilon_0$ for all $\gamma_i \subset \Gamma_i$. Furthermore, by Theorem 3.1.4 in [10], Γ' is torsion-free. Notice that N' does not have finite volume (since it is the geometric limit of infinite volume hyperbolic 3-manifolds). The covering theorem then guarantees that the covering of N' by $N_\infty = \mathbf{H}^3/\Gamma$ is finite-to-one on each end of $(N_\infty)_\varepsilon^0$ and hence that the entire covering is finite-to-one. Thus, Γ has finite index in Γ'.

Suppose there exists $\gamma \in \Gamma' - \Gamma$. Then there exists n such that γ^n is an element in Γ and if $\beta \in \Gamma$ and $\beta = \gamma^n$ then β does not have a n^{th} root in Γ. Let $\gamma^n = \rho_\infty(g)$ and $\{\gamma_k\}$ be a sequence of elements of Γ_k converging to γ, then γ_k^n converges to γ^n. Since g does not have a n^{th} root, $\gamma_k^n \neq \rho_k(g)$, which implies that $\gamma_k^{-n}\rho_k(g)$ is a sequence of non-trivial elements of Γ_k converging to the identity. However, this would contradict the fact that $d(x_0, \gamma_i(x_0)) \geq \varepsilon_0$ for all $\gamma_i \subset \Gamma_i$.

Thus every convergent subsequence of $\{\Gamma_i\}$ converges to Γ. Therefore, since the space of closed subsets of $Isom_+(\mathbf{H}^3)$ is compact in the Chabauty topology, we see that $\{\Gamma_i\}$ converges to Γ. □

Theorem 9.2 is most useful in situations where one knows that the algebraic limit must be topologically tame, for example when Γ satisfies Condition B. Notice that Γ may satisfy Condition B even when no element of the sequence $\{\Gamma_i\}$ does. Even when $\rho_i(\pi_1(M))$ satisfies Condition B, Theorem 9.2 contains some new information when Γ contains "new" parabolics.

The arguments in the proof of Theorem 9.2 can be used to give roughly half the proof of strong convergence in Thurston's theorem 9.1. However, one needs to make use of pleated surfaces to handle the geometrically finite ends of N_∞. To prove that N_∞ is homeomorphic to N_i, in Theorem 9.1, one needs to check that the induced homotopy equivalence from N_i to N_∞ preserves peripheral subgroups and then apply Waldhausen's theorem.

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Note added in proof. In this brief note we explain how to complete the proof of the Filling Theorem in the case that the surface S is a four-punctured sphere. As before, we first consider two U-nice useful simplicial hyperbolic surfaces $g: S \to U_P$ and $h: S \to U_P$. Let g and h have associated triangulations T_g and T_h and distinguished edges e_g and e_h respectively. We assume that each puncture of S is the endpoint of only one edge of T_g and only one edge of T_h. We may construct a finite sequence $(T_0, e_0), \ldots, (T_m, e_m)$ such that T_i is a triangulation of S with one vertex and e_i is an edge of T_i where $T_0 = T_g$, $e_0 = e_g$, $T_m = T_h$ and $e_m = e_h$. We may assume that T_i is obtained from T_{i-1} by making an elementary move and that e_i is the (unique) edge of T_i such that $e_i \cup v$ is a non-peripheral closed curve. We cannot be assured that the elementary move does not affect the distinguished edge, since a triangulation with one vertex of a four-times-punctured sphere has only one compact edge e such that $e \cup v$ is non-peripheral.

The proof proceeds as in Section 6 except that we need to alter the construction of the continuous family of simplicial hyperbolic surfaces originating at g which either ends at h or terminates by meeting the boundary of U_P. We do this by augmenting lemma 5.3 with lemma 9.3 and by replacing lemmas 5.4 and 5.5 with the lemmas 9.4 and 9.5 stated below.

LEMMA 9.1. *Let S be a four-times-punctured sphere and let $h: S \to N$ be a useful simplicial hyperbolic surface with associated triangulation T and distinguished edge $\hat{e}$. Let e_1, e_2, e_3 and e_4 bound a quadrilateral in T with diagonal $e_5 = \hat{e}$. Let e_6 be the other diagonal of this quadrilateral and T' the triangulation obtained by replacing e_5 with e_6. Assume that $h(e_6 \cup v)$ is homotopic to a closed geodesic. Then we may construct a continuous family of simplicial hyperbolic surfaces with at most two vertices (each having property NLSC) joining h to a useful simplicial hyperbolic surface h' with associated triangulation T' and distinguished edge e_6.*

Proof of 9.1. Let e_7 and e_8 be edges of T joining (distinct) punctures of S to v which are not contained in the quadrilateral bounded by e_1, e_2, e_3 and e_4. Let $\tilde{e}_7$ and $\tilde{e}_8$ be lifts of e_7 and e_8 to $\tilde{S}$ which share an endpoint $\tilde{v}$. Let $\tilde{h}: \tilde{S} \to \mathbf{H}^3$ be a lift of h and let $\tilde{h}(\tilde{e}_7 \cup \tilde{v} \cup \tilde{e}_8)^*$ denote the infinite geodesic properly homotopic to $\tilde{h}(\tilde{e}_7 \cup \tilde{v} \cup \tilde{e}_8)$. Let $\tilde{e}_5$ be a lift of e_5 which has an endpoint at $\tilde{v}$. If $g \in \pi_1(S)$ takes one endpoint of $\tilde{e}_5$ to the other, then $\tilde{h}(\tilde{e}_5)$ is a segment in the axis A of $h_*(g)$. Let R_0 be the unique common perpendicular joining A to $\tilde{h}(\tilde{e}_7 \cup \tilde{v} \cup \tilde{e}_8)^*$. Let R_1 be a segment of A connecting $\tilde{h}(\tilde{v})$ to $R_0 \cap A$. We now drag h along (the projection to N of) $R_0 \cup R_1$ to obtain a continuous family $G: S \times [0, 1] \to N$ of simplicial hyperbolic surfaces with associated triangulation T (each satisfying NLSC).

Now let $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4$ and $\tilde{e}_6$ be lifts of e_1, e_2, e_3, e_4 and e_6 to $\tilde{S}$ such that $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$, and $\tilde{e}_4$ form a quadrilateral with diagonals $\tilde{e}_5$ and $\tilde{e}_6$. Let $\tilde{G}_1(\tilde{e}_6)^*$ denote the geodesic arc joining the endpoints of $\tilde{G}_1(\tilde{e}_6)$. Notice that $\tilde{G}_1(\tilde{e}_1)$, $\tilde{G}_1(\tilde{e}_2)$, $\tilde{G}(\tilde{e}_3)$, $\tilde{G}_1(\tilde{e}_4)$, $\tilde{G}_1(\tilde{e}_5)$ and $\tilde{G}_1(\tilde{e}_6)^*$ form a tetrahedron X in $\mathbf{H}^3$. Let v' denote the intersection of e_5 and e_6 and let $\tilde{v}' = \tilde{e}_5 \cap \tilde{e}_6$. Let R_2 be a geodesic in X joining $\tilde{G}_1(\tilde{v}')$ to $\tilde{G}_1(\tilde{e}_6)^*$. We obtain a new triangulation $\bar{T}$ of S by adding e_6 to T (which now has an additional vertex at $v' = e_5 \cap e_6$.) We can drag G_1 along (the projection to N of) R_2 to obtain a continuous family $H: S \times [1, 2] \to N$ of simplicial hyperbolic surfaces with associated triangulation $\bar{T}$ (each satisfying NLSC.) Notice that H_2 maps e_6 to the geodesic arc $G_1(\tilde{e}_6)^*$, and we may remove the vertex v' and the edge e_5. Viewed in this manner, H_2 is a simplicial hyperbolic surface with associated triangulation T'.

Let $g' \in \pi_1(S)$ be the element which takes one endpoint of $\tilde{e}_6$ to the other and let A' be the axis of $h_*(g')$. Let R_3 be the unique common perpendicular joining A' and $\tilde{h}(\tilde{e}_7 \cup \tilde{v} \cup \tilde{e}_8)^*$. Let R_4 be the segment of $\tilde{h}(\tilde{e}_7 \cup \tilde{v} \cup \tilde{e}_8)^*$ joining $\tilde{H}_2(\tilde{v})$ to $\tilde{h}(\tilde{e}_7 \cup \tilde{v} \cup \tilde{e}_8)^* \cap R_3$. We drag H_2 along (the projection to N of) $R_3 \cup R_4$ to obtain a continuous family $J: S \times [2, 3] \to N$ of simplicial hyperbolic surfaces with associated triangulation T' (each satisfying NLSC) such that J_3 is a useful hyperbolic surface with distinguished edge e_6.

We may concatenate G, H and J to obtain the desired family. □

We may similarly prove the following analogues of Lemma 5.4 and 5.5 which suffice to complete the proof of the Filling theorem in the case when S is a four-times-punctured sphere.

LEMMA 9.2. *Let S be a four-times-punctured sphere and let $h: S \to N$ be a useful simplicial hyperbolic surface with associated triangulation T and distinguished edge $\hat{e}$. Let e_1, e_2, e_3 and e_4 bound a quadrilateral in T with diagonal $e_5 = \hat{e}$. Let e_6 be the other diagonal of this quadrilateral and T' the triangulation obtained by replacing e_5 with e_6. Assume that $h(e_6 \cup v)$ is an accidental parabolic. Then there exists a continuous family $H: S \times [0, \infty) \to N$ of simplicial hyperbolic surfaces with only one vertex (which satisfies NLSC) such that the length of $H_t(e_6 \cup v)$ decreases and converges to 0.*

LEMMA 9.3. *Let S be a four-times-punctured sphere and let $h: S \to N$ be a useful hyperbolic surface with associated triangulation T and distinguished edge $\hat{e}$. Let e_1, e_2, e_3 and e_4 form a quadrilateral with diagonal $e_5 = \hat{e}$ such that the "other diagonal" e_6 determines a compressible curve $e_6 \cup v$. Then there exists a continuous family $H: S \times [0, r) \to N$ of simplicial hyperbolic surfaces with at most two vertices (each satisfying NLSC) such that the length of $H_t(e_6)$ decreases and converges to 0.*