The Existence Theorems of the Random Solutions for Random Hammerstein Type Nonlinear Integral Equations

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Abstract—This paper deals with the existence theorems of random solutions of random Hammerstein type nonlinear integral equations. These theorems are proved by using the random fixed-point theorems of cone expansion and compression of random operator discussed by Li and Sheng [1]. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords—Random integral equation, Random solution, Random fixed-point.

1. INTRODUCTION

Let \((\Omega, \mathcal{U}, \mu)\) be a complete measure space, \(E\) a separable infinite dimensional real Banach space, \((E, \beta)\) a measurable space, where \(\beta\) denotes the \(\sigma\)-algebra of all Borel subsets generated by all open subsets in \(E\), \(P\) a cone \([2]\) in \(E\), \(P_r = \{x \in P, \|x\| < r\}\), \(P_r = \{x \in P, \|x\| = r\}\), \(\partial P_r = \{x \in P, \|x\| = r\}\), \(P_{r_1, r_2} = \{x \in P, 0 < r_2 < \|x\| < r_1\}\). A mapping \(T : \Omega \times P_r \rightarrow P_r\) is called a random operator for each \(x \in P_r\), \(T(x, x)\) is measurable.

A measurable map \(\phi : \Omega \rightarrow P_r\) is a random fixed-point of the random operator \(T\) if \(T(\omega, \phi(\omega)) = \phi(\omega)\) for each \(\omega \in \Omega\).

**LEMMA 1.1.** (See [1].) Let \(T : \Omega \times P_r \rightarrow P_r\) be a random completely continuous operator and \(0 < r_2 < r_1 < r\) such that

\[
\begin{align*}
(i) & \quad \left\{ \begin{array}{l}
(\omega, x) \in \Omega \times \partial P_{r_1} \Rightarrow T(\omega, x) \notin x, \\
(\omega, x) \in \Omega \times \partial P_{r_2} \Rightarrow T(\omega, x) \notin x;
\end{array} \right. \\
(ii) & \quad \left\{ \begin{array}{l}
(\omega, x) \in \Omega \times \partial P_{r_1} \Rightarrow T(\omega, x) \notin x, \\
(\omega, x) \in \Omega \times \partial P_{r_2} \Rightarrow T(\omega, x) \notin x.
\end{array} \right.
\end{align*}
\]

Then \(T\) has at least a random fixed-point \(x_0(\omega) \in P_{r_1, r_2}\) where \(r_2 < \|x_0(\omega)\| < r_1\).

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Let $\mathbb{R}^N$ be an $N$-dimensional Euclidean space, $G$ be a bounded closed domain in $\mathbb{R}^N$, $C(G)$ separable Banach space of all continuous functions on $G$, $(C(G), \beta)$ be a measurable space, $\beta\sigma$-algebra of all Borel subsets generated by all open subsets in $C(G)$. Some ideas and results can be found in [3].

We consider a general polynomial type random nonlinear Hammerstein integral equation

$$
\phi = \int_G k(\omega, x, y) f(y, \phi) \, dy = A(\omega, \phi) = A(\omega) \phi,
$$

where $f(x, u) = \sum_{i=1}^{n} a_i(x) u^{a_i}, a_i > 0 (i = 1, 2, \ldots, n)$, and $\phi : \Omega \times G \rightarrow C(G)$ denotes $\phi(\omega)(x)$, since for fixed $x \in G \phi : \Omega \rightarrow C(G)$ is a random variable with values in $C(G)$. For a fixed $\omega \in \Omega$, $\phi(\omega) : G \rightarrow \mathbb{R}$ is a continuous function.

We prove existence theorems of random solutions of integral equation (1.1).

### 2. EXISTENCE THEOREMS

**Theorem 2.1.** Suppose that

(i) $k(\omega, x, y)$ is a nonnegative bounded random continuous kernel such that

$$
\int_G k(\omega, x, y) \, dx > 0, \quad \text{for all } (\omega, y) \in \Omega \times G,
$$

(ii) $a_i(x) \geq 0$, for all $x \in G$, $a_i(x) \in C(G)$, $0 < a_i < 1$ ($i = 1, 2, \ldots, n$)

and there exists some $a_{i_0}$ satisfying $\inf_{x \in G} a_{i_0}(x) > 0$. Then equation (1.1) has at least one nonnegative, not identically vanishing random continuous solution $\phi(\omega, x) \in C(G)$.

**Proof.** First, we construct a cone in $(C(G), \| \cdot \|_C)$

$$
P = \{ \phi \mid \phi \in C(G), \phi(x) \geq 0, \| \phi \|_L \geq \beta M^{-1} \| \phi \|_C \}, \quad (2.1)
$$

where

$$
\beta = \inf_{y \in G} \int_{\omega \in \Omega} k(\omega, x, y) \, dx > 0,
$$

$$
M = \sup_{(x, y) \in \Omega \times G} k(\omega, x, y),
$$

$$
\| \phi \|_C = \max_{\omega \in \Omega} \| \phi(\omega) \|,
$$

$$
\| \phi \|_L = \left( \int_G \| \phi(x) \|^p \, dx \right)^{1/p}.
$$

It is easy to show that $P$ is a cone.

We next consider three steps to prove the theorem.

(i) We prove that for each $\omega \in \Omega$, $A(\omega)(\cdot) : P \rightarrow P$ is a completely continuous operator. In fact, $\phi \in P$, and

$$
\| A(\omega) \phi \|_L = \int_G (A(\omega) \phi) \, dx = \int_G dx \left( \int_G k(\omega, x, y) f(y, \phi(y)) \, dy \right) \geq \beta \left\| \bar{f} \phi \right\|_L,
$$

where $\bar{f} \phi$ denote $f(x, \phi(x))$. On the other hand, $A(\omega) \phi \leq M \| \bar{f} \phi \|_L$ and hence,

$$
\| A(\omega) \phi \|_L \geq \beta M^{-1} \| A(\omega) \phi \|_C.
$$

Hence, $A(\omega) \phi \in P$, that is, $A(\omega)(\cdot) : P \rightarrow P$, for all $\omega \in \Omega$. 

It is easy to see that \( \tilde{f} : P \to L \) is a continuous bounded operator. Then, for any fixed \( \omega \in \Omega \), \( A(\omega, \cdot) : P \to P \) is a completely continuous operator.

(ii) We prove that \( A(\omega)\phi : \Omega \to P \) is a random operator for all \( \phi \in P \).

Set \( Q = C(G \times G) \), \( Q \) is a separable real Banach space with a norm

\[
\| K(x, y) \| = \max_{x, y \in G} | K(x, y) |, \quad \text{for all } K(x, y) \in Q.
\]

Since \( k(\omega, x, y) \) is a random continuous kernel for all \( \omega \in \Omega \), \( k(\omega, x, y) \in Q \). The kernel \( k(\omega, x, y) \) may be seen as mapping \( k(\omega) : \Omega \to Q \), \( k(\omega) \) is a \( Q \)-valued random variable.

We first recall the following known propositions and a corollary.

1. Suppose that \( X \) is a separable Banach space, \( x(\omega) : \Omega \to X \) is an \( X \)-valued random variable if and only if for every linear functional \( x^* \in D \), \( x^*(x(\omega)) \) is a real-valued random variable (where set \( D \subset X^* \) is said to be a total set on \( X \), i.e., if for every \( x \neq 0 \), \( x \in X \), there exists one \( x^* \in D \) such that \( x^*(x) \neq 0 \)).

2. In a separable Banach space, random variable \( \Rightarrow \) weakly random variable.

3. Let \( X \) be a separable Banach space, \( X_0^* \) be a total set on \( X \), \( x : \Omega \to X \), then \( x(x) \) is a weakly random variable \( \Rightarrow \) for all \( x^* \in X_0^* \), \( x^*(x(\omega)) \) is a real-valued random variable.

**Corollary.** In a separable Banach space, \( x(\omega) : \Omega \to X \) is \( X \)-random variable if and only if for all \( x^* \in X_0^* \), \( x^*(x(\omega)) \) is a real-valued random variable. We write

\[
\tilde{f}(y) = f(y, \phi(y)) \in C(G), \quad \phi \in C(G).
\]

We then set

\[
\tilde{A}(\omega) \tilde{f} = A(\omega) \phi = \int_G k(\omega, x, y) \tilde{f}(y) \, dy.
\]

Obviously, for all \( \omega \in \Omega \), \( \tilde{A}(\omega) : C(G) \to C(G) \) is completely continuous.

We construct two total sets on \( Q \).

(a) If \( (x_0, y_0) \in G \times G \), set \( g_{x_0, y_0}(k) = k(x_0, y_0) \), for all \( k(x, y) \in Q \). Hence, \( Q_1^* = \{ g_x, y : x, y \in G \} \) is a total set on \( Q \) and for all \( \omega \in \Omega \), \( g_{x_0, y_0}(\omega, x, y) = k(\omega, x, y) \).

(b) If \( x_0 \in G \), \( f_0 \in C(G) \), set

\[
H_{x_0, f_0}(k) = \int_G k(x_0, y) f_0(y) \, dy, \quad \text{for all } k(x, y) \in Q.
\]

Hence, \( Q_2^* = \{ H_{x, f} : x \in G, \ f \in C(G) \} \) is a total set on \( Q \) and

\[
H_{x, f}(k(\omega, x, y)) = \int_G k(\omega, x, y) \tilde{f}(y) = \tilde{A}(\omega) \tilde{f} = A(\omega) \phi.
\]

From known propositions \( Q_1^* \) and \( Q_2^* \), we know that \( k(\omega, x, y) : \Omega \to Q \) is a random variable, where \( k \) is a random kernel and

\[
\Rightarrow \text{ for all } x \in G, \text{ for all } f \in C(G), \Phi(\omega, x) = \int_G k(\omega, x, y) f(y) \, dy \text{ is a random variable},
\]

\[
\Rightarrow \text{ for all } f \in C(G), \Phi(\omega, x) = \int_G k(\omega, x, y) f(y) \, dy \text{ is a } C(G) \text{-valued random variable},
\]

\[
\Rightarrow \text{ for all } f \in C(G), \tilde{A}(\omega) f \text{ is a } C(G) \text{-valued random variable},
\]

\[
\Rightarrow \tilde{A} : \Omega \times C(G) \to C(G) \text{ is a random operator}.
\]

In particular, we take \( \tilde{f}(y) = f(y, \phi(y)) \), \( \tilde{A}(\omega) \tilde{f} = A(\omega) \phi \) is a \( C(G) \)-valued random operator.

Then \( \phi \in P \subset C(G) \), \( A(\omega) \phi : \Omega \times P \to P \) is a random completely continuous operator.

(iii) We prove that \( A(\omega) \phi \), i.e., \( A(\omega, \phi) \) satisfies conditions of Lemma 1.1.
Suppose that $P_r$ is a bounded open set in $P(r > 0)$

$$0 < r < (\tau_0 \beta)^{1 - a_{i_0}}^{-1}, \quad \tau_0 = \inf_{x \in G} a_{i_0}(x) > 0.$$

(2.2)

We prove that

for all $(\omega, \phi) \in \Omega \times \partial P_r \Rightarrow A(\omega, \phi) \notin \phi.$

(2.3)

In fact, if there is $(\omega_0, \phi_0) \in \Omega \times \partial P_r$ such that

$$A(\omega_0, \phi_0) \leq \phi_0, \quad 0 \leq \phi_0(x) \leq r$$

so that

$$\phi_0(x) \geq A(\omega_0, \phi_0(x)) \geq \int_G k(\omega_0, x, y) a_{i_0}(y) [\phi_0(y)]^{a_{i_0}} dy \geq \frac{\tau_0}{r^{1 - a_{i_0}}} \int_G k(\omega_0, x, y) \phi_0(y) dy,$$

(2.4)

since

$$\int_G \phi_0(x) dx > 0, \quad r \leq (\tau_0 \beta) (1 - a_{i_0})^{-1}.$$

This contradicts (2.2), then (2.3) holds.

On the other hand, we take $R > r$ such that

$$R > M \sum_{i=1}^n \|a_i\| L R^{a_i}.$$

(2.5)

We prove that

$$(\omega, \phi) \in \Omega \times \partial P_R \Rightarrow A(\omega, \phi) \notin \phi.$$

(2.6)

This contradicts (2.4), then (2.5) holds.

From (2.3), (2.5), and Lemma 1.1, we know that $A(\omega, \phi)$ has random fixed-point $\phi(\omega, x) \in P_R \setminus \overline{P}_r$, i.e., random Hammerstein equation (1.1) has a nonnegative random continuous solution which does not vanish identically.

**Theorem 2.2.** Suppose that

(i) $k(\omega, x, y)$ is a nonnegative bounded random continuous kernel, that is,

$$\int_G k(\omega, x, y) \, dx > 0, \quad \text{for all } (\omega, y) \in \Omega \times G.$$

(ii) $a_i(x) > 0$, $a_i(x) \in C(G)$, $a_i > 1$ ($i = 1, 2, \ldots, n$) and there exist some $a_{i_1}$, satisfying

$$\inf_{x \in G} a_{i_1}(x) > 0.$$

Then equation (1.1) has at least one nonnegative, not identically vanishing random continuous solution $\phi(\omega, x)$.

**Proof.** By the same method used for Theorem 2.1, we may prove that $A : \Omega \times P \rightarrow P$ is a random completely continuous operator, where $P$ denotes a cone (2.1) in $C(G)$. We take

$$R \geq \beta^{(-a_{i_1}+1)/(a_{i_1}-1)} \tau_1^{(-(a_{i_1}-1)^{-1})} \left( M \text{ mes } G \right)^{(a_{i_1})/(a_{i_1}-1)},$$

(2.6)
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where

\[ \tau_1 = \inf_{x \in G} a_i(x) > 0, \quad M = \sup_{(x,y) \in G \times G} k(\omega, x, y), \]

and mes \( G \) represents the measure of \( G \).

First, we prove that

\[ (\omega, \phi) \in \Omega \times \partial P_R \Rightarrow A(\omega, \phi) \notin \phi. \tag{2.7} \]

In fact, there exists \( (\omega_2, \phi_2) \in \Omega \times \partial P_R \) such that \( A(\omega_2, \phi_2) \leq \phi_2 \), hence,

\[ (\text{mes} G) \| A(\omega_2, \phi_2) \|_C \geq \int_G A(\omega_2, \phi_2) \, dx \]

\[ \geq \int_G k(\omega, x, y) a_i(y) \, d\mu(\omega) \int_G \phi_2(y) \, dy \geq \beta \tau_1 \int_G [\phi_2(y)]^{a_i} \, dy. \]

By Theorem 192 [4], we have

\[ \left( \int_G [\phi_2(x)]^{a_i} \, dx \right)^{\frac{1}{a_i}} \geq (\text{mes} G)^{\frac{1}{a_i}} - 1 \int_G \phi_2(x) \, dx, \]

since \( \phi_2 \in P, \int_G \phi_2(x) \, dx \geq \beta M^{-1} \| \phi_2 \|_C \). Hence,

\[ R = \| \phi_2 \|_C \geq \| A(\omega_2, \phi_2) \|_C \geq \beta^{a_i + 1} \tau_1 (M \text{ mes} G)^{-a_i}, \]

\[ \| \phi_2 \|_C^{a_i} = \beta^{a_i + 1} \tau_1 (M \text{ mes} G)^{-a_i} R^{a_i}. \]

This contradicts (3.6) and so (2.7) holds.

On the other hand, we take \( r \) such that \( 0 < r < R \) and

\[ M \sum_{i=1}^n \| a_i \|_L \, r^{a_i} < 1. \tag{2.8} \]

We prove that

\[ (\omega, \phi) \in \Omega \times \partial P_r \Rightarrow A(\omega, \phi) \notin \phi. \tag{2.9} \]

In fact, there exists \( (\omega_3, \phi_3) \in \Omega \times \partial P_r \) such that \( A(\omega_3, \phi_3) \geq \phi_3(x) \). Hence,

\[ r = \| \phi_3 \|_C \leq \| A(\omega_3, \phi_3) \|_C \leq M \sum_{i=1}^n \| a_i \|_L \, \| \phi_3 \|_C^{a_i} = M \sum_{i=1}^n \| a_i \|_L \, r^{a_i}. \]

This contradicts (2.8) and so (2.9) holds.

From (2.7), (2.9), and Lemma 1.1, we know \( A(\omega, \phi) \) has at least one random fixed-point \( \phi(\omega, x) \in C(G) \). For any \( \omega \in \Omega, \phi(\omega, x) \in P_{P_r} \), denotes \( \phi(\omega, x) \in C(G) \) and is nonnegative, but does not vanish identically.

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