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Operator Splitting Methods for American Option Pricing

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Abstract—We propose operator splitting methods for solving the linear complementarity problems arising from the pricing of American options. The space discretization of the underlying Black-Scholes equation is done using a central finite-difference scheme. The time discretization as well as the operator splittings are based on the Crank-Nicolson method and the two-step backward differentiation formula. Numerical experiments show that the operator splitting methodology is much more efficient than the projected SOR, while the accuracy of both methods are similar.

Keywords—American option, Operator splitting method, Time discretization, Linear complementarity problem.

1. INTRODUCTION

Options give a right to buy (call option) or sell (put option) an underlying asset, which can be a stock, for a given price (strike/exercise price). The writer of option, who sold the option, has the obligation to buy or sell the underlying asset if the holder of option chooses to exercise the option. A European option can be exercised only when the option expires, while an American option can be exercised any time before the expiry date. Thus, American options are more flexible and, hence, they are generally more valuable.

The seminal paper [1] describes the Black-Scholes parabolic partial differential equation which gives the value of European options. One of the first papers studying the valuation of American options is [2]. Since then there has been a lot of research on pricing options; see [3] and references therein.

For European options, there exist analytical formulas to calculate their price. American options have an additional constraint for the value of the option, and due to this, they lead to free boundary problems. In general, numerical methods have to be employed to compute the value of American options. In this paper, we consider methods based on a finite-difference discretization

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of the Black-Scholes equation for pricing American options. Typically, the space discretization is performed using central finite-difference schemes. We will also use this type of discretization in the following.

The most common time discretization for the Black-Scholes equation is the Crank-Nicolson method. It is second-order accurate with respect to the size of the time step, but it has poor stability properties which lead to oscillations in the price function. The two-step backward differentiation formula (BDF) time discretization is also second-order accurate in time but it has better stability properties [4], while being slightly more complicated to use since it is a multistep method. We consider both methods in this paper.

These traditional time discretizations lead to a linear complementarity problem at each time step. It is much more difficult to construct efficient solution methods for linear complementarity problems than for linear problems. The most commonly used method is the projected successive over relaxation (PSOR) method [3,5,6]. It is rather inefficient for finer space discretization. In order to avoid the solution of linear complementarity problems, we propose time discretization schemes based on operator splitting. These techniques are often used in computational fluid dynamics; see [7,8] and references therein. The basic idea is to decouple problematic operators into separate fractional time steps in the discretization. In our case, we decouple the Black-Scholes operator and the constraint for the value of the option. This splitting leads to the solution of a linear problem and to a simple correction step at each time step. Thus, the arising subproblems can be solved much more easily and efficiently than the original problem. In the numerical experiments, we also show that the accuracy of the split and nonsplit versions of the discretizations are essentially the same.

The outline of the paper is the following. First, we describe the linear complementarity problem giving the value of the American options. Then, the space and time discretizations are considered. After this, the operator splittings are described for the Crank-Nicolson and BDF time discretizations. Numerical experiments demonstrate the efficiency and accuracy of the proposed approach. Finally, some conclusions are given as well as a proposal for future research.

2. MODEL FOR AMERICAN OPTIONS

We consider an American put option which gives the right to sell the underlying asset with the exercise price at any time before the expiry time of the option. In the following, we denote the exercise price by E, the expiry time by T, the risk free interest rate by r, and the volatility by σ. The price v of the option can be obtained from the solution of the linear complementarity problem

$$\begin{aligned}
&\lambda = \frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + rx\frac{\partial v}{\partial x} - rv, && x > 0 \text{ and } t \in [0,T],\\
&[v - (E - x)] \cdot \lambda = 0, && x > 0 \text{ and } t \in [0,T],\\
&v - (E - x) \geq 0, \quad \lambda \geq 0, && x > 0 \text{ and } t \in [0,T],\\
&v = \max(E - x, 0), && x > 0 \text{ and } t = T,\\
&v = E, && x = 0 \text{ and } t \in [0,T],\\
&v \to 0, && x \to \infty \text{ and } t \in [0,T],
\end{aligned} \tag{1}$$

where t is the time and x is the value of the underlying asset [3]. One should note that we have a final condition in (1) and the integration in time is done backwards. The auxiliary variable λ in (1) forces the value of the option to be higher than $E - x$.

In order to obtain an equation in a finite domain, the domain $(0, \infty)$ is truncated to be $(0, cE)$, where c is typically three or four. The boundary condition at $x = cE$ is chosen to be $v = 0$. Normally, this truncation of the domain leads to a negligible error in the value of the option [9].

3. SPACE DISCRETIZATION

For the space discretization, we use a uniform grid on interval $(0, cE)$ with $n+1$ grid points. The grid step size is denoted by Δx. Let the value of v at grid point $x_i = i\Delta x$ be denoted by v_i, $i = 0, \dots, n$. By using central finite differences, we get

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} \approx \frac{1}{2(\Delta x)^2}\sigma^2 x_i^2 (v_{i-1} - 2v_i + v_{i+1}) \tag{2}$$

and

$$rx\frac{\partial v}{\partial x} \approx \frac{1}{2\Delta x} r x_i (v_{i+1} - v_{i-1}). \tag{3}$$

By substituting $x_i = i\Delta x$ to approximations (2) and (3), we obtain

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + rx\frac{\partial v}{\partial x} - rv \approx \frac{1}{2}\left(\sigma^2 i^2 - ri\right) v_{i-1} - \left(\sigma^2 i^2 + r\right) v_i + \frac{1}{2}\left(\sigma^2 i^2 + ri\right) v_{i+1}. \tag{4}$$

Let us denote by A the matrix arising from the finite-difference scheme (4). Then, the semidiscrete form of the PDE in (1) reads

$$\frac{\partial v}{\partial t} + Av - \lambda = 0, \qquad t \in [0, T]. \tag{5}$$

It can be shown [10] that this discretization leads A to be an M-matrix if the condition $\sigma^2 > r$ is verified, and this guarantees that there are no oscillations in the solution due to the space discretization. If $\sigma^2 > r$ does not hold, an M-matrix can still be obtained by adding some artificial diffusion into the Black-Scholes operator.

4. TIME DISCRETIZATIONS

We use a uniform time step Δt. The solution v at time $t = k\Delta t$ is denoted by $v^{(k)}$. Similarly, the auxiliary variable λ at time $t = k\Delta t$ is denoted by $\lambda^{(k)}$. The nodal values at $t = T$ are given by $v_i^{(m)} = \max(E - x_i, 0)$, where $m = T/\Delta t$.

A typical time stepping scheme for American option pricing, in which $v^{(k)}$ and $\lambda^{(k)}$ are computed from $v^{(k+1)}$, is given by

$$\begin{gathered} \frac{1}{\Delta t}\left(v^{(k+1)} - v^{(k)}\right) + A\left((1-\alpha)v^{(k+1)} + \alpha v^{(k)}\right) - \lambda^{(k)} = 0, \\ \left[v_i^{(k)} - (E - x_i)\right] \cdot \lambda_i^{(k)} = 0, \quad v_i^{(k)} \ge E - x_i, \quad \lambda_i^{(k)} \ge 0, \qquad i = 1, 2, \dots, n-1. \end{gathered} \tag{6}$$

Here, the parameter α is chosen to be between zero and one. The following three choices lead to well-known schemes: $\alpha = 1$ gives the implicit backward Euler method, $\alpha = 1/2$ gives the Crank-Nicolson method, and $\alpha = 0$ gives the explicit forward Euler method. Out of all possible values of α, only the Crank-Nicolson method is second-order accurate in time. The implicit Euler method has good stability properties, while the Crank-Nicolson method has poor stability properties, since it is not L-stable [4].

The two-step backward differentiation formula (BDF) is second-order accurate in time and L-stable [4]. In [11], the BDF was used for pricing American options. The BDF for (5), in which $v^{(k)}$ and $\lambda^{(k)}$ are computed from $v^{(k+1)}$ and $v^{(k+2)}$, is given by

$$\begin{gathered} \frac{1}{\Delta t}\left(\frac{4}{3}v^{(k+1)} - v^{(k)} - \frac{1}{3}v^{(k+2)}\right) + \frac{2}{3}\left(Av^{(k)} - \lambda^{(k)}\right) = 0, \\ \left[v_i^{(k)} - (E - x_i)\right] \cdot \lambda_i^{(k)} = 0, \quad v_i^{(k)} \ge E - x_i, \quad \lambda_i^{(k)} \ge 0, \qquad i = 1, 2, \dots, n-1. \end{gathered} \tag{7}$$

The time integration has to be started with some other method, since the BDF is a two-step scheme. We will use the implicit Euler method for the first time step. This is likely to reduce the accuracy of the solution, but the asymptotic second-order accuracy is still maintained.

The arising problems (6) and (7) are linear complementarity problems. Special methods have been developed for their solution. The projected SOR (PSOR) method is the simplest and the most commonly used among these methods [3,5,6].

5. OPERATOR SPLITTINGS

The idea of our splittings is to treat the Black-Scholes operator in one fractional time step and the constraint $v_i^{(k)} \geq E - x_i$ in another fractional time step. For example, it is typical to treat the incompressibility constraint in the Navier-Stokes equations in a similar manner; see [7,8] and references therein. For the discretizations (6) and (7), the first fractional step reads

$$\frac{1}{\Delta t}\left(v^{(k+1)} - \tilde{v}^{(k)}\right) + A\left((1-\alpha)v^{(k+1)} + \alpha\tilde{v}^{(k)}\right) - \lambda^{(k+1)} = 0 \tag{8}$$

and

$$\frac{1}{\Delta t}\left(\frac{4}{3}v^{(k+1)} - \tilde{v}^{(k)} - \frac{1}{3}v^{(k+2)}\right) + \frac{2}{3}\left(A\tilde{v}^{(k)} - \lambda^{(k+1)}\right) = 0, \tag{9}$$

respectively. The intermediate vector $\tilde{v}^{(k)}$ can be efficiently solved from (8) and (9) by using the LU decomposition.

The second fractional step enforces the constraint by projecting the solution to be feasible and updates the auxiliary variable λ. This step associated to (8) and (9) is given by

$$\begin{gathered}\frac{1}{\Delta t}\left(\tilde{v}^{(k)} - v^{(k)}\right) + \lambda^{(k+1)} - \lambda^{(k)} = 0, \\ \left[v_i^{(k)} - (E - x_i)\right] \cdot \lambda_i^{(k)} = 0, \quad v_i^{(k)} \geq E - x_i, \quad \lambda_i^{(k)} \geq 0, \qquad i = 1, 2, \dots, n-1,\end{gathered} \tag{10}$$

and

$$\begin{gathered}\frac{1}{\Delta t}\left(\tilde{v}^{(k)} - v^{(k)}\right) + \frac{2}{3}\left(\lambda^{(k+1)} - \lambda^{(k)}\right) = 0, \\ \left[v_i^{(k)} - (E - x_i)\right] \cdot \lambda_i^{(k)} = 0, \quad v_i^{(k)} \geq E - x_i, \quad \lambda_i^{(k)} \geq 0, \qquad i = 1, 2, \dots, n-1,\end{gathered} \tag{11}$$

respectively. The component pairs $v_i^{(k)}$ and $\lambda_i^{(k)}$ from (10) and (11) can be solved easily, since they are not coupled with other components.

6. NUMERICAL EXPERIMENTS

We consider an American put option with the following parameters: the volatility $\sigma = 0.4$, the risk free interest rate $r = 0.1$, the exercise price $E = 10.0$, and the expiry time $T = 0.5$. In our experiments, the computational domain $(0, cE) \times [0, T]$ is $(0, 50) \times [0, 0.5]$. We compare the accuracy of all proposed discretizations. The number n of space steps and the number m of time steps are chosen in such a way that almost the best possible accuracy is reached with a given number mn of space-time grid points.

In Table 1, we give the maximum errors for the option price v and the CPU times on a HP J5600 workstation for the Crank-Nicolson-based time discretizations. The same information for the BDF-based time discretizations are given in Table 2. In the PSOR method, the relaxation parameter is one leading to the projected Gauss-Seidel method and the stopping criterion is chosen experimentally in such a way that the total error is 10% larger than the discretization error. The operator splitting is denoted by O-S in the tables and it uses the LU decomposition to solve the tridiagonal systems of linear equations in the first fractional step.

7. CONCLUSIONS

The proposed operator splittings for American options lead to solution of tridiagonal systems of linear equations and simple projections. The numerical experiments demonstrate that the solution of these problems is much faster for larger problems than the use of the PSOR method to solve the original linear complementarity problems; moreover, operator splitting does not reduce essentially the accuracy of solutions. With the PSOR method, the CPU time is increased by a factor of eight when both space and time discretizations are made twice finer. The corresponding

Table 1. The CPU times and the maximum errors for the Crank-Nicolson-based schemes.

		PSOR		O-S	
n	m	Error	CPU	Error	CPU
16	2	1.20515E–01	0.000	1.28368E–01	0.000
32	4	4.50055E–02	0.001	4.46548E–02	0.000
64	8	9.39104E–03	0.005	8.23145E–03	0.001
128	16	2.45729E–03	0.017	1.34804E–03	0.004
256	32	9.09189E–04	0.119	1.45181E–03	0.014
512	64	1.80041E–04	1.302	3.20324E–04	0.053
1024	128	7.21728E–05	11.944	1.33409E–04	0.209
2048	256	2.45297E–05	111.962	5.95396E–05	0.852
4096	512	1.06021E–05	1 038.110	3.30749E–05	3.333

Table 2. The CPU times and the maximum errors for the BDF-based schemes.

		PSOR		O-S	
n	m	Error	CPU	Error	CPU
16	2	1.48356E–01	0.000	1.63957E–01	0.000
32	4	4.31706E–02	0.001	4.23987E–02	0.000
64	8	1.29812E–02	0.009	1.01632E–02	0.001
128	16	3.08468E–03	0.022	1.92143E–03	0.004
256	32	1.09742E–03	0.194	7.43285E–04	0.013
512	64	3.52142E–04	1.716	2.89327E–04	0.050
1024	128	1.54448E–04	15.012	1.18621E–04	0.195
2048	256	5.77862E–05	145.808	4.21406E–05	0.781
4094	512	2.54980E–05	1 268.679	1.99552E–05	3.058

factor for operator splitting is four. It would be possible to improve the efficiency of the PSOR method by tuning the relaxation parameter, but still it would not be competitive with the operator splitting approach. In the PSOR method, it is not easy to tune the relaxation parameter and to choose a suitable tolerance in the stopping criterion. Due to this, the operator splitting is much easier to use, since it is a parameter free method. The accuracy of the Crank-Nicolson and BDF-based methods were similar in numerical experiments.

An interesting topic for future research is to study operator splittings for pricing American options using higher-dimensional models, for example, the Black-Scholes equation with a stochastic volatility. These problems are computationally much more expensive to solve and it is not easy to design an efficient solution procedure without relying on operator splitting.

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