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New Results on the Stability of Discrete-Time Systems and Applications to Control Problems

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The aim of this article is to present some new stability sufficient conditions for discrete-time nonlinear systems. It shows how to use nonnegative semi-definite functions as Lyapunov functions instead of positive definite ones for studying the stability of a given system. Several examples and some applications to control theory are presented to illustrate the various theorems. © 1998 Academic Press

1. INTRODUCTION

This paper deals with the stability of nonlinear discrete-time systems that can be described by an autonomous difference equation

$$x(k+1) = f(x(k)), \qquad f(0) = 0. \tag{1}$$

Stability is a very important property in control system design. Some of the most important results in stability theory have been known for many decades. Well-known stability criteria for linear systems were developed a long time ago. Stability of nonlinear systems can be studied via linearization but the general and the most powerful technique is Lyapunov's second method. This method actually has its origin in energy considerations. It consists in the use of an auxiliary function, which generalizes the role of energy in mechanical systems. For differential equations the method has

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been used since 1893 [7], while for difference equations its use is more recent and can be summarized as follows [2, 3, 5]:

THEOREM 1.1 (Lyapunov). *The null solution, or the equilibrium state at the origin of system* (1) *is stable if there is some neighborhood of the origin where a positive definite function* $V(x)$ *exists such that its difference* $\Delta V(x) = V(f(x)) - V(x)$ *along the solutions of* (1) *is negative semi-definite in that region.*

THEOREM 1.2 (Lyapunov). *The null solution, or the equilibrium state at the origin of system* (1), *is asymptotically stable if there is some neighborhood of the origin where a positive definite function* $V(x)$ *exists such that its difference* $\Delta V(x)$ *along the solutions of* (1) *is negative definite in that region.*

The existence of a positive definite function, whose difference is negative definite, is actually a necessary and sufficient condition for the asymptotic stability. However, it is often difficult to find such a function. Thanks to LaSalle's invariance principle [6, 4], the assumption on the difference of the Lyapunov function in the asymptotic stability theorem has been considerably relaxed:

THEOREM 1.3. *The null solution of system* (1) *is asymptotically stable if there is some neighborhood* $\mathscr{D}$ *of the origin where a positive definite function* $V(x)$ *exists such that its difference* $\Delta V(x)$ *along the solutions of* (1) *is negative semi-definite in* $\mathscr{D}$, *and such that no solution of* (1) *can stay forever in* $\{x \in \mathscr{D} | \Delta V(x) = 0\}$, *other than the trivial solution.*

Theorem 1.3 is a generalization of the original Lyapunov's asymptotic stability theorem. It is very useful in practice, for it does not require the definiteness of the difference of the Lyapunov function and so it is easier to find Lyapunov functions satisfying the assumptions of Theorem 1.3 than it is to find Lyapunov functions which satisfy the assumptions of Theorem 1.2.

Our contribution can be seen as a continuation of the works summarized in the paragraph above. The primary objective of this paper is to give a new generalization of Lyapunov's theorems. We do not only relax the definiteness requirement on the difference of the Lyapunov function, but also on the Lyapunov function used in the stability theorem as well as in the asymptotic stability theorem. We show how the results of Lyapunov can apply when the Lyapunov function is only semi-definite. Our result can be formulated as follows: The null solution of system (1) is stable if it is asymptotically stable with respect to perturbations belonging to the set where the Lyapunov function V vanishes. It is asymptotically stable if it is asymptotically stable with respect to perturbations belonging to the largest invariant set contained in the set where the difference ΔV vanishes. The natural interest is that it is often much easier to find a nonnegative Lyapunov function satisfying the conditions of our results than it is to find a positive definite one which satisfies the assumptions of Theorems 1.1, 1.2, and 1.3. For a simple illustration, consider the following 2-dimensional system [6]:

$$\begin{cases} x(k+1) = \dfrac{ay(k)}{1+x(k)^2}, \\ y(k+1) = \dfrac{bx(k)}{1+y(k)^2}, \\ a \text{ and } b \text{ are two constants.} \end{cases} \tag{2}$$

LaSalle [6] has applied Theorem 1.3 with the definite Lyapunov function $V = x^2 + y^2$. He has shown that the null solution of system (2) is globally asymptotically stable if $a^2 \leq 1$, $b^2 \leq 1$, and $a^2 + b^2 < 2$. By using the results of this paper with the semi-definite function $V = (xy)^2$, one can easily prove that system (2) is globally asymptotically stable if and only if $|ab| < 1$. Section 3 contains the main stability theorems of this paper as well as some illustrating examples and remarks.

The last section of this paper is devoted to investigating the feedback stabilization of nonlinear control systems that can be described by

$$\begin{cases} x(k+1) = f(x(k), u(k)), \quad f(0,0) = 0, \\ y(k) = h(x(k)), \end{cases} \tag{3}$$

where $x(k)$ and $y(k)$ denote the state and the measurable output, respectively, and $u(k)$ is the input or the control.

The stabilization of nonlinear control systems has become, during the last two decades, one of the most important problems in control theory and engineering design: whatever the control system performance criterion may be, one must check that the resulting system is stable. System (3) is said to be state feedback stabilizable if there exists a continuous feedback law $u(x)$ in such a way that the closed-loop system $x(k+1) = f(x(k), u(x(k)))$ is asymptotically stable. When the complete state measurement is not available, one has to compute a stabilizing control using only the measurable output $y(k)$. If there exists an output feedback law $u(y)$ in such a way that the closed-loop system $x(k+1) = f(x(k), u(y(k))) = f(x(k), u(h(x(k))))$ is asymptotically stable then system (3) is said to be output feedback stabilizable. However, it is rarely possible to stabilize a given system by means of an output feedback control. The alternative way is to achieve the stabilization of system (3) by using dynamic output feedback. This consists in constructing an observer for system (3), i.e., a dynamical system $\hat{x}(k+1) = g(\hat{x}(k), y(k), u(k))$ which produces an estimate $\hat{x}(k)$ of the state $x(k)$ and computing a feedback law $u(\hat{x}(k))$ in such a way that the null solution of the composite system

$$\begin{cases} x(k+1) = f(x(k), \alpha(\hat{x}(k))), \\ e(k+1) = f(x(k), \alpha(\hat{x}(k))) - g(\hat{x}(k), h(x(k)), \alpha(\hat{x}(k))) \end{cases}$$

is asymptotically stable.

As an application of the theorems of Section 3, we give a sufficient and necessary condition for the asymptotic stability of systems in cascade form

$$\begin{cases} x(k+1) = f(x(k), y(k)), \\ y(k+1) = g(y(k)). \end{cases} \tag{4}$$

This allows us to achieve the stabilization of control systems

$$\begin{cases} x(k+1) = F(x(k), y(k), u(k), v(k)), \\ y(k+1) = g(y(k), v(k)), \end{cases} \tag{5}$$

by means of a decentralized feedback law $(u(x), v(y))$.

We derive also a sufficient condition for the feedback stabilization of control systems (3) that are free dynamics stable, i.e., $x(k+1) = f(x(k), 0)$ is stable but not asymptotically stable. The last paragraph of this section deals with the stabilization of system (3) by an estimated state feedback. We show that if there exists an observer which produces an estimate $\hat{x}(k)$ of the state $x(k)$ of system (3) and if some conditions hold then it is possible to stabilize system (3) by an estimated state feedback law $u(\hat{x})$. An application to bilinear systems is given.

2. NOTATIONS AND PRELIMINARIES

Let us consider a system of difference equations

$$\begin{cases} x(k+1) = f(x(k)), \quad x \in \mathscr{U}, \\ f(0) = 0, \end{cases} \tag{6}$$

where $\mathscr{U}$ is a neighborhood of the origin in $\mathbb{R}^n$ and $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function.

For each $p \in \mathscr{U}$, let $f^k(p)$ denote the value at time k of the solution of (6) starting at p. We recall that $f^k(p) = f(f^{k-1}(p))$, $f^0(p) = p$.

First of all, we give some usual notations and standard definitions:

$\mathbb{N}$ is the set of nonnegative integers, $\mathbb{R}$ is the set of real numbers, and $M_n(\mathbb{R})$ is the set of $n \times n$ square matrices.

$\| \ \|$ denotes a norm on $\mathbb{R}^n$ and also its associated norm on $M_n(\mathbb{R})$.

$B_\epsilon = \{x \in \mathbb{R}^n\colon \|x\| < \epsilon\}$, $\overline{B_\epsilon} = \{x \in \mathbb{R}^n\colon \|x\| \le \epsilon\}$, $S_\epsilon = \{x \in \mathbb{R}^n\colon \|x\| = \epsilon\}$.

$f^+(p) = \{f^k(p), k \in \mathbb{N}\}$.

$L^+(p)$ is the ω-limit set of p.

In general we are interested in points $x_0 \in \mathscr{U}$ such that $f(x_0) = x_0$. They are usually called an *equilibrium point* of the system. Corresponding to each equilibrium point x_0, we have a constant solution $f^k(x_0) \equiv x_0$ of (6).

DEFINITION 2.1. Let $x_0 \in \mathscr{U}$ be an equilibrium point. System (6) is Lyapunov stable at x_0 (or x_0 is a Lyapunov stable equilibrium point for (6)), if for each $\epsilon > 0$ there is a positive δ such that for each $x \in \mathscr{U}$ with $\|x - x_0\| < \delta$, one has $\|f^k(x) - x_0\| < \epsilon$ for all $k \in \mathbb{N}$.
System (6) is unstable at x_0 if it is not Lyapunov stable at x_0.

Let $\mathscr{A}$ be the set of points $p \in \mathscr{U}$ for which

$$\lim_{k \to +\infty} f^k(p) = x_0$$

holds, for all solutions $f^k(p)$ issuing from p. $\mathscr{A}$ is called the *region of attraction*, or *domain of attraction* of x_0. An equilibrium point is said to be *attractive* if it is an interior point of its region of attraction. We say also that system (6) is attractive at x_0. In general, an attractive equilibrium point is not necessarily Lyapunov stable (an example can be found in [2, p. 170]).

DEFINITION 2.2. x_0 is an asymptotically stable equilibrium point for system (6) if it is Lyapunov stable and attractive.

In the sequel, we will take $x_0 = 0$.

DEFINITION 2.3. A set Y is invariant if $f(Y) = Y$, positively invariant if $f(Y) \subset Y$, and negatively invariant if $Y \subset f(Y)$.

For any positively invariant set Y, $\mathscr{A}_Y$ will denote the relative domain of attractivity i.e., $\mathscr{A}_Y = \mathscr{A} \cap Y$.

DEFINITION 2.4. Let Y be a closed positively invariant set such that $0 \in Y$. The origin is said to be:

(a) Y-stable if $\forall \epsilon > 0, \exists \delta > 0$: $f^+(B_\delta \cap Y) \subset B_\epsilon$.

(b) Y-asymptotically stable if it is Y-stable and there exists $\delta > 0$ such that

$$\lim_{k \to +\infty} f^k(x) = 0, \qquad \forall x \in Y \cap B_\delta.$$

DEFINITION 2.5. A control system $x(k+1) = F(x(k), u(k))$ is stabilizable if there exists a feedback control law $u(x)$ in such a way that the closed-loop system $x(k+1) = F(x(k), u(x(k)))$ is asymptotically stable.

DEFINITION 2.6. A real valued function V is a Lyapunov function for system (6) in a neighborhood $\mathscr{U}$ of the origin if $V(0) = 0$ and $V(f(x)) - V(x) \leq 0$ for all $x \in \mathscr{U}$.

In the sequel, if the system (6) has a nonnegative Lyapunov function defined in a neighborhood of the origin $\mathscr{V} \subset \mathscr{U}$, we will denote by G_0 the set where V vanishes, G the set where the difference of V along the solutions of the system vanishes, and G^* the largest positively invariant set contained in G.

One can easily show that G_0, G^*, and G are closed sets, G_0 is positively invariant, and

$$G_0 \subseteq G^* \subseteq G.$$

3. MAIN RESULTS

In the first part of this section, we give the stability theorems. The second subsection contains some remarks and examples.

3.1. *Stability Theorems*

The first theorem concerns the Lyapunov stability.

THEOREM 3.1. *If there exist a neighborhood $\mathscr{V} \subset \mathscr{U}$ of the origin and a function $V \in C^0(\mathscr{V}, \mathbb{R})$ such that*

(1) $V(x) \geq 0$ *for all* $x \in \mathscr{V}$ *and* $V(0) = 0$,

(2) $\Delta V(x) = V(f(x)) - V(x) \leq 0$ *for all* $x \in \mathscr{V}$,

(3) *the origin is G_0-asymptotically stable, where*: $G_0 = \{x \in \mathscr{V}: V(x) = 0\}$,

then the origin is Lyapunov stable.

Proof. Suppose that the origin is not stable. Then there exists $\epsilon > 0$ for which it is possible to construct a sequence of initial conditions $(x_n)_{n \in \mathbb{N}} \subset B_\epsilon$ satisfying $\lim_{n \to \infty} x_n = 0$ and such that for each $n \in \mathbb{N}$, the positive trajectory $f^+(x_n)$ does not stay within B_ϵ, i.e., $\{f^k(x_n), k \in \mathbb{N}\} \not\subseteq B_\epsilon$. In other words, for each initial condition x_n, there exists a set of positive integers $\mathscr{K}_n \subset \mathbb{N}$ such that $f^k(x_n) \notin B_\epsilon$ for all $k \in \mathscr{K}_n$. Let k_n be the smallest element of $\mathscr{K}_n$; k_n is simply the first exit-time from B_ϵ for the solution issued from x_n (the solution does not leave B_ϵ before the time k_n). The sequence $(k_n)_{n \in \mathbb{N}} \subset \mathbb{N}$ satisfies

$$\left\{x_n, f(x_n), \dots, f^{k_n - 1}(x_n)\right\} \subset B_\epsilon \qquad \text{and} \qquad \left\|f^{k_n}(x_n)\right\| \geq \epsilon, \forall n \in \mathbb{N}. \tag{7}$$

First of all, let us remark that (7) implies the property

$$\left(\left\| f^k(x_n)\right\| < \epsilon, \forall k \in \{0, \dots, q\}\right) \Leftrightarrow q < k_n. \tag{8}$$

It must be emphasized, however, that the definition of k_n does not give any information about $f^k(x_n)$ for $k > k_n$: it is possible to have $f^k(x_n) \in B_\epsilon$ as well as to have $f^k(x_n) \notin B_\epsilon$.

We take ϵ sufficiently small in order to have $\overline{B_\epsilon} \cap G_0 \subset \mathscr{A}_{G_0}$.

The origin is G_0-asymptotically stable, so there exists $N \in \mathbb{N}$ such that the solutions of (6) satisfy

$$\left\| f^k(y)\right\| < \frac{\epsilon}{2}, \qquad \forall k \geq N \text{ and } \forall y \in \overline{B_\epsilon} \cap G_0. \tag{9}$$

(Thanks to the compacity of $\overline{B_\epsilon} \cap G_0$, one can easily check that the integer N can be chosen independently of z.)

The continuity of the solutions with respect to initial conditions ensures the existence of $\delta > 0$ such that

$$\forall (x, y) \in \overline{B_\epsilon} \times \overline{B_\epsilon}, \qquad \|x - y\| < \delta \Rightarrow \left\| f^k(x) - f^k(y)\right\| < \frac{\epsilon}{2}, \forall k \leq N. \tag{10}$$

The sequence $(x_n)_{n \in \mathbb{N}}$ tends to the origin as n tends to $+\infty$, so there exists $n_0 \in \mathbb{N}$ such that $\|x_n\| < \delta$ for all $n \geq n_0$. Thus, by (10), one has

$$\left\| f^m(x_n)\right\| < \frac{\epsilon}{2}, \qquad \forall m \in \{0, \dots, N\} \text{ and } \forall n \geq n_0.$$

So, by (8), this implies that $N < k_n$ for all $n \geq n_0$. Therefore, we have $0 < k_n - N < k_n$ for all $n \geq n_0$. Taking into account (7), we get

$$\left\| f^{k_n - N}(x_n)\right\| < \epsilon, \qquad \forall n \geq n_0.$$

Therefore, the sequence $(u_n)_{n \geq n_0}$ defined by $u_n = f^{k_n - N}(x_n)$ has a convergent subsequence say $(u_{\phi(n)})_{n \geq n_0}$. Let $z = \lim_{n \to +\infty} u_{\phi(n)} \in \overline{B_\epsilon}$. Since V is assumed to be continuous, we have

$$\begin{aligned} 0 \leq V(z) &= \lim_{n \to +\infty} V(u_{\phi(n)}) = \lim_{n \to +\infty} V\left(f^{k_{\phi(n)} - N}(x_{\phi(n)})\right) \\ &\leq \lim_{n \to +\infty} V(x_{\phi(n)}) = 0. \end{aligned}$$

Hence, z belongs to $\overline{B_\epsilon} \cap G_0$ and then (9) yields

$$\| f^N(z) \| < \frac{\epsilon}{2}. \tag{11}$$

Since $z = \lim_{n \to +\infty} f^{k_{\phi(n)} - N}(x_{\phi(n)})$, there exists $p \geq n_0$ such that $\|z - f^{k_p - N}(x_p)\| < \delta$. So, by (10), we get

$$\left\| f^N(z) - f^N\left(f^{k_p - N}(x_p)\right) \right\| < \frac{\epsilon}{2}. \tag{12}$$

Finally, the combination of (11) and (12) leads to

$$\left\| f^{k_p}(x_p) \right\| < \epsilon,$$

which is a contradiction to (7). This ends the proof of Theorem 3.1. ∎

Now we state and prove the asymptotic stability theorem.

THEOREM 3.2. *If there exist a neighborhood* $\mathscr{V} \subset \mathscr{U}$ *of the origin and a function* $V \in C^0(\mathscr{V}, \mathbb{R})$ *such that*

(1) $V(x) \geq 0$ *for all* $x \in \mathscr{V}$ *and* $V(0) = 0$,

(2) $\Delta V(x) = V(f(x)) - V(x) \leq 0$ *for all* $x \in \mathscr{V}$,

(3) 0 *is* G^**-asymptotically stable where* G^* *is the largest positively invariant set contained in* $G = \{x \in \mathscr{V} : V(f(x)) - V(x) = 0\}$,

then the origin is asymptotically stable.

Proof. The set $G_0 = \{x \in \mathscr{V} : V(x) = 0\}$ is positively invariant, so it is contained in G^*. All the assumptions of Theorem 3.1 are satisfied which implies that the origin is stable, that is, for any positive δ there exists a positive number γ such that any solution of (6) which starts in B_γ remains in B_δ for all integers n.

Let $\mathscr{A}_{G^*}$ be the domain of attractivity relative to G^*. We choose $\delta > 0$, such that $\overline{B_\delta} \cap G^* \subset \mathscr{A}_{G^*}$. To show the attractivity of the origin, we shall prove that B_γ is contained in the domain of attractivity, i.e.,

$$\forall x \in B_\gamma, \qquad \lim_{k \to +\infty} f^k(x) = 0. \tag{13}$$

Let $x_0 \in B_\gamma$ and let ϵ be any positive real number. Thanks to the stability of the origin, it is possible to find $\eta > 0$ in such a way

$$f^n(B_\eta) \subset B_\epsilon, \qquad \forall n \in \mathbb{N}. \tag{14}$$

Since $\overline{B_\delta} \cap G^* \subset \mathscr{A}_{G^*}$, there exists $N \in \mathbb{N}$ such that

$$\| f^n(y) \| < \frac{\eta}{2} \qquad \forall n \geq N, \forall y \in \overline{B_\delta} \cap G^*. \tag{15}$$

The continuity of the solutions ensures the existence of $\alpha > 0$ such that

$$\forall (x, y) \in \overline{B_\delta} \times \overline{B_\delta}, \qquad \|x - y\| < \alpha \Rightarrow \| f^n(x) - f^n(y) \| < \frac{\eta}{2} \; \forall n \leq N. \tag{16}$$

Now, let y be an element of $L^+(x_0)$. According to the LaSalle Invariance Principle, y belongs to $\overline{B_\delta} \cap G^*$, so, by (15), we have

$$\| f^n(y) \| < \frac{\eta}{2}, \qquad \forall n \geq N. \tag{17}$$

On the other hand $y \in L^+(x_0)$, hence

$$\exists p \in \mathbb{N}, \qquad \| f^p(x_0) - y \| < \alpha. \tag{18}$$

Using (18), (16), and (17) we get

$$\| f^{N+p}(x_0) \| < \frac{\eta}{2} + \frac{\eta}{2} = \eta. \tag{19}$$

From (14), it follows

$$\left\| f^n\left(f^{N+p}(x_0)\right) \right\| < \epsilon, \qquad \forall n \in \mathbb{N}.$$

This proves that $\lim_{k \to +\infty} f^k(x_0) = 0$. So we have shown that there exists a positive real number γ such that B_γ is contained in the domain of attractivity, and thus Theorem 3.2 is established. ∎

Now suppose that the system (6) is defined on $\mathbb{R}^n$ and there exists a nonnegative function $V \in C^0(\mathbb{R}^n, \mathbb{R}^+)$ which is a Lyapunov function for (6) that is $\Delta V(x) = V(f(x)) - V(x) \leq 0$ for all $x \in \mathbb{R}^n$. As above, G^* denotes the largest positively invariant set contained in $G = \{x \in \mathbb{R}^n : \Delta V(x) = V(f(x)) - V(x) = 0\}$. One can ask the following: Does the global asymptotic stability of the system restricted to the positively invariant set G^* imply the global asymptotic stability of system (6)? The answer is unfortunately no as it can be shown in Example 3.3 below. Nevertheless, we have the following global result which is a consequence of the LaSalle Invariance Principle and Theorem 3.2.

THEOREM 3.3. *If there exists a function* $V \in C^0(\mathbb{R}^n, \mathbb{R}^+)$ *satisfying*

(1) $V(x) \geq 0$ *for all* $x \in \mathbb{R}^n$ *and* $V(0) = 0$,

(2) $\Delta V(x) = V(f(x)) - V(x) \leq 0$ *for all* $x \in \mathbb{R}^n$,

(3) 0 *is* G^**-globally asymptotically stable where* G^* *is the largest positively invariant set contained in* $G = \{x \in \mathbb{R}^n: V(f(x)) - V(x) = 0\}$,

(4) *All the solutions of system* (6) *are bounded*,

then the origin is globally asymptotically stable.

3.2. *Remarks and Examples*

Remark 3.1. Theorem 3.2 states that the asymptotic stability of the origin is equivalent to its G^*-asymptotic stability. This equivalence is no more true for the Lyapunov stability as it can be shown thanks to the following example:

EXAMPLE 3.1.

$$\begin{cases} x(k+1) = x(k) + y^2(k), \\ y(k+1) = y(k), \\ (x, y) \in \mathbb{R}^2. \end{cases} \tag{20}$$

Let $V(x, y) = y^2$. We have $G_0 = \{(x, y) \in \mathbb{R}^2: y = 0\}$.

The origin is G_0-stable but the solution of the system is $f^n(x, y) = (x + ny^2, y)$, which tends to infinity for all initial data (x, y). Thus the system is unstable. This shows that G_0-stability of the origin is not sufficient to get the stability of the origin with respect to arbitrary perturbations.

EXAMPLE 3.2. It is known that if f is a $\mathscr{C}^1$ function and the linearization of system (6) at zero, namely

$$A = \frac{\partial f}{\partial x}(0) \tag{21}$$

is stable (that is, the eigenvalues λ_i of the matrix A lie in the open unit disk: $|\lambda_i| < 1$), then system (6) is locally asymptotically stable and when the linearization (21) has at least one eigenvalue λ outside the closed unit disk (i.e., $|\lambda| > 1$) then system (6) is unstable. But, when the linearized system is critical, that is, the matrix A has all its eigenvalues inside the closed unit disk with at least one eigenvalue λ which satisfies $|\lambda| = 1$, then one cannot conclude about the stability of system (6), that is, the zero solution of (6) may be stable or unstable. So the results of this article can be helpful to study the stability properties of systems whose linearization is critical. For instance, consider the example

$$\begin{cases} x(k+1) = y(k), \\ y(k+1) = \dfrac{y(k)}{1+\beta x(k)^2}, \\ (x(k), y(k)) \in \mathbb{R}^2, \quad \beta > 0. \end{cases} \tag{22}$$

The linearized system around the equilibrium $(0,0)$ is

$$\begin{cases} x(k+1) = y(k), \\ y(k+1) = y(k). \end{cases} \tag{23}$$

Here $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. The linearized system is critical (here it is stable but not asymptotically stable), so the linearization techniques do not allow us to conclude about the stability of system (22). Let $V(x,y) = y^2$. We have $\Delta V = y^2[(1/(1+\beta x(k)^2)) - 1] \leq 0$, so V is a nonnegative Lyapunov function for system (22). Moreover, we have $G^* = G_0$ and the origin is G^*-asymptotically stable. Hence, by Theorem 3.2, the zero solution of (22) is asymptotically stable.

EXAMPLE 3.3. Consider a system described by

$$\begin{cases} x(k+1) = \dfrac{x(k)}{2} + \dfrac{3}{2}\left(y(k)x(k)^2\right), \\ y(k+1) = \dfrac{y(k)}{2}, \\ (x(k), y(k)) \in \mathbb{R}^2. \end{cases} \tag{24}$$

If we take $V(x,y) = y^2$ then $\Delta V(x,y) = -3/4y^2 \leq 0$.

$G_0 = \{(x,y) \in \mathbb{R}^2 \colon V(x,y) = 0\} = \{(x,0) \colon x \in \mathbb{R}\} = G$ and so $G_0 = G^*$. The origin is G^*-globally asymptotically stable but the system is not globally asymptotically stable. Indeed, one can see that the set

$$\left\{(x,y) \in R^2 \colon xy = 1\right\}$$

is invariant, and so global asymptotic stability cannot be expected.

EXAMPLE 3.4. Consider a system described by

$$\begin{cases} x(k+1) = Ax(k) + f(x(k), y(k)), \\ y(k+1) = By(k) + g(x(k), y(k)), \\ (x,y) \in \mathbb{R}^n \times \mathbb{R}^p, \end{cases} \tag{25}$$

where $A \in M_n(\mathbb{R})$ and $B \in M_p(\mathbb{R})$ are two constant square matrices such that all the eigenvalues of A belong to $\{z \in \mathbb{C}: |z| = 1\}$ while all the eigenvalues of B are in $\{z \in \mathbb{C}: |z| < 1\}$, f and g are two functions of class C^2 with $f(0,0) = 0$, $f'(0,0) = 0$, $g(0,0) = 0$, $g'(0,0) = 0$. (Here, f' is the Jacobian matrix of f.)

J. Carr [1, Theorem 8, p. 35] has established that system (25) has a center manifold $y = h(x)$ with h of class C^2 and $h(0) = 0$, $h'(0) = 0$. He has also proved the following result

THEOREM 3.4 [1]. *If the zero solution of*

$$x(k+1) = Ax(k) + f(x(k), h(x(k))) \tag{26}$$

is asymptotically stable then so is the zero solution of system (25).

The proof proposed in [1] is very clever but here, using Theorem 3.2, we can give a simpler proof.

Indeed, the matrix B has all its eigenvalues with modulus < 1, so there exists a symmetric positive definite matrix $Q \in M_p(\mathbb{R})$ such that $B^T Q B - Q = -I_p$, where I_p is the identity matrix of $M_p(\mathbb{R})$.

Let $V = [y - h(x)]^T Q [y - h(x)]$. The difference of V along the solutions of (25) is

$$\begin{aligned}\Delta V = &\left(By + g(x,y) - h(Ax + f(x,y))\right)^T \\ &\times Q\left(By + g(x,y) - h(Ax + f(x,y))\right) \\ &- [y - h(x)]^T Q [y - h(x)].\end{aligned}$$

On the one hand f and g are supposed to be C^2, so one can write

$$f(x,y) = f(x, h(x)) + F(x,y)[y - h(x)],$$

$$g(x,y) = g(x, h(x)) + G(x,y)[y - h(x)],$$

$$\begin{aligned}&h(Ax + f(x,y)) \\ &\quad = h(Ax + f(x, h(x))) + H(x,y) \cdot F(x,y) \cdot [y - h(x)].\end{aligned}$$

The matrices F, G, and H are defined by

$$F(x,y) = \int_0^1 \frac{\partial f}{\partial y}(x, sy + (1-s)h(x))\,ds,$$

$$G(x,y) = \int_0^1 \frac{\partial g}{\partial y}(x, sy + (1-s)h(x))\,ds,$$

$$H(x,y) = \int_0^1 \frac{\partial h}{\partial x}(Ax + f(x, h(x)) + sF(x,y) \cdot [y - h(x)])\,ds.$$

On the other hand the set $y = h(x)$ is positively invariant, so

$$Bh(x) + g(x, h(x)) - h(Ax + f(x, h(x))) = 0.$$

Thus

$$\begin{aligned}\Delta V = \big(&B[y - h(x)] + G(x, y)[y - h(x)] \\ &- H(x, y)F(x, y)[y - h(x)]\big)^T Q \\ &\times \big(B[y - h(x)] + G(x, y)(y - h(x)) \\ &- H(x, y)F(x, y)[y - h(x)]\big) \\ &- [y - h(x)]^T Q[y - h(x)] \\ = {}& [y - h(x)]^T \big[B^T QB - Q\big][y - h(x)] \\ &+ [y - h(x)]^T \epsilon(x, y)[y - h(x)],\end{aligned}$$

where

$$\begin{aligned}\epsilon(x, y) &\\ = {}& B^T Q[G(x, y) - H(x, y)F(x, y)] \\ &+ [G(x, y) - H(x, y)F(x, y)]^T QB \\ &+ [G(x, y) - H(x, y)F(x, y)]^T Q[G(x, y) - H(x, y)F(x, y)].\end{aligned}$$

So the difference of V along the solutions of (25) becomes

$$\Delta V = (y - h(x))^T \big(-I_p + \epsilon(x, y)\big)(y - h(x)).$$

The functions F, G, and H are of class C^1 and vanish at the origin, so the matrix $\epsilon(x, y)$ is symmetric, C^1 in (x, y) and vanishes at the origin. Hence, in a neighborhood of the origin, one has

$$\Delta V \leq \frac{-1}{2} |y - h(x)|^2.$$

Thus Theorem 3.2 can apply to finish the proof. ∎

4. APPLICATIONS TO CONTROL PROBLEMS

4.1. *Stability and Stabilization of Cascade Systems*

In this section we use the theorems of Section 3 to study a special class of nonlinear discrete-time systems called Cascade systems. The continuous-time systems of this form have been studied by many authors (an extensive bibliography can be found in [8, 9]). Consider the following cascade nonlinear system:

$$\begin{cases} x(k+1) = f(x(k), y(k)), \\ y(k+1) = g(y(k)), \\ (x(k), y(k)) \in \mathbb{R}^n \times \mathbb{R}^m, \end{cases} \tag{27}$$

where f and g are continuous functions, $f(0,0) = 0$, and $g(0) = 0$.

Consider then the systems

$$\begin{cases} x(k+1) = f(x(k), 0), \\ x(k) \in \mathbb{R}^n. \end{cases} \tag{28}$$

$$\begin{cases} y(k+1) = g(y(k)), \\ y(k) \in \mathbb{R}^m. \end{cases} \tag{29}$$

Using Theorem 3.2, we can state the following

PROPOSITION 4.1. *The system* (27) *is asymptotically stable if and only if the subsystems* (28) *and* (29) *are asymptotically stable*.

Proof. ("If") By the converse Lyapunov theorem, the asymptotic stability of system (29) guarantees the existence of a Lyapunov function $V(y)$ defined on a neighborhood $\mathscr{Y}$ of the origin of $\mathbb{R}^m$ and satisfying

$$V(y) > 0, \forall y \neq 0, \qquad V(0) = 0,$$

and

$$V(g(y)) - V(y) < 0 \; \forall y \neq 0.$$

For the overall system (27), this function (considered as a function of (x, y)) is a nonnegative function. Its difference along the solutions of (29) is nonpositive and vanishes on the set $G = \{(x, y)\colon \Delta V(x, y) = 0\} = \{(x, 0), x \in \mathbb{R}^n\}$. Here $G^* = G = G_0$. Moreover, $x = 0$ is assumed to be an asymptotically stable equilibrium point of $x(k+1) = f(x(k), 0)$, so the origin is G^*-asymptotically stable. Thus, all the hypotheses of Theorem 3.2 are satisfied which implies that the origin is an asymptotically stable equilibrium point of the system (27).

("Only if") Suppose $(x, y) = (0, 0)$ is an asymptotically stable equilibrium point of (27). Then $x = 0$ is an asymptotically stable equilibrium point of (28) because $\{(x, 0), x \in \mathbb{R}^n\}$ is a positively invariant set for (27) and $y = 0$ is an asymptotically stable equilibrium point of (29) because $\{(y, 0), y \in \mathbb{R}^m\}$ is a positively invariant set for (27). ∎

The above result can be easily generalized by induction to relate the stability properties of the large-scale system

$$\begin{cases} z_1(k+1) = f_1(z_1(k)), \\ z_2(k+1) = f_2(z_1(k), z_2(k)), \\ \quad\vdots \\ z_p(k+1) = f_p\big(z_1(k), z_2(k), \dots, z_p(k)\big), \\ z_i(k) \in \mathbb{R}^{m_i}, \quad i = 1, \dots, p, \end{cases} \tag{30}$$

to those of a collection of isolated subsystems

$$z_i(k+1) = f_i(0, \dots, 0, z_i(k)). \tag{31}$$

COROLLARY 4.1. *The origin $z = 0$ is an asymptotically stable equilibrium point of* (30) *if and only if $z_i = 0$ is an asymptotically stable equilibrium point of* (31), *for all $i \in \{1, \dots, p\}$.*

From Proposition 4.1 and Corollary 4.1 we derive a stabilization result for control systems described by

$$\begin{cases} x(k+1) = F(x(k), y(k), u(k), v(k)), \\ y(k+1) = G(y(k), v(k)), \end{cases} \tag{32}$$

where $z(k) = (x(k), y(k))$ is the state vector of the system and $w(k) = (u(k), v(k)) \in \mathbb{R}^p \times \mathbb{R}^q$ is the control. F and G are continuous, $F(0,0,0,0) = 0$, and $G(0,0) = 0$.

The problem addressed here is to find conditions under which it is possible to stabilize system (32) by a "decentralized" feedback control $w(z) = (u(x), v(y))$. Proposition 4.1 allows us to establish the following

COROLLARY 4.2. *The cascade control system* (32) *is stabilizable by means of a decentralized feedback law $w(z(k)) = (u(x(k)), v(y(k)))$ if and only if the systems $x(k+1) = F(x(k), 0, u(k), 0)$ and $y(k+1) = G(y(k), v(k))$ are stabilizable.*

In particular, when $G(y,u) = Ay + Bu$ and $F(x(k), y(k), u(k), v(k)) = f(x(k), y(k))$ we rediscover the following classical result

COROLLARY 4.3. *If the system $x(k+1) = f(x(k), 0)$ is asymptotically stable and the pair (A, B) is a stabilizable pair of matrices then the system*

$$\begin{cases} x(k+1) = f(x(k), y(k)), \\ y(k+1) = Ay(k) + Bu(k), \end{cases}$$

is stabilizable by a linear feedback law $u = Ky$.

We would like to mention here that the result given in Corollary 4.3 can also be proved by means of the center manifold machinery as it has done for the continuous-time systems. However, the center manifold theorems cannot be used to achieve the stabilization of system (32) because no assumptions are made on the linear part of the functions F and G.

4.2. *Stabilization of Nonlinear Systems with Stable Free Dynamics*

Consider a discrete-time control nonlinear system described by

$$x(k+1) = f(x(k), u(k)), \tag{33}$$

where $x(k) \in \mathbb{R}^n$ is the state of the system at time k, $u \in \mathbb{R}^m$ is the control, and f: $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a C^2 function satisfying $f(0,0) = 0$. The problem addressed here is how to find a feedback control which stabilizes the system at its equilibrium point. To be more precise we recall the following definition:

DEFINITION 4.1. System (33) is said to be stabilizable if there exists a continuous mapping u: $\mathbb{R}^n \to \mathbb{R}^m$ satisfying $u(0) = 0$ and such that the closed loop system $x(k+1) = f(x(k), u(x(k)))$ is asymptotically stable at the origin.

Thanks to Theorem 3.1 and Theorem 3.2 we shall develop a machinery to construct a stabilizing feedback for systems of the form (33) that are Lyapunov stable (but not asymptotically stable) when the control is identically null. Furthermore, we make the following assumption:

($\mathscr{H}$) There exist a neighborhood $\mathscr{V}$ of the origin and a function $V \in C^2(\mathbb{R}^n, \mathbb{R})$ satisfying $V(x) \geq 0$ for all $x \in \mathscr{V}$, $V(0) = 0$, and $V(f(x,0)) \leq V(x)$ for all $x \in \mathscr{V}$.

Before stating our stabilization result we need to introduce the following notations: As in Section 3, G_0 denotes the set where V vanishes, i.e., $G_0 = \{x \in \mathscr{V} | V(x) = 0\}$. We define $\tilde{f}$: $\mathbb{R}^n \to \mathbb{R}^n$ to be the C^2 function defined on $\mathbb{R}^n$ by

$$\tilde{f}(x) = f(x, 0). \tag{34}$$

For all $k \in \mathbb{N}$, we define recursively $\tilde{f}^k$: $\mathbb{R}^n \to \mathbb{R}^n$ by

$$\tilde{f}^0(x) = x,$$
$$\tilde{f}^k(x) = \tilde{f}\big(\tilde{f}^{k-1}(x)\big), \qquad \text{for } k \geq 1.$$

$\tilde{V}$: $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and φ: $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ are defined respectively by

$$\tilde{V}(x,u) = V(f(x,u)), \tag{35}$$

$$\varphi(x,u,v) = \int_0^1 (1-t) v^T(x) \frac{\partial^2 \tilde{V}}{\partial u^2}(x, tu(x)) v(x)\, dt. \tag{36}$$

For a fixed $\eta > 0$, let $K_1(x)$ and $K_2(x)$ be any nonnegative continuous real valued functions satisfying $K_1(x) + K_2(x) \neq 0$, $\forall x \in \mathscr{V}$ and

$$K_1(x) \geq \sup_{\|u\| \leq \eta,\ \|v\|=1} |\varphi(x,u,v)|, \qquad \forall x \in \mathscr{V}. \tag{37}$$

$$K_2(x) \geq \left\| \frac{\partial V}{\partial x}\big(\tilde{f}(x)\big) \frac{\partial f}{\partial u}(x,0) \right\|, \qquad \forall x \in \mathscr{V}, \tag{38}$$

and set

$$K(x) = \frac{\eta}{\eta K_1(x) + K_2(x)}. \tag{39}$$

Now we can state our stabilization result:

THEOREM 4.1. *Under the assumption* ($\mathscr{H}$), *if the unforced system* $x(k+1) = f(x(k), 0)$ *is* G_0*-asymptotically stable and the set*

$$W = \Big\{ x \in \mathscr{V} \,|\, V\big(\tilde{f}^{k+1}(x)\big) - V\big(\tilde{f}^{k}(x)\big) = 0 \text{ and } \frac{\partial V}{\partial x}\big(\tilde{f}^{k+1}(x)\big) \frac{\partial f}{\partial u}\big(\tilde{f}^{k}(x), 0\big) = 0,\ \forall k \in \mathbb{N} \Big\}$$

is reduced to G_0 *then, for any positive constant* η, *system* (33) *is asymptotically stabilizable by means of the continuous feedback law*

$$u(x) = -K(x) \left(\frac{\partial V}{\partial x}\big(\tilde{f}(x)\big) \frac{\partial f}{\partial u}(x,0) \right)^T \tag{40}$$

which satisfies

$$\|u(x)\| \leq \eta, \qquad \forall x \in \mathscr{V}.$$

Proof. If one computes the difference of the Lyapunov function V along the solutions of the closed-loop system (33)–(40), one gets from (35) and the Taylor expansion formula

$$\begin{aligned}
\Delta V(x) &= V(f(x, u(x))) - V(x) \\
&= \tilde{V}(x, u(x)) - V(x) \\
&= \tilde{V}(x, 0) - V(x) + \frac{\partial \tilde{V}}{\partial u}(x,0) u(x) \\
&\quad + \int_0^1 (1-t) u^T(x) \frac{\partial^2 \tilde{V}}{\partial u^2}(x, tu(x)) u(x)\, dt.
\end{aligned}$$

Notice that

$$\tilde{V}(x,0) = V\big(\tilde{f}(x)\big), \tag{41}$$

and

$$\frac{\partial \tilde{V}}{\partial u}(x,0) = \frac{\partial V}{\partial x}\big(\tilde{f}(x)\big)\frac{\partial f}{\partial u}(x,0).$$

So, from (36) and (40), one gets

$$\Delta V(x) = V\big(\tilde{f}(x)\big) - V(x) - \frac{1}{K(x)}u^T(x)u(x) + \varphi(x,u(x),u(x)).$$

It follows that, for $x \in \mathscr{V}$ such that $u(x) = 0$, one has

$$\Delta V(x) = V\big(\tilde{f}(x)\big) - V(x),$$

and otherwise, $\varphi(x,u,v)$ being homogeneous of degree 2 with respect to v, one gets

$$\Delta V(x) = V\big(\tilde{f}(x)\big) - V(x) - \frac{1}{K(x)}\|u(x)\|^2 + \|u(x)\|^2\varphi\left(x, u(x), \frac{u(x)}{\|u(x)\|}\right).$$

From (37)–(38)–(39) and (40), one has for any $x \in \mathscr{V}$, $\|u(x)\| \leq \eta$, and so one can deduce that $\Delta V(x) \leq 0$.

It remains to prove the asymptotic stability of the origin. To this end, by Theorem 3.2, it is sufficient to show that the origin is G^*-asymptotically stable where G^* is the largest invariant set contained in the locus

$$G = \big\{x \in \mathscr{V} | \Delta V(x) = V(f(x,u(x))) - V(x) = 0\big\}.$$

We have

$$\begin{aligned}\Delta V(x) = 0 &\Leftrightarrow V\big(\tilde{f}(x)\big) - V(x) = 0 \text{ and } u(x) = 0\\ &\Leftrightarrow V\big(\tilde{f}(x)\big) - V(x) = 0 \text{ and } \frac{\partial V}{\partial x}\big(\tilde{f}(x)\big)\frac{\partial f}{\partial u}(x,0) = 0.\end{aligned} \tag{42}$$

Let x be any element of G^* and $x(k)$ be the solution of the closed-loop system (33)–(40) with $x(0) = x \in G^*$. Since G^* is positively invariant for the closed-loop system, one has $x(k) \in G^*$ for all $k \geq 0$. Now $u(x) \equiv 0$ on G^*, so the closed-loop system (33)–(40) on G^* is governed by

$$x(k+1) = f(x(k), 0) = \tilde{f}(x(k)). \tag{43}$$

Therefore, the solution issued from x is $x(k) = \tilde{f}^k(x)$. Hence, by (42), x satisfies

$$V\big(\tilde{f}^{k+1}(x)\big) - V\big(\tilde{f}^k(x)\big) = 0$$

and

$$\frac{\partial V}{\partial x}\big(\tilde{f}^{k+1}(x)\big)\frac{\partial f}{\partial u}\big(\tilde{f}^k(x), 0\big) = 0, \forall k \in \mathbb{N}.$$

This implies that $x \in W$. Thus $G^* \subseteq W$. By hypothesis, the set W is assumed to be equal to G_0, so we have $G^* = G_0$. On the other hand system (43) is supposed to be G_0-asymptotically stable, so the origin is G^*-asymptotically stable for (33)–(40). This completes the proof of Theorem 4.1. ∎

Remark 4.1. When system (33) is such that the unforced system $x(k+1) = f(x(k), 0)$ is not stable, the above stabilization procedure can still apply if one can find a C^2 preliminary feedback $u = \beta(x)$ in such a way that $x(k+1) = f(x(k), \beta(x))$ is stable. In fact, one can write $f(x, u) = f(x, \beta(x) + v)$. Therefore, if the conditions of Theorem 4.1 are satisfied with $\tilde{f}(x) = f(x, \beta(x))$, then the feedback law $u(x) = \beta(x) - K(x)((\partial V/\partial x)(\tilde{f}(x))(\partial f/\partial u)(x, \beta(x)))^T$ stabilizes system (33).

EXAMPLE 4.1. To illustrate the stabilization procedure of Theorem 4.1, let us consider the following nonlinear control system evolving on $\mathbb{R}^3$:

$$\begin{cases} x_1(k+1) = \dfrac{x_1(k)}{2} + x_2(k) + x_2^2(k) + x_3^2(k), \\ x_2(k+1) = \big(x_1(k) + 2x_3^2(k)\big)u_1(k), \\ x_3(k+1) = x_3(k) + \dfrac{x_1^4(k)}{1 + x_2^2(k)}u_2(k). \end{cases} \tag{44}$$

The unforced dynamic system

$$\begin{cases} x_1(k+1) = \dfrac{x_1(k)}{2} + x_2(k) + x_2^2(k) + x_3^2(k), \\ x_2(k+1) = 0, \\ x_3(k+1) = x_3(k) \end{cases}$$

is Lyapunov-stable but not asymptotically stable. By using Theorem 4.1, we shall compute a stabilizing feedback law $u(x)$ (here $x = (x_1, x_2, x_3)$). Let V be the nonnegative function $V(x) = x_2^2 + x_3^2$. Here $G_0 = \{x_2 = x_3 = 0\}$.

We have $\Delta V(x) = -x_2^2 \leq 0$, so V is a Lyapunov function for the unforced system. On the one hand, it is obvious that the origin is G_0-asymptotically stable. On the other hand, one may easily check that $W = \{x_2 = x_3 = 0\} = G_0$. So, one can apply Theorem 4.1 to system (44) and the procedure for the computation of a stabilizer feedback is the following.

With the same notations as above, simple computations give

$$\varphi(x,u,v) = \left((x_1 + 2x_3^2)v_1\right)^2 + \left(\left(\frac{x_1^4}{1 + x_2^2}\right)v_2\right)^2,$$

so one can take $K_1(x) = (x_1 + 2x_3^2)^2 + (x_1^4/(1 + x_2^2))^2$.

$$\frac{\partial V}{\partial x}\left(\tilde{f}(x)\right)\frac{\partial f}{\partial u}(x,0) = \left(0, \frac{2x_1^4 x_3}{1 + x_2^2}\right),$$

so one can take $K_2(x) = 1 + x_1^8 x_3^2$. Thus, we get the following bounded stabilizer,

$$u(x) = \left(0, -\frac{2\eta x_1^4 x_3\left(1 + x_2^2\right)}{\eta\left(x_1^8 + \left(1 + x_2^2\right)^2\left(x_1 + 2x_3^2\right)^2\right) + \left(1 + x_2^2\right)^2\left(1 + x_1^8 x_3^2\right)}\right)^T,$$

where η is an arbitrary positive real constant.

4.3. *A Separation Principle*

We consider input-output nonlinear systems described by

$$\begin{cases} x(k+1) = f(x(k), u(k)) \\ y(k) = h(x(k)), \end{cases} \tag{45}$$

where $x(k) \in \mathbb{R}^n$ is the state of the system at time k and $y(k)$ is the measurable output of the system.

$f\colon \mathbb{R}^n \to \mathbb{R}^n$ and $h\colon \mathbb{R}^n \to \mathbb{R}^m$ are C^0 functions satisfying $f(0,0) = 0$, $h(0) = 0$.

An observer for (45) is a dynamical system whose inputs are the inputs and the outputs of system (45) and which produces an estimate $\hat{x}(k)$ of the state $x(k)$ in such a way $x(k) - \hat{x}(k) \to 0$ as $k \to +\infty$ and the estimation error must be small if it starts small.

Here, we assume that (45) admits an observer of the form

$$\hat{x}(k = 1) = g(\hat{x}(k), y(k), u(k)), \tag{46}$$

where g is continuous and $g(0,0,0) = 0$. Let $e(k) = x(k) - \hat{x}(k)$ be the estimation error. The error equation is

$$e(k+1) = f(x(k), u(k)) - g(\hat{x}(k), y(k), u(k)). \tag{47}$$

System (46) is assumed to be an observer for (45), so g and f satisfy $f(x,u) = g(x, h(x), u)$ for all x and all admissible inputs u. Furthermore the null solution $e(k) \equiv 0$ of (47) is asymptotically stable. The problem addressed here is the following: Is it possible to stabilize system (45) by an estimated state feedback? More precisely, suppose there exists a continuous feedback law $\alpha(x)$ defined in a neighborhood of the origin in such a way that the origin is an asymptotically stable equilibrium point for the closed-loop system

$$x(k+1) = f(x(k), \alpha(x(k))). \tag{48}$$

Is the equilibrium $(x, e) = (0, 0)$ of

$$\begin{cases} x(k+1) = f(x(k), \alpha(\hat{x}(k))), \\ e(k+1) = f(x(k), \alpha(\hat{x}(k))) - g(\hat{x}(k), h(x(k)), \alpha(\hat{x}(k))) \end{cases} \tag{49}$$

asymptotically stable?

Without any supplementary conditions, this question remains an open problem. With the help of Theorem 3.2, we are able to give the following result which solves the problem for a wide class of nonlinear control systems (including bilinear systems).

PROPOSITION 4.2. *Suppose system* (46) *is an observer for system* (45) *and there exists an autonomous Lyapunov continuous function W for the error equation* (47) *satisfying*

(i) $W(0) = 0$, $W(e) > 0$ *for all* $e \neq 0$ *in a neighborhood* $\mathscr{E}$ *of the origin* $(e = 0)$.

(ii) *The difference* $W(f(x,u) - g(x - e, h(x), u)) - W(e)$ *is negative definite in* $e \in \mathscr{E}$ *for all* $x \in \mathbb{R}^n$ *and all admissible controls* u.

Then the problem of stabilization by an estimated state feedback for system (45) *is solvable. In other words if the origin* $x = 0$ *is an asymptotically stable equilibrium point for system* (48) *then* $(x, e) = (0, 0)$ *is also an asymptotically stable equilibrium point for system* (49).

Proof. To show that system (49) is asymptotically stable, we define the function $V(x, e) = W(e)$. V satisfies the assumptions (1) and (2) of Theorem 3.2. It remains to show that the origin is G^*-asymptotically stable. Here, we have $G_0 = G = G^* = \{(x, 0), x \in \mathscr{V}\}$. So, on G^*, we have $\hat{x} = x$ and thus the system is governed by

$$x(k+1) = f(x(k), \alpha(x(k))),$$

which is assumed to be asymptotically stable. This proves that assumption (3) of Theorem 3.2 is satisfied which ends the proof of Proposition 4.2. ∎

EXAMPLE 4.2. Consider the following bilinear control system

$$\begin{cases} x(k+1) = Ax(k) + uBx(k), \\ y(k) = Cx(k), \end{cases} \tag{50}$$

where $x(k) \in \mathbb{R}^n$ is the state of the system at time $k \in \mathbb{N}$, $y(k) \in \mathbb{R}^m$ is the measurable output of the system, and $u(k) \in U \subset \mathbb{R}$ is the input or the control. A, B, and C are constant matrices of appropriate dimensions. We assume that the pair (A, C) is observable. A candidate observer for system (50) is the system

$$\hat{x}(k+1) = A\hat{x}(k) + uB\hat{x}(k) + K(y(k) - C\hat{x}(k)). \tag{51}$$

Indeed the error equation is

$$e(k+1) = (A - KC)e(k) + uBe(k). \tag{52}$$

According to linear systems theory, the observability of the pair (A, C) ensures the existence of a matrix K in such a way that all the eigenvalues of the matrix $\hat{A} = A - KC$ lie inside the open unit disk. Thus there exists a symmetric positive definite matrix P such that $\hat{A}^T P \hat{A} - P = -I_n$ (I_n is the $n \times n$ identity matrix and M^T denotes the transpose matrix of M). To prove the asymptotic stability of (52), we use the following candidate Lyapunov function $W(e) = e^T P e$. Its difference along the solutions of (52) is

$$\begin{aligned} \Delta W(e) &= (e^T \hat{A}^T + ue^T B^T) P(\hat{A}e + uBe) - e^T Pe \\ &= e^T \Big(\hat{A}^T P \hat{A} - P + u\big(\hat{A}^T PB + B^T P \hat{A}\big) + u^2 B^T PB \Big) e \\ &= -\|e\|^2 + e^T \Big(u\big(\hat{A}^T PB + B^T P\hat{A}\big) + u^2 B^T PB \Big) e \\ &\leq -\|e\|^2 \big(1 - 2\|u\|\,\|A\|\,\|P\|\,\|B\| - u^2 \|B\|^2 \|P\|\big). \end{aligned}$$

Thus for $\|u\| \leq \eta$ a constant which depends on $\|A\|$, $\|P\|$, and $\|B\|$, we have $\Delta W(e) \leq -\gamma \|e\|^2$ where γ is a positive constant. This proves that system (51) is an observer for system (50) and moreover the function W satisfies the conditions of Proposition 4.2.

Now if system (50) is stabilizable by a state feedback law $u = \alpha(x)$ which satisfies $\|\alpha(x)\| \leq \eta$ for all $x \in \mathbb{R}^n$ (this can be realized if, for example, system (50) satisfies the assumptions of Theorem 4.1) then all the hypotheses of Proposition 4.2 are satisfied and so system (50) can be stabilized thanks to an estimated state feedback $u = \alpha(\hat{x})$ where the estimation $\hat{x}$ is given by the observer (51).

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