Discrete Mathematics 84 (1990) 303–307
North-Holland


COMMUNICATION

# SOME PROGRESS IN THE PACKING OF EQUAL CIRCLES IN A SQUARE

Michel MOLLARD and Charles PAYAN
*LSD[2] (IMAG), BP 53X, 38041 Grenoble Cedex, France*

**Communicated by C. Benzaken**
Received 19 April 1990

The problem of the densest packing of $n$ equal circles in a square has been solved for $n < 10$ in [4, 6]; and some solutions have been proposed for $n \geq 10$. In this paper we give some better packings for $n = 10$, 11, 13 and 14.

## 1. Introduction

Consider a finite family of $n$ circular disks, each of diameter one, whose interiors are pairwise disjoint and contained in a square $S$. A classical problem is to find the smallest side $s$ of such a square. This is clearly equivalent to maximizing the minimum pairwise distance $m$ among $n$ points in a unit square and we have $m = 1/(s-1)$.

This problem has been solved for $n \leq 9$ [4, 6]. For $n \leq 27$ efficient arrangements are given by Goldberg [4]. The case $n = 10$ has been successively improved by Schaer [7], Milano [5] and Valette [8] (see *Fig. 1a* and *1b*).

In this paper we give a better solution for $n = 10$, which, following the tradition instituted by the previous authors, we think to be optimal, and some better arrangements for $n = 11$, 13 and 14.

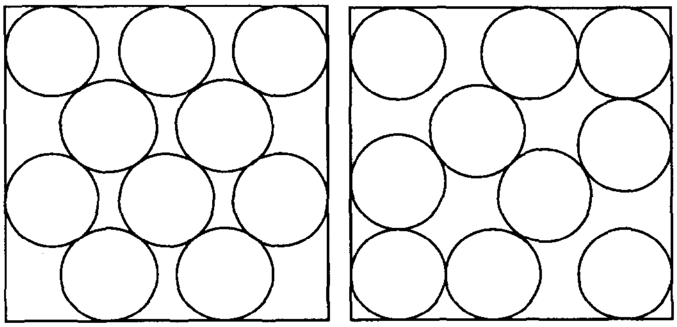
## 2. Packing 10 circles

In a square $ABCD$ of side $s-1$ let us define (whenever possible) the points $P_1$, $P_2, \ldots, P_9$ as shown in *Fig. 2* where $P_1 = A$, $P_3$ is a point of $AD$ at distance $x$ from $D$, and $P_2$, $P_4$, $P_5$, $P_6$, $P_7$, $P_8$ are on the boundary in such a way that the eight distances $d(P_2, P_3)$, $d(P_3, P_4)$, $d(P_4, P_5)$, $d(P_5, P_6)$, $d(P_6, P_7)$, $d(P_7, P_8)$, $d(P_8, P_9)$ and $d(P_9, P_1)$ are equal to 1. Let $y = d(P_2, P_9)$.

It is not difficult to show that for some fixed $s$, the distance $y$ is a continuous function of $x$, and for some fixed $x$, this distance is an increasing function of $s$. Numerical calculation gives that:

$$\text{for } x = 0 \quad \text{and} \quad s = 3.365 \quad y > 1.0045 \quad \text{and}$$
$$\text{for } x = 0.1 \quad \text{and} \quad s = 3.390 \quad y < 0.967.$$

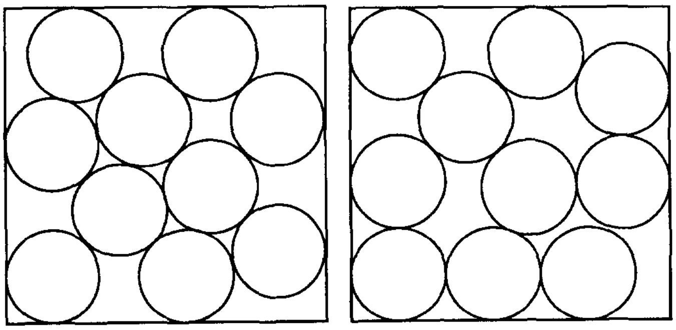
Elsevier Science Publishers B.V. (North-Holland)



Goldberg's solution
s=3.4
m≃0.41667

Schaer's solution
s≃3.38356
m≃0.41954

Fig. 1a.



Milano's solution
s≃3.38016
m≃0.42014

Valette's solution
s≃3.374227
m≃0.421190

Fig. 1b.

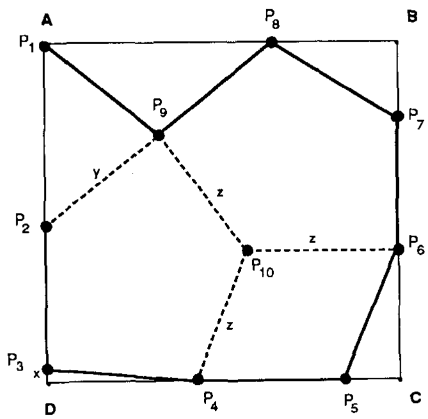



Fig. 2.

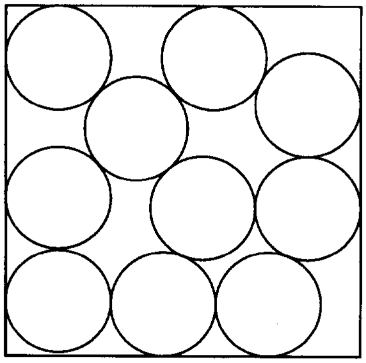


Fig. 3.

It follows that for any value of $s$ in [3.365, 3.390] we have

for $x = 0$ $\quad y > 1$ and
for $x = 0.1$ $\quad y < 1$.

Then for any value of $s$ in [3.365, 3.390] there exists an $x_{\text{opt}}$ in [0, 0.1] such that $y = 1$. In the following we will choose $P_3$ such that $d(P_2, P_9) = 1$.

Let $P_{10}$ be the point equidistant from $P_4$, $P_6$ and $P_9$ and let $z$ be $d(P_{10}, P_4)$.

$z$ is a continuous and increasing function of $s$; and solving the equation $z = 1$ for $s$ in [3.365, 3.390] gives $s = 3.37372076\ldots$ (this is obtained with $x = 0.02724496\ldots$).

Corresponding to this value of $s$ we obtain $m = 0.42127954\ldots$.

This packing is shown in *Fig. 3.*

## 3. Packing 11, 13 or 14 circles

The arrangements shown in *Fig. 4* are given by Goldberg [4] and we propose the better packings shown in *Fig. 5.* In fact our packing of 13 circles can be

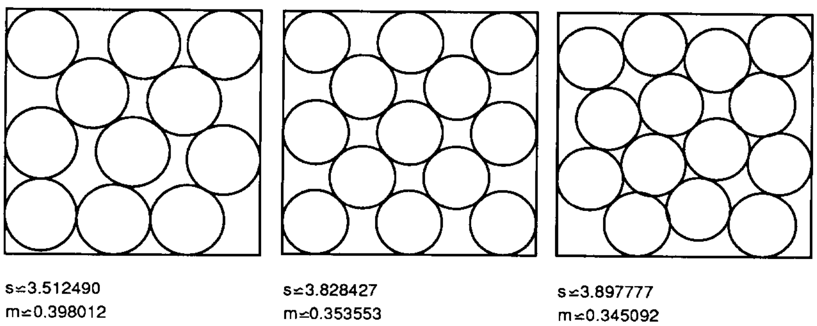
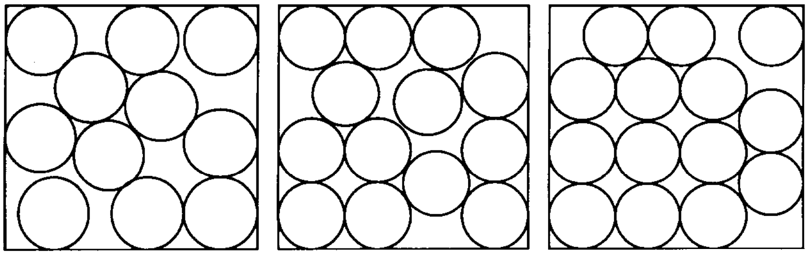



Fig. 4.



$s=\frac{6+2\sqrt{2}+\sqrt{6}+\sqrt{2+4\sqrt{2}}}{4}=3.511254\ldots$
$m=0.398207\ldots$

$s=2+\sqrt{3}=3.732\ldots$
$m=\frac{\sqrt{3}-1}{2}=0.366025\ldots$

$s=3+\frac{\sqrt{3}}{2}=3.866025\ldots$
$m=\frac{8-2\sqrt{3}}{13}=0.348915\ldots$

Fig. 5.

improved as shown in *Fig. 6.* In this figure plain lines represent unit distances and $X$ is a point of $DC$ at distance $x$ from $D$.

By adjusting the values of $s$ and $x$ we can make the distances $d$ and $e$ equal to 1, the other distances being greater than 1 and we obtain the packing shown in *Fig. 7.*

The corresponding values are:

$$x = 0.029018318\ldots,\quad f = 1.00018318\ldots,\quad g = 1.00019284\ldots,$$
$$s = 3.731523914\ldots,\quad m = 0.366096007\ldots.$$

The research of these arrangements has been facilitated by the use of Cabri-Géomètre, a software for Geometry developed in our laboratory. Elementary objects of Cabri-Géomètre can be linked by geometrical relations which remain when moving any basic point [1–3].

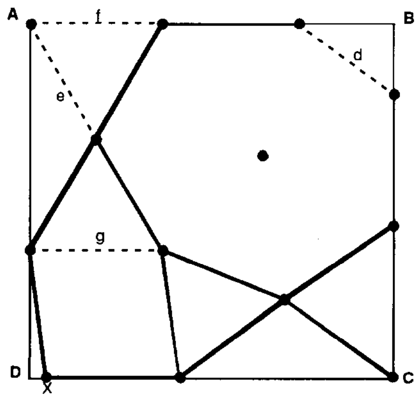



Fig. 6.

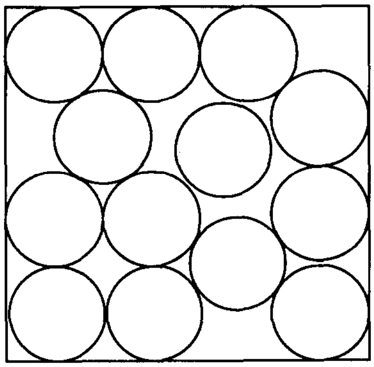


Fig. 7.

## References


[1] Y. Baulac and J.-M. Laborde, Informatique et apprentissage de la géométrie, TSI 8 (1989) 387–391.

[2] Y. Baulac, Un micromonde de géométrie, Cabri-Géomètre, Thèse de l'Université Joseph Fourier, Grenoble (1989).

[3] Y. Baulac, F. Bellemain and J.-M. Laborde, Cabri-Géomètre, un logiciel d'aide à l'apprentissage de la géométrie, logiciel et manuel d'utilisation (Cedic-Nathan, Paris, 1988).

[4] M. Goldberg, The packing of equal circles in a square, Math. Mag. 43 (1970) 24–30.

[5] R. Milano, Configurations optimales de disques dans un polygone regulier, Mémoire de Licence, Université Libre de Bruxelles (1987).

[6] J. Schaer, The densest packing of 9 circles in a square, Canad. Math. Bull. 8 (3) (1965) 273–277.

[7] J. Schaer, On the packing of ten equal circles in a square, Math. Mag. 44 (1971) 139–140.

[8] G. Valette, A better packing of ten equal circles in a square, Discrete Math. 76 (1989) 57–59.