The Enveloping Algebra of the Lie Superalgebra osp(1, 2)

GEORGES PINCZON

Laboratoire de Physique-Mathématique, UA CNRS 1102, Université de Bourgogne, BP 138, 21004 Dijon Cedex, France

Communicated by Walter Feit
Received August 8, 1988

INTRODUCTION AND MAIN RESULTS

Let $G = G_0 \oplus G_1$ be a Lie superalgebra, and $U$ the enveloping algebra of the Lie algebra $G_0$. $G$ also has an enveloping algebra $\mathcal{V}$, which is a $\mathbb{Z}_2$-graded associative algebra; $U$ is an entire subalgebra of $\mathcal{V}$, and $\mathcal{V}$ is a finite type free $U$-module. This is easily seen from the Poincaré-Birkhoff-Witt (P.B.W.) theorem, which holds not only for Lie algebras, but also for Lie superalgebras.

Unfortunately, many examples are known of superalgebras $G$ having a non entire enveloping algebra, and this is a little disheartening, because computations become very involved!

For instance, construction, or even complete classification of primitive (or prime) ideals of $U$ has been carried out for some series or particular cases of Lie algebras $G_0$ (e.g., [6]), but there is no similar result in the case of superalgebras. Finally, the relation between primitive (or prime) ideals of $\mathcal{V}$ and ideals of $U$, which is highly related to the reduction of irreducible $\mathcal{V}$-module into $U$-modules, does not seem very clear.

In this paper, we give a very detailed study of the enveloping algebra $\mathcal{V}$ of the orthosymplectic complex superalgebra $G = \text{osp}(1, 2)$. In this case, $G_0 = \text{sl}(2)$, and the enveloping algebra of $G_0$ is very well known (e.g., [5, 11]). $G$ is simple and also semi-simple (in the sense of [8]), and it is exactly an analogue for superalgebras of the Lie algebra $\text{sl}(2)$; irreducible Harish Chandra modules of $G$ are known [3]. As a first result, our study will provide an example of a simple superalgebra having an entire enveloping algebra, since we shall show that it is the case if $G = \text{osp}(1, 2)$. We shall then give a complete detailed classification of all primitive (or prime) ideals of $\mathcal{V}$, and compare with the well-known classification of $U$ [10]. This will reveal very deep differences and singularities between Lie algebras and Lie superalgebras, even in the case of the simplest ones!
Let us now give our main results (unexplained notations and conventions can be found in Sections (0) and (1)).

In Section (1), we study some first consequences of the relation

$$4Q^2 - (8C - 1)Q + 2C(2C - 1) = 0$$

which holds in $\mathcal{V}$, between the respective Casimir operators $Q$ and $C$ of $U$ and $\mathcal{V}$. In our opinion, this relation is the key to the structure of $\mathcal{V}$. A general result of [7], which is true for any superalgebra, states that irreducible $\mathcal{V}$-modules always split into a finite sum of indecomposable $U$-modules. In the case of $G = \text{osp}(1,2)$, we prove:

**Theorem 1.** Let $V = V_0 \oplus V_1$ be an irreducible $\mathcal{V}$-module; then $V_0$ and $V_1$ are irreducible $U$-modules.

We then make precise what happens if we forget the $Z_2$-grading of $\mathcal{V}$:

**Theorem 2.** Let $V$ be an ordinary (not a priori $Z_2$-graded) finite dimensional $\mathcal{V}$-module. There exists a grading $V_0 \oplus V_1$ for which $V$ is a $Z_2$-graded $\mathcal{V}$-module.

As a corollary, Burnside's theorem holds for finite dimensional $Z_2$-irreducible $\mathcal{V}$-modules.

In Section (2), we prove:

**Theorem 3.** $\mathcal{V}$ is entire.

The standard Lie algebras argument (which is the P.B.W. theorem) does not work, and our proof makes use of a new filtration of $\mathcal{V}$; the corresponding grading operation is interpreted as a simple Inonu–Wigner contraction process.

Let us briefly discuss a more general problem: which simple Lie superalgebras do have an entire enveloping algebra? It can be checked that such a superalgebra, if classical, must be of type $\text{osp}(1,2n)$; when $n = 1$, the answer is given by Theorem 1. It would be interesting to find the answer for general $n$.

Now, we come back to $G = \text{osp}(1,2)$. In Section (3), we reduce the adjoint representation of $G$ in $\mathcal{V}$. This will be the key to finding primitive ideals. Let us recall the well-known classification of irreducible representations of $G_0$ and $G$ (e.g., [8]). We denote by $D(h)$ (resp. $D(h)$), $2h \in \mathbb{N}$, the irreducible $G_0$-module (resp. $G$-module) of dimension $(2h + 1)$ (resp. $(4h + 1)$). For $D(h)$ (resp. $\mathcal{D}(h)$), the value of the Casimir operator $Q$ (resp. $C$) is $q = h(h+1)$ (resp. $\lambda = h(2h+1)/2$). As a $G_0$-module, $D(h)$ splits into $D(h) \oplus D(h - \frac{1}{2})$, if $h \neq 0$, and the values of $Q$ are $q_0 = h(h + 1)$ and $q_1 = (h - \frac{1}{2})(h + \frac{1}{2})$. $\mathcal{D}(0) = D(0)$ is the trivial representation. We prove:
THEOREM 4. There exists a submodule $\mathcal{H}$ of the adjoint representation of $G$ in $\mathcal{V}$ such that
\[ \mathcal{V} = \mathcal{H} \oplus (C - \lambda) \mathcal{V}, \quad \forall \lambda \in \mathbb{C}. \]

One has $\mathcal{H} = \sum_{n \geq 0} H_n \oplus V_{n+2}$, where $H_n \cong \mathcal{D}(n)$, and $V_{n+2} \cong \mathcal{D}(n+\frac{1}{2})$. We note that all finite dimensional irreducible representations appear in the reduction.

In Sections (4), (5), (6), and (7), we study the quotient algebra $\mathcal{B}_\lambda = \mathcal{V}/(C - \lambda) \mathcal{V}$, $\lambda \in \mathbb{C}$, and our results lead to a complete classification of prime and primitive ideals of $\mathcal{V}$, which is proved in Section (8):

Given $\lambda \in \mathbb{C}$, we denote by $\mathcal{J}_\lambda = (C - \lambda) \mathcal{V}$. When $\lambda = h(2h + 1)/2$, $2h \in \mathbb{N}$, we denote by $\mathcal{J}_\lambda$ the kernel of $\mathcal{D}(h)$. When $\lambda = -1/16$, we denote by $\mathcal{J}$ the two sided ideal $\mathcal{J} = (Q - 3C) \mathcal{V} + \mathcal{J}_{-1/16}$.

THEOREM 5. (1) When $\lambda \neq -1/16$, $\mathcal{J}_\lambda$ is maximal if and only if $\lambda \neq h(2h + 1)/2$, $2h \in \mathbb{N}$. $\mathcal{J}_\lambda$ is never completely prime.

(2) When $\lambda = h(2h + 1)/2$, $2h \in \mathbb{N}$, $\mathcal{J}_\lambda$ is maximal, and one has $\mathcal{J}_\lambda = [\sum_{n \geq 2h + 1} (H_n \oplus V_{n+1})] \oplus \mathcal{J}_\lambda$. $\mathcal{J}_\lambda$ is completely prime if and only if $\lambda = 0$. $\mathcal{J}_\lambda$ is primitive.

(3) When $\lambda = -1/16$, $\mathcal{J}_\lambda$ is not semi-prime (and a fortiori not primitive).

(4) $\mathcal{J}$ is primitive, maximal, and completely prime. The quotient algebra $\mathcal{V}/\mathcal{J}$ is the Weyl algebra. One has $\mathcal{J} = (\sum_{n \geq 0} V_{n+2}) \oplus \mathcal{J}_{-1/16}$.

(5) Let $\mathcal{J}$ be a prime (resp. primitive, resp. completely prime) non zero ideal of $\mathcal{V}$; then $\mathcal{J}$ is one of the prime (resp. primitive, resp. completely prime) ideals in the above list.

(6) Any prime non zero ideal of $\mathcal{V}$ is primitive.

Note that $\{0\}$ is completely prime (Theorem 3) but not primitive. In order to compare, let us recall the corresponding classification for $U$ (e.g., [10]):

Let $I_q = (Q - q) U$, $q \in \mathbb{C}$, $J_q = \text{Ker } D(h)$ when $q = h(h + 1)$, $2h \in \mathbb{N}$; then $I_q$ is always primitive and completely prime, maximal if $q \neq h(h + 1)$, $2h \in \mathbb{N}$, contained in $J_q$ otherwise. Primitive (resp. prime non zero) ideals of $U$ are exactly the ideals in this list.

What are the principal differences between $U$ and $\mathcal{V}$? A striking one is the existence of the metaplectic case $\lambda = -1/16$, which is completely singular: for instance, $\mathcal{J}_{-1/16}$ is not even semi-prime! A second one is the existence of only three completely prime ideals in $\mathcal{V}$, (namely $\{0\}$, $\mathcal{J}_0$, and $\mathcal{J}$) and the fact that $\mathcal{J}_\lambda$ is never completely prime. This means that the fact that $\mathcal{V}$ is entire is completely lost in most irreducible infinite dimensional representations. It is not at all the case for $U$, which has, from this viewpoint, a much more rigid entire structure than $\mathcal{V}$.
Finally, in (9), we compute the Krull dimension of $\mathcal{V}(G)$, when $G$ is a general Lie superalgebra. Let us note that we have to define two notions of Krull dimension for $\mathcal{V}(G)$: the first one is the deviation of the ordered set of $Z_2$-graded left ideals in $\mathcal{V}(G)$, let us say $sK \dim \mathcal{V}(G)$; the second one is the ordinary Krull dimension $K \dim \mathcal{V}(G)$, defined as the deviation of the ordered set of ordinary left ideals in $\mathcal{V}(G)$, (see, e.g., [6]). Obviously $sK \dim \mathcal{V}(G) \leq K \dim \mathcal{V}(G)$. Moreover, using the canonical filtration of $\mathcal{V}(G)$, we define the associated graded algebra $Gr \mathcal{V}(G)$, and it is known that $K \dim \mathcal{V}(G) \leq K \dim Gr \mathcal{V}(G)$ [6]. If $\{X_1, \ldots, X_n\}$ is a basis of $G_0$, and $\{Y_1, \ldots, Y_p\}$ a basis of $G_1$, by the P.B.W. theorem, $Gr \mathcal{V}(G) = \mathbb{C}[X_1, \ldots, X_n] \otimes A(Y_1, \ldots, Y_p)$. Since $A(Y_1, \ldots, Y_p)$ is artinian, $K \dim Gr \mathcal{V}(G) = K \dim \mathbb{C}[X_1, \ldots, X_n] = n$. So the inequality $K \dim \mathcal{V}(G) \leq \dim G_0$ holds, as was remarked in [2]. Our result gives a better estimate:

**Theorem 6.** One has $sK \dim \mathcal{V}(G) = K \dim \mathcal{V}(G) = K \dim \mathcal{V}(G_0)$.

Specializing to $G = osp(1, 2)$, we obtain $K \dim \mathcal{V}(G) = 2$.

0. **General Conventions**

All (Lie, or Lie super, or associative) algebras considered in this paper are algebras over the field of complex numbers $\mathbb{C}$. Associative algebras always have a unit element. When (Lie or associative) $Z_2$-graded algebras are concerned, all considered objects are implicitly assumed (if the contrary is not mentioned) to be $Z_2$-graded (so modules (or representations) are $Z_2$-graded modules, submodules are homogeneous submodules, ideals are homogeneous ideals, irreducibility is $Z_2$-irreducibility, etc.).

Given an (ordinary, or $Z_2$-graded) associative algebra, we use the following terminology:

- $A$ is a semi-prime algebra if any two sided nilpotent ideal of $A$ vanishes. $A$ is a prime algebra if the product of two non zero two sided ideals of $A$ is a non zero ideal of $A$. $A$ is entire if it has no zero divisors. $A$ is primitive if it has a faithful irreducible representation. $A$ is quasi-simple if its only two sided ideals are $\{0\}$ and $A$.

- Given a two sided ideal $I \neq A$ of $A$, we say that:
  - $I$ is a semi-prime (resp. prime, resp. completely prime, resp. primitive) ideal if $A/I$ is a semi-prime (resp. prime, resp. entire, resp. primitive) algebra.

Let us recall that primitive ideals are prime (e.g., [6]). Completely prime ideals are prime, and prime ideals are semi-prime. Moreover, a maximal two sided ideal is primitive.
1. Remarks about Representations of the Orthosymplectic Superalgebra osp(1, 2)

We define the orthosymplectic super-algebra $G = G_0 \oplus G_1$ by the respective bases \{ $Y, F, G$ \} of $G_0$, \{ $E_+, E_-$ \} of $G_1$, and the commutation rules

$$\begin{align*}
[Y, E_\pm] &= \pm E_\pm/2, \quad [F, E_+] = [G, E_-] = 0, \\
[F, E_-] &= -E_+, \quad [G, E_+] = -E_-
\end{align*}$$

(1.1)

$$\begin{align*}
[E_+, E_+] &= F, \quad [E_-, E_-] = -G, \quad [E_+, E_-] = Y.
\end{align*}$$

$G$ is a simple superalgebra, and $G_0 = sl(2)$. Denote by $\mathcal{V} = \mathcal{V}(G)$ (resp. $U = \mathcal{V}(G_0)$) the enveloping algebra of $U$ (resp. $G_0$). By the P.B.W. theorem, $\mathcal{V}$ is a finite type free $U$-module, and since $U$ is noetherian, $\mathcal{V}$ is a noetherian $U$-module, and, a fortiori, a noetherian algebra.

The $\mathbb{Z}_2$-grading $G = G_0 \oplus G_1$ induces a natural $\mathbb{Z}_2$-grading $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1$, so $\mathcal{V}$ is a $\mathbb{Z}_2$-graded associative algebra. Note that $\mathcal{V}_0 \neq U$ (e.g., $E_+ E_- \not\in \mathcal{V}_0$ and $E_+ E_- \not\in U$). Any $G$-module is a $\mathcal{V}$-module, and conversely, so we shall make no difference between the two notions.

We introduce the Casimir element $Q = GF + Y + Y^2 = FG - Y + Y^2$ and the center $Z = \mathbb{C}[Q]$ of $U$. We denote by $C$ the Casimir element of $\mathcal{V}$, $C = Q - (E_+ E_- E_+ E_-)/2$. $Q$ and $C$ are related, as shown by the following proposition:

(1.2) Proposition. $4Q^2 - (8C - 1)Q + 2C(2C - 1) = 0$.

Proof. Let us note $[E_+, E_-]_\mathcal{V} = E_+ E_- - E_- E_+$. One has $[C, Q] = [C, [E_+, E_-]_\mathcal{V}] = [Q, [E_+, E_-]_\mathcal{V}] = 0$ and therefore, the relation to be proved can be written as

$$Q = [E_+, E_-]_\mathcal{V} + [E_+, E_-]^2_\mathcal{V}.$$

Using $[E_+ E_-]_\mathcal{V} = 2E_+ E_- - Y$, and $[Y, E_+ E_-] = 0$, we have to establish

$$Q = -4E_+^2 E_-^2 - Y + Y^2$$

which is valid since $F = 2E_+^2$ and $G = -2E_-^2$.

Q.E.D.

It is well-known [7] (and easily obtained from the fact that $\mathcal{V}$ is a noetherian $U$-module) that any finitely-generated $\mathcal{V}$-module splits into a direct sum of a finite number of indecomposable $U$-modules. For irreducible $\mathcal{V}$-modules, one can be much more precise:
(1.3) **Theorem 1.** Let \( V = V_0 \oplus V_1 \) be an irreducible \( \mathcal{Y} \)-module; then \( V_0 \) and \( V_1 \) are irreducible \( U \)-modules.

**Proof.** Assume that \( V' \) is a \( U \)-invariant subspace of \( V_0 \), and let \( W = V' + E_+ V' + E_- V' \).

From the commutation rules (1, 0), it is obvious that \( W \) is invariant under \( Y, F \) and \( G \).

From the relations

\[
E_+ E_- = Q - C + Y/2, \quad E_- E_+ = C - Q + Y/2,
\]

and Quillen's lemma [3], which gives that \( C \) is a scalar operator, we deduce that \( E_+ E_- V' \subset V' \), \( E_- E_+ V' \subset V' \), and therefore \( E_+ \) maps \( E_- V' \) into \( V' \) and \( E_- \) maps \( E_+ V' \) into \( V' \). Moreover, since \( E_+^2 = F/2, E_-^2 = -G/2 \), we obtain that \( E_+ \) maps \( E_+ V' \) into \( V' \) and \( E_- \) maps \( E_- V' \) into \( V' \). Therefore \( W \) is \( \mathcal{Y} \)-stable, so \( W = \{0\} \) or \( V' \), \( V' = \{0\} \) or \( V_0 \), and \( V_0 \) is \( U \)-irreducible. By the same arguments, \( V_1 \) is \( U \)-irreducible. Q.E.D.

(1.4) Let us assume that \( V = V_0 \oplus V_1 \) is an irreducible \( \mathcal{Y} \)-module. By Quillen's lemma (e.g., [3]) \( C = \lambda \text{Id}_V, \lambda \in \mathbb{C} \). Using (1.2) and (1.3), we see that \( Q|_{V_i} = q_i \text{Id}_{V_i}, i = 0 \) or 1, where \( \{q_0, q_1\} \) is the set of solutions of the equation \( 4x^2 - (8\lambda - 1)x + 2\lambda(2\lambda - 1) = 0 \), namely \( \{q_0, q_1\} = (8\lambda - 1 \pm \sqrt{1 + 16\lambda})/8 \).

(1.5) A representation of \( G \) or \( G_0 \) is a Harish Chandra module if \( Y \) is diagonal, with finite dimensional eigenvalues. All irreducible Harish-Chandra modules of \( G_0 \) are computed in [9]. All irreducible Harish-Chandra modules of \( G \) are computed in [3]. Obviously, finite dimensional \( G \) or \( G_0 \) modules are Harish-Chandra modules, and it is known that they split into a direct sum of irreducible submodules (e.g., [8] for the case of \( G \)). This is no longer true for infinite dimensional Harish-Chandra modules of \( G \) or \( G_0 \) [3].

(1.6) In our conventions, we have assumed that modules over \( \mathbb{Z}_2 \)-graded algebras are always \( \mathbb{Z}_2 \)-graded. In this subsection, let us see what happens if we forget that \( \mathcal{Y} \) is a \( \mathbb{Z}_2 \)-graded algebra and consider ordinary (so a priori not \( \mathbb{Z}_2 \)-graded) \( \mathcal{Y} \)-modules.

(1.7) **Theorem 2.** Let \( V \) be an ordinary (not necessarily \( \mathbb{Z}_2 \)-graded) finite dimensional \( \mathcal{Y} \)-module. Then there exists a grading \( V = V_0 \oplus V_1 \) for which \( V \) is a \( \mathbb{Z}_2 \)-graded \( \mathcal{Y} \)-module.

**Proof:** As a \( G_0 \)-module, \( V \) splits into a direct sum of irreducibles, so \( Q \) is diagonal. Using (1.2) it results that \( C \) is also diagonal. We write \( V = \)
ENVELOPING ALGEBRA OF \( \mathfrak{osp}(1, 2) \)

\[ \sum_{\lambda \in \Lambda} W_\lambda, \] where \( W_\lambda = \{ v | C v = \lambda v \} \neq \{ 0 \} \). Obviously, \( W_\lambda \) is a \( \mathcal{V} \)-module, and if we prove (1.7) for \( W_\lambda \), it will be true in general. So we can assume that \( C = \lambda \text{Id}_V \). Using (1.4), we see that \( V = S_q \oplus S_{q'} \), where \( \{ q, q' \} = (8 \lambda - 1 \pm \sqrt{1 + 16 \lambda})/8 \), \( S_q \) and \( S_{q'} \) are the eigenspaces of \( Q \) of eigenvalue \( q \) and \( q' \), respectively. Note that \( q \neq q' \), since \( q = q' \) implies \( C = -1/16 \), and then \( q = q' = -\frac{3}{16} \), which does not correspond to any finite dimensional representation of \( G_0 \).

From the definition (1.0), one has \( C = Q - (E_+ E_- - E_- E_+)/2 \), and we compute

\[
2E_+ C = E_+ Q + QE_+ + E_+/4,
\]

so

\[
2E_+ C = E_+ Q + QE_+ + E_+/4, \tag{1.7.1}
\]

and by similar computations

\[
2E_- C = E_- Q + QE_- + E_-/4 \tag{1.7.2}
\]

Therefore, if \( V \in S_q \), we obtain

\[
QE_+ V = (2\lambda - q - 1/4) E_+ V = q'E_+ V,
Q E_- V = (2\lambda - q - 1/4) E_- V = q'E_- V,
\]

and the same formulae exchanging \( q \) and \( q' \). This proves that \( E_\pm \) maps \( S_q \) into \( S_{q'} \) and \( S_{q'} \) into \( S_q \). Moreover, \( S_q \) and \( S_{q'} \) are obviously \( G_0 \) stable, so if we set \( V_0 = S_q \), \( V_1 = S_{q'} \), for the grading \( V = V_0 \oplus V_1 \), \( V \) is a \( Z_2 \)-graded \( \mathcal{V} \)-module.

(1.7.3) COROLLARY. Let \( V \) be a \( Z_2 \)-graded irreducible finite dimensional \( \mathcal{V} \)-module. Then \( V \) is an ordinary irreducible \( \mathcal{V} \)-module.

Proof. Let \( V = V_0 \oplus V_1 \). As a \( G_0 \)-module, \( V \) is semi-simple with isotypical components \( V_0 \) and \( V_1 \) (1.6). If \( W \) is a (not necessarily homogeneous) \( G \)-submodule of \( V \), \( W \) must reduce on the isotypical components, so \( W \) is homogeneous, and \( W = \{ 0 \} \) or \( V \). Q.E.D.

(1.7.4) As a consequence of (1.7), (1.7.3), and (1.5), any finite dimensional \( \mathcal{V} \)-module reduces into a direct sum of irreducible submodules. We shall make free use of this property in the remainder of the paper. Moreover, by (1.7.3), Burnside's theorem holds for (\( Z_2 \)-graded) finite dimensional irreducible \( \mathcal{V} \)-modules. This means that if \( (\pi, V) \) is a \( \mathcal{V} \)-module of type \( B(h) \), then \( \pi(\mathcal{V}) = \text{End} \ V \).
2. Proof of Theorem 3

Obviously, \( \mathcal{V} \) is a filtered algebra, and by P.B.W., \( \text{Gr } \mathcal{V} \) is isomorphic to \( \mathbb{C}[Y, F, G] \otimes \mathcal{A}(E_+, E_-) \), which is not entire, and so the standard proof cannot be applied. Our proof will make use of a less refined filtration, but first, we need some computational lemmas:

(2.1) Lemma. The following formulas hold:

\[
E_+^n Y^n = (Y - \alpha/2)^n E_+^\alpha, \quad E_-^n Y^n = (Y + \beta/2)^n E_-^\beta \quad \alpha, \beta, n \in \mathbb{N}
\]

\[
E_+ E_+^{2n} = E_+^{2n} E_+ + (n/2) E_+^{2n-1},
\]

\[
E_- E_-^{2n} = -E_-^{2n-1} E_- + (Y - (n/2)) E_-^{2n-1} \quad n \in \mathbb{N}
\]

\[
E_+ E_-^{2n} = E_-^{2n} E_+ - (n/2) E_-^{2n-1},
\]

\[
E_- E_-^{2n} = -E_-^{2n} E_- + (Y + (n/2)) E_-^{2n} \quad n \in \mathbb{N}
\]

\[
E_+^\alpha E_-^\beta = \pm E_+^\alpha E_-^\beta + \sum_{i < \beta \atop j < \alpha} P_{ij}(Y) E_+^i E_-^j,
\]

\[
\alpha, \beta \in \mathbb{N}, \quad \text{where } P_{ij}(Y) \in \mathbb{C}[Y].
\]

Proof. \( E_+ Y = (Y - 1/2) E_+ \), \( E_- Y = (Y + 1/2) E_- \), we check by induction. \( E_+ Y^n = (Y - 1/2)^n E_+ \), \( E_- Y^n = (Y + 1/2)^n E_- \), and then we have the two first relations. Using \( E_- F = FE_- + E_+ \), we obtain by induction \( E_- F^n = F^n E_- + nF^{n-1} E_+ \). Since \( F = 2E_+^2 \), we deduce the third relation, and then the fourth one. The two next ones are deduced from the same sort of arguments.

The last formula is then true for \( \alpha = 1 \), and easily obtained in general by an induction using the first proved formulas. Q.E.D.

(2.2) Proposition. The set \( \{ Y^n E_+^\alpha E_-^\beta, n, \alpha, \beta \in \mathbb{N} \} \) is a basis of \( \mathcal{V} \).

Proof. By the P.B.W. theorem, \( \{ Y^n E_+^a E_-^b, n \in \mathbb{N}, a \) and \( b = 0 \) or \( 1 \} \) is a basis of \( \mathcal{V} \). Since \( F = 2E_+^2 \), \( G = -2E_-^2 \), the result follows by straightforward computations using Lemma (2.1). Q.E.D.

(2.3) We now define a new filtration \( \\{ \mathcal{V}_k, k \in \mathbb{N} \} \) of \( \mathcal{V} \) by

\[
\mathcal{V}_k = \left\{ u = \sum \lambda_{n\alpha} Y^n E_+^\alpha E_-^\beta, \text{with } \alpha + \beta \leq k \right\}.
\]

Actually, it is not obvious that such a definition gives a filtration, and this will be demonstrated in the following proposition:

(2.3.1) Proposition. One has \( \mathcal{V}_k \mathcal{V}_{k'} \subset \mathcal{V}_{k+k'}, \forall k, k' \in \mathbb{N}. \)
Proof. It will be enough to prove (2.3) for monomials:

\[ u = Y^n E^n_+ E^n_-, \quad v = Y^n' E^n'_+ E^n'_-, \quad \alpha + \beta \leq k, \quad \alpha' + \beta' \leq k' \]

\[ uv = Y^n \left( Y + \frac{\beta - \alpha}{2} \right)^n E^n_+ (E^n_- E^n_+)^n E^n_- \]

\[ = Y^n \left( Y + \frac{\beta - \alpha}{2} \right)^n E^n_+ (\pm E^n_+ E^n_- + \sum_{i < z, j < \beta} P_{ij}(Y) E^n_i E^n_j) E^n_- \]  \hspace{1cm} \text{by (2.1).}

Therefore, \( uv \in \mathcal{V}_{(k+k')} \). Q.E.D.

(2.3.2) We now introduce \( \mathcal{V} = \text{Gr} \mathcal{V} \), by \( \mathcal{V} = \sum_{k \geq 0} \mathcal{V}^k \), where \( \mathcal{V}^0 = \mathcal{V}_{(0)} \), and \( \mathcal{V}^k = \mathcal{V}_{(k)} \mathcal{V}_{(k-1)} \) if \( k \geq 1 \). Given \( u \in \mathcal{V} \), there exists a unique \( k \) such that \( u \in \mathcal{V}_{(k)} \) and \( u \notin \mathcal{V}_{(k-1)} \); we set \( \tilde{u} = u + \mathcal{V}_{(k-1)} \). The product on \( \mathcal{V} \) is now defined by \( \tilde{u} \cdot \tilde{u}' = uu' + \mathcal{V}_{(k+k-1)} \). \( \mathcal{V} \) is a graded associative algebra. From (2.2), we see that the set \( \{ Y^n E^n_+, i \in \mathbb{N}, \alpha, \beta \in \mathbb{N} \} \) is a basis of \( \mathcal{V} \). Let us now characterize \( \mathcal{V} \) a little more:

From the relation \( E_+ E_+ = Y - E_+ E_- \), we deduce \( E_+ E_- = -E_+ E_- \), and therefore \( [E_+, E_-] = 0 \). Similarly, one easily shows that the following commutation rules hold in \( \mathcal{V} \):

\[
\begin{align*}
[Y, E_\pm] &= \pm E_\mp, \quad [F, E_+] = [F, E_-] = [G, E_-] = \{G, E_+\} = 0 \\
[E_+, E_+] &= F, \quad [E_-, E_-] = -G, \quad [E_+, E_-] = 0.
\end{align*}
\]

(2.3.3)

Let us denote by \( \mathcal{G} = \mathcal{G}_0 + \mathcal{G}_1 \) this new superalgebra, with basis \( \{ Y, F, G \} \) for \( \mathcal{G}_0 \), and \( \{ E_+, E_- \} \) for \( \mathcal{G}_1 \). Using the above basis of \( \mathcal{V} \) and the P.B.W. theorem, one checks that \( \mathcal{V} = \mathcal{V}(\mathcal{G}) \). Therefore, filtering \( \mathcal{V} \) and taking the associated graded algebra \( \mathcal{V} \), we have obtained the enveloping algebra of the superalgebra \( \mathcal{G} \) defined by the commutation rules (2.3.3).

(2.3.4) Here is an interesting question: what is the relation between \( \mathcal{G} \) and \( \mathcal{G} \)? Let us answer in the following way:

We start with the superalgebra \( \mathcal{G} \) and use a simple Inonu–Wigner contraction. We set \( Y' = Y, F' = \varepsilon G, E'_{\pm} = \varepsilon E_{\pm}, \varepsilon \in \mathbb{R} \). We then take the limit \( \varepsilon \to 0 \) in the commutation rules, and obtain a contraction of \( \mathcal{G} \), which is obviously the superalgebra \( \mathcal{G} \). So, our operation of filtering and then grading can be interpreted as an Inonu–Wigner contraction process.

(2.4) Now we come back to our initial purpose:

(2.4.1) Theorem 3. \( \mathcal{V} \) is entire. The only invertible elements of \( \mathcal{V} \) are non vanishing scalars.
Proof. In order to prove the first claim, using standard results, we only have to prove that $\tilde{\mathcal{V}}$ is entire.

Since $[E_+, E_-] = 0$, we deduce that $E^\beta_+ E^\alpha_- = (-1)^{\beta x} E^\alpha_+ E^\beta_-$. Therefore

$$\tilde{\mathcal{V}}^n E^\alpha_+ E^\beta_- \tilde{\mathcal{V}}^n E^\alpha_+ E^\beta_- = (-1)^{\beta x} \tilde{\mathcal{V}}^n (\tilde{\mathcal{V}}^{-\alpha/2 + \beta/2}) E^\alpha_+ + z E^\beta_- + \beta' = 0.$$ 

Let us now assume that $u = \sum_{p=0, \ldots, k} u_p$, $u' = \sum_{p=0, \ldots, k} u'_p$ and $uu' = 0$. Using the grading, we obtain $u_k u' = 0$. We decompose on the basis:

$$u_k u'_k = \sum_{\alpha + \beta = k} \lambda_{n_0 \alpha \beta} \tilde{\mathcal{V}}^n E^\alpha_+ E^\beta_- \sum_{\alpha' + \beta' = k'} \lambda'_{n_0' \alpha' \beta'} \tilde{\mathcal{V}}^n E^\alpha_+ E^\beta_- = 0.$$ 

Let us denote by $\deg_{\tilde{\mathcal{V}}}$ the degree of elements in $\tilde{\mathcal{V}}$ with respect to $\tilde{\mathcal{V}}$ in the decomposition on the canonical basis $\{ \tilde{\mathcal{V}}^n E^\alpha_+ E^\beta_-, n, \alpha, \beta \in \mathbb{N} \}$.

So, if $\deg_{\tilde{\mathcal{V}}} u_k = n_0$, and $\deg_{\tilde{\mathcal{V}}} u'_k = n'_0$, one has

$$u_k = \tilde{\mathcal{V}}^{n_0} \sum_{\alpha + \beta = k} \lambda_{n_0 \alpha \beta} E^\alpha_+ E^\beta_- + v, \quad \text{with } \deg_{\tilde{\mathcal{V}}} v < n_0,$$

$$u'_k = \tilde{\mathcal{V}}^{n'_0} \sum_{\alpha' + \beta' = k'} \lambda'_{n'_0 \alpha' \beta'} E^\alpha_+ E^\beta_- + v', \quad \text{with } \deg_{\tilde{\mathcal{V}}} v' < n'_0,$$

therefore

$$u_k u'_k = \tilde{\mathcal{V}}^{n_0 + n'_0} \sum_{\alpha + \beta = k} \lambda_{n_0 \alpha \beta} \lambda'_{n'_0 \alpha' \beta'} (-1)^{\beta x} E^\alpha_+ + z E^\beta_- + \beta' + w,$$

where $\deg_{\tilde{\mathcal{V}}} w < n_0 + n'_0$, and we deduce that

$$\sum_{\alpha + \beta = k} \lambda_{n_0 \alpha \beta} \lambda'_{n'_0 \alpha' \beta'} (-1)^{\beta x} \tilde{\mathcal{V}}^{n_0 + n'_0} E^\alpha_+ + z E^\beta_- + \beta' = 0.$$ 

Let us now denote by $\deg_{E_+}$ the degree of an element in $\tilde{\mathcal{V}}$ with respect to $E_+$.

So, if $\deg_{E_+} (\sum_{\alpha + \beta = k} \lambda_{n_0 \alpha \beta} \tilde{\mathcal{V}}^{n_0} E^\alpha_+ E^\beta_-) = x_0$, and $\deg_{E_+} (\sum_{\alpha' + \beta' = k'} \lambda'_{n'_0 \alpha' \beta'} \tilde{\mathcal{V}}^{n'_0} E^\alpha_+ E^\beta_-) = x'_0$, we set $\lambda_{n_0, x} = \lambda_{n_0, x, k - z}$, $\lambda'_{n'_0, x'} = \lambda'_{n'_0, x', k' - z'}$, and obtain

$$\sum_{\alpha = 0, \ldots, x_0} (-1)^{(k - z) x'} \lambda_{n_0, x} \lambda'_{n'_0, x'} \tilde{\mathcal{V}}^{n_0 + n'_0} E^\alpha_+ + z E^\beta_- + k_0 = 0.$$
Now, in this summation, there is exactly one term of $\deg_{E_+}$ equal to $\alpha_0 + \alpha'_0$ which is obtained if $\alpha = \alpha_0$ and $\alpha' = \alpha'_0$. If now we assume that both $u$ and $u'$ do not vanish, we are led to a contradiction, because this assumption implies that both $\lambda_{\alpha_0}$ and $\lambda'_{\alpha'_0}$ do not vanish, and the above equality implies that $\lambda_{\alpha_0} \lambda' = 0$. Therefore $\mathcal{V}$ is entire.

We now prove the second claim.

Given $u \in \mathcal{V}$, we set $\deg(u) = \inf\{k/u \in \mathcal{V}(k)\}$. Since $\mathcal{V} = \text{Gr} \mathcal{V}$ is entire, one has $\deg(uu') = \deg(u) + \deg(u')$ when $u \neq 0$ and $u' \neq 0$. The second claim follows.

Q.E.D.

(2.4.2) Corollary. Let $u$ be a non scalar element of $\mathcal{V}$. There exists a maximal left ideal $M$ such that $u \in M$.

(2.4.3) Remark. Alternatively, Theorem 3 can be deduced from a result of [11] about the Ramond–Schwarz superalgebras.

3. REDUCTION OF THE ADJOINT REPRESENTATION

The adjoint representation of $G$ in $\mathcal{V}$ is defined by

$$ X \in G, \quad u \in \mathcal{V}, \quad \text{ad} \, X(u) = [X, u]. $$

We introduce the standard filtration $\mathcal{V}'_n$ of $\mathcal{V}$. From the P.B.W. theorem one has $\text{dim} \, \mathcal{V}'_n / \mathcal{V}'_{n-1} = n^2 + (n+1)^2$, $n \geq 1$.

We consider the subspaces $H_0 = \mathbb{C}$, $H_1 = G$, $H_2$ = the submodule of $\mathcal{V}'_2$ generated by $F_2$. Obviously, $H_0 \simeq \mathcal{D}(0)$, $H_1 \simeq \mathcal{D}(1)$, and $H_2 \simeq \mathcal{D}(2)$, since $F_2$ is a dominant weight vector of weight 2.

Now we set $\theta_0 = Q - 3C$, $\theta_{1/2} = FE_+ - YE_+ + 3/4E_+ + GE_+ + 3/4E_-$, and $\theta_{-1/2} = YE_+ - GE_+ + 3/4E_-$, and $V_2 = \text{span}(\theta_0, \theta_{1/2}, \theta_{-1/2})$.

(3.1) Lemma. $V_2$ is a $G$-submodule of $\mathcal{V}'_2$, and $V_2 \simeq \mathcal{D}(1/2)$. $\mathcal{V}_1$ and $\mathcal{V}_2$ reduce as

$$ \mathcal{V}_1' = H_1 \oplus H_0 \simeq \mathcal{D}(1) \oplus \mathcal{D}(0) $$

$$ \mathcal{V}_2' = H_2 \oplus V_2 \oplus CH_0 \oplus \mathcal{V}_1 \simeq \mathcal{D}(2) \oplus \mathcal{D}(1/2) \oplus \mathcal{D}(0) \oplus \mathcal{V}_1. $$

Proof. One has $\text{ad} \, E_+ (\theta_0) = \text{ad} \, E_+ (Q) = \theta_1$, $\text{ad} \, E_- (\theta_0) = \text{ad} \, E_- (Q) = \theta_2$. Moreover $\text{ad} \, E_+ (\theta_{1/2}) = (\text{ad} \, E_+)^2 (Q) = [[E_+ E_+], Q] - [E_+ [E_+, Q]] = -(\text{ad} \, E_+)^2 (Q)$, therefore $\text{ad} \, E_+ (\theta_{1/2}) = 0$, and by the same argument $\text{ad} \, E_- (\theta_{-1/2}) = 0$.

Now,

$$ \text{ad} \, E_+ (\theta_{1/2}) = \text{ad} \, E_+ (\text{ad} \, E_+ (Q)) = [[E_-, E_+], Q] - [E_+, [E_-, Q]] $$

$$ = - \text{ad} \, E_+ (\text{ad} \, E_- (Q)) = - \text{ad} \, E_+ (\theta_{-1/2}). $$
And finally

\[ \text{ad } E_-(\theta_{1/2}) = [E_-, FE_+ - YE_+ + \frac{1}{3}E_+] \]

\[ = E_-, E_+ - FG - \frac{1}{3}E_+ E_+ - Y^2 + \frac{1}{3}Y \]

\[ = E_-, E_+ - (Q + Y - Y^2) - \frac{1}{3}Y + \frac{1}{3}E_+ E_+ - Y^2 + \frac{1}{3}Y, \]

since \( Q = FG - Y + Y^2 \)

\[ = \frac{1}{3}E_+ E_+ - Q - \frac{1}{3}Y = \frac{1}{3}(Q - C + \frac{1}{3}Y) - Q - \frac{1}{3}Y, \]

since \( C = Q - \frac{1}{3}(2E_+ E_+ - Y) \)

\[ = \frac{1}{3}(Q - 3C) = \frac{1}{3}\theta_0. \]

This proves that \( V_2 \) is \( G \)-stable, and since \( \theta_{1/2} \) is a dominant weight vector of weight \( \frac{1}{3} \), which generates \( V_2 \), we obtain \( V_2 \cong \mathcal{D}(\frac{1}{3}) \).

The sum \( H_2 \oplus V_2 \) is direct because \( H_2 \) and \( V_2 \) are irreducible inequivalent \( G \)-modules, the sum \( \mathcal{V}_1 \oplus CH_0 \) is obviously direct because \( C = Q - \frac{1}{3}(E_+ E_+ - E E_+) \) is not in \( \mathcal{V}_1 \), and the sum \( H_2 \oplus V_2 \oplus \mathcal{V}_1 \oplus CH_0 \) is direct because \( H_2 \) is of type \( \mathcal{D}(2) \), \( V_2 \) is of type \( \mathcal{D}(\frac{1}{3}) \), and \( \mathcal{V}_1 \oplus CH_0 \) is of type \( \mathcal{D}(1) \oplus \mathcal{D}(0) \).

Finally, \( \dim(H_2 \oplus V_2 \oplus CH_0) = 9 + 3 + 1 = 13 = \dim \mathcal{V}_2 - \dim \mathcal{V}_1 \).

Q.E.D.

We need some more notations:

We denote by \( H_n \), \( n \geq 0 \), the \( G \)-submodule of \( \mathcal{V}_n \) generated by \( F^n \); by \( V_n \), \( n \geq 2 \), the \( G \)-submodule of \( \mathcal{V}_n \) generated by \( F^{n-2}\theta_{1/2} \). From \( \text{ad } E_+(F^n) = \text{ad } E_+(F^{n-2}\theta_{1/2}) = 0 \), and \( \text{ad } Y(F^n) = nF^n \), \( \text{ad } Y(F^{n-2}\theta_{1/2}) = (n - \frac{3}{2}) F^n - \frac{3}{2}\theta_{1/2} \), we see that \( H_n \cong \mathcal{D}(n) \), and \( V_n \cong \mathcal{D}(n - \frac{3}{2}) \). Finally, we set \( V_0 = V_1 = \{0\} \).

\[ (3.2) \text{ Proposition.} \] The reduction of the adjoint representation of \( G \) on \( \mathcal{V}_n \) is given by

\[ \mathcal{V}_n = \mathcal{V}_{n-1} \oplus A_n, \]

where

\[ A_n = H_n \oplus V_n \oplus CH_{n-2} \oplus CV_{n-2} \oplus C^2H_{n-4} \oplus C^2V_{n-4} \oplus \ldots \]

\[ \oplus \begin{cases} C^{n/2}H_0 \oplus C^{n/2}V_0, & \text{if } n \text{ is even} \\ C^{(n-1)/2}H_1 \oplus C^{(n-1)/2}V_1, & \text{if } n \text{ is odd} \end{cases} \]

As a \( G \)-module, \( A_n \) reduces as

\[ A_n = \mathcal{D}(n) \oplus \mathcal{D}(n - \frac{3}{2}) \oplus \mathcal{D}(n - 2) \oplus \mathcal{D}(n - \frac{5}{2}) \oplus \ldots \]

\[ \oplus \begin{cases} \mathcal{D}(0), & \text{if } n \text{ is even} \\ \mathcal{D}(1), & \text{if } n \text{ is odd} \end{cases} \]
Proof. The result is true for \( n = 0, 1, 2 \), from (3.1). By induction, we assume that it is true up to rank \( (n - 1) \).

First, we note that the map \( u \to C u \) is a G-morphism, injective since \( \mathcal{V} \) is entire, and so

\[
CA_{n-2} = C(H_{n-2} \oplus V_{n-2} \oplus CH_{n-4} \oplus CV_{n-4} \oplus \cdots)
\]

\[
= CH_{n-2} \oplus CV_{n-2} \oplus C^2H_{n-4} \oplus C^2V_{n-4} \oplus \cdots
\]

\[
\simeq \mathcal{D}(n-2) \oplus \mathcal{D}(n-\frac{3}{2}) \oplus \mathcal{D}(n-4) \oplus \mathcal{D}(n-\frac{7}{2}) \oplus \cdots
\]

\[
\bigoplus_{n \text{ even}} \mathcal{D}(0) \quad \text{if} \quad n \text{ is even}
\]

\[
\bigoplus_{n \text{ odd}} \mathcal{D}(1) \quad \text{if} \quad n \text{ is odd}.
\]

Second, we note that \( H_n \oplus V_n \) is direct, since \( H_n \simeq \mathcal{D}(n) \), \( V_n \simeq \mathcal{D}(n-\frac{3}{2}) \).

Now, we have to prove that \( \mathcal{V} = \mathcal{V}_{n-1} \oplus CA_{n-2} \oplus H_n \oplus V_n \). Assuming that \( u \in \mathcal{V}_{n-1} \cap CA_{n-2} \), we get \( u = Ca \), \( a \in A_{n-2} \); if \( a \neq 0 \), then \( \mathcal{C}a = 0 \), and we get a contradiction to (2.4.1), so \( a = u = 0 \).

We now compute \( (H_n \oplus V_n) \cap (\mathcal{V}_{n-1} \oplus CA_{n-2}) \). \( H_n \simeq \mathcal{D}(n) \), \( V_n \simeq \mathcal{D}(n-\frac{3}{2}) \), and such types do not appear in the reduction of \( \mathcal{V}_{n-1} \oplus CA_{n-2} \), so the intersection vanishes.

We have proved that the sum is direct, and finally it remains to show the equality with \( \mathcal{V}_n \). We compute:

\[
\dim H_n + \dim V_n + \dim A_{n-2} = (4n + 1) + (4n - 5) + (n - 2)^2 + (n - 1)^2
\]

\[
= n^2 + (n + 1)^2
\]

\[
= \dim \mathcal{V}_n - \dim \mathcal{V}_{n-1}.
\]

Q.E.D.

(3.3) Corollary. The center of \( \mathcal{V} \) is the polynomial algebra \( \mathbb{C}[C] \).

(3.4) Theorem 4. Define \( \mathcal{H} = \sum_{n \geq 0} (H_n \oplus V_n) \). \( \mathcal{H} \) is a G-submodule of \( \mathcal{V} \) (for the adjoint representation) and reduces as \( \mathcal{H} = \sum_{n \geq 0} \mathcal{D}(n) \oplus \sum_{n \geq 0} \mathcal{D}(n + \frac{1}{2}) \). Given any complex number \( \lambda \), denote by \( \mathcal{J}_\lambda \) the two sided ideal \( \mathcal{I}_\lambda = (C - \lambda) \mathcal{V} \). Then \( \mathcal{V} = \mathcal{H} \oplus \mathcal{J}_\lambda \).

Proof. The decomposition of (3.2) can be written

\[
\mathcal{V} = \sum_{p \geq 0} C^p \mathcal{H}.
\]

This sum being direct, it is obvious from (2.4.1) that the sum \( \sum_{p \geq 0} (C - \lambda)^p \mathcal{H} \) is also direct. We prove by induction that \( \sum_{p \geq 0, \ldots, n} C^p \mathcal{H} \subset \sum_{p = 0, \ldots, n} (C - \lambda)^p \mathcal{H} \). It is true if \( n = 0 \), we assume that it is true for \( (n - 1) \), and consider \( x \in \sum_{p = 0, \ldots, n} C^p \mathcal{H} \). We have

\[
x = C^n a + \xi, \quad \text{with} \quad \xi \in \sum_{p = 0, \ldots, n - 1} C^p \mathcal{H} \subset \sum_{p = 0, \ldots, n - 1} (C - \lambda)^p \mathcal{H}, \quad \text{and} \quad a \in \mathcal{H}.
\]
Moreover

\[ x = (C - \lambda)^n a + \left( \xi - \sum_{k=0}^{n-1} (-1)^{n-k} C^n(\xi - k)^{k} a \right) \]

\[ = (C - \lambda)^n a + \xi', \]

with \( \xi' \in \sum_{p=0}^{n-1} C^p H \subset \sum_{p=0}^{n-1} (C - \lambda)^p H \). Therefore, we obtain that \( \sum_{p=0}^{n-1} C^p H \subset \sum_{n \in \mathbb{N}} (C - \lambda)^p H \), and since \( \forall nu \in N \), since \( V = \sum_{p \geq 0} C^p H \), we conclude that \( V = \sum_{p \geq 0} (C - \lambda)^p H \). Q.E.D.

4. The Quotient Algebra \( \mathcal{B}_\lambda \)

Let us note \( \mathcal{B}_\lambda = V / (C - \lambda) V \), and similarly \( B_q = U / (Q - q) U \), \( \lambda, q \in \mathbb{C} \). The structure of \( B_q \) is well known (e.g., [5, 1]).

From the natural \( Z_2 \)-grading \( V = V_0 \oplus V_1 \), we get a \( Z_2 \)-grading \( \mathcal{B}_\lambda = \mathcal{B}_{\lambda_0} \oplus \mathcal{B}_{\lambda_1} \). Given \( u \in V \), we denote by \( \bar{u} \) its canonical image in \( \mathcal{B}_\lambda \). Using (1.2), we see that the equality

\[ (\bar{Q} - q_0)(\bar{Q} - q_1) = 0 \] (4.1)

holds in \( \mathcal{B}_\lambda \), if \( \{q_0, q_1\} \) is the set of solutions of

\[ 4x^2 - (8\lambda - 1)x + 2(2\lambda - 1) = 0. \]

If \( \lambda \neq -\frac{1}{16} \), one has \( q_0 \neq q_1 \); if \( \lambda = -\frac{1}{16} \), one has \( q_0 = q_1 = -\frac{3}{16} \). As a consequence of (4.1), \( \mathcal{B}_\lambda \) is never an entire algebra; equivalently, the ideal \( (C - \lambda) V \) is not completely prime.

(4.2) The reduction of the adjoint action of \( G \) in \( \mathcal{B}_\lambda \) gives

\[ \mathcal{B}_\lambda = \sum_{n \geq 0} \tilde{H}_n \oplus \sum_{n \geq 2} \tilde{V}_n, \]

where \( \tilde{H}_n \simeq \mathcal{D}(n) \) is generated, as a \( G \)-module, by \( \tilde{F}^n \), and \( \tilde{V}_n \simeq \mathcal{D}(n - \frac{1}{2}) \) is generated by \( \tilde{F}^{n-\frac{1}{2}} \tilde{G}_{1/2} \) (see Section 3).

(4.3) Proposition. Assume that \( \lambda \neq -\frac{1}{16} \), then \( \mathcal{B}_{\lambda_0} \simeq B_{q_0} \times B_{q_1} \). As an algebra, \( \mathcal{B}_{\lambda_0} \) is generated by \( G_0 \); as a \( \mathcal{B}_{\lambda_0} \)-module, \( \mathcal{B}_\lambda \) is generated by 1, \( E_+ \), and \( E_- \).

Proof. Since \( q_0 \neq q_1 \), we can define \( l_0 = (\bar{Q} - q_1)/(q_0 - q_1) \), \( l_1 = (\bar{Q} - q_0)/(q_1 - q_0) \) and check that

\[ l_0 + l_1 = 1, \quad l_0 \cdot l_1 = 0. \]
From these identities, we deduce $1^2_0 = 1_0$, $1^2_1 = 1_1$. We define linear mappings $P_0$ and $P_1$ by

$$P_0(u) = 1_0 u, \quad P_1(u) = 1_1 u, \quad \text{for } u \in \mathcal{B}_{\lambda_0}.$$ 

Obviously, $P_0$ and $P_1$ are projections, and $\mathcal{B}_{\lambda_0} = P_0(\mathcal{B}_{\lambda_0}) \oplus P_1(\mathcal{B}_{\lambda_0})$. Let

$$A_{\lambda_0} = P_0(\mathcal{B}_{\lambda_0}), \quad A_{\lambda_1} = p_1(\mathcal{B}_{\lambda_0}).$$

Using the P.B.W. theorem and the relation $\bar{E}_+ \bar{E}_- = \bar{Q} - \lambda + \frac{1}{2} \bar{Y}$, we see that $\mathcal{B}_{\lambda_0}$ is generated, as an algebra, by $\bar{G}_0$, and that $\mathcal{B}_{\lambda_1}$ is generated (as a $\mathcal{B}_{\lambda_0}$-module) by $1$, $\bar{E}_+$, and $\bar{E}_-$. Moreover, one has

$$P_i(uv) = P_i(u) P_i(v), \quad \forall i = 0, 1, \quad u, v \in \mathcal{B}_{\lambda_0}.$$

It then results that $A_{\lambda_0}$ and $A_{\lambda_1}$ are unitary subalgebras of $\mathcal{B}_{\lambda_0}$, with respective neutral element $1_0 = P_0(1)$ and $1_1 = P_1(1)$. Moreover $A_{\lambda_0} \cdot A_{\lambda_1} = 0$ and $\mathcal{B}_{\lambda_0}$ is the direct sum of the two ideals $A_{\lambda_0}$ and $A_{\lambda_1}$. We compute $P_0(\bar{Q}) = 1_0 1_1 + q_0 1_0 = q_0 1_1$, $P_1(\bar{Q}) = 1_1 1_1 + q_1 1_1 = q_1 1_1$.

So the canonical morphism $\varphi_0$ (resp. $\varphi_1$) from $U$ onto $A_{\lambda_0}$ (resp. $A_{\lambda_1}$) defined by the relations $\varphi_0(Y) = P_0(\bar{Y})$, $\varphi_0(F) = P_0(\bar{F})$, $\varphi_0(G) = P_0(\bar{G})$ (resp. $\varphi_1(Y) = P_1(\bar{Y})$, $\varphi_1(F) = P_1(\bar{F})$, $\varphi_1(G) = P_1(\bar{G})$) induces a morphism (that we denote by the same letter) $\varphi_0$ (resp. $\varphi_1$) from $B_{\lambda_0}$ (resp. $B_{\lambda_1}$) onto $A_{\lambda_0}$ (resp. $A_{\lambda_1}$). Now, it remains to prove that $\varphi_0$ and $\varphi_1$ are one to one.

Using (3.2) and (3.3), we easily deduce that the adjoint representation of $G_0$ on $\mathcal{B}_{\lambda_0}$ reduces as $\mathcal{B}_{\lambda_0} \simeq \sum_{n \geq 0}(D(n) \oplus D(n))$. It is well-known that the adjoint representation of $G_0$ on $B_q$ reduces as $B_q \simeq \sum_{n \geq 0} D(n)$.

Actually, $A_{\lambda_0}$ and $A_{\lambda_1}$ are $G_0$ submodules of $\mathcal{B}_{\lambda_0}$ (for the adjoint representation); moreover, one has $\mathcal{B}_{\lambda_0} = A_{\lambda_0} \oplus A_{\lambda_1}$, and they are quotients respectively of the $G_0$-modules $B_{\lambda_0}$ and $B_{\lambda_1}$, so, using the above reductions, we obtain that necessarily $A_{\lambda_0} \simeq \sum_{n \geq 0} D(n)$, $A_{\lambda_1} \simeq \sum_{n \geq 0} D(n)$, so $\varphi_0$ and $\varphi_1$ are one to one.

**Q.E.D.**

*(4.4) COROLLARY.* Setting $\mathcal{J}_2 = (C - \lambda) \mathcal{Y}$, $J_2 = \mathcal{J}_2 \cap U$, and assuming that $\lambda \neq -\frac{1}{16}$, one has $J_\lambda = (Q - q_0)(Q - q_1)U$.

**Proof.** From (4.3), $U/J_\lambda \simeq B_{\lambda_0} \simeq B_{\lambda_0} \simeq B_{q_0} \times B_{q_1}$, so $J_\lambda = (Q - q_0) \mathcal{Y} \cap (Q - q_1) \mathcal{Y} = (Q - q_0)(Q - q_1) \mathcal{Y}$, because $q_0 \neq q_1$. **Q.E.D.**

*(4.5) Remark.* Non trivial relations hold in the $\mathcal{B}_{\lambda_0}$-module $\mathcal{B}_{\lambda_1}$ between $\bar{E}_+$ and $\bar{E}_-$. For instance, one has $FE_- = E_+ F - E_+$ and $F = 2E_+^2$, so $FE_- = (2E_- E_+ - 1) E_+$, but $E_+ E_- = \lambda - \bar{Q} + \frac{1}{2} \bar{Y}$, so we obtain $\bar{F} E_- = (2\lambda - 1 - 2\bar{Q} + \bar{Y}) E_+ = 0$. 

**Q.E.D.**
5. IDEALS OF THE ALGEBRA \( \mathcal{B}_\lambda, \lambda = h(2h + 1)/2, 2h \in \mathbb{N} \)

In this section, we assume that \( \lambda = h(2h + 1)/2, 2h \in \mathbb{N} \). There exists exactly one irreducible finite dimensional representation \( \pi \) of \( G \) for which \( \pi(C) = h(2h + 1)/2 \); it is the representation \( \mathcal{D}(h) \). Any representation of \( G \) can be reduced as a sum of irreducible ones, so any finite dimensional \( \mathcal{B}_\lambda \)-module is semi-simple and isotypical of type \( \mathcal{D}(h) \).

(5.1) PROPOSITION. Let \( \mathcal{L} \) be the kernel of the representation \( \mathcal{D}(h) \) of \( \mathcal{B}_\lambda \). Then one has \( \mathcal{L} = \sum_{n \geq 2h + 1} \widetilde{H}_n \oplus \sum_{n \geq 2h + 2} \widetilde{V}_n \) and \( \dim[\mathcal{B}_\lambda/\mathcal{L}] = (4h + 1)^2 \).

**Proof.** Let \( W = W_0 \oplus W_1 \) be a space on which \( G \) acts by \( \mathcal{D}(h) \). As \( G_0 \)-modules, one has \( W_0 \simeq D(h) \) and \( W_1 \simeq D(h - \frac{1}{2}) \), so the operator \( F^{2h+1} \) vanishes on \( W \).

Moreover \( F^{2h+1} \) and \( \mathcal{F}^{2h+1} \) are in \( \mathcal{L} \). But \( \mathcal{L} \) is a two sided ideal, so it is also a \( G \)-module for the adjoint action, so \( \widetilde{H}_n \subset \mathcal{L}, \forall n \geq 2h + 1 \), and \( \widetilde{V}_n \subset \mathcal{L}, \forall n \geq 2h + 2 \). We conclude that \( \mathcal{L} = \sum_{n \geq 2h + 1} \widetilde{H}_n \oplus \sum_{n \geq 2h + 2} \widetilde{V}_n \).

By Burnside's theorem (1.7.5), \( \dim(\mathcal{B}_\lambda/\mathcal{L}) = (4h + 1)^2 \). But \( \dim(\sum_{n < 2h} \widetilde{H}_n \oplus \sum_{n < 2h + 1} \widetilde{V}_n) = (4h + 1)^2 \). So we obtain \( \mathcal{L} = \sum_{n \geq 2h + 1} \widetilde{H}_n \oplus \sum_{n \geq 2h + 2} \widetilde{V}_n \).

Q.E.D.

(5.2) Up to now, we only know one primitive ideal of \( \mathcal{B}_\lambda \): the ideal \( \mathcal{L} \) of (5.1), which is of finite codimension, and related to the irreducible finite dimensional representation \( \mathcal{D}(h) \). Using Burnside's theorem (1.7.5), we see that \( \mathcal{L} \) is completely prime if and only if \( \lambda = 0 \). In general, \( \mathcal{L} \) is prime.

On the other hand, from [3, Theorem 4.2.13]), \( \mathcal{B}_\lambda \) has a large number of infinite dimensional irreducible representations. Let us compute the kernel of these representations:

(5.2.1) PROPOSITION. \( \mathcal{L} \) is the only non trivial two sided ideal of \( \mathcal{B}_\lambda \). Any irreducible infinite dimensional representation is faithful, and \( \mathcal{B}_\lambda \) is a primitive algebra.

**Proof.** Let \( \mathcal{J} \neq \{0\} \) be a two sided ideal of \( \mathcal{B}_\lambda \). First, we show that \( \dim[\mathcal{B}_\lambda/\mathcal{J}] < +\infty \):

Since \( \mathcal{J} \) is a \( G \)-module for the adjoint action, there must exist \( n \) such that \( \widetilde{H}_n \subset \mathcal{J} \) or \( \widetilde{V}_{n+2} \subset \mathcal{J} \).

In the first case, \( \mathcal{F}^m \in \mathcal{J} \), so \( \mathcal{F}^m \in \mathcal{J}, \forall m \geq n \), and since the \( G \)-module \( \widetilde{H}_m \) is generated from \( \mathcal{F}_m \), we obtain that \( \sum_{m \geq n} \widetilde{H}_m \subset \mathcal{J} \). Moreover, \( \mathcal{F}^n \theta_{1/2} \in \mathcal{J} \),
\(\forall m \geq n\), so by the same argument, \(\sum_{m \geq n} \tilde{V}_{m+2} \subset \mathcal{J}\). From these inclusions, we conclude that \(\dim[\mathcal{B}/\mathcal{J}] < +\infty\).

In the second case, \(\tilde{F}^n \tilde{\theta}_{1/2} = F^n \text{ad } E_+ (\tilde{\theta}_{0}) \in \mathcal{J}\), so \(\text{ad } E_- (\tilde{F}^n \text{ad } E_+ (\tilde{\theta}_{0})) \in \mathcal{J}\). From the proof of (3.1), one has \(\text{ad } E_- \text{ad } E_+ (\tilde{\theta}_{0}) = +\frac{1}{2} \tilde{\theta}_{0}\), so

\[\text{ad } E_- (\tilde{F}^n \text{ad } E_+ (\tilde{\theta}_{0})) = n\tilde{E}_+ \tilde{F}^{n-1} \text{ad } E_+ (\tilde{\theta}_{0}) + \frac{1}{2} \tilde{F}^n \tilde{\theta}_{0} \in \mathcal{J}.
\]

Therefore, we obtain \(\tilde{F}^{n+1} \tilde{\theta}_{0} \tilde{\theta}_{0}^{-1} \in \mathcal{J}\).

Note that using (1.2), one has \(4 \tilde{\theta}_{0}^2 + (16C + 1) \tilde{\theta}_{0} + C(16C + 1) = 0\). So, if \(\lambda \neq 0\), \(\tilde{\theta}_{0}\) is invertible in \(\mathcal{B}_\lambda\), and its inverse is given by

\[
\tilde{\theta}_{0}^{-1} = -\lambda^{-1}[1 + 4\tilde{\theta}_{0}/(16\lambda + 1)].
\]

Therefore, \(\tilde{F}^{n+1} \tilde{\theta}_{0} \tilde{\theta}_{0}^{-1} = \tilde{F}^{n+1} \in \mathcal{J}\), we are back to the first case, and \(\dim[\mathcal{B}/\mathcal{J}] < +\infty\).

Now, let us assume that \(\lambda = 0\).

By (1.7.2), \(2E_+ C = E_+ Q + QE_+ + \frac{1}{2} E_+\).

Since \(\lambda = 0\), we have \(\tilde{\theta}_{0} = \tilde{\bar{Q}}\), so we obtain \(\tilde{E}_+ = -4(\tilde{E}_+ \tilde{\theta}_{0} + \tilde{\theta}_{0} \tilde{E}_+)\) (1.7.1) and \(\tilde{F} = 2\tilde{E}_+^2 = -8(\tilde{E}_+ \tilde{\theta}_{0} \tilde{E}_+ + \tilde{\theta}_{0} \tilde{E}_+^2)\).

We compute \(u = \tilde{E}_+ (\tilde{F}^{n+1} \tilde{\theta}_{0}) \tilde{E}_+ + (\tilde{F}^{n+1} \tilde{\theta}_{0}) \tilde{E}_+^2 \in \mathcal{J}\) and obtain

\[
u = \tilde{F}^{n+1} (\tilde{E}_+ \tilde{\theta}_{0} \tilde{E}_+ + \tilde{\theta}_{0} \tilde{E}_+^2) = \frac{1}{2} \tilde{F}^{n+2} \in \mathcal{J},
\]

so we come back to the first case, and conclude that \(\dim[\mathcal{B}/\mathcal{J}] < +\infty\).

If we assume that \(\mathcal{J} \neq \mathcal{B}_\lambda\), since \(\mathcal{B}_\lambda\) is noetherian, we can find a maximal left ideal \(M\) which contains \(\mathcal{J}\). The \(\mathcal{B}_\lambda\)-module \(\mathcal{B}_\lambda/M\) is irreducible and finite dimensional, so it is isomorphic to \(\mathcal{D}(h)\), and its kernel is \(\mathcal{L}\). But \(\mathcal{J}\), being a two sided ideal contained in \(M\), is contained in the kernel of \(\mathcal{B}_\lambda/M\), so \(\mathcal{J} \subset \mathcal{L}\).

Let us introduce the finite dimensional algebra \(\mathcal{C} = \mathcal{B}_\lambda/\mathcal{J}\). Any \(\mathcal{C}\)-module is obviously a \(\mathcal{B}_\lambda\)-module, so \(\mathcal{C}\) has only one irreducible module, namely \(\mathcal{D}(h)\). Moreover, since any finite dimensional \(\mathcal{V}\)-module is semi-simple (see (1.7.5)), any finite dimensional \(\mathcal{C}\)-module is semi-simple, and by the way, \(\mathcal{C}\) as a module over itself is semi-simple. This proves that \(\mathcal{C}\) is a simple algebra.

Consequently, the only two sided ideals of \(\mathcal{C}\) are the trivial ones. Therefore \(\mathcal{L}/\mathcal{J} = \{0\}\) or \(\mathcal{C}\). If \(\mathcal{L}/\mathcal{J} = \{0\}\), we get \(\mathcal{L} = \mathcal{J}\). \(\mathcal{L}/\mathcal{J} = \mathcal{C}\) is excluded, because \(\mathcal{L} \neq \mathcal{B}_\lambda\) implies that \(\dim \mathcal{L}/\mathcal{J} < \dim[\mathcal{B}/\mathcal{J}]\). Q.E.D.

6. IDEALS OF THE ALGEBRA \(\mathcal{B}_\lambda\), \(\lambda \neq h(2h + 1)/2, 2h \in \mathbb{N}\) AND \(\lambda \neq -\frac{1}{16}\)

In this section, we assume that \(\lambda \neq h(2h + 1)/2, 2h \in \mathbb{N}\), and \(\lambda \neq -\frac{1}{16}\).

(6.1) LEMMA. If \(\lambda \neq h(2h + 1)/2, 2h \in \mathbb{N}\), \(\mathcal{B}_\lambda\) has no finite dimensional representation.
Proof. Let \( V = V_0 \oplus V_1 \) be a finite dimensional representation of \( \mathcal{B}_2 \). Then \( V \) is a \( G \)-module, so it reduces to a direct sum of irreducible submodules. But \( C \) acts on \( V \) as the scalar \( \lambda \), and \( C \) acts on an irreducible module \( \mathcal{V}(h) \) as the scalar \( h(2h + 1)/2 \), \( 2h \in \mathbb{N} \), so there is a contradiction. Q.E.D.

(6.2) Proposition. If \( \lambda \neq h(2h + 1)/2 \), \( 2h \in \mathbb{N} \), and \( \lambda \neq -\frac{1}{16} \), \( \mathcal{B}_2 \) is a quasi-simple algebra.

(6.3) Corollary. Under the same assumptions, \( \mathcal{B}_2 \) is a primitive algebra.

Proof. Let \( \mathcal{I} \neq \{0\} \) be a two-sided ideal of \( \mathcal{B}_2 \). Using arguments similar to the proof of (5.2.1), one obtains \( \dim[\mathcal{B}_2/\mathcal{I}] < +\infty \).

If \( \mathcal{I} \neq \mathcal{B}_2 \), \( \mathcal{B}_2/\mathcal{I} \) is a finite dimensional \( \mathcal{B}_2 \)-module for the canonical left action, but such modules do not exist by (6.1). So \( \mathcal{I} = \mathcal{B}_2 \). Q.E.D.

7. THE QUOTIENT ALGEBRA \( \mathcal{B}_{-1/16} \) AND THE WEYL ALGEBRA

Let \( \mathcal{B}_{-1/16} = \mathcal{V}/(C + \frac{1}{16}) \mathcal{V} \). In \( \mathcal{B}_{-1/16} \), (1.2) becomes
\[
(\bar{Q} + \frac{3}{16})^2 = 0.
\]

(7.2) Lemma. \( \mathcal{B}_{-1/16} \) has no finite dimensional representation.

Proof. See (6.1).

Our first proposition shows that \( \mathcal{B}_{-1/16} \) is very different from the general \( \mathcal{B}_2 \), \( \lambda \neq -1/16 \).

(7.3) Proposition. Let \( \mathcal{I} = (\bar{Q} + \frac{3}{16}) \mathcal{B}_{-1/16} \); then \( \mathcal{I} \) is a two-sided ideal of \( \mathcal{B}_{-1/16} \). One has \( \mathcal{I}^2 = \{0\} \), so \( \mathcal{B}_{-1/16} \) is neither a semi-prime algebra nor a primitive algebra.

Proof. By (1.7.1) and (1.7.2), one has \( 2E_+ C = E_+ Q + QE_+ + \frac{1}{4}E_+ \), \( 2E_- C = E_- Q + QE_- + \frac{1}{4}E_- \), so:
\[
E_+(Q - 3C) = -(Q - 3C)E_+ - (16C + 1)E_+/4, \quad E_- (Q - 3C) = -(Q - 3C)E_- - (16C + 1)E_-/4.
\]

From (7.3.1), we deduce that \( \mathcal{I} = (Q - 3C) \mathcal{V} + (C + \frac{1}{16}) \mathcal{V} \) is a two sided ideal of \( \mathcal{V} \), and therefore \( \mathcal{I} \) is a two sided ideal of \( \mathcal{B}_{-1/16} \).

Moreover, using (7.3.1) once more and (7.1), we see that given a product \( (\bar{Q} + \frac{3}{16}) b(\bar{Q} + \frac{3}{16}) b' \) in \( \mathcal{B}_{-1/16} \), there exists \( b'' \in \mathcal{B}_{-1/16} \) such that \( (\bar{Q} + \frac{3}{16}) b(\bar{Q} + \frac{3}{16}) b' = (\bar{Q} + \frac{3}{16})^2 b'' b' = 0 \), so \( \mathcal{I}^2 = \{0\} \).
Finally, \( \mathcal{B} \neq \mathcal{B}_{-1/16} \): if not, there would exist \( b \) such that \((\mathcal{Q} + \frac{3}{16})b = 1\), and then \( \mathcal{Q} + \frac{3}{16} = (\mathcal{Q} + \frac{3}{16})^2 b = 0 \), which is a contradiction. This proves that \( \mathcal{B}_{-1/16} \) is not a semi-prime algebra, and, a fortiori, is not a primitive algebra [6]. Q.E.D.

(7.4) Let us now give a brief introduction to the Weyl algebra, e.g. [4]. We introduce the space \( V = \mathbb{C}[x] \), graded by \( V_0 = \text{span}\{x^n, n \in \mathbb{N}\} \), \( V_1 = \text{span}\{x^{2n+1}, n \in \mathbb{N}\} \), and the two linear operators \( p = \frac{d}{dx} \) and \( q = x \). One has \([p, q] = 1\). We denote by \( A \) the algebra generated by \( p \) and \( q \), which is naturally graded as \( A = A_0 \oplus A_1 \) by the graduation inherited from the graduation of \( V \). Given \( a \in A_i \), \( i = 0 \) or \( 1 \), we set \( \deg a = i \). The set \( \{p^nq^m, \ n, m \in \mathbb{N}\} \) (resp. \( \{p^nq^m, n + m \text{ even}\}, \text{ resp. } \{p^nq^m, n + m \text{ odd}\} \) is a basis of \( A \) (resp. of \( A_0 \), resp. of \( A_1 \)). From Lie’s viewpoint, \( A \) can be given the two following structures:

- the Lie algebra structure defined by \([a, b]_{\mathcal{L}} = ab - ba, \ a, b \in A\),
- the super algebra structure defined by \([a, b] = ab - (-1)^{\deg a \deg b}ba, \ a \in A_{\deg a}, \ b \in A_{\deg b}\).

We define \( y = \frac{1}{4}[p, q], \ f = \frac{1}{2}q^2, \ g = -\frac{1}{2}p^2, \ e_+ = \frac{1}{2}q, \ e_- = \frac{1}{2}p \). The subspace of \( A \) generated by these elements is a super algebra isomorphic to \( G \) which we denote by the same letter. Since \( G \) generates the algebra \( A \), we deduce a morphism \( \phi \) from \( \mathcal{Y} \) onto \( A \). It is easily seen that

\[
\phi(Q) = gf + y + y^2 = -\frac{1}{16}, \quad \text{and} \quad \phi(C) = \phi(Q) - \frac{1}{2}[e_+, e_-]_{\mathcal{L}} = -\frac{1}{16}.
\]

Therefore, \( \phi \) induces a morphism (which we denote by the same letter) from \( \mathcal{B}_{-1/16} \) onto \( A \).

We define a filtration \( A_k, \ k \in \mathbb{N} \), by \( A_k = \text{span}\{p^nq^m, n + m \leq k\} \). One has \([A_k, A_{k'}]_{\mathcal{L}} \subset A_{k+k' - 2}\), and therefore \([G_0, A_k]_{\mathcal{L}} = [G_0, A_{k-1}]_{\mathcal{L}} \subset A_k\). Moreover, from the formulæ \([p, q^m]_{\mathcal{L}} = mq^{m-1}, \ [q, p^n]_{\mathcal{L}} = -np^{n-1} \), we deduce

\[
[p, p^nq^m] = p^{n+1}q^m - (\mathcal{L})^{n+m}p^{n+1}q^m + m(-1)^{n+m}p^nq^{m+1}.
\]

Therefore \([p, A_{2k}] \subset A_{2k}\), and the same holds if \( p \) is replaced by \( q \), so \([G_1, A_{2k}] \subset A_{2k}\). This proves that \( A_{2k} \) is a \( G \)-submodule of the representation \( \pi \) defined by

\[X \in G, \ a \in A, \quad \pi(X)(a) = [X, a] \]

Let us denote by \( W_k \) the \( G \)-submodule of \( A_{2k} \) generated by \( F^k = \frac{1}{2}q^{2k} \). Since \( F^k \) is a dominant weight vector of weight \( k \), we obtain \( W_k \cong \mathcal{D}(k) \).
(7.4.1) Lemma. The reduction of the representation \( \pi \) on \( A_{2k} \) is

\[
A_{2k} = \sum_{i=0}^{k} W_i \cong \sum_{i=0}^{k} D(i).
\]

Proof. The result is true for \( A_0 \cong D(0) \) and \( A_2 \cong D(1) \oplus D(0) \), so, by induction, we assume that it holds for \( A_{2k-2} \).

The \( G \)-submodule \( W_k \subset A_{2k} \) is isomorphic to \( D(k) \), so \( \dim W_k = 4k + 1 \), and \( W_k \cap A_{2k-2} = \{0\} \) since the reduction of \( A_{2k-2} \) does not contain \( D(k) \). But \( \dim A_{2k} - \dim A_{2k-2} = 4k + 1 = \dim W_k \), so \( A_{2k} = A_{2k-2} \oplus W_k \).

Q.E.D.

(7.5) Proposition. As in (7.3), let \( \mathcal{J} = (\mathcal{O} + \frac{1}{16}) \mathcal{B}_{-1/16} \), and let \( \phi \) be the canonical morphism from \( \mathcal{B}_{-1/16} \) onto \( A \) defined in (7.4). Then \( \ker \phi = \mathcal{J} = \sum_{n \geq 2} V_n \), so \( \mathcal{B}_{-1/16}/\mathcal{J} \cong A \).

Proof. We have a \( G \)-morphism \( \phi : \mathcal{B}_{-1/16} \cong \sum_{n \geq 0} D(n) \oplus \sum_{n \geq 2} D(n - \frac{3}{2}) \) onto \( A \cong \sum_{n \geq 0} D(n) \), so necessarily \( \ker \phi = \sum_{n \geq 2} V_n \). Since \( (\mathcal{O} + \frac{1}{16}) \in \ker \phi \), one has \( \mathcal{J} \subset \ker \phi \).

One has \( \mathcal{O} + \frac{1}{16} = \theta_n \), and from (3.1), as a \( G \)-module for the adjoint action, \( \mathcal{V}_n \) is generated by \( F^{n-2}\theta_1/2 = F^{n-2} \text{ad } E_+ (\theta_0) \). \( \mathcal{J} \) being a two sided ideal (7.3), it results that \( \mathcal{V}_n \subset \mathcal{J}, \forall n \geq 2 \), and then that \( \ker \phi = \mathcal{J} \). Q.E.D.

(7.6) Corollary. \( \mathcal{J} \) is a primitive and completely prime ideal of \( \mathcal{B}_{-1/16} \).

Proof. Since \( A \cong \mathcal{B}_{-1/16}/\mathcal{J} \), \( \mathcal{J} \) is primitive from the given definition of \( A \). It is well known that \( A \) is entire, so \( \mathcal{J} \) is completely prime. Q.E.D.

(7.7) Proposition. \( \mathcal{J} \) is the only non trivial two sided ideal of \( \mathcal{B}_{-1/16} \).

Proof. Let \( \mathcal{J} \) be a two sided ideal of \( \mathcal{B}_{-1/16} \), and let us assume that \( \mathcal{J} \neq \{0\} \). \( \mathcal{J} \cap \mathcal{J} \) is also a two sided ideal, and if \( \mathcal{J} \cap \mathcal{J} \neq \{0\}, \) since \( \mathcal{J} \cap \mathcal{J} \) is a \( G \)-module for the adjoint action, there must exist \( n \) such that \( \mathcal{V}_{n+2} \subset \mathcal{J} \cap \mathcal{J} \). So \( F^n\theta_1/2 \in \mathcal{J} \cap \mathcal{J} \), therefore \( F^n\theta_1/2 \in \mathcal{J} \cap \mathcal{J}, \forall m \geq n \), and \( \sum_{m \geq n} \mathcal{V}_{n+2} \subset \mathcal{J} \cap \mathcal{J} \). Since \( \mathcal{J} = \sum_{m \geq 0} \mathcal{V}_{n+2} \), we have \( \dim[\mathcal{B}/\mathcal{J} \cap \mathcal{J}] < +\infty \).

If \( \mathcal{J} \cap \mathcal{J} = \mathcal{J} \cap \mathcal{J} \), \( \mathcal{J} / \mathcal{J} \cap \mathcal{J} \) is a finite dimensional \( \mathcal{B}_{-1/16} \)-module for the canonical left action, so there is a contradiction with (7.2), and necessarily \( \mathcal{J} \cap \mathcal{J} = \mathcal{J} \). If we assume that the inclusion \( \mathcal{J} \subset \mathcal{J} \) is strict, there must exist \( k \) such that \( \mathcal{H}_k \subset \mathcal{J} \), so \( F^k \in \mathcal{J} \), and \( F^m \in \mathcal{J}, \forall m \geq k \). Since \( \mathcal{H}_m \) is generated (as a \( G \)-module for the adjoint action) by \( F^m \), we can conclude that \( \sum_{m \geq k} \mathcal{H}_m \subset \mathcal{J} \).

Now \( \mathcal{J} = \sum_{m \geq 0} \mathcal{V}_{m+2} \subset \mathcal{J} \), \( \sum_{m \geq k} \mathcal{H}_m \subset \mathcal{J} \), therefore \( \dim[\mathcal{B}_{-1/16}/\mathcal{J}] < +\infty \). If \( \mathcal{J} \neq \mathcal{B}_{-1/16} \), \( \mathcal{B}_{-1/16}/\mathcal{J} \) is a finite dimensional \( \mathcal{B}_{-1/16} \)-module for the canonical left action, and such modules do not exist by (7.2), so \( \mathcal{J} = \mathcal{B}_{-1/16} \).
Let us now assume that $\mathcal{F} \cap \mathcal{J} = \{0\}$. If $\mathcal{J} \neq \{0\}$, since $\mathcal{J}$ is a $G$-module for the adjoint action, there exists $n$ such that $\tilde{H}_n \subset \mathcal{J}$, so $\tilde{F}^n \in \mathcal{J}$. But then $\tilde{F}^n \tilde{G}_{1/2} \in \mathcal{J} \cap \mathcal{F}$, and we find a contradiction. So $\mathcal{F} = \{0\}$. Q.E.D.

8. PROOF OF THEOREM 5

In this section, we prove Theorem 5 (see the introduction for notations).

(8.1) Proof of Theorem 5. (1), (2), (3), (4) are direct consequences of the results in Sections 4, 5, 6, 7.

In order to prove (5), we need the following lemma:

**Lemma.** Let $\lambda \in \mathbb{C}$, then $\bigcap_{n \geq 0} (C - \lambda)^n \mathcal{V} = \{0\}$.

**Proof.** From the decomposition $\mathcal{V} = \bigoplus_{n \geq 0} (C - \lambda)^n \mathcal{H}$ of (3.3), we deduce that $\mathcal{V} = \mathcal{H} \oplus (C - \lambda) \mathcal{H} \oplus \cdots \oplus (C - \lambda)^{n-1} \mathcal{H} \oplus (C - \lambda)^n \mathcal{V}$, for any $n$. Now, from the first decomposition, given $u \in \mathcal{V}$, $u \neq 0$, there exists $n_0$ such that $u \in \bigoplus_{n = 0}^{n_0} (C - \lambda)^n \mathcal{H}$, and so far, there exists $n_0$ such that $u \notin (C - \lambda)^{n_0} \mathcal{V}$.

Q.E.D.

**Proof of (5).** Let $I \neq \{0\}$ be a prime ideal of $\mathcal{V}$. Then there exists a maximal left ideal $M$ such that $I \subset M$. The $\mathcal{V}$-module $\mathcal{V}/M$ is irreducible, and $I$ is contained in its kernel $I_0$. From Sections (5), (6), and (7), we see that there are three cases:

(1) $I_0 = (C - \lambda) \mathcal{V}$, $\lambda \neq -\frac{1}{16}$. In this case, let $J = \{ j \in \mathcal{V}, (C - \lambda) j \in I \}$; then $J$ is a two sided ideal of $\mathcal{V}$ containing $I$. Denote by $\tilde{I}$ and $\tilde{J}$ the canonical images of $I_0$ and $J$ in the prime algebra $\mathcal{V}/I$. Since $\tilde{I} \tilde{J} = \{0\}$, one must have $\tilde{I} = \{0\}$, or $\tilde{J} = \{0\}$. But $\tilde{I} = \{0\}$ implies that $I = I_0$, and $\tilde{J} = \{0\}$ implies $I = (C - \lambda) I$, so $\tilde{I} = (C - \lambda) \tilde{I}$ for $n$, and $I \subset \bigcap_{n \geq 0} (C - \lambda)^n \mathcal{V} = \{0\}$ by Lemma (8.1), so we are done.

(2) $I_0 = \text{Ker} D(h)$, $2h \in \mathbb{N}$. Set $\lambda = h(2h + 1)/2$ and $J = \{ j \in \mathcal{V}, (C - \lambda) j \in I \}$. Denote by $\tilde{I}$ and $\tilde{J}$ the canonical images of $I$ and $J$ in the prime algebra $\mathcal{V}/I$. Since $\tilde{I} \tilde{J} = \{0\}$, one has $\tilde{I} = \{0\}$ or $\tilde{J} = \{0\}$. If $\tilde{I} = \{0\}$, we get $I = (C - \lambda) I$, so $I \subset \bigcap_{n \geq 0} (C - \lambda)^n \mathcal{V} = \{0\}$ and we have a contradiction. If $\tilde{J} = \{0\}$, we obtain $\tilde{J} \subset I$. Denote by $\tilde{I}$ and $\tilde{J}$ the canonical images of $I$ and $I_0$ in the algebra $\tilde{B} = \mathcal{V}/\tilde{J}$; from (5.2.1) we obtain either $\tilde{I} = \{0\}$, and then $I = \tilde{J}$, or $\tilde{J} = \tilde{I}_0$, and then $I = I_0$.

(3) $I_0 = \mathcal{J}$. Once more, we introduce $J = \{ j \in \mathcal{V} / (C + \frac{1}{16}) j \in I \}$, and using the same arguments as in the preceding case, we conclude that $\mathcal{J} \subset I$. Denote by $\tilde{I}$ and $\tilde{J}$ the canonical images of $I_0$ and $I$ in the algebra $\tilde{B}_{-1/16} = \mathcal{V}/\tilde{J}$, using (7.7), we obtain either $\tilde{J} = \tilde{I}_0$, and then
$I = I_0$, or $\overline{I} = \{0\}$, and then $I = \mathcal{I}_{1/16}$, which is excluded since $\mathcal{I}_{1/16}$ is not prime.

Actually, we have proved that a prime ideal is $\{0\}$, or a prime ideal in the list. A non-zero primitive (resp. completely prime) ideal is prime, so it appears in the list. Q.E.D.

Proof of 6. All ideals in the list are primitive except $\mathcal{I}_{1/16}$, which is not prime. Q.E.D.

(8.2) Remark. $\{0\}$ is completely prime by Theorem 1, but cannot be primitive by Quillen's lemma.

(8.3) Remark. In [2], the following questions were raised: given a superalgebra $G = G_0 \oplus G_1$, $\mathcal{V}$ its enveloping algebra, $U$ the enveloping algebra of $G_0$, $\mathcal{J}$ a two sided ideal of $\mathcal{V}$, $J = \mathcal{J} \cap \mathcal{V}$, if $\mathcal{J}$ is prime, is $J$ prime? if $\mathcal{J}$ is primitive, is $J$ primitive, prime?

Now we set $G = \text{osp}(1, 2)$, and we answer these questions in the negative. Take $\mathcal{J} = \mathcal{J}_2$, $\lambda \neq -\frac{1}{16}$, then $\mathcal{J}$ is primitive, so $\mathcal{J}$ is prime. Using (4.4) $J = (Q - q_0)(Q - q_1)U$, with $q_0 \neq q_1$, which is obviously not prime, and a fortiori not primitive.

9. KRULL DIMENSION OF A SUPERALGEBRA

In this section, we assume that $G = G_0 \oplus G_1$ is a (general) superalgebra, we denote by $\mathcal{V}$ its enveloping algebra, and by $U$ the enveloping algebra of its even Lie subalgebra $G_0$.

(9.1) Theorem 6. One has $sK \dim \mathcal{V} = K \dim \mathcal{V} = K \dim U$.

Proof. First, we prove that $K \dim U \leq sK \dim \mathcal{V}$:

By the P.B.W. theorem, there exists a finite homogeneous basis $\{u_i, i = 0, ..., p\}$ of the right $U$-module $\mathcal{V}$. Given a left ideal $M$ in $U$, we set

$$\mathcal{M} = \sum_{i = 0, ..., p} u_i M.$$ 

Obviously, $\mathcal{M}$ is a $\mathbb{Z}_2$-graded left ideal in $\mathcal{V}$. Let us denote by $I$ the set of left ideals of $U$, by $\mathcal{I}$, the set of graded left ideals in $\mathcal{V}$, and by $\phi$ the mapping from $I$ into $\mathcal{I}$, defined by $\phi(M) = \mathcal{M}$.

It is immediate that $\phi$ is an increasing mapping for the natural ordering. Let us show that it is strictly increasing:

We assume that $M \subset M'$, and $\mathcal{M} = \mathcal{M}'$. Writing $M' = M \oplus V$, we deduce that $\mathcal{M}' = \mathcal{M} \oplus \sum_{i = 0, ..., p} u_i V$, so $V = \{0\}$, and $M = M'$.

Now, we can use [6, 3.5.2] and conclude that $K \dim U \leq sK \dim \mathcal{V}$. We then prove that $K \dim \mathcal{V} \leq K \dim U$: 
Using once more the P.B.W. theorem, there exists a finite basis \{u_i, i = 0, \ldots, p\} of the left \( U \)-module \( \mathcal{V} \). We fix \( u_0 = 1 \). Any element \( u \) in \( \mathcal{V} \) can be written

\[
u = \sum_{i=0}^{p} \alpha_i(u) u_i, \quad \alpha_i(u) \in U, \quad i = 0, \ldots, p.
\]

Given an ordinary left ideal \( \mathcal{M} \) of \( \mathcal{V} \), we define

\[
0 \leq k < p, \quad \mathcal{M}_k = \{ m \in \mathcal{M} / \alpha_{k+1}(m) = \alpha_k(m) = \cdots = \alpha_{p}(m) = 0 \},
\]

and \( \mathcal{M}_p = \mathcal{M} \).

Let \( \mathcal{I} \) be the set of ordinary left ideals in \( \mathcal{V} \).

We set \( f_k(\mathcal{M}) = \{ m \in \mathcal{M}_k / \alpha_{k+1}(m) = \alpha_k(m) = \cdots = \alpha_{p}(m) = 0 \} \).

Let \( \mathcal{M}_0 = \mathcal{M}_0 \). So we assume proved that \( \mathcal{M}_k = \mathcal{M} \).

Let now \( m' \in \mathcal{M}_{k+1} \); we have \( m' = \alpha_0(m') + \cdots + \alpha_{k+1}(m') u_{k+1} \). Since \( f_{k+1}(\mathcal{M}') = f_{k+1}(\mathcal{M}) \), there exists \( m \in \mathcal{M}_{k+1} \) such that \( m = \alpha_0(m) + \cdots + \alpha_{k+1}(m' u_{k+1} \). We deduce that \( m - m \in \mathcal{M}_k \), but since \( \mathcal{M}_k = \mathcal{M}_k \), it results that \( m' \in \mathcal{M} \), and so \( \mathcal{M}_k = \mathcal{M}_{k+1} \). Therefore \( \mathcal{M} = \mathcal{M}' \).

Now, we use [6, 3.5.2] and obtain that \( K \dim \mathcal{V} \leq \text{dev}(I^{p+1}) \), where \( \text{dev}(I^{p+1}) \) is the deviation of the ordered set \( I^{p+1} \) (see, e.g., [6]). By [6, 3.5.3] \( \text{dev}(I^{p+1}) = \text{dev} I \), but \( \text{dev} I = K \dim U \), so \( K \dim \mathcal{V} \leq K \dim U \).

Since \( sK \dim \mathcal{V} \leq K \dim \mathcal{V} \), the proof is complete.

Q.E.D.

(9.2) COROLLARY. If \( G \) is the orthosymplectic superalgebra \( \text{osp}(1, 2) \), one has \( K \dim \mathcal{V} = 2 \).

Proof. It is known that \( K \dim U = 2 \) [12].

(9.3) PROPOSITION. \( K \dim \mathcal{B}_\lambda = 1, \forall \lambda \in \mathbb{C} \).

Proof. Using [6, 3.5.11], we obtain \( K \dim \mathcal{B}_\lambda < K \dim \mathcal{V} = 2 \). So we have to prove that \( \mathcal{B}_\lambda \) is not artinian. But this last claim is obvious, since \( \mathcal{B}_\lambda \) has an infinite number of inequivalent irreducible representations by [3, 4.2.13].

Q.E.D.

ACKNOWLEDGMENTS

I thank Professor Christian Fronsdal, and especially Professor Moshe Flato for several enlightening discussions.
Note added in proof. Since the acceptance of this paper, the following result was proved and kindly communicated to me by Professor M. Duflo:

The enveloping algebra of a complex superalgebra $G$ is entire if and only if the equation $[X, X'] = 0, X \in G_1$, has the only solution $X = 0$.

As a consequence, a simple complex classical Lie superalgebra $G$ has an entire enveloping algebra if and only if $G$ is of type $osp(1, 2n)$, as conjectured in the Introduction of the present paper.

The proof makes use of a result of M. Aubry and J. A. Lemaire (J. Pure Appl. Algebra 38 (1985), 159) together with an adapted Inonu–Wigner contraction (see, e.g., Section 3).

REFERENCES

5. J. Dixmier, Quotients simples de l'algèbre enveloppante de $sl(2)$, J. Algebra 24 (1973), 551.