Split decomposition over an abelian group, Part 2: Group-valued split systems with weakly compatible support

Andreas Dress

Department for Combinatorics and Geometry, CAS-MPG Partner Institute for Computational Biology, Shanghai Institutes for Biological Sciences, Chinese Academy of Sciences, Shanghai, China
Max Planck Institute for Mathematics in the Sciences, Inselstrasse 22–26, D 04103 Leipzig, Germany

Abstract

Split-decomposition theory deals with relations between \( \mathbb{R} \)-valued split systems and metrics. In a previous publication (the first of a series of papers on split decomposition over an abelian group), a general conceptual framework has been set up to study these relationships from an essentially algebraic point of view, replacing metrics by certain more general, appropriately defined multivariate maps, and considering group-valued split systems that take their values in an arbitrary abelian group. Here, we make use of this set up and establish the principal results of split-decomposition theory regarding split systems with weakly compatible support within this new algebraic framework.

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1. Why is decomposition theory of interest in phylogenetic analysis?

Given a collection \( X \) of (biological) species, it is one of the most basic tasks in phylogenetic analysis to identify all monophyletic clades in \( X \), i.e., all subsets \( C \) of \( X \) that consist of all species in \( X \) that are offspring of a single ancestral species while none of the species in the complement \( X - C \) of \( C \) have evolved from that ancestral species. However, as Charles Darwin put it in his treatise THE DESCENT OF MAN, as we have no record of the lines of descent, the pedigree can be discovered only by the degrees of resemblance between the beings which are to be classed. That is, all that we commonly can rely on to identify the collection of all monophyletic clades in \( X \) is information about how distinct, or how similar, the present-day species are that make up the set \( X \).

Consequently, a standard assumption in phylogenetic analysis is that, together with a finite set \( X \) of species, we are given a metric \( D \) defined on \( X \)\(^1\) that quantifies that degree of resemblance between the species contained in \( X \). And the task one has to address can then be described as that of designing a method for deriving a phylogenetic \( X \)-tree \( T = T(D) \) from these data that – at least approximatively – represents the metric \( D \). That is, one wants to find a finite edge-weighted and \( X \)-labeled tree \( T = (V, E, ℓ; ϕ) \) consisting of

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\(^1\) That is, a map \( D : X \times X \to \mathbb{R} \) for which \( D(x, x) = 0 \) and \( D(x, y) \leq D(x, z) + D(y, z) \) hold for all \( x, y, z \in X \).
(i) a finite vertex set V,
(ii) an edge set \( E \subseteq \binom{V}{2} \).
(iii) a weight map \( \ell : E \to \mathbb{R}_{>0} \) from \( E \) into the set \( \mathbb{R}_{>0} \) of positive real numbers,
(iv) a labeling map \( \varphi : X \to V \) from \( X \) into \( V \) such that (at least) every vertex of degree 1 or 2 in \( V \) is contained in the image set \( \varphi(X) \) of the labeling map \( \varphi \), and the distance \( D(x, y) \) of any two taxa \( x, y \in X \) coincides – at least approximatively – with the length \( \ell_T(x, y) \) of the unique path in \( T \) from \( \varphi(x) \) to \( \varphi(y) \) (measured in terms of the weight map \( \ell \)).

Remarkably, denoting the set consisting of all splits of a set \( X \), i.e., the set of all 2-element subsets \( \{A, B\} \) of the power set \( \mathcal{P}(X) \) of \( X \) for which \( A \cap B = X \) and \( A \cap B = \emptyset \) hold, by \( \delta(X) \), this task is simply equivalent to finding a map \( \Sigma \) from
\[
\delta^*(X) := \{ S \in \delta(X) : S \neq \{X, \emptyset\} \}
\]
into the set \( \mathbb{R}_{>0} \) of non-negative real numbers such that
(i) the distance \( D(x, y) \) of \( x \) and \( y \) coincides – again at least approximatively – with the sum
\[
\Sigma(x : y) := \sum_{\{A, B\} \in \delta^*(X) : x \in A, y \in B} \Sigma([A, B])
\]
(ii) and any two splits in the support
\[
\text{supp}(\Sigma) := \{ \{A, B\} \in \delta^*(X) : \Sigma([A, B]) \neq 0 \}
\]
of \( \Sigma \) are compatible, i.e., one of the four intersections \( A \cap A', A \cap B', B \cap A', B \cap B' \) is empty for any two splits \( \{A, B\} \) and \( \{A', B'\} \) in \( \text{supp}(\Sigma) \).

This fact was probably, in one or the other disguise, folklore already in the mid-twentieth century; it was stated explicitly – more or less just as stated above – by Peter Buneman around 1970 (see for instance [8]); and it has been one of the fundamental insights on which much further development of computational phylogenetics was based.

However, in that development, it soon turned out that it might be worthwhile to consider, more generally, arbitrary maps \( \Sigma \) from \( \delta^*(X) \) into \( \mathbb{R}_{>0} \), and to relate these maps to certain phylogenetic networks, the so-called \( X \)-nets, that can be used to represent ambiguous phylogenetic signals (cf. [10]). And to consider, even more generally, maps \( \Sigma \) from \( \delta^*(X) \) into arbitrary abelian groups and to relate properties of their traces to corresponding properties of \( \Sigma \) and its support (see for example [3–5,7,11]). Here, we will report the results obtained following this line of thought.

2. An algebraic approach to split-decomposition theory

Split-decomposition theory, as developed in [3], deals with relations between real-valued split systems and metrics. Here, we generalize parts of this theory, using the concepts introduced in [11], the first of a series of papers on split decomposition over an abelian group (cf. also [5,16,17]). More specifically, we replace, as suggested by that paper,
(i) real-valued split systems by group-valued split systems that take their values in an arbitrary abelian group \( \mathcal{A} \),
(ii) metrics by certain multivariate maps that also take their values in that same group \( \mathcal{A} \), — actually, we replace metrics by all of those three or four kinds of multivariate maps that were introduced in [11], systematizing various definitions and concepts that had been proposed and studied in, e.g., [1,5,17,21],
(iii) and the canonical trace homomorphism by various such homomorphisms that associate to any map from the set \( \delta^*(X) \) of all proper splits of \( X \) into the non-negative real numbers the corresponding multivariate maps.

Within this new conceptual framework, we will establish the principal results of split-decomposition theory regarding weakly compatible split systems.

We will begin by recalling the basic big commutative diagram depicted below in which all the considerations presented in [11] culminated. In its center, one finds the group \( \delta^*(X) \) of all \( \mathcal{A} \)-valued split systems defined on (the set \( \delta^*(X) \) of all proper splits of) \( X \) from which various arrows representing the trace homomorphisms emanate. These trace homomorphisms point to the trace groups, that is, to all those groups whose elements are made up by the various multivariate maps under consideration representing the potential traces of \( \mathcal{A} \)-valued split systems. In turn, these trace groups are connected by a

\[
\sigma_{\text{mult}}(\Sigma) := \sum_{S \in \delta^*(X)} \Sigma(S) \delta_S
\]
of the associated split metrics
\[
\delta_S : X \times X \to \mathbb{R} : (x, y) \mapsto \delta_S(x, y) := \delta_{\Sigma(x,y)}
\]
where \( S(z) \) denotes, for any \( z \in X \) the unique subset \( A \) or \( B \) in the split \( S = \{A, B\} \) that contains \( z \).
double-circuit of arrows representing mutually inverse group isomorphisms including for example, in case the group \( A \) has no 2-torsion, the celebrated Farris transform (cf. e.g. [15] for a recent review).

We will then show that, restricting attention to group-valued split systems with weakly compatible support, we can derive the principal results that were established in [3] in case \( A \) is the additive group of real numbers, as well as some further results that had not yet been established even in this very particular situation. That is, given a weakly compatible split system \( \mathcal{R} \),

- we will show that any two \( A \)-valued split systems whose support is contained in \( \mathcal{R} \) coincide if and only if their traces coincide in one or, as well, in all of the trace groups under consideration,
- we will characterize the subgroups consisting of all those multivariate maps that are derived, via the various trace homomorphisms, from split systems with support in \( \mathcal{R} \),
- and we will investigate how all these subgroups are related to each other via the respective group isomorphisms in the basic commutative diagram.

Of course, we will shortly introduce and then freely use the notations, definitions, and results established in [11].

3. Notations, definitions, and results from the first part

Given a non-empty set \( X \) and an (additively written) abelian group \( A \) with neutral element \( 0_A \), we denote throughout

1. by \( \mathcal{S}(X) \) the set of all \( X \)-splits, i.e., of those 2-subsets \( \{ A, B \} \) of the power set \( \mathcal{P}(X) \) of \( X \) for which \( A \cup B = X \) and \( A \cap B = \emptyset \) hold,
2. by \( \mathcal{S}^+(X) \) the set of all proper splits of \( X \), i.e., all those splits \( \{ A, B \} \) in \( \mathcal{S}(X) \) for which \( A, B \neq \emptyset \) holds,
3. by \( \mathcal{S}(x) \), for every split \( S = \{ A, B \} \in \mathcal{S}(X) \) and every element \( x \in X \), that subset \( A \) or \( B \), in \( S \) that contains \( x \),
4. by \( \mathcal{S}_A \), for every subset \( A \) of \( X \), the associated split \( S_A := \{ A, X - A \} \),
5. by \( \mathcal{R}(Y|Z) \) or also by \( \mathcal{R}(y_1, \ldots, y_i|z_1, \ldots, z_j) \), for any subset \( \mathcal{R} \) of \( \mathcal{S}(X) \) and any two subsets \( Y = \{ y_1, \ldots, y_i \} \) and \( Z = \{ z_1, \ldots, z_j \} \), the set of all splits \( S \in \mathcal{R} \) that separate \( Y \) from \( Z \), i.e., the set of all splits \( S = \{ A, B \} \) in \( \mathcal{R} \) with, say, \( Y \subseteq A \) and \( Z \subseteq B \),
6. by \( \mathcal{R}(A) \), for any subset \( \mathcal{R} \) of \( \mathcal{S}(X) \), the group of all maps \( \Sigma \) from \( \mathcal{S}(X) \) into \( A \) whose support \( \text{supp}(\Sigma) = \{ S \in \mathcal{S}(X) : \Sigma(S) \neq 0_A \} \) is contained in \( \mathcal{R} \),
7. by \( \mathcal{S}(X | A) \) and by \( \mathcal{S}^+(X | A) \) the groups \( \mathcal{R}(A) \) for \( \mathcal{R} := \mathcal{S}(X) \) and \( \mathcal{R} := \mathcal{S}^+(X) \), respectively,
8. by \( \Sigma_+ \), for any map \( \Sigma \) in \( \mathcal{S}(X | A) \) and any subset \( \mathcal{R} \) of \( \mathcal{S}(X) \), the sum

\[
\Sigma_+ := \sum_{S \in \mathcal{R}} \Sigma(S)
\]

over all the values that the map \( \Sigma \) attains at the splits in \( \mathcal{R} \),
9. by \( \Sigma_+(Y|Z) \) or \( \Sigma_+(y_1, \ldots, y_i|z_1, \ldots, z_j) \), for any two subsets \( Y = \{ y_1, \ldots, y_i \} \) and \( Z = \{ z_1, \ldots, z_j \} \), the sum \( \Sigma_+(\mathcal{S}(Y|Z)) \),
10. by \( \mathcal{L}_2(X | A) \) the groupoid canonically associated with the set \( X \) whose elements are the pairs \( xy := (x, y) \) of elements from \( X \) while a product \( xy * uv \) of any two such pairs \( xy, uv \) is defined if and only if \( y = u \) holds in which case \( xy * uv \) is defined as \( xy * uv := xy \),
11. by \( \mathcal{P}(X | A) \) the group of "bilinear" symmetric maps defined on \( \mathcal{S}(X) \) with values in \( A \), i.e., the group of maps

\[
A : \mathcal{S}(X) \times \mathcal{S}(X) \to A : (xy, uv) \mapsto A(xy * uv)
\]

for which

\[
\]

and

\[
A(xy : uv) = A(uv : xy)
\]

hold for all \( x, y, z, u, v \) in \( X \), noting that, for example, the identity

\[
A(xy : uv) + A(xu : vy) + A(xv : yu) = 0_A
\]

holds for all \( A \in \mathcal{L}_2(X | A) \) and all \( x, y, u, v \) in \( X \),
12. by \( \binom{x}{z} \), for any positive integer \( k \), the set of all non-empty subsets of \( X \) of cardinality at most \( k \), and by \( \mathcal{P}_{\leq k}(X | A) \) the group consisting of all maps from \( \binom{x}{z} \) into \( A \),
13. by \( \Pi(x_1, x_2, \ldots, x_i) \), for every map \( \Pi \in \mathcal{P}_{\leq k}(X | A) \), all \( i \in \{1, \ldots, k\} \), and all \( x_1, x_2, \ldots, x_i \in X \), the group element \( \Pi((x_1, x_2, \ldots, x_i)) \) and further, for every \( j \in \{1, \ldots, i\} \), by \( \Pi(x_1, x_2, \ldots, x_i : j) \) the sum over all terms of the form \( \Pi((x_l : \ell \in I)) \) with \( I \in \binom{\ell}{1, 2, \ldots, i} \):

\[
\Pi(x_1, x_2, \ldots, x_i : j) := \sum_{I \in \binom{\ell}{1, 2, \ldots, i}} \Pi((x_l : \ell \in I))
\]
by \( P_{2 \leq 2}(X|A) \) the subgroup of \( P_{2 \leq 2}(X|A) \) consisting of all maps \( \Pi \) in \( P_{2 \leq 2}(X|A) \) for which \( \Pi(x) = 0 \) and \( \Pi(xyz : 3) \in 2A := \{2\alpha : \alpha \in A\} \) hold for all \( x, y, z \in X \).

by \( P_{3 \leq 3}(X|A) \) the subgroup of \( P_{3 \leq 3}(X|A) \) consisting of all maps \( \Psi \) in \( P_{3 \leq 3}(X|A) \) for which \( \Psi(x) = 0_A \) and \( \Psi(xuv : 3) = \Psi(xuv : 2) \) hold for all \( x, y, u, v \in X \).

and the same holds for just all group homomorphisms except the trace homomorphisms in case \( \sigma \in \Sigma \) consisting of all maps in \( P_{2 \leq 2}(X|A) \) that vanish on any subset \( \{x, y\} \) in \( \left( X \leq 2 \right) \) with \( z \in \{x, y\} \).

Then, given any fixed element \( z \in X \), there are well-defined canonical group homomorphisms specified below that fit into one big commutative diagram (so, actually, there is one such diagram for each point \( z \in X \):

\[
\begin{array}{ccccccccc}
L_2(X|A) & \xrightarrow{\lambda_2} & P_{2 \leq 2}(X, z | A) & \xrightarrow{\delta^*(X|A)} & P_{3 \leq 3}(X|A) & \xrightarrow{\psi_2} & L_2(X|A) \\
\sigma_{bil} & & \sigma_2 & & \sigma_3 & & \sigma_{bil} \\
\psi_{bil} & & \psi_2 & & \psi_{bil} & & \psi_{bil} \\
\pi_{bil} & & \pi_2 & & \pi_{bil} & & \pi_{bil} \\
\end{array}
\]

In this diagram, the solid arrows represent group homomorphisms that are defined for any abelian group \( A \) while the broken arrows represent group homomorphisms that are defined only in case the abelian group \( A \) does not contain any element of order 2. The trace homomorphisms \( \sigma_{bil}, \sigma_2, \sigma_3 \) emanating from the group \( \delta^*(X|A) \) in the center of the diagram are split-surjective,

and the same holds for the group homomorphisms \( \sigma_2 \) in case \( A \) does not contain any element of order 2. All group homomorphisms in-between the "trace groups" \( L_2(X|A), P_{2 \leq 2}(X, z | A), \) and \( P_{3 \leq 3}(X|A) \) are group isomorphisms, and the same holds for just all group homomorphisms except the trace homomorphisms in case \( A \) does not contain any element of order 2. Furthermore, all pairs of parallel arrows pointing in opposite directions represent mutually inverse group isomorphisms (if defined).

The trace homomorphisms \( \sigma_{bil}, \sigma_2, \sigma_3 \) and the group isomorphisms \( \lambda_2, \lambda_3, \lambda_2, \lambda_2, \psi_{bil}, \psi_2, \psi_{bil}, \psi_2, \pi_{bil}, \pi_{bil}, \pi_2 \) in this diagram are defined as follows:

\[
\begin{align*}
\sigma_{bil} : & \delta^*(X|A) \rightarrow L_2(X|A) : \Sigma \mapsto \left( \Sigma^{(bil)} : (xy, uv) \mapsto \Sigma_+(xy|yu) - \Sigma_+(xu|yv) \right), \\
\sigma_2 : & \delta^*(X|A) \rightarrow P_{2 \leq 2}(X|A) : \Sigma \mapsto \left( \Sigma^{(2)} : [a, b, c] \mapsto \Sigma_+(ab|c) + \Sigma_+(bc|a) + \Sigma_+(ca|b) \right), \\
\sigma_3 : & \delta^*(X|A) \rightarrow P_{3 \leq 3}(X|A) : \Sigma \mapsto \left( \Sigma^{(3)} : [a, b, c] \mapsto \Sigma_+(ab|c) \right), \\
\lambda_2 : & L_2(X|A) \rightarrow P_{2 \leq 2}(X|A) : \Lambda \mapsto \left( \Lambda^{(2)} : [a, b] \mapsto \Lambda(ab : ba) \right), \\
\lambda_3 : & L_2(X|A) \rightarrow P_{3 \leq 3}(X|A) : \Lambda \mapsto \left( \Lambda^{(3)} : [a, b, c] \mapsto \Lambda(ab : bc) + \Lambda(bc : ca) + \Lambda(ca : ab) \right), \\
\lambda_2 : & L_2(X|A) \rightarrow P_{2 \leq 2}(X, z | A) : \Lambda \mapsto \left( \Lambda^{(2)} : [a, b] \mapsto \Lambda(az : zb) \right), \\
\lambda_3 : & L_2(X|A) \rightarrow P_{3 \leq 3}(X, z | A) : \Lambda \mapsto \left( \Lambda^{(3)} : [a, b, c] \mapsto \Lambda(ab : ba) \right), \\
\psi_{bil} : & P_{2 \leq 2}(X, z | A) \rightarrow L_2(X|A) : \Pi \mapsto \left( \Pi^{(bil)} : (xy, uv) \mapsto \Pi(xy) + \Pi(yu) - \Pi(xu) - \Pi(yv) \right), \\
\psi_2 : & P_{3 \leq 3}(X|A) \rightarrow L_2(X|A) : \Psi \mapsto \left( \Psi^{(bil)} : (xy, uv) \mapsto \Psi(xy) - \Psi(xu) - \Psi(yu) + \Psi(yv) \right), \\
\psi_{bil} : & P_{3 \leq 3}(X|A) \rightarrow L_2(X|A) : \Psi \mapsto \left( \Psi^{(bil)} : (xy, uv) \mapsto \Psi(xy) - \Psi(xu) - \Psi(yu) + \Psi(yv) \right). \\
\end{align*}
\]

\[3\] Recall that a group homomorphism \( \alpha \) from a group \( G \) into a group \( G' \) is split-surjective (or split-injective, respectively) if there exists a group homomorphism \( \alpha' \) back from \( G' \) into \( G \) such that \( \alpha \circ \alpha' = \text{Id}_G \) (or \( \alpha' \circ \alpha = \text{Id}_G \) holds.)
Given any weakly compatible split system with given weakly compatible support are split-injective

Recall that split-decomposition theory, as developed in [4], deals with the map

$$\sigma_2 : \delta^*(X|A) \to \mathcal{P}_2(X|A) : \Sigma \mapsto \left( \Sigma^{(2)} : \{a, b\} \mapsto \Sigma_2(a|b) \right)$$

in case $A$ is the additive group of real numbers and $X$ is finite. One of its most basic results refers to weakly compatible system $\mathcal{R}$ of $X$-splits, i.e., subsets $\mathcal{R}$ of $\delta^*(X)$ such that, for any three splits $\{A_1, B_1\}, \{A_2, B_2\}, \{A_3, B_3\}$ in $\mathcal{R}$, at least one of the four intersections

$$A_1 \cap A_2 \cap A_3, \quad A_1 \cap B_2 \cap B_3, \quad B_1 \cap A_2 \cap B_3, \quad B_1 \cap B_2 \cap A_3$$

and, hence, also one of the four intersections $B_1 \cap B_2 \cap B_3, B_1 \cap A_2 \cap A_3, A_1 \cap B_2 \cap A_3, A_1 \cap A_2 \cap B_3$ is empty. The result in [4] states that in that case, restricting $\sigma_2$ to the sub-vectorspace $\mathcal{R}(\mathbb{R})$ of $\delta^*(X|\mathbb{R})$ consisting of all maps $\Sigma$ in $\delta^*(X|\mathbb{R})$ whose support $\supp(\Sigma)$ is contained in $\mathcal{R}$, induces an injective homomorphism

$$\sigma_2^R : \mathcal{R}(\mathbb{R}) \to \mathcal{P}_2(X|\mathbb{R})$$

into the real vectorspace $\mathcal{P}_2(X|\mathbb{R})$.

Here, we want to establish a variant of this result that holds for every abelian group $A$, provided one uses – instead of the group $\mathcal{P}_2(X|A)$ – either one of the groups $\mathcal{L}_2(X|A), \mathcal{P}_{2}(X, z | A)$, or $\mathcal{P}_3(X|A)$. More precisely, our result reads as follows:

**Theorem 4.1.** Given any weakly compatible split system $\mathcal{R} \subseteq \delta^*(X)$, restricting the surjective group homomorphisms $\sigma_{\text{bil}} : \delta^*(X|A) \to \mathcal{L}_2(X|A)$, $\sigma_3 : \delta^*(X|A) \to \mathcal{P}_3(X|A)$, and $\sigma_2 : \delta^*(X|A) \to \mathcal{P}_{2}(X, z | A)$ to the subgroup $\mathcal{R}(A)$ of $\delta^*(X|A)$ consisting of all maps $\Sigma$ in $\delta^*(X|A)$ whose support $\supp(\Sigma)$ is contained in $\mathcal{R}$ yields split-injective group homomorphisms

$$\sigma^R_{\text{bil}} : \mathcal{R}(A) \to \mathcal{L}_2(X|A),$$

$$\sigma^R_3 : \mathcal{R}(A) \to \mathcal{P}_3(X|A),$$

and

$$\sigma^R_2 : \mathcal{R}(A) \to \mathcal{P}_{2}(X, z | A)$$

from $\mathcal{R}(A)$ into $\mathcal{L}_2(X|A), \mathcal{P}_3(X|A)$, and $\mathcal{P}_{2}(X, z | A)$, respectively. In other words, these restrictions map $\mathcal{R}(A)$ onto a direct summand of $\mathcal{L}_2(X|A), \mathcal{P}_3(X|A)$, or $\mathcal{P}_{2}(X, z | A)$, and there exist group homomorphisms

$$\lambda^R : \mathcal{L}_2(X|A) \to \mathcal{R}(A),$$

$$\psi^R : \mathcal{P}_3(X|A) \to \mathcal{R}(A),$$

and

$$\zeta^R : \mathcal{P}_{2}(X, z | A) \to \mathcal{R}(A)$$

such that

$$\lambda^R \circ \sigma_{\text{bil}}^R = \psi^R \circ \sigma_3^R = \zeta^R \circ \sigma_2^R = \text{id}_{\mathcal{R}(A)}.$$

holds.

In particular, assuming as above that $\mathcal{R}$ is a weakly compatible system, these three restrictions $\sigma_{\text{bil}}^R, \sigma_3^R, \sigma_2^R$ are group isomorphisms if and only if $\mathcal{R}$ is a weakly compatible system of $X$-splits of maximal cardinality, that is, a weakly compatible split system $\mathcal{R}$ with $\#\mathcal{R} = \left( \frac{|X|}{2} \right)$ (or, equivalently, if and only if there exists a cyclic graph$^4$ $C = (X, E)$ with vertex set $X$ and

---

$^4$That is, a finite connected graph all of whose vertices have degree $\leq 2$. 
edge set $E$ so that $\mathcal{R}$ coincides with the set $\mathcal{R}_c$ of all splits $S = \{A, B\}$ of $X$ for which two distinct edges $e, f \in E$ exist so that $A$ and $B$ are the vertex sets of the — exactly two — connected components of the subgraph $C^{e,f} := (X, E - \{e, f\})$ of $C$, cf. [4]).

In view of all the isomorphisms in the commutative diagram above, we only need to establish this for the map $\sigma_{\text{bil}}^\mathcal{R}$ in which case our claim follows quite easily from the following two observations:

**Lemma 4.2.** Given any weakly compatible system $\mathcal{R} \subseteq \Sigma^+(X)$ of $X$-splits and any split $S = \{A, B\}$ in $\mathcal{R}$, there exist always four (not necessarily distinct) elements $a, a' \in A$ and $b, b' \in B$ such that $S$ is the only split in $\mathcal{R}(ab' | ab')$ while there is no split in $\mathcal{R}(ab | a'b)$.

More specifically, given any two subsets $Y, Z$ of $X$ with $| \mathcal{R}(Y')| = \mathcal{R}(Y'|Z)$, there exist always elements $y, y' \in Y$ with $S = \mathcal{R}(\{y, y'\}|Z)$.

**Proof.** In a slightly round-about way, this can be deduced from the results that were established already in [4].

To give a more direct proof, note first that the first assertion follows directly from the second one by applying that one twice: First, put $Y := A$ and $Z := B$ to obtain two elements $a, a' \in A$ with $S = \mathcal{R}(\{a, a'\}|Z)$. And then put $Y := B$ and $Z := \{a, a'\}$ to obtain elements $b, b' \in B$ with $S = \mathcal{R}(\{a, a'\}|\{b, b'\}) = \mathcal{R}(ab' | ab')$. And note finally that, by definition of weak compatibility, $\mathcal{R}(ab' | ab') \neq \emptyset$ implies that either $\mathcal{R}(ab | a'b) = \emptyset$ or $\mathcal{R}(ab' | ab) = \emptyset$ must hold. So, by switching the labels for the two elements $b, b'$ in $B$ if required, we can find elements $a, a' \in A$ and $b, b' \in B$ such that (i) $S$ is the only split in $\mathcal{R}(ab' | ab')$ and (ii) $\mathcal{R}(ab | a'b)$ is empty.

To establish the second assertion, note first that there is nothing to show in case $\# Y \leq 2$. Otherwise, let $Y'$ denote a smallest subset of $Y$ with $\mathcal{R}(Y'|Z) = \{S\}$, i.e., a subset of $Y$ such that $\mathcal{R}(Y' - \{y\}|Z) \neq \{S\}$ holds for all $y \in Y'$, and note that $\# Y' \geq 2$ must hold: Indeed, given any element $z \in Z$ and three distinct elements $y_1, y_2, y_3 \in Y$, at least one of the three sets $\mathcal{R}(y_1y_2|y_3)$, $\mathcal{R}(y_2y_3|y_1)$, $\mathcal{R}(y_3y_1|y_2)$ must be empty (as $\mathcal{R}$ was assumed to be weakly compatible). Furthermore, $\mathcal{R}(Y' - \{y\}|Z) = \mathcal{R}(Y'|Z) \cup \mathcal{R}(Y' - \{y\}|Z \cup \{y\})$ holds for all $Y', Z \subseteq X$ and $y \in Y'$. So, if there were three distinct elements $y_1, y_2, y_3 \in Y'$, $\mathcal{R}(Y' - \{y\}|Z) = \mathcal{R}(Y'|Z) \cup \mathcal{R}(Y' - \{y\}|Z \cup \{y\}) = \mathcal{R}(Y'|Z) = \{S\}$ would hold for at least that index $i \in \{1, 2, 3\} = \{i, j, k\}$ with $\mathcal{R}(y_1y_k|y_3) = \emptyset$. Clearly, this contradicts our choice of $Y'$.

To state our next result, note first that one can associate, to any $\mathcal{R}$-indexed family $F = \{(y_5, y'_5; z_5, z'_5)\}_{S \in \mathcal{R}}$ of quadruples $(y_5, y'_5; z_5, z'_5) \in X^4$, a group homomorphism

$$\lambda^F : L_2(X|A) \to \mathcal{R}(A) : A \mapsto \Sigma_{(A, F)}$$

from $L_2(X|A)$ back into $\mathcal{R}(A)$ by defining the $\lambda^F$-image of an element $A$ in $L_2(X|A)$ to be the map $\Sigma_{(A, F)}$ from $\Sigma^+(X)$ into $A$ that maps any split $S \in \mathcal{R}$ onto $\Lambda(y_5z'_5; z_5y'_5)$, and any split $S \in \Sigma^+(X) - \mathcal{R}$ onto the neutral element $0_A$ in $A$:

$$\Sigma_{(A, F)}(S) := \Lambda(y_5z'_5; z_5y'_5) \quad \text{for every split } S \in \mathcal{R}$$

and

$$\Sigma_{(A, F)}(S) := 0_A \quad \text{for every split } S \in \Sigma^+(X) - \mathcal{R}.$$ 

Then, the following holds:

**Lemma 4.3.** Given any collection $\mathcal{R} \subseteq \Sigma^+(X)$ of proper splits of $X$ such that, for every split $S \in \mathcal{R}$, there exists a quadruple $(y, y'; z, z') \in X^4$ of elements $y, y', z, z'$ in $X$ with

$$\{S\} = \mathcal{R}(yy'|zz') \quad \text{and} \quad \mathcal{R}(yz|y'z') = \emptyset,$$

the restriction

$${\sigma}_{\text{bil}}^\mathcal{R} : \mathcal{R}(A) \to L_2(X|A)$$

of the map $\sigma_{\text{bil}}^\mathcal{R} : \Sigma^+(X|A) \to L_2(X|A)$ to $\mathcal{R}(A)$ is split-injective, i.e., it maps $\mathcal{R}(A)$ onto a direct summand of $L_2(X|A)$.

More specifically, choosing a family $F = F_\mathcal{R} = \{(y_5, y'_5; z_5, z'_5)\}_{S \in \mathcal{R}}$ of such quadruples $(y_5, y'_5; z_5, z'_5) \in X^4$, one for each split $S$ in $\mathcal{R}$,

$$\Sigma = \lambda^F (\sigma_{\text{bil}}^\mathcal{R} (\Sigma))$$

holds, for every map $\Sigma$ in $\mathcal{R}(A)$, for the associated group homomorphism $\lambda^F$, implying in particular that $L_2(X|A)$ is the direct sum of the image of $\sigma_{\text{bil}}^\mathcal{R}$ and the kernel of the group homomorphism $\lambda^F$. 

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This follows easily from the fact that in case
for some map \(\Sigma \in \delta^*(X|A)\) whose support \(\Sigma\) is weakly compatible, and any element \(z \in X\), then we have
\[
\Sigma = \Sigma' \iff \sigma_{\text{bil}}(\Sigma) = \sigma_{\text{bil}}(\Sigma') \iff \sigma_2(\Sigma) = \sigma_2(\Sigma') \iff \sigma_3(\Sigma) = \sigma_3(\Sigma').
\]
In particular, one has \(\Sigma = \Sigma'\) for any two maps \(\Sigma_1, \Sigma_2 \in \delta^*(X|A)\) with \(\sigma_{\text{bil}}(\Sigma_1) = \sigma_{\text{bil}}(\Sigma_2)\), respectively, for which \(\Sigma_1, \Sigma_2\) are compatible split systems. Furthermore, if \(A\) is a group without 2-torsion, all the assertions above remain true if one replaces the group homomorphism \(\sigma_{\text{bil}}\) by the group homomorphism \(\sigma_2\).

Another corollary deals with the question of whether there are any characteristic properties of maps in \(P_{\leq 2}(X, z|A)\), \(L_2(X|A)\), or \(P_{\leq 3}(X|A)\) that are images of maps \(\Sigma \in \delta^*(X|A)\) with weakly compatible support. In many papers, cf. [1–3, 5–6, 9, 13, 17] (see also [12], the third part of these notes), it has been shown that maps in \(L_2(X|A)\), \(P_{\leq 3}(X|A)\), or \(P_{\leq 2}(X, z|A)\) that are images of maps with compatible support share certain very specific properties. Remarkably, it is easy to see that there are no such characteristic properties, i.e., any map in \(L_2(X|A), P_{\leq 3}(X|A), \) or \(P_{\leq 2}(X, z|A)\) is the image of a map with weakly compatible support:

\[\text{Corollary 4.5. Every map } \Lambda \in L_2(X|A) \text{ is of the form } \Lambda = \sigma_{\text{bil}}(\Sigma) \text{ for some map } \Sigma \in \delta^*(X|A) \text{ whose support supp}(\Sigma) \text{ is a weakly compatible split system, e.g., a split system contained in the split system } R_{\mathbb{C}} \text{ for some cyclic graph } C \text{ with vertex set } X. \]

\[\text{Proof. This follows easily from the fact that in case } A = \mathbb{Z}, \text{ the four groups } R(\mathbb{Z}), L_2(X|\mathbb{Z}), P_{\leq 3}(X|\mathbb{Z}), \text{ and } P_{\leq 2}(X, z|\mathbb{Z}) \text{ have the same rank } \left(\begin{array}{c} \mathbb{Z}^3 \\ \mathbb{Z}^2 \end{array}\right), \text{ that the split injective group homomorphisms from } R(\mathbb{Z}) \text{ into } L_2(X|\mathbb{Z}), P_{\leq 3}(X|\mathbb{Z}), \text{ and } P_{\leq 2}(X, z|\mathbb{Z}) \text{ must therefore be isomorphisms, and that, in case of an arbitrary abelian group } A, \text{ the corresponding (split injective) group homomorphisms from } R(A) \text{ into } L_2(X|A), P_{\leq 3}(X|A), \text{ and } P_{\leq 2}(X, z|A), \text{ respectively, can be canonically identified with the group homomorphisms one obtains by forming the tensor product with } A \text{ over } \mathbb{Z} \text{ of these group homomorphisms from } R(\mathbb{Z}) \text{ into } L_2(X|\mathbb{Z}), P_{\leq 3}(X|\mathbb{Z}), \text{ and } P_{\leq 2}(X, z|\mathbb{Z}). \]

5. The image of restrictions of trace homomorphisms to \(\Lambda\)-valued split systems with given weakly compatible support

In view of these results, it is apparently also of interest to characterize, for any split system \(R \subseteq \delta^*(X)\) of \(X\), the elements \(\Lambda\) in the image \(\sigma_{\text{bil}}(R(\Lambda)) \subseteq L_2(X|\Lambda)\) of the subgroup \(R(\Lambda)\) of \(\delta^*(X|A)\) in \(L_2(X|\Lambda)\). Clearly, we must have \(\Lambda(y_1z_1' : y_1'z_1) = \Lambda(y_2z_2' : y_2'z_2')\) for any such \(\Lambda\) and all elements \(y_1, z_1, y_1', z_1', y_2, z_2, y_2', z_2'\) in \(X\) for which \(\Lambda(y_1y_1' : z_1z_1') = \Lambda(y_2y_2' : z_2z_2')\) and \(\Lambda(y_1y_1' : y_1z_1') = \Lambda(y_2y_2' : y_2z_2')\) hold. In particular, \(\Lambda(yz' : y') = 0\) must hold for all \(y, z, y', z' \in X\) with \(R(yz' : y') = R(yz : y) = 0\).

Conversely, if \(\Lambda\) satisfies the assumptions of Lemma 4.3, some family \(F = F_R = \{(y_5, y_5' : z_5, z_5')\}_{R \subseteq \delta^*(X)}\) as above has been chosen, and some map \(\Lambda \in L_2(X|\Lambda)\) is given for which \(\Lambda(y_1z_1' : z_1y_1') = \Lambda(y_2z_2' : z_2y_2')\) holds for all elements \(y_1, z_1, y_1', z_1', y_2, z_2, y_2', z_2'\) in \(X\) for which
\[
\Lambda(y_1y_1' : z_1z_1') = \Lambda(y_2y_2' : z_2z_2').
\]
and \(\#R(y_1y_1' : z_1z_1') \leq 1\), the only candidate for a map in \(R(\Lambda)\) that could have \(\Lambda\) as its \(\sigma_{\text{bil}}\)-image is the map \(\chi^F(\Lambda) = \Sigma_{(\Lambda,F)}\) as defined in the proof of Lemma 4.3 that would, moreover, be uniquely defined by \(R\) and \(\Lambda\) in this case, and independent of the choice of \(F = F_R\).

In this context, the following result therefore seems to be of some interest:

**Theorem 5.1.** (i) Given a weakly compatible split system \(R \subseteq \delta^*(X)\), a map \(\Lambda \in L_2(X|\Lambda)\) is contained in the corresponding direct summand \(\sigma_{\text{bil}}(R(\Lambda))\) of \(L_2(X|\Lambda)\) if and only if
\[ \Lambda(y_1z'_1 : z_1y'_1) = \Lambda(y_2z'_2 : z_2y'_2) \]
holds for all elements \( y_1, z_1, y'_1, z'_1, y_2, z_2, y'_2, z'_2 \) in \( X \) for which
\[ \mathcal{R}(y_1y'_1 | z_1z'_1) = \mathcal{R}(y_2y'_2 | z_2z'_2), \quad \mathcal{R}(y_1z_1 | y'_1z'_1) = \mathcal{R}(y_2z_2 | y'_2z'_2) = \emptyset, \]
and \# \mathcal{R}(y_1y'_1 | z_1z'_1) \leq 1 hold.

(ii) If, furthermore, \( \mathcal{R} \subseteq \delta^*(X) \) is not only a weakly compatible, but a compatible split system, one has \( \Lambda \in \sigma_{bd}( \mathcal{R}(A) ) \) for some \( \Lambda \in \mathcal{L}_2(X|A) \) if and only if \( \Lambda(yz' : zy') = 0_A \) holds for all elements \( y, z, y', z' \) in \( X \) for which \( \mathcal{R}(yy'| zz') = \mathcal{R}(yz| y'z') = \emptyset \) holds.

**Proof.** It was noted already above that the conditions stated in (i) and (ii) are necessary for a map \( \Lambda \in \mathcal{L}_2(X|A) \) to be contained in \( \sigma_{bd}( \mathcal{R}(A) ) \) and that the condition stated in (i) implies that, whatever family \( \mathcal{F} = \mathcal{F}_R \) one may choose to construct a group homomorphism
\[ \lambda^\mathcal{F} : \mathcal{L}_2(X|A) \to \mathcal{R}(A) \subseteq \delta^*(X|A), \]
the image \( \lambda^\mathcal{F}(A) \in \sum_{\delta^*(X|A)} \) of \( A \) as constructed in the proof of Lemma 4.3 is independent of the choice of \( \mathcal{F} \) implying in particular that
\[ \Lambda(yz' : y'z) = 0_A \]
holds for \( y, z, y', z' \) in \( X \) for which \( \mathcal{R}(yy'| zz') = \mathcal{R}(yz| y'z') = \emptyset \) holds (just choose any \( x \) in \( X \) and note that, in view of \( \mathcal{R}(yy'| zz') = \mathcal{R}(xx|xx) = \emptyset \) and \( \mathcal{R}(yz| y'z') = \mathcal{R}(xx|xx) = \emptyset \), our assumption implies that \( \Lambda(yz' : y'z) = \Lambda(xx : xx) = 0_A \) must hold) and that, more generally,
\[ \Lambda(yz' : y'z) = \Lambda'(yz' : y'z) \]
must hold for the map \( \Lambda' := \sigma_{bd}(\sum_{\delta^*(X|A)}) \) for all \( y, z, y', z' \) in \( X \) for which
\[ N(yz' : y'z) = N(yy' : y'z)_R := \#(\mathcal{R}(yy'| zz') \cup \mathcal{R}(yz| y'z')) \]
does not exceed 1. We may therefore proceed by induction with respect to \( N(yy' : y'z) \).

To reduce repetitive arguments, note first that, by assumption,
\[ \mathcal{R}(ab|cd), \mathcal{R}(ad|bc) \neq \emptyset \Rightarrow \mathcal{R}(ac|bd) = \emptyset \]
holds for any 4 points \( a, b, c, d \) in \( X \), implying in turn that
\[ \mathcal{R}(ab|cd), \mathcal{R}(ae|bd) \neq \emptyset \Rightarrow \mathcal{R}(ad|be) = \mathcal{R}(ac|bd) = \emptyset \]
holds for any 5 points \( a, b, c, d, e \) in \( X \) because \( \mathcal{R}(ab|cd), \mathcal{R}(ae|bd) \neq \emptyset \) implies \( \mathcal{R}(ab|ce), \mathcal{R}(ae|bc) \neq \emptyset \) as well as \( \mathcal{R}(ab|de), \mathcal{R}(ae|bd) \neq \emptyset \) and, therefore, \( \mathcal{R}(ac|be) = \mathcal{R}(ad|be) = \emptyset \) which readily implies our claim in view of \( \mathcal{R}(ac|be) = \mathcal{R}(ac|bd) \cup \mathcal{R}(ac|bd) \) and \( \mathcal{R}(ad|be) = \mathcal{R}(ad|be) \cup \mathcal{R}(ad|be) \).

Next, observe that it will be sufficient to deal with the two cases

(i) \( \mathcal{R}(yy'| zz') \neq \emptyset, \) \( \mathcal{R}(yz| y'z') \neq \emptyset, \)
(ii) \( \mathcal{R}(yy'| zz') = \emptyset \) and \# \( \mathcal{R}(yz| y'z') \geq 2 \).

In the first case, both, \( N(yz : y'z) \) and \( N(yy' : zz') \) must be positive, and we must have \( \mathcal{R}(yz| y'z) = \emptyset \) as well as
\[ N(yz : y'z) + N(yy' : zz') = N(yy' : y'z), \]
and, hence,
\[ N(yy' : zz') < N(yy' : y'z). \]
So, induction implies that
\[ \Lambda(yz : y'z) = \Lambda'(yz : y'z) \quad \text{and} \quad \Lambda(yy' : zz') = \Lambda'(yy' : zz') \]
and, therefore, also
\[ \Lambda(yz : y'z) = - \Lambda(yy' : zz') - \Lambda(yz : z'y') \]
\[ = - \Lambda'(yy' : zz') - \Lambda'(yz : z'y') \]
\[ = \Lambda'(yz : y'z) \]
must hold in this case, as claimed.

In case (ii), some \( x \in X \) with \( \mathcal{R}(yxz| y'z') \), \( \mathcal{R}(yxz| y'z') \neq \emptyset \) must exist (just, choose any two distinct splits \( S = \{A, B\} \) and \( S' = \{A', B'\} \) in the set \( \mathcal{R}(yz| y'z') \) with, say, \( y, z \in A \cap A' \) and any element \( x \in A \Delta A' \)). Further,
\[ \mathcal{R}(xyy'| zz') = \mathcal{R}(yy'| xxz') = \emptyset \]
must hold for any such element \( x \) in view of our assumption \( \mathcal{R}(yy'| xxz') = \emptyset \).
Now, we distinguish the three subcases

((Ia)): \( R(xy'y'z) = R(xy'yz') = \emptyset \),
((Ib)): \( R(xy'y'z') = R(xz'|yy'z) = \emptyset \),
((Ic)): neither ((Ia)) nor ((Ib)) holds.

Further, exchanging \( y \) with \( z \) and \( y' \) with \( z' \) in Case ((Ib)), we may assume by symmetry that either (Ia) or (Ic) holds.

In Case (Ia), we have
\[
R(yz'y') = R(xyz|yz') \cup R(yz|yz')
\]
and therefore – in view of \( R(xz'|y') \cap R(yz|xy') \subseteq R(xy'|x'y') \cap R(z|xy') = \emptyset \), \( \emptyset \neq R(yz|x'y') \subseteq R(xz|y'z') \), and \( \emptyset \neq R(yz|x'y') \subseteq R(yz|xy') \) – also
\[
0 < \#R(xz|y'z') \cap R(yz|xy') < \#R(yz'y') = N(yz': y'z),
\]
as well as
\[
R(xy'|zz') = R(xyy'|zz') \cup R(xy'|yzz') = \emptyset
\]
and
\[
R(yy'|xz) = R(yy'|z)|xz) \cup R(yy'|xxz') = \emptyset,
\]
implying that
\[
N(xy : y'z) = \#R(yz|xy') + \#R(yy'|xz) = \#R(yz|xy') < N(yz': y'z)
\]
and
\[
N(xy' : y'z') = \#R(xz|x'y') + \#R(xy'|zz') = \#R(xz|x'y') < N(yz': y'z)
\]
and therefore, by induction,
\[
\Lambda(xy : y'z) = \Lambda'(xy : y'z) \quad \text{and} \quad \Lambda(xz' : y'z) = \Lambda'(xz' : y'z)
\]
must hold. Thus,
\[
\Lambda(yz : y'z') = \Lambda'(xy : y'z) + \Lambda(xz' : y'z)
\]
\[
= \Lambda'(xy : y'z) + \Lambda'(xz' : y'z)
\]
\[
= \Lambda'(yz : y'z)
\]
(3)
must also hold in this case, as claimed.

In other words, our claim holds by induction in case
\[
R(xyz|y'z') \cap R(yz|xy'z') \neq \emptyset
\]
and
\[
R(xyy'|zz') = R(xyy'|zz') = R(xz|yy'z') = R(xy'|yzz') = \emptyset.
\]
Finally, in Case (Ic), we may assume without loss of generality that
\[
R(xz|yy'z') \neq \emptyset
\]
holds which, in view of \( R(yz|xy'z') \neq \emptyset \) and Assertion (2) above implies that
\[
R(yz|xy'z') = R(xz'|yy'z) = R(y'z'|xy) = \emptyset
\]
must hold. So, our assumption that (Ib), i.e., \( R(xy|yy'z') \cap R(xz'|yy'z) = \emptyset \), does not hold, implies that also \( R(xz'|yy'z) \neq \emptyset \) and thus, in view of \( R(xz|yy'z') \neq \emptyset \), also
\[
R(xy|yy'z') = R(xy|yy'z') = R(xy|yy'z') = \emptyset
\]
and, in view of \( R(xyz|y'z') \neq \emptyset \) (by the choice of \( x \)), also
\[
R(z'y|xy') = R(z'y|xy') = R(z'y|xy') = \emptyset
\]
holds. Thus, we have
\[
N(yz : y'z') = \#R(yz'|y'z') + \#R(yy'|zz')
\]
\[
= \#R(xyz'|y'z') + \#R(yz'|xy') + \#R(xyy'|zz') + \#R(yy'|xxz') = 0
\]
and, therefore, \( \Lambda(yz : y'z') = \Lambda'(yz : y'z') = 0 \), and, hence, also
\[
\Lambda(yz' : y'z) = -\Lambda(yy' : zz') - \Lambda(yz' : z'y) = -\Lambda(zz' : yy')
\]
and
\[
\Lambda'(yz' : y'z) = -\Lambda'(yy' : zz') - \Lambda'(yz : z'y') = -\Lambda'(zz' : yy')
\]
in this case.

Furthermore, we now have \( \mathcal{R}(xzy'y') \), \( \mathcal{R}(zy'yx') \neq \emptyset \) and \( \mathcal{R}(xzy'yz') = \mathcal{R}(xyz'|yxz') = \mathcal{R}(xy'|zyzz') = \emptyset \). So, noting that also
\[
N(yy' : zz') = \#\mathcal{R}(yz|y'z') + \#\mathcal{R}(yz'|y'z)
\]
holds and replacing \( y \) by \( z \) and \( z \) by \( y \) in the argument above leading to Formula (3), we can use induction as before, yielding that
\[
\Lambda(zz' : y'y) = \Lambda'(zz' : y'y)
\]
must hold. Yet, together with
\[
\Lambda(yz' : y'z) = -\Lambda(zz' : yy') \quad \text{and} \quad \Lambda'(yz' : y'z) = -\Lambda'(zz' : yy'),
\]
this implies that also
\[
\Lambda(yz' : y'z) = -\Lambda(zz' : yy') = -\Lambda'(zz' : yy') = \Lambda'(yz' : y'z)
\]
must hold. Altogether, this implies the first assertion.

To also establish the second assertion, let us now assume that \( \mathcal{R} \) is a compatible split system. The necessity of the conditions listed in (ii) is obvious. Conversely, Assertion (i) implies that all we have to show is that
\[
\Lambda(y_1z_1' : z_1y_1') = \Lambda(y_2z_2' : z_2y_2')
\]
holds for all elements \( y_1, z_1, y_1', z_1', y_2, z_2, y_2', z_2' \) in \( X \) for which
\[
\mathcal{R}(y_1y_1'|z_1z_1') = \mathcal{R}(y_2y_2'|z_2z_2'), \quad \mathcal{R}(y_1z_1|y_1'z_1') = \mathcal{R}(y_2z_2|y_2'z_2') = \emptyset,
\]
and \( \#\mathcal{R}(y_1y_1'|z_1z_1') \leq 1 \) hold. By assumption, this holds indeed in case \( \#\mathcal{R}(y_1y_1'|z_1z_1') = 0 \). So, assume without loss of generality that \( \{S\} = \mathcal{R}(y_1z_1|y_1'z_1') = \mathcal{R}(y_2z_2|y_2'z_2') \) holds for some \( S = \{A, A'\} \) in \( \mathcal{R} \) and some elements \( y_1, z_1, y_2, z_2 \in A \) and \( y_2', z_2', z_2' \in A' \), and note first that this implies that also
\[
\mathcal{R}(y_1y_1'|z_1z_1') = \mathcal{R}(y_2y_2'|z_2z_2') = \mathcal{R}(y_1z_1|y_1'z_1') = \mathcal{R}(y_2z_2|y_2'z_2') = \emptyset
\]
must hold. It suffices to show that \( \Lambda(y_1z_1' : z_1y_1') = \Lambda(y_2z_2' : z_2y_2') \) holds in this case.

To simplify discussions, we will now use the well known fact (see, e.g., [22]) that, given a compatible split system \( \mathcal{R} \subset \mathcal{S}(X) \), there exists an (essentially unique) tree \( T = (V, E) \) with vertex set \( V \) and edge set \( E \) and a map \( \varphi : X \to V \) such that every vertex in \( V \) - \( \varphi(X) \) has degree at least 3, and there exists a one-to-one correspondence between \( \mathcal{R} \) and \( E \) such that an edge \( e \in E \) corresponds to a split \( S = \{A, A'\} \) in \( \mathcal{R} \) if and only if \( e \) separates every \( \varphi(a) \) in \( A \) from every \( \varphi(a') \) in \( A' \) in which case one also writes \( S = S_e \) and \( e = e_5 \). Recall also that
\[
\mathcal{R}(a_1a_1'|a_2'a_2') = \{S_e : e \in E \text{ separates } \varphi(a_1), \varphi(a_2) \text{ from } \varphi(a_1'), \varphi(a_2')\}
\]
holds for all \( a_1, a_2, a_1', a_2' \in X \) in this case.

Consequently, our assumptions above imply that either
\[
\mathcal{R}(y_1z_1|y_1'z_1') = \mathcal{R}(y_2z_2|y_2'z_2') = \emptyset
\]
or
\[
\mathcal{R}(y_1z_1|y_1'z_1') = \mathcal{R}(y_2z_2|y_2'z_2') = \{S\}
\]
must hold implying that we need to establish Eq. (4) only in case \( y_1' = y_2', z_1' = z_2' = z' \).

Furthermore, it follows also that \( \{S\} \) coincides either with \( \mathcal{R}(y_1y_2|y'z') \) or with \( \mathcal{R}(z_1z_2|y'z') \) implying that we need to establish Eq. (4) only in case, say, \( z_1 = z_2 = z \). However, in this case, \( \mathcal{R}(y_1y_2|y_2z) \) must be empty because any split in \( \mathcal{R}(y_1y_2|y_2z) \) would also separate \( y_2 \) as well as \( z \) from \( y' \) and \( z' \) and be distinct from \( S \), and \( \mathcal{R}(y_2y_2'|y_1z) \) must also be empty because any split in \( \mathcal{R}(y_2y_2'|y_1z) \) would also separate \( y_1 \) as well as \( z \) from \( y' \) and \( z' \) and be distinct from \( S \). Thus, we have
\[
\mathcal{R}(y_1y_2|y_2z) = \emptyset
\]
and therefore, by assumption, \( \Lambda(y_1y_2 : zy') = 0 \) implying that also
\[
\Lambda(y_1z_1' : zy') = \Lambda(y_2z_2' : zy')
\]
must hold as claimed.

Thus, we have
\[
\Lambda(y_1z_1' : zy') = \Lambda(y_2z_2' : zy') = \Lambda(y_1y_2 : zy') \neq 0,
\]
and therefore, we have
\[
\Lambda(y_1z_1' : zy') = \Lambda(y_2z_2' : zy') = \Lambda(y_1y_2 : zy').
\]
Remark 5.2. It is easy to see that the condition in the first assertion cannot be replaced by the condition stated in the second one as there are non-maximal weakly compatible split systems $R$ such that $R(ab|cd) = R(ac|bd) = \emptyset$ never holds for any four distinct elements $a, b, c, d \in X$. E.g., put $X := \{0, 1, 2, 3, 4\}$ and let $R$ denote the set consisting of the four splits $S_{1,2}, S_{2,3}, S_{3,4}, S_{4,1}$.

It follows from the results from [11] recalled in Section 3 that Theorem 5.1 implies – and is indeed equivalent with each of the first two of – the following three corollaries:

**Corollary 5.3.** (i) Given a weakly compatible split system $R \subseteq \delta^*(X)$, a map $\Pi \in \mathcal{P}_{\leq 2}(X, z \mid A)$ is contained in the direct summand $\sigma_2(R(A))$ of $\mathcal{P}_{\leq 2}(X, z \mid A)$ if and only if
\[
\Pi(y_1'y_1') + (z'_{2}z'_{1}z'_{2}) - \Pi(y_2z_2) - \Pi(z'_{2}z'_{2})
\]
holds for all elements $y_1, y_1', y_1', z_2, z_2', z_2' \in X$ for which
\[
\Psi(y_1'y_1'|z_2z_2') = \Psi(y_2z_2|z_2z_2'), \quad \Psi(y_1z_1'|z_2z_2') = \emptyset,
\]
and $\#\Psi(y_1'y_1'|z_2z_2') \leq 1$ hold.

(ii) If, furthermore, $R \subseteq \delta^*(X)$ is not only a weakly compatible, but a compatible split system, one has $\Pi \in \sigma_2(R(A))$ for some $\Pi \in \mathcal{P}_{\leq 2}(X, z \mid A)$ and some $z \in X$ if and only if
\[
\Pi(yz) + (z'y') = \Pi(yy') + (z'z)
\]
holds for all elements $y, y', z' \in X$ for which $\Psi(yy'|zz') = \Psi(yz|y'z') = \emptyset$ holds.

**Corollary 5.4.** (i) Given a weakly compatible split system $R \subseteq \delta^*(X)$, a map $\Psi \in \mathcal{P}_{3}(X\mid A)$ is contained in the direct summand $\sigma_3(R(A))$ of $\mathcal{P}_{3}(X\mid A)$ if and only if
\[
\Psi(y_1'y_1'|z_2z_2') - \Psi(y_2z_2|z_2z_2') = \Psi(y_2z_2|z_2z_2') - \Psi(z_2z_2) + \Psi(z_2z_2')
\]
holds for all elements $y_1, y_1', z_2, z_2', z_2' \in X$ for which
\[
\Psi(y_1'y_1'|z_2z_2') = \Psi(y_2z_2|z_2z_2'), \quad \Psi(y_1z_1'|z_2z_2') = \emptyset,
\]
and $\#\Psi(y_1'y_1'|z_2z_2') \leq 1$ hold.

(ii) If, furthermore, $R \subseteq \delta^*(X)$ is not only a weakly compatible, but a compatible split system, one has $\Psi \in \sigma_3(R(A))$ for some $\Psi \in \mathcal{P}_{3}(X\mid A)$ if and only if
\[
\Psi(yz') + \Psi(z'y') = \Psi(yz') + \Psi(z'z)
\]
holds for all elements $y, z, y', z' \in X$ for which $\Psi(yy'|zz') = \Psi(yz|y'z') = \emptyset$ holds.

**Corollary 5.5.** (i) Provided the group $A$ does not contain any element of order 2, then, given a weakly compatible split system $R \subseteq \delta^*(X)$, a map $\Pi \in \mathcal{P}_{\leq 2}(X\mid A)$ is contained in the direct summand $\sigma_2(R(A))$ of $\mathcal{P}_{\leq 2}(X\mid A)$ if and only if
\[
\Pi(yz_1) + (z'y_1') - \Pi(yz_1') = \Pi(y_2z_2) + (z_2y_2') - \Pi(z_2y_2') - \Pi(z_2z_2)
\]
holds for all elements $y_1, y_1', z_1, y_2, z_2, y_2', z_2' \in X$ for which
\[
\Psi(y_1'y_1'|z_2z_2') = \Psi(y_2z_2|z_2z_2'), \quad \Psi(y_1z_1'|z_2z_2') = \emptyset,
\]
and $\#\Psi(y_1'y_1'|z_2z_2') \leq 1$ hold.

(ii) If, furthermore, $R \subseteq \delta^*(X)$ is not only a weakly compatible, but a compatible split system, one has $\Pi \in \sigma_2(R(A))$ for some $\Pi \in \mathcal{P}_{\leq 2}(X\mid A)$ if and only if
\[
\Pi(yz) + (z'y') = \Pi(yy') + (z'z)
\]
holds for all elements $y, z, y', z' \in X$ for which $\Psi(yy'|zz') = \Psi(yz|y'z') = \emptyset$ holds.

**Remark 5.6.** It follows in particular that, e.g., $\Pi(ab) = \Pi(a'b')$ must hold, for any $\Pi \in \sigma_2(R(A))$, for all $a, a', b, b' \in X$ with $R(ab|z) = R(a'b'|z)$ and, therefore, for any $\Pi \in \mathcal{P}_{\leq 2}(X, z \mid A)$ that satisfies the condition stated in Corollary 5.3. However, there does not seem to be a simpler argument for establishing this than the inductive procedure used in the proof of Theorem 5.1. This is particularly remarkable as the condition that $\Pi(ab) = \Pi(a'b')$ holds for all $a, a', b, b' \in X$ with $R(ab|z) = R(a'b'|z)$ is not even sufficient for a map $\Pi \in \mathcal{P}_{\leq 2}(X, z \mid A)$ to be contained in $\sigma_2(R(A))$ (unless $A$ is the trivial group): Consider for instance the example $X := \{1, 2, 3, 4\}$ and
\[
R := \{S_{1,2}, S_{1,4}\},
\]
put $z := 1$, and let $\Pi$ denote the map in $\sigma_2(R(A))$ that maps the subsets in $X_{\leq 2}$ containing $z$ and the subset $\{2, 4\}$ onto the neutral element 0$_A$ and every other subset in $X_{\leq 2}$ onto one fixed element in $\mathcal{A}$ distinct from 0$_A$. Then, $\Pi(ab) = \Pi(a'b')$. 

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holds for all \( a, a', b, b' \in X \) with \( R(ab|z) = R(a'b'|z) \) (as \( R(ab|z) = \emptyset \) holds exactly for all \( \{a, b\} \in \left( \begin{array}{c} X \\ \leq 2 \end{array} \right) \)) while, in view of \( R(3|1) = \{S_{1,2}, S_{1,4}\}, R(4|1) = \{S_{1,2}\} \) and \( R(2|1) = \{S_{2,3}\} \), one must have \( \Pi(2) = \Pi(1) + \Pi(3) \) for all \( \Pi \) in \( \sigma_2(R(A)) \).

Similar considerations apply as well for the maps in the groups \( \mathcal{P}_{3}(X|A) \) and \( \mathcal{P}_{2}(X|A) \).

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References