Available online at www.sciencedirect.com

ScienceDirect

Nuclear Physics B 910 (2016) 273–292

NUCLEAR PHYSICS B

www.elsevier.com/locate/nuclphysb

Exploring Lovelock theory moduli space for Schrödinger solutions

Dileep P. Jatkar *, Nilay Kundu

Harish-Chandra Research Institute, Chhatnag Road, Jhunsi, Allahabad 211019, India

Received 14 February 2016; received in revised form 10 May 2016; accepted 6 July 2016

Available online 14 July 2016

Editor: Leonardo Rastelli

Abstract

We look for Schrödinger solutions in Lovelock gravity in $D > 4$. We span the entire parameter space and determine parametric relations under which the Schrödinger solution exists. We find that in arbitrary dimensions pure Lovelock theories have Schrödinger solutions of arbitrary radius, on a co-dimension one locus in the Lovelock parameter space. This co-dimension one locus contains the subspace over which the Lovelock gravity can be written in the Chern–Simons form. Schrödinger solutions do not exist outside this locus and on this locus they exist for arbitrary dynamical exponent z. This freedom in z is due to the degeneracy in the configuration space. We show that this degeneracy survives certain deformation away from the Lovelock moduli space.

1. Introduction

The Schrödinger solutions belong to a class of solutions to the gravitational equations of motion which asymptotically do not preserve the Lorentz symmetry. They, however, do respect some non-relativistic symmetries. The deviation from the relativistic symmetry is parametrized by the Schrödinger scaling exponent z, or the dynamical exponent.

* Corresponding author.
E-mail addresses: dileep@hri.res.in (D.P. Jatkar), nilaykundu@hri.res.in (N. Kundu).

http://dx.doi.org/10.1016/j.nuclphysb.2016.07.008
0550-3213/

The Schrödinger solution was first obtained by Son [1] as well as Balasubramanian and Mcgreevy [2]. They assumed the stress tensor consisting of the cosmological constant term and the pressure-less dust. The Schrödinger solution possesses the Galilean boost invariance by assigning a specific transformation property to one of the light-like directions [1,2] (non-relativistic metrics in higher derivative gravity were discussed in [3]).

In this note we analyze this question in more detail by spanning the entire coupling parameter space of Lovelock theories in various dimensions. Up to four space–time dimensions, the Lovelock action is identical to the Einstein Hilbert action with the cosmological constant, but from five dimensions onwards the Lovelock action has additional Gauss–Bonnet term in the action. This term can be added in four dimensions as well but being total derivative term it does not contribute to the dynamics. In five and higher dimensions the Gauss–Bonnet term does contribute to the dynamics. Similarly the cubic order Lovelock term can be added from six dimensions onwards but it contributes to dynamics only from seven dimensions onwards.

We will show that the Schrödinger metric is generically not a solution to the Lovelock equations of motion, however, it exists as a solution on a co-dimension 1 locus in the Lovelock coupling space. We show that the Schrödinger solution exists precisely on the same locus on which the Lifshitz solution is known to exist.[1] In our computation we restrict ourselves to the Lovelock terms up to cubic order in the curvature tensor but we generalize our analysis to arbitrary dimensions. The co-dimension 1 locus on which we get the Schrödinger solution is interesting from another point of view. It is known that the Lovelock theories can be written in terms of the parity preserving Chern–Simons theory. However, this representation exists only for specific values of the Lovelock couplings. The Chern–Simons formulation exists at a point on this co-dimension 1 locus on which we find the Schrödinger solutions. We present these solutions in the Chern–Simons gauge field forms as well.

The Schrödinger solutions are relevant from the point of view of application to holographically dual condensed matter physics systems. It then naturally raises a question of relevance of these higher dimensional solutions to $2 + 1$ and $3 + 1$ dimensional condensed matter systems. In this regard it is worth pointing out that unlike the AdS and Lifshitz holography which relates D dimensional theory of gravity to $D - 1$ dimensional field theory, the Schrödinger holography relates D dimensional theory of gravity to $D - 2$ dimensional field theory. Therefore, $4 + 1$ and $5 + 1$ dimensional Lovelock theories are relevant to $2 + 1$ and $3 + 1$ dimensional boundary physics. Higher dimensional theories can be dimensionally reduced to lower dimensional theories. Such higher dimensional theories typically give rise to scalar-tensor theories of gravity which are either referred to as Galileon or Horndeski theories [5–8]. For example, let us consider $D = d + n + 1$ dimensional theory of gravity with the cosmological constant, the Einstein–Hilbert, the Gauss–Bonnet term

$$S = \int d^D x \sqrt{-g}\,[R - 2\Lambda + a_2\mathcal{L}_2] \ , \tag{1.1}$$

where $\mathcal{L}_2$ is the Gauss–Bonnet term. We will dimensionally reduce it down to $d + 1$ dimensions by using an n-dimensional compact manifold $\tilde{K}_n$ such that

$$ds_D^2 = d\bar{s}_{d+1}^2 + e^{\phi} d\tilde{K}_n^2 \ . \tag{1.2}$$

This is a simple but consistent diagonal toroidal compactification which gives rise to one extra scalar degree of freedom, that is the size of the internal space. All terms with a tilde refer to

[1] For closely related solutions of Kasner type in the Lovelock theory, see [4].

internal n dimensional space, while terms with a bar refer to the $d+1$ dimensional space–time. As we integrate out the internal space the effective action looks like [9]

$$\begin{aligned}\bar{S}_{(d+1)} = \int d^{d+1}x\sqrt{-\bar{g}}\, e^{\frac{n}{2}\phi}\Bigg\{ & \bar{R} - 2\Lambda + a_2\bar{\mathcal{G}} + \frac{n}{4}(n-1)(\partial\phi\partial\phi) \\ & - a_2 n(n-1)\bar{G}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \\ & - \frac{a_2}{4}n(n-1)(n-2)\left[(\partial\phi\partial\phi)\nabla^2\phi - \frac{(n-1)}{4}(\partial\phi\partial\phi)^2\right] \\ & + e^{-\phi}\tilde{R}\big[1 + a_2\bar{R} + 4a_2(n-2)(n-3)(\partial\phi\partial\phi)\big] \\ & + a_2\tilde{\mathcal{G}}e^{-2\phi}\Bigg\}, \end{aligned} \tag{1.3}$$

where $\bar{\mathcal{G}}$ is $d+1$ dimensional Gauss–Bonnet term, $\tilde{\mathcal{G}}$ is n dimensional Gauss–Bonnet term and $\bar{G}^{\mu\nu}$ is the Einstein tensor of $d+1$ dimensional space. The effective action written above can be related to the so called Galileon action, with the Galileon field is realized as the scalar parametrizing the volume of the internal space. As we have reduced from the Einstein–Gauss–Bonnet action, in the reduced action all the terms are up to quartic order in derivatives of the metric or the scalar or of both, but the equations of motion following from it will still be of second order. The term of the form $(\partial\phi\partial\phi)\nabla^2\phi$ is often called the DGP term [10] appearing in the decoupling limit of the DGP model, and the term of the form $(\partial\phi\partial\phi)^2$ is the standard Galileon term [5]. We will argue that the Schrödinger solution exists in a co-dimension 1 subspace of Lovelock moduli space, for example, if we restrict to only the Gauss–Bonnet term then it exists on a subspace which relates a_2 to Λ. We will get back to the issue of dimension reduction in this context in the discussion section. As long as $n \leq 2$, neither the DGP term nor the Galileon term appears in the dimensionally reduced theory. In addition, for Ricci-flat compact spaces the lower dimensional action, up to the addition of higher derivative curvature terms, has a familiar form.

This note is organized in the following manner. We will first give basics of the Lovelock theory in arbitrary dimensions. Most of the information in this section is not new but is useful to fix the notation. In the next section we will look at various solutions to the Lovelock equations of motion. It is well known that the AdS solution generically exists for arbitrary values of Lovelock couplings in any dimension. This feature is not shared by the Schrödinger solution. We present the solution for general value of D and in explicit form for dimensions $D = 5, 6, 7$. We also comment on the solutions with anisotropic scaling in spatial direction and their relation to AdS $\times$ R type solutions. In section 4, we analyze branches of the AdS solution [11,12]. Our interest in presenting this result is to emphasize that the non-relativistic solutions exist only when the discriminant vanishes. Degeneracy in the configuration space [13,14] has been well studied for Lifshitz solutions [15,16]. We show that this degeneracy is responsible for unconstrained z for Schrödinger case as well. Unlike Lifshitz, in the case of Schrödinger solutions this degeneracy extends beyond the Lovelock moduli space. In this sense our results provide a template for a dynamical exponent z for which all values are equally likely. Any suitable value of z then can be obtained by either appropriately modifying the couplings in this theory or by adding new interactions.

It is known that the Lovelock theory in odd space–time dimensions can be written in the Chern–Simons form and in even space–time dimensions in the Born–Infeld form exactly when the discriminant vanishes [11]. We discuss the relation between vanishing discriminant and locus of non-relativistic solutions in section 5 and write down the Schrödinger solution in the Chern–Simons gauge field form in odd space–time dimensions and in the Born–Infeld gauge field form in even space–time dimensions. Finally, we point out the relation with the causality and stability constraints obtained in the Lovelock theories in higher dimensions [17–19]. Finally we summarize our results and speculate about their applications. Various technical details are relegated to Appendix A. Appendix A.1 contains details of Lovelock equations of motion. Appendix A.2 recounts details of AdS and Lifshitz solutions, which are given for the purpose of comparison with the Schrödinger solution. Appendix A.3 contains spin connections and curvature tensors for Schrödinger solution.

2. The Lovelock gravity theory

Let us consider following action

$$I = \frac{1}{16\pi G}\int_M d^D x \sum_{p=0}^{[(D-1)/2]} a_p \mathcal{L}_p \ , \tag{2.1}$$

where G is the D dimensional Newton's constant, a_p are coupling constants with $a_0 = -2\Lambda = (D-1)(D-2)/\ell^2$, $a_1 = 1$, a_2 is the Gauss–Bonnet coupling etc.,[2] and $\mathcal{L}_p$ are terms in the Lagrangian density of the Lovelock action,

$$\mathcal{L}_p = \frac{1}{2^p}\sqrt{-g}\,\delta^{\nu_1\nu_2\cdots\nu_{2p}}_{\mu_1\mu_2\cdots\mu_{2p}} R^{\mu_1\mu_2}_{\nu_1\nu_2}\cdots R^{\mu_{2p-1}\mu_{2p}}_{\nu_{2p-1}\nu_{2p}} \ , \tag{2.2}$$

where $\delta^{\nu_1\nu_2\cdots\nu_{2p}}_{\mu_1\mu_2\cdots\mu_{2p}}$ is totally antisymmetric product of $2p$ Kronecker deltas normalized to take values 0 and ± 1, and hence is completely antisymmetric in all its upper and lower indices separately. It can also be considered as the determinant of a $(2p \times 2p)$ matrix whose (ij)-th element is given by $\delta^{\nu_i}_{\mu_j}$.

We are using notation of [11] and the equation of motion can be written in the compact form as

$$E^\nu_\mu = \sum_{p=0}^{[(D-1)/2]} \frac{a_p}{2^p}\delta^{\nu\nu_1\nu_2\cdots\nu_{2p}}_{\mu\mu_1\mu_2\cdots\mu_{2p}} R^{\mu_1\mu_2}_{\nu_1\nu_2}\cdots R^{\mu_{2p-1}\mu_{2p}}_{\nu_{2p-1}\nu_{2p}} = 0 \ . \tag{2.3}$$

Writing the Lovelock terms in this fashion makes it obvious that up to $D = 4$ only relevant couplings are a_0 and a_1. The Gauss–Bonnet term with the coupling a_2 is topological in $D = 4$ but it is dynamical in $D = 5, 6$. In these dimensions we will explore the parameter space spanned by a_0 and a_2 to find the range of values for which the Lifshitz solution is possible. In $D = 6$, the term with coupling a_3 can be written down but like the Gauss–Bonnet in $D = 4$, this term is topological in $D = 6$ and hence does not affect the equation of motion. However, this term becomes relevant in $D > 6$. We will explore the three dimensional parameter space spanned by a_0, a_2 and a_3 and find conditions for Lifshitz solutions.

We will start with the study of solution to the $D = 5$ equations of motion. We therefore write the action of pure gravity in D-dimensions as

$$I = \int d^D x\, \sqrt{-g}\, [R - 2\Lambda + \mathcal{L}_{hd}] \tag{2.4}$$

[2] It is important to note that the mass dimensions of various parameters appearing in the action, eq. (2.1) are as follows $[G] = D-2$, $[\Lambda] = [a_0] = 2$, $[a_1] = 0$, $[a_2] = -2$, $[a_3] = -4$, $\cdots$, and the parameter ℓ has dimensions of length.

where Λ is the cosmological constant and $\mathcal{L}_{hd}$ is the Lagrangian for the higher derivative terms of the Lovelock form.

As mentioned earlier in $D = 5$ space–time dimensions the higher derivative Lagrangian density contains the quadratic Lovelock term, also known as the Gauss–Bonnet term, which appears in the Lagrangian with the coupling constant, a_2,

$$\begin{aligned}\mathcal{L}_{hd} &= a_2\mathcal{L}_2, \quad \text{where}\\ \mathcal{L}_2 &= (R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}).\end{aligned} \tag{2.5}$$

For $D = 7$ space–time dimensions we will have, in addition to the Gauss–Bonnet term, the cubic Lovelock term $\mathcal{L}_3$,

$$\mathcal{L}_{hd} = a_2\mathcal{L}_2 + a_3\mathcal{L}_3 \tag{2.6}$$

where a_3 is the coupling constant of the cubic Lovelock term. The cubic term is explicitly written as

$$\begin{aligned}\mathcal{L}_3 = {} & 2R^{\mu\nu\sigma\kappa}R_{\sigma\kappa\rho\tau}R^{\rho\tau}{}_{\mu\nu} + 8R^{\mu\nu}{}_{\sigma\rho}R^{\sigma\kappa}{}_{\nu\tau}R^{\rho\tau}{}_{\mu\kappa} + 24R^{\mu\nu\sigma\kappa}R_{\sigma\kappa\nu\rho}R^{\rho}_{\mu}\\ & + 3RR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 24R^{\mu\nu\sigma\kappa}R_{\sigma\mu}R_{\kappa\nu} + 16R^{\mu\nu}R_{\nu\sigma}R^{\sigma}_{\mu}\\ & - 12RR^{\mu\nu}R_{\mu\nu} + R^3.\end{aligned} \tag{2.7}$$

The equation of motion that follows from here is as written below,

$$G^{(1)}_{\mu\nu} + a_2G^{(2)}_{\mu\nu} + a_3G^{(3)}_{\mu\nu} - \Lambda g_{\mu\nu} = 0 \tag{2.8}$$

where the explicit forms of $G^{(1)}_{\mu\nu}$, $G^{(2)}_{\mu\nu}$ and $G^{(3)}_{\mu\nu}$ are given in Appendix A.1. The equations of motion in $D = 5$ are obtained by setting $a_3 = 0$ in (2.8). We will now study specific solutions to the equations of motion.

3. Solutions to the Lovelock gravity

In this section we will analyze solutions to the Lovelock gravity equations of motion. In Appendix A.2 we will summarize known results about the AdS and Lifshitz solutions. In particular the AdS solution is possible for generic values of the Lovelock couplings a_2, a_3, etc. On the other hand the Lifshitz solutions exist only on the co-dimension one subspace. As we will see below the Schrödinger solutions can be obtained only on the same co-dimension one locus in the parameter space. Furthermore, as we will see, on this locus the solution admits arbitrary dynamical scaling exponent z, which is also true for the Lifshitz solutions.

3.1. The Schrödinger solution

We consider the Schrödinger space–times as solutions of the higher derivative Lovelock gravity theories. This is an example of non-relativistic space–time, known apart from the Lifshitz solution. We will look for the Schrödinger solution to the Lovelock gravity equations of motion in arbitrary dimensions but we will restrict to terms cubic in curvature tensor. Generalization to higher order Lovelock terms is tedious but straightforward.

The metric ansatz for the Schrödinger solution looks like

$$ds^2 = L_{\rm sch}^2\left[-\frac{dt^2}{r^{2z}} + \frac{dr^2}{r^2} + \frac{2}{r^2}dt d\xi + \frac{1}{r^2}\sum_{i=1}^{D-3} dx_i^2\right]. \tag{3.1}$$

Note that this metric for the Schrödinger solution also has two parameters, z which is the Schrödinger exponent and $L_{\rm sch}$ which is the "Schrödinger radius". We will first state the results for arbitrary D but restricting up to cubic Lovelock terms and then write down explicit expressions for $D = 5, 6, 7$.

The Schrödinger space–time solution in general D dimensions exists subject to following two constraints,

$$\begin{aligned}\Lambda &= -\frac{(D-1)(D-2)}{4L_{\rm Sch}^2}\left(1-(D-3)(D-4)(D-5)(D-6)\frac{a_3}{L_{\rm sch}^4}\right),\\ a_2 &= \frac{L_{\rm Sch}^2}{2(D-3)(D-4)} + \frac{3(D-5)(D-6)}{2L_{\rm Sch}^2}a_3 .\end{aligned} \tag{3.2}$$

The dynamical exponent z is unconstrained. If we eliminate $L_{\rm sch}$, then it gives one relation between the parameters in the Lovelock action. Thus the Schrödinger solutions exist on co-dimension one subspace of the Lovelock moduli space.

In $D = 5$ space–time we have the Gauss–Bonnet term in the Lovelock action besides the Einstein–Hilbert and the cosmological constant term. The constraint (3.2) corresponds to

$$\Lambda = -\frac{3}{L_{\rm Sch}^2} \text{ and } a_2 = \frac{L_{\rm Sch}^2}{4} \implies a_2\Lambda = -3/4 . \tag{3.3}$$

The non-zero components of the Ricci tensor and the Ricci scalar R for the metric of Schrödinger space–time are given by $R_{tt} = 2\left(z^2+1\right)/r^{2z}$, $R_{t\xi} = R_{rr} = R_{x_i x_i} = -4/r^2$; and $R = -20/L_{\rm Sch}^2$. In $D = 6$ space–time the Gauss–Bonnet term is important but the curvature cubed Lovelock term being a total derivative is not. The constraint eq. (3.2) becomes $\Lambda = -5/L_{\rm Sch}^2$ and $a_2 = L_{\rm Sch}^2/12$. We again write down the components of the Ricci tensor $R_{tt} = \left(2z^2+z+2\right)r^{-2z}$, $R_{t\xi} = R_{rr} = R_{x_i x_i} = -5/r^2$ and the Ricci scalar $R = -(30/L_{\rm Sch}^2)$.

In $D = 7$ space–time apart from the Gauss–Bonnet term, the cubic order Lovelock term will also be important and hence the action will contain three parameters, Λ, a_2 and a_3. The constraint (3.2) in this case takes the form

$$\Lambda = -\frac{15}{2L_{\rm sch}^2}\left(1-\frac{24}{L_{\rm sch}^4}a_3\right) \text{ and } a_2 = \frac{L_{\rm sch}^2}{24} + 3\frac{a_3}{L_{\rm sch}^2} . \tag{3.4}$$

The Ricci tensor components are $R_{tt} = 2\left(z^2+z+1\right)r^{-2z}$, $R_{t\xi} = R_{rr} = R_{x_i x_i} = -6/r^2$, and the Ricci Scalar becomes $R = -(42/L_{\rm Sch}^2)$.

3.2. Other solutions

Lifshitz solutions,[3] where the time coordinate (t) scaled differently compared to the other coordinates, are known to be solutions of Lovelock gravity. As an alternative one can consider a

[3] See eq. (A.7) for Lifshitz metric in Appendix A.2, where we discuss Lifshitz solutions briefly.

different version of the Lifshitz solutions where instead of the time coordinate one of the spatial coordinates may scale differently compared to others. We call it "spatial Lifshitz" space–time. More specifically, we take in $D = 5$ dimensions the following metric

$$ds^2 = L_{\text{Lif}}^2 \left(\frac{-dt^2 + dr^2 + dx_1^2 + dx_2^2}{r^2} + \frac{dx_3^2}{r^{2z}} \right). \tag{3.5}$$

It is obvious from the metric in eq. (3.5) that the x_3 coordinate scales differently compared to the other coordinates with a Lifshitz exponent parametrized by z.

Following the similar procedure as in the previous two subsections, we can obtain this "spatial Lifshitz" as a solution to the Lovelock action eq. (2.4). In $D = 5$, for the Gauss–Bonnet case eq. (2.5), we find the solution to be identical to both the Schrödinger case eq. (3.3), and the Lifshitz case eq. (A.9). Similarly, for $D = 7$ dimensions in the cubic Lovelock theory we find this "spatial Lifshitz" as a solution, again, identical to both the Schrödinger case eq. (3.4), and the Lifshitz case eq. (A.10). In all these cases, this solution exists at the same point in the coupling space at which the Schrödinger and Lifshitz solutions exist. Thus we see that the special point continues to be relevant as long as one direction, whether spatial or temporal, has anisotropic scaling property. In case of multiple anisotropic directions also one can show that such solutions exist at special points in the coupling space but generically this point is different from the one under consideration.

It is also worth mentioning that the situation here is analogous to what happens in the case of Schrödinger solutions discussed earlier, namely the dynamical exponent z remains unconstrained for this solution as well. This, in particular, allows us to consider a special case when the dynamical exponent for the "spatial Lifshitz" solution vanishes, *i.e.*, $z = 0$. A vanishing z in eq. (3.5) corresponds to the metric in the x_3 direction being invariant under scaling of the radial coordinate. Since the metric in the directions transverse to x_3 is simple AdS metric in the Poincaré coordinates, we obtain a $AdS_4 \times R$ solution of the form

$$ds^2 = L^2 \left(\frac{-dt^2 + dr^2 + dx_1^2 + dx_2^2}{r^2} + dx_3^2 \right). \tag{3.6}$$

This kind of anisotropic solution was studied earlier in [20], where additional matter in the form of a linearly varying dilaton was coupled to two derivative Einstein gravity with negative cosmological constant to construct such anisotropic solutions.

Finally we would like to mention that Lovelock equations of motion do not support black brane solutions with either Lifshitz or Schrödinger asymptotic.

We illustrate this in five dimensions by considering a finite temperature metric ansatz for the Lifshitz solutions of the following form

$$ds^2 = L_{\text{lif}}^2 \left[-\frac{f(r)dt^2}{r^{2z}} + \frac{dr^2}{r^2 f(r)} + \frac{1}{r^2} \sum_{i=1}^{3} dx_i^2 \right] \tag{3.7}$$

with $f(r) = 1 + c_1 r^{c_2}$, where c_1 and c_2 are constants. The equations of motion are solved by this ansatz if $z = 1$ and $c_2 = -2$ with arbitrary c_1. That is, finite temperature black brane solutions with only AdS asymptotic are allowed. A similar analysis can be done for Schrödinger solutions and again we find that there are no finite temperature Schrödinger black brane solutions to eq. (3.3).

4. The phase-space of solutions

In this section we analyze the parameter space of the higher derivative theory and understand in some more detail the phase space of the solutions we obtained in the previous section. We are working with the action eq. (2.4) and the number of parameters of the theory depend on D. For $D \leq 4$, the cosmological constant Λ is the only parameter. For $D = 5, 6$ we have Λ and the Gauss–Bonnet coupling constant a_2 and from $D = 7$ onwards we also have a_3 the cubic term coupling. While AdS solution is parametrized only by its radius L_{AdS}, the Schrödinger and Lifshitz solution has two parameters, radius L_{Sch}, respectively L_{Lif} and the dynamical exponent z. Note that the Schrödinger solutions obtained in the last section and the Lifshitz solutions exist on the locus which does not pass through the origin of the Lovelock moduli space. Thus these solutions would cease to exist if we turn off higher derivative couplings and they cannot be obtained perturbatively in Lovelock couplings.

4.1. The phase-space of solutions in $D = 5$

The AdS solution in $D = 5$ is written in eq. (A.5). One can re-express it as L_{AdS} being determined in terms of Λ.

$$\frac{1}{L_{\text{AdS}}^2} = \frac{6 \pm \sqrt{36 + 48 a_2 \Lambda}}{24 a_2} \tag{4.1}$$

which indicates that there are two branches for the AdS solution with different AdS-radius. This AdS solution exists when the term within the square root is non-negative

$$\Lambda \geq -\frac{3}{4a_2}. \tag{4.2}$$

There is one more constraint on the parameters for the existence of AdS solution coming from the demand that $L_{\text{AdS}}^2 > 0$, which is $a_2 > 0$.

The two branches of the AdS solutions meet at a point in the 2-dimensional phase-space spanned by the parameters $\{\Lambda, \ a_2\}$, given by $\Lambda = -3/(4a_2)$. The Schrödinger solution and the Lifshitz solution in $D = 5$ eq. (3.3) and eq. (A.9) exist at this point in the phase-space with arbitrary value of the dynamical exponent z. We will discuss the relation between unconstrained z and the degeneracy of the configuration space later in this section.

4.2. The phase-space of solutions in $D = 7$

The AdS solution in $D = 7$ dimensions is given in eq. (A.6). The AdS radius squared L_{AdS}^2 can be expressed in terms of the parameters Λ, a_2 and a_3. From eq. (A.6) it is easy to see that one has to solve a cubic equation for L_{AdS}^2 and the solutions are

$$\begin{aligned} L_{\text{AdS}}^2 &= \frac{540 a_2 \Lambda + s\Lambda(s\Lambda - 15) + 225}{3 s \Lambda^2} \\ L_{\text{AdS}}^2 &= -\frac{540 a_2 \Lambda + (s\Lambda + 15)^2}{6\Lambda^2 s} \pm \frac{i\left(540 a_2 \Lambda - s^2\Lambda^2 + 225\right)}{2\sqrt{3}\Lambda^2 s} \end{aligned} \tag{4.3}$$

where

$$s^3 = \frac{-135\left(6\Lambda\left(15a_2 + 6a_3\Lambda - \sqrt{180a_2a_3\Lambda + 36a_3^2\Lambda^2 + 50a_3 - 240a_2^3\Lambda - 75a_2^2}\right) + 25\right)}{\Lambda^3}. \tag{4.4}$$

One can see that the last two roots are complex conjugate of each other. Now demanding that the term within the square root in eq. (4.4) vanishes, *i.e.*,

$$180a_2a_3\Lambda + 36a_3^2\Lambda^2 + 50a_3 - 240a_2^3\Lambda - 75a_2^2 = 0, \tag{4.5}$$

and also using $\Lambda = \frac{1}{L_{\text{AdS}}^2}\left(-15 + 180\frac{a_2}{L_{\text{AdS}}^2} - 360\frac{a_3}{L_{\text{AdS}}^4}\right)$, one obtains a relation between a_2 and a_3

$$a_2 = \frac{L_{\text{AdS}}^2}{24} + 3\frac{a_3}{L_{\text{AdS}}^2} \implies \Lambda = -\frac{15}{2L_{\text{AdS}}^2}\left(1 - \frac{24a_3}{L_{\text{AdS}}^4}\right). \tag{4.6}$$

These relations are same as those encountered in our study of Schrödinger and Lifshitz solutions in $D = 7$. For these values, the imaginary parts of the second and third roots in eq. (4.3) vanish and they become equal, whereas the first root remains different,

$$L_{\text{AdS}}^2 = \frac{-2\sqrt{60a_2\Lambda + 25} - 5}{\Lambda}, \; L_{\text{AdS}}^2 = \frac{\sqrt{60a_2\Lambda + 25} - 5}{\Lambda}, \; L_{\text{AdS}}^2 = \frac{\sqrt{60a_2\Lambda + 25} - 5}{\Lambda}. \tag{4.7}$$

Interestingly, there is yet another choice which simplifies solutions to the cubic equation. Consider the choice $a_2 = -5/(12\Lambda)$, and set the discriminant equal to zero then the quantity s in (4.4) vanishes and all three roots of the cubic become equal to $L_{\text{AdS}}^2 = -5/\Lambda$ This point is on the co-dimension one locus and admits solutions of Schrödinger and Lifshitz kind.

4.3. *Degeneracy of the configuration space*

We will now take up the issue of unconstrained dynamical exponent in the Lifshitz and the Schrödinger metrics in eq. (A.7) and eq. (3.1) respectively. It is known in the literature, see [13,14], that in pure Lovelock theories, unconstrained dynamical exponent z of the Lifshitz solutions follows from the existence of degeneracy of the configuration space. The degeneracy of the configuration space corresponds to complete arbitrariness in specifying the metric component $g_{tt}(r)$ on this locus. Since this metric component is completely unconstrained, it naturally follows that the dynamical exponent is not constrained. While this result is known for the Lifshitz metrics, we believe our results for the Schrödinger metrics are new. More specifically, in $D = 5$ space–time dimensions, the statement of degeneracy in configuration space amounts to the fact that a metric ansatz of the following form for the Schrödinger geometry

$$ds^2 = L_{\text{sch}}^2\left[-f(r)\frac{dt^2}{r^{2z}} + \frac{dr^2}{r^2} + \frac{2}{r^2}dt\,d\xi + \frac{1}{r^2}\sum_{i=1}^{2}dx_i^2\right], \tag{4.8}$$

happens to be a solution for the action in eq. (2.4) with any arbitrary choice of the function $f(r)$, at the same locus in the parameter space, eq. (3.3).

The degeneracy of the configuration space was studied mostly in the context of the Chern–Simons representation of the Lovelock theory. Since the Chern–Simons action does not have any free parameters, this representation exists only at a point in the Lovelock moduli space. However, in dimensions $D > 6$ the Lifshitz and Schrödinger solutions exist on a subspace which extends way beyond the Chern–Simons point. In order to understand the relation between the degeneracy of the configuration space and the special locus better, we deform the five dimensional Lovelock theory by adding R^2 and $R_{\mu\nu}R^{\mu\nu}$ terms to it. Since neither Lifshitz nor Schrödinger solution exists in the Lovelock moduli space away from this locus, only way to study dependence of the degeneracy on the couplings is to expand the coupling constant space by adding new terms.

Let us deform the Lovelock action in $D = 5$ (2.4) by adding R^2 and $R_{\mu\nu}R^{\mu\nu}$ terms. This deformation of the Lovelock theory is given in terms of two parameters b_1 and b_2,

$$I = \int d^5x \sqrt{-g} \left[R - 2\Lambda + a_2(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}) + b_1 R^2 + b_2 R_{\mu\nu}R^{\mu\nu}\right]. \tag{4.9}$$

We are now considering the most general action of gravity up to quadratic order in curvatures in $D = 5$. This action now contains 4 parameters, Λ, a_2, b_1 and b_2.

The Lifshitz solution, eq. (A.7), as is known in the literature [14], occurs at

$$\begin{aligned} \lambda &= -\frac{3}{L_{\text{Lif}}^2} - b_1 \frac{2z(z+3)(z(z+3)+6)}{L_{\text{Lif}}^4} - b_2 \frac{z(z+3)\left(z^2+3\right)}{L_{\text{Lif}}^4}, \\ a_2 &= \frac{L_{\text{Lif}}^2}{4} - b_1(z(z+3)+6) - \frac{b_2}{2}\left(z^2+3\right). \end{aligned} \tag{4.10}$$

When $b_1 = b_2 = 0$ we get back eq. (A.9), the Lifshitz solution exists for non-zero b_1, b_2 but with fixed dynamical exponent z, determined by the parameters of the theory. This agrees with [14] that the degeneracy in the configuration space for the Lifshitz solution occurs only at

$$\lambda = -\frac{3}{L_{\text{Lif}}^2}, \quad a_2 = \frac{L_{\text{Lif}}^2}{4}, \quad b_1 = b_2 = 0. \tag{4.11}$$

Interestingly, the Schrödinger solution with the metric ansatz, eq. (3.1) in $D = 5$, for a general theory of higher derivative gravity beyond Gauss–Bonnet theory with action eq. (4.9), is obtained as

$$\begin{aligned} \Lambda &= -\frac{3}{L_{\text{sch}}^2} - b_1 \frac{80}{L_{\text{sch}}^4} + b_2 \frac{4\left(3z^2-7\right)}{L_{\text{sch}}^4}, \\ a_2 &= \frac{L_{\text{sch}}^2}{4} - 10b_1 + b_2\left(z^2-3\right). \end{aligned} \tag{4.12}$$

We recover eq. (3.3), as expected, when we put $b_1 = b_2 = 0$. But, it is interesting to notice that in eq. (4.12), we get a solution with unconstrained z when $b_2 = 0$ but $b_1 \neq 0$. Which in turn means, if we go beyond the Gauss–Bonnet theory and deform it with only R^2 term, but with no $R_{\mu\nu}R^{\mu\nu}$ term, we still obtain a Schrödinger solution with arbitrary dynamical exponent. It is then natural to ask, if the degeneracy of the configuration space is still present at this locus in parameter space beyond Gauss–Bonnet point, and it indeed turns out to be true. More specifically, a Schrödinger metric with the ansatz

$$ds^2 = L_{\rm sch}^2\Big[-f(r)\frac{dt^2}{r^{2z}} + \frac{dr^2}{r^2} + \frac{2}{r^2}dt d\xi + \frac{1}{r^2}\sum_{i=1}^{2} dx_i^2\Big], \tag{4.13}$$

is a solution to the equations of motion obtained from the action in eq. (4.9) with $b_2 = 0$, for

$$\Lambda = -\frac{3}{L_{\rm sch}^2} - b_1\frac{80}{L_{\rm sch}^4}, \quad a_2 = \frac{L_{\rm sch}^2}{4} - 10b_1 \tag{4.14}$$

with arbitrary $f(r)$. We thus conclude that the degeneracy in configuration space and the solutions with arbitrary dynamical exponent belong to the same locus on the parameter space. The special locus in Gauss–Bonnet theory on which both Lifshitz and Schrödinger solutions co-exist also a Chern–Simons description but the degeneracy of the configuration space of Schrödinger solution is neither confined to Chern–Simons description nor to the Lovelock subspace. Although, we have carried out the study of degeneracy of the configuration space in $D = 5$ for Gauss–Bonnet theory and its deformation to more general quadratic curvature theories, similar analysis can be done in $D > 5$ dimensions.

5. Lovelock gravity as *AdS* Chern–Simons gravity and Born–Infeld gravity

The Lovelock theory has the property that the action has general covariance and the field equations contain at most two derivatives of the metric. We parametrize the Lovelock theory using a set of real coefficients a_p, $p = 0, 1, \cdots, [D/2]$ which are coupling constants of the higher derivative terms. It is convenient to adopt the first order approach, with the dynamical variables being the vielbein, $e^a = e^a_\mu dx^\mu$, and the spin connection, $\omega^{ab} = \omega^{ab}{}_\mu \, dx^\mu$, obeying first order equations of motion. It is straightforward to solve the vanishing of the torsion for the connection and eliminate them by writing them in terms of the vielbeins to obtain the standard second order form in terms of metric.

The action is constructed as a polynomial of degree $[D/2]$ in $R^{ab} = (1/2)R^{ab}{}_{\mu\nu}\, dx^\mu \wedge dx^\nu$ and

$$I = \frac{1}{16\pi G}\int_M d^D x \sum_{p=0}^{[D/2]} a_p \mathcal{L}_p, \quad \text{where } \mathcal{L}_p = \epsilon_{a_1\cdots a_D} R^{a_1 a_2}\cdots R^{a_{2p-1}a_{2p}}\, e^{a_{2p+1}}\cdots e^{a_D}. \tag{5.1}$$

Imposing the condition that the theory possesses maximum possible degrees of freedom determines all Lovelock couplings in terms Λ and G_N. The action in odd dimensions can then be written as a Chern–Simons action with *AdS*, *dS* or Poincaré symmetry [21–23], and in even dimensions as a Born–Infeld like action [24].[4]

5.1. Odd dimensions: Lovelock gravity as Chern–Simons gravity

5.1.1. The Chern–Simons theory

It is well known that gravity in $(2 + 1)$ dimensions can equivalently be written as a Chern–Simons theory for the gauge groups $ISO(2, 1)$ or $SO(2, 2)$, but with no propagating bulk degrees

[4] Though having explicit torsion in the Lagrangian for $D = 4k - 1$ is possible with the same requirements, we will not consider them here.

of freedom. In higher dimensions, $D = 2n - 1$, $n \geq 2$, the essential idea for constructing a Chern–Simons theory is to utilize the fact that there exists a $2n$-form in $D = 2n$,

$$Q_{2n}(\mathbf{A}) = Tr[\mathbf{F}^n] = Tr[\underbrace{\mathbf{F} \wedge \mathbf{F} \wedge \cdots \wedge \mathbf{F}}_{n-times}]\,. \tag{5.2}$$

This form is closed, *i.e.* $dQ_{2n} = 0$, where $\mathbf{A}$ is the Lie Algebra valued connection 1-form $\mathbf{A} = A^a_\mu \mathbf{T}_a dx^\mu$ and $\mathbf{F} = d\mathbf{A} + \mathbf{A} \wedge \mathbf{A}$ is the corresponding field strength or curvature 2-form, with $\mathbf{T}_a$ being the generators of the Lie algebra g of the gauge group $\mathbf{G}$ [25]. The fact that Q_{2n} is closed leads to the existence of a $(2n-1)$-form L^{2n-1}_{CS} such that

$$dL^{2n-1}_{CS} = Q_{2n} = Tr[\mathbf{F}^n] \tag{5.3}$$

which can always be solved as

$$L^{2n-1}_{CS}(\mathbf{A}) = \frac{1}{(n+1)!}\int_0^1 dt\; Tr\big[\mathbf{A}(td\mathbf{A} + t^2\mathbf{A}^2)^{n-1}\big] + \alpha \tag{5.4}$$

with α being some arbitrary closed $(2n-1)$-form. This way one constructs a Chern–Simons Lagrangian $L^{2n-1}_{CS}(\mathbf{A})$ in $D = 2n - 1$ dimensions with an action

$$I_{CS}(\mathbf{A}) = \int_{M_{2n-1}} L^{2n-1}_{CS}(\mathbf{A}). \tag{5.5}$$

5.1.2. Connection with the Lovelock gravity

In odd dimensions, *i.e.*, $D = 2n - 1$, it was argued that the requirement of having maximum possible degrees of freedom fixes the Lovelock coefficients as [25]

$$a_p = \frac{\kappa L^{2p-D}}{D - 2p}\binom{n-1}{p}, \quad 0 \leq p \leq n-1 \tag{5.6}$$

leaving the action depending on only two parameters, gravitational constant κ and the cosmological constant Λ.[5] The precise connection of Lovelock theories with Chern–Simons gravity theories in odd dimensions $(D = 2n - 1)$ is that the Lagrangian for the Lovelock theory can be cast as a Chern–Simons theory for the group *AdS*. This can be demonstrated through the packaging of the Lovelock vielbeins e^a and connections ω^{ab} in the following connection 1-form as

$$W^{AB} = \begin{bmatrix} \omega^{ab} & \frac{e^a}{L} \\ -\frac{e^a}{L} & 0 \end{bmatrix} \tag{5.7}$$

where the indices $a, b = 1, \cdots, D$ and $A, B = 1, \cdots, D + 1$. Note that the A, B-indices are raised or lowered with respect to the *AdS* metric

$$\Pi_{AB} = \begin{bmatrix} \eta_{ab} & 0 \\ 0 & -1 \end{bmatrix}. \tag{5.8}$$

[5] Note that L is the length parameter related to cosmological constant as $\Lambda = \pm\frac{(D-1)(D-2)}{2L^2}$, where as the Newton's constant G_N is related to κ through $\kappa^{-1} = 2(D-2)!\Omega_{D-2}G_N$.

This connection defines a curvature 2-form, also called the *AdS* curvature, as

$$F^{AB} = dW^{AB} + W^{A}{}_{C} \wedge W_{C}{}^{B} = \begin{bmatrix} R^{ab} + \frac{e^a \wedge e^b}{L^2} & \frac{T^a}{L} \\ -\frac{T^a}{L} & 0 \end{bmatrix} \tag{5.9}$$

where $R^{ab} = d\omega^{ab} + \omega^{a}{}_{c} \wedge \omega_{c}{}^{b}$ is the curvature 2-form for the 1-form ω^{ab}, which is $(2n-1)$-dimensional and not to be confused with the $2n$-dimensional *AdS* curvature 2-form F^{AB}. T^a is the torsion form and setting it to zero corresponds to imposing torsion-free constraint.

Next, using the invariant tensor for the *AdS* group $\epsilon_{A_1\cdots A_{2n}}$ along with the $2n$-dimensional AdS curvature F^{AB} one constructs the Euler form in $2n$-dimension

$$\mathcal{E}_{2n} = \epsilon_{A_1\cdots A_{2n}} F^{A_1A_2} \cdots F^{A_{2n-1}A_{2n}}. \tag{5.10}$$

Using the Bianchi identity for the *AdS* curvature F^{AB} we can show that this Euler density is closed, $d\mathcal{E}_{2n} = 0$. Using explicit form of F^{AB} given in eq. (5.9) one can then write the $(2n-1)$-form L_{CS}^{2n-1} in terms of the $(2n-1)$-dimensional curvature 2-form R^{ab} and the vielbeins e^a,

$$L_{CS}^{2n-1} = \sum_{p=0}^{[D/2]} a_p\, \epsilon_{a_1\cdots a_D} R^{a_1a_2} \cdots R^{a_{2p-1}a_{2p}}\, e^{a_{2p+1}} \cdots e^{a_D}\ , \tag{5.11}$$

and $dL_{CS}^{2n-1} = \mathcal{E}_{2n}$. The coefficients a_p are completely fixed here due to the relation between the Chern–Simons density and the Euler density and they turn out to be exactly same as those in eq. (5.6). The field equations obtained from the action in eq. (5.5) are

$$\begin{aligned} \epsilon_{a_1a_2a_3\cdots\alpha_{2n-1}} F^{a_2a_3} \cdots F^{a_{2n-2}a_{2n-1}} &= 0, \\ \epsilon_{a_1a_2a_3\cdots\alpha_{2n-1}} F^{a_3a_4} \cdots F^{a_{2n-3}a_{2n-2}} T^{a_{2n-1}} &= 0. \end{aligned} \tag{5.12}$$

5.2. *Even dimensions: Lovelock gravity as Born–Infeld gravity*

As we have seen in odd dimensions there are gravity actions which are invariant not just under Lorentz group but also under some its extensions, e.g. *AdS* group $SO(D-1,2)$. On the contrary, this is not possible in even dimensions, $D = 2n$. However the requirement of having maximum possible number of degrees of freedom fixes the Lovelock coefficients as [25]

$$a_p = \kappa \binom{n}{p}, \quad 0 \le p \le n. \tag{5.13}$$

The Lovelock action depends on two constants only, the gravitational constant and the cosmological constant, and the Lagrangian, given in eq. (5.1), becomes

$$\mathcal{L} = \frac{\kappa}{2n} \epsilon_{a_1\cdots a_D} F^{a_1a_2} \cdots F^{a_{D-1}a_D} \tag{5.14}$$

which is pfaffian of the two form $F^{ab} = R^{ab} + \frac{e^a e^b}{L^2}$ and can be cast in Born–Infeld form [25]

$$\mathcal{L} = 2^{n-1}(n-1)!\kappa \sqrt{det\left(R^{ab} + \frac{e^a e^b}{L^2}\right)}. \tag{5.15}$$

It is important to note that the two forms F^{ab} are no longer a part of any AdS curvature. The field equations in even dimensions take the form

$$\begin{aligned} \epsilon_{ab_1\cdots b_{D-1}} F^{b_1 b_2} \cdots F^{b_{D-3}b_{D-2}}\, e^{b_{D-1}} &= 0 \\ \epsilon_{aba_3\cdots a_D} F^{a_3 a_4} \cdots F^{a_{D-3}a_{D-2}} T^{a_{D-1}} e^{a_D} &= 0. \end{aligned} \tag{5.16}$$

5.3. Schrödinger space–time as a solution to Chern–Simons gravity in $D = 5$ dimensions

We will now explicitly show that the Schrödinger solution obtained earlier from the Lovelock action can also be seen as a solution to the Chern–Simons gravity in odd dimensions. We will work in $D = 5$ dimensions. The metric in 5-dimension looks like

$$ds^2 = L_{\rm sch}^2 \left[-\frac{dt^2}{r^{2z}} + \frac{dr^2}{r^2} + \frac{2}{r^2} dt d\xi + \frac{1}{r^2}(dx^2 + dy^2) \right]. \tag{5.17}$$

We make the following choice for vielbeins corresponding to the metric in eq. (5.17),

$$e^1_t = -\frac{L_{\rm sch}}{r^z},\ e^1_\xi = L_{\rm sch} r^{z-2},\ e^3_\xi = L_{\rm sch} r^{z-2},\ e^2_r = e^4_x = e^5_y = \frac{L_{\rm sch}}{r}. \tag{5.18}$$

The spin connections ω^{ab} and the AdS curvature F^{AB} for the Schrödinger metric are listed in Appendix A.3. It is easy to see that the AdS curvature F^{AB} does indeed satisfy the field equations (5.12).

5.4. Relation with causality and stability constraints

Stability analysis of Lovelock theories in higher dimensions has been carried out in the past [17–19]. These studies derive constraints on the values of Gauss–Bonnet coupling (a_2) and the cubic Lovelock coupling (a_3) by demanding causality and stability condition on the solutions of the Lovelock theory in $D = 7$. These two conditions are satisfied in a region in the neighborhood of the origin of the (a_2, a_3) plane and at an isolated point, which in our choice of normalization corresponds to ($a_2 = L^2/36$, $a_3 = L^4/648$). The Lovelock parameters (a_2, a_3) used in this paper are related to the parameters (β_2, β_3) or (λ_1, λ_2) used in [17] in the following way $a_2 = \beta_2 L^2 = (\lambda_1/12)L^2$ and $a_3 = \beta_3 L^4 = (\lambda_2/72)L^4$. In terms of these parameters the isolated point, mentioned above, corresponds to ($\lambda_1 = 1/3$, $\lambda_2 = 1/9$).[6] It is interesting to note that this isolated apex point in the phase diagram, see Fig. 1 in [19], is also the same point where we have the Chern–Simons representation for the Lovelock theory. The Schrödinger and Lifshitz solutions exist only at this point in the Lovelock coupling space, which presumably also implies that they also satisfy the causality and stability constraints. It would be interesting to check this explicitly for these solutions.

6. Discussion

We studied the coupling constant parameter space of Lovelock gravity theories in arbitrary dimensions, while restricting our analysis to the Lovelock terms up to cubic in curvatures. We

[6] Note that in [17] the cosmological constant is taken to be $\Lambda = -15/L^2$ and relation between β_i and λ_i is given for $L = 1$. This reduces the coupling parameter space from three to two dimensions, therefore the Schrödinger or Lifshitz solutions exist only at point in the reduced parameter space (β_1, β_2) or (λ_1, λ_2).

demonstrated that Schrödinger solutions exist on co-dimension 1 subspace in the parameter space. Similar results for Lifshitz solutions already exist in the literature. Interestingly, both the solutions exist on the same locus. We found that on this locus, both Schrödinger and Lifshitz exponents were completely unconstrained. Even if we couple the Maxwell or Yang–Mills fields to these Lovelock theories, the Schrödinger and the Lifshitz moduli space would continue to be the same co-dimension 1 subspace.

As already mentioned earlier in the introduction, Schrödinger holography relates a theory of gravity to field theories living on a co-dimension 2 subspace. Therefore, $D = 5$ and $D = 6$ dimensional Schrödinger geometries are directly relevant for studying field theories with this symmetry in $2 + 1$ and $3 + 1$ dimensions. However, higher dimensional Schrödinger geometries in $D \geq 7$ dimensions are also relevant for studying field theory systems in lower dimensions, since the higher dimensional Schrödinger space–times can be first dimensionally reduced to lower dimensions and then we can analyze those dimensionally reduced theories. As was pointed out in eq. (1.3), the dimensional reductions do generally lead to effective theories in the lower dimensions similar to Galileon type theories. In fact, in the case when $n = 1$ in eq. (1.3), that is, starting from $D = d + 2$ dimensions we come down to $D = d + 1$ dimensions, the effective action takes even simpler form

$$\bar{S}_{(d+1)} = \int d^{d+1}x\sqrt{-\bar{g}}\, e^{\frac{\phi}{2}} \left\{ \bar{R} - 2\Lambda + a_2\bar{\mathcal{G}} \right\} \tag{6.1}$$

and for the case when $n = 2$, with simple toroidal reduction it becomes

$$\bar{S}_{(d+1)} = \int d^{d+1}x\sqrt{-\bar{g}}\, e^{\frac{\phi}{2}} \left\{ \bar{R} - 2\Lambda + a_2\bar{\mathcal{G}} + \frac{1}{2}\bar{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2a_2\bar{G}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \right\}. \tag{6.2}$$

Since these effective actions are obtained by dimensionally reducing the higher dimensional theories, any solution of the higher dimensional theories continues to solve the equations of motion obtained from the reduced action.

We have seen in section 3.1 that for a specific value for the Gauss–Bonnet coupling constant a_2, given in eq. (3.3), we obtain a Schrödinger solution in $D = 5$ dimensions with an unconstrained exponent z. Similarly, starting in $D = 7$ dimensions and for the sake of simplicity allowing only the Gauss–Bonnet term, that is, assuming $a_3 = 0$, we can perform a toroidal compactification over a 2 dimensional internal manifold and the resulting effective theory in $D = 5$ dimension comes with action given in eq. (6.2). The Schrödinger solution in $D = 7$ dimension, given in eq. (3.4) with $a_3 = 0$, also becomes a solution to the dimensionally reduced effective theory eq. (6.2), with unconstrained dynamical exponent z for these particular values of the parameters Λ and a_2. Therefore, from the view point of effective lower dimensional theories with the action motivated by the dimensional reductions from higher dimensional theories, we can take a phenomenological bottom-up approach to study various aspects of field theoretical systems with Schrödinger symmetries. In this regard, our study in this paper provides a template for studying non-relativistic field theories with arbitrary dynamical exponent via holography, in theories of gravity coupled to matter systems through non-trivial but specific values of couplings of the Galileon terms, as dictated by eq. (3.3) and eq. (3.4). For $n \leq 2$ there is further simplification because in that case neither the DGP term nor the Galileon type terms appear in the $d + 1$ dimensional theory as is evident from eq. (6.1) and eq. (6.2). Starting from these dimensionally reduced theories, one can then deform them appropriately by adding new terms to the action and engineer non-relativistic solutions with particular fixed values of the dynamical exponent z.

Analysis of these deformations leading to specific values of z relevant for application to, say, the condensed matter systems is beyond the scope of this investigation.

We also pointed that the co-dimension 1 subspace of the Lovelock theories on which Schrödinger and Lifshitz solutions exist also supports Chern–Simons formulation in odd space–time dimensions and Born–Infeld formulation in even space–time dimensions. We cast our non-relativistic metric in the gauge connection form suitable for these formulations. At this point it is interesting to note that [22,24] these gauge connection formulations have natural super-symmetric extension. It would be interesting to explore super symmetric non-relativistic solutions in the Lovelock theories.

It is known that the Chern–Simons point in the coupling space of the Lovelock theories is maximally symmetric. However, most of the studies at this point are concentrated either on the AdS type solutions or on the black brane solutions. Neither of these solutions can shed direct light on the possible values of the dynamical exponent z that appears in the Schrödinger or Lifshitz solutions. Unconstrained dynamical exponent is related to the degeneracy of the configuration space [13,14], which was studied in the context of Chern–Simons formulation. We studied modification of the Gauss–Bonnet theory by doing general deformation using terms quadratic in curvature and found that the degeneracy of the configuration space in case of the Schrödinger solution is not confined to the Chern–Simons point but extends in the direction orthogonal to the Lovelock moduli space corresponding to deformation by the Ricci scalar squared term. Thus the unconstrained dynamical exponent is a result of the degeneracy and it has weak dependence on the special locus in the Lovelock moduli space in the case of the Schrödinger solutions.

Another point worth mentioning is that we do not find hyper-scaling violating solutions anywhere in the Lovelock moduli space, nor do we find black brane solutions with either Schrödinger or Lifshitz type scaling. This in turn means it is harder to turn on temperature in these geometries, however, these shortcomings can be remedied by deforming away from the Lovelock moduli space.

Acknowledgements

We would like to thank Sujay Ashok, Anirban Basu, Jyotirmoy Bhattacharya, Sayantani Bhattacharyya, Anshuman Maharana, K. Narayan, Gautam Mandal, S. Kalyana Rama, Ashoke Sen and Sandip P. Trivedi for discussion and useful suggestions. One of us (DPJ) would like to thank IMSc for hospitality during the course of this work.

Appendix A

A.1. Explicit form of terms in the equation of motion

In this appendix we write down the explicit forms of $G^{(1)}_{\mu\nu}$, $G^{(2)}_{\mu\nu}$ and $G^{(3)}_{\mu\nu}$ appearing in the equation of motion eq. (2.8). These terms come from the Einstein term, curvature squared and cubic terms respectively in the Lovelock action, eq. (2.1). The term $G^{(1)}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is the standard Einstein tensor coming from $\mathcal{L}_0$. The term $G^{(2)}_{\mu\nu}$, coming from $\mathcal{L}_1$ is given by

$$G^{(2)}_{\mu\nu} = 2(R_{\mu\sigma\kappa\tau}R_\nu{}^{\sigma\kappa\tau} - 2R_{\mu\rho\nu\sigma}R^{\rho\sigma} - 2R_{\mu\rho}R^\rho_\nu + RR_{\mu\nu}) - \frac{1}{2}g_{\mu\nu}\mathcal{L}_2, \tag{A.1}$$

and, the third term, $G^{(3)}_{\mu\nu}$, comes from the cubic term $\mathcal{L}_2$ of the Lovelock action,

$$
\begin{aligned}
G^{(3)}_{\mu\nu} = & \, 3R^2 R_{\mu\nu} - 12 R_{\mu\rho} R^{\rho}_{\nu} - 12 R_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} + 24 R_{\mu\alpha} R^{\alpha\beta} R_{\nu\beta} - 24 R^{\alpha}_{\mu} R^{\beta\sigma} R_{\alpha\beta\sigma\nu} \\
& + 3 R_{\mu\nu} R^{\alpha\beta\sigma\kappa} R_{\alpha\beta\sigma\kappa} - 12 R_{\mu\alpha} R_{\nu\beta\sigma\kappa} R^{\alpha\beta\sigma\kappa} - 12 R R_{\mu\sigma\nu\kappa} R^{\sigma\kappa} \\
& + 6 R R_{\mu\alpha\beta\sigma} R_{\nu}{}^{\alpha\beta\sigma} + 24 R_{\mu\alpha\nu\beta} R^{\alpha}_{\sigma} R^{\beta\sigma} + 24 R_{\mu\alpha\beta\sigma} R^{\beta}_{\nu} R^{\alpha\sigma} \\
& + 24 R_{\mu\alpha\nu\beta} R^{\alpha\sigma\beta\kappa} R_{\sigma\kappa} - 12 R_{\mu\alpha\beta\sigma} R^{\kappa\alpha\beta\sigma} R_{\kappa\nu} - 12 R_{\mu\alpha\beta\sigma} R^{\alpha\kappa} R_{\nu\kappa}{}^{\beta\sigma} \\
& + 24 R_{\mu}{}^{\alpha\beta\sigma} R^{\kappa}_{\beta} R_{\sigma\kappa\nu\alpha} - 12 R_{\mu\alpha\nu\beta} R^{\alpha}{}_{\sigma\kappa\rho} R^{\beta\sigma\kappa\rho} \\
& - 6 R_{\mu}{}^{\alpha\beta\sigma} R_{\beta\sigma}{}^{\kappa\rho} R_{\kappa\rho\alpha\nu} - 24 R_{\mu\alpha}{}^{\beta\sigma} R_{\beta\rho\nu\lambda} R_{\sigma}{}^{\lambda\alpha\rho} - \frac{1}{2} g_{\mu\nu} \mathcal{L}_3 \, .
\end{aligned}
\tag{A.2}
$$

A.2. AdS and Lifshitz solutions in Lovelock gravity

Here we will list AdS and Lifshitz solutions to the Lovelock theories in various dimensions. These results are presented here so that we can compare them with the Schrödinger solutions in the main text. The metric ansatz for the AdS solution is

$$
ds^2 = L^2_{\mathrm{AdS}} \left(-u^2 dt^2 + \frac{du^2}{u^2} + u^2 \sum_{i=1}^{D-2} dx_i^2 \right), \tag{A.3}
$$

where L_{AdS} is the AdS radius. The higher derivative terms modify the AdS solution by changing the relation between the AdS radius and the cosmological constant term in the Lagrangian. This modification depends on the dimensions and on the number of Lovelock terms that are turned on. The AdS solution in general D dimensions for Lovelock Lagrangians up to cubic in curvature invariants gives

$$
\Lambda = -\frac{(D-1)(D-2)}{2L^2_{\mathrm{AdS}}} \left[1 - \frac{(D-3)(D-4)a_2}{L^2_{\mathrm{AdS}}} - \frac{(D-3)(D-4)(D-5)(D-6)a_3}{L^4_{\mathrm{AdS}}} \right] . \tag{A.4}
$$

The equations of motion in $D = 5$, namely eq. (2.8) with $a_3 = 0$ gives rise to one condition between the variables Λ, L_{AdS} and a_2,

$$
\Lambda = \frac{1}{L^2_{\mathrm{AdS}}} \left(-6 + 12 \frac{a_2}{L^2_{\mathrm{AdS}}} \right) \implies \frac{1}{L^2_{\mathrm{AdS}}} = \frac{1}{4a_2} \left(1 \pm \sqrt{1 + \frac{4a_2\Lambda}{3}} \right) . \tag{A.5}
$$

In the Gauss–Bonnet theory there are two branches of AdS solutions corresponding to two signs of the square-root in eq. (A.5). These two branches merge when $a_2\Lambda = -3/4$. The Lifshitz and Schrödinger solutions exist precisely at these values of the couplings.

In $D = 7$ the Lagrangian has three parameters, Λ, a_2 and a_3. The equation of motion, eq. (2.8), is solved by the AdS metric ansatz provided the following condition is satisfied,

$$
\Lambda = \frac{1}{L^2_{\mathrm{AdS}}} \left(-15 + 180 \frac{a_2}{L^2_{\mathrm{AdS}}} - 360 \frac{a_3}{L^4_{\mathrm{AdS}}} \right) . \tag{A.6}
$$

This condition can be inverted to write L_{AdS} in term of the couplings in the Lagrangian.

We will now consider Lifshitz solutions to the Lovelock equations of motion. We consider the metric ansatz for the Lifshitz space–time as follows[7]

$$ds^2 = L_{\rm Lif}^2\left(-\frac{dt^2}{r^{2z}} + \frac{dr^2 + \sum_{i=1}^{D-2} dx_i^2}{r^2}\right). \tag{A.7}$$

The parameter z in eq. (A.7), which in principle can take any real value, is called the Lifshitz exponent. Asymptotic symmetries are non-relativistic whenever $z \neq 1$. The parameter $L_{\rm Lif}$ is Lifshitz radius. The Lifshitz solutions are parametrized by the set of parameters $\{z,\ L_{\rm Lif}\}$.

The Lifshitz solution in D dimensional cubic Lovelock theory exists if [14,29] (see also [30])

$$\begin{aligned}\Lambda &= -\frac{(D-1)(D-2)}{4L_{\rm Lif}^2}\left(1 - \frac{(D-3)(D-4)(D-5)(D-6)a_3}{L_{\rm Lif}^4}\right),\\ a_2 &= \frac{L_{\rm Lif}^2}{2(D-3)(D-4)} + \frac{3(D-5)(D-6)}{2L_{\rm Lif}^2}a_3\,.\end{aligned} \tag{A.8}$$

In $D = 5$, the Lifshitz solution exists whenever following relations are valid,

$$\Lambda = -\frac{3}{L_{\rm Lif}^2} \quad \text{and} \quad a_2 = \frac{L_{\rm Lif}^2}{4}\,. \tag{A.9}$$

Notice that these relations do not contain z, *i.e.*, if (A.9) is satisfied then the Lifshitz solution exists with arbitrary dynamical exponent z. If we eliminate $L_{\rm Lif}$ in eq. (A.9) then the Lifshitz solution exists only if $a_2\Lambda = -3/4$.

In $D = 7$ the Lifshitz solution has two conditions which relates three coupling parameters appearing in the Lagrangian:

$$\Lambda = -\frac{15}{2L_{\rm Lif}^2}\left(1 - 24\frac{a_3}{L_{\rm Lif}^4}\right) \quad \text{and} \quad a_2 = \frac{L_{\rm Lif}^2}{24} + 3\frac{a_3}{L_{\rm Lif}^2}\,, \tag{A.10}$$

as in five dimensions, we do not get any condition on z. These values of Λ and a_2 are also related to the locus in the parameter space at which three AdS branches merge.

A.3. Spin connections, and F^{AB} for Schrödinger solutions

In this appendix we will follow the discussion of section 5.1 and 5.2 to compute the spin connection 1-form ω^{ab} by solving the torsion-free condition $T^a = de^a + \omega^a_b \wedge e^b = 0$. Using the connection we will then compute the corresponding curvature 2-form R^{ab} and finally the AdS curvature 2-form F^{AB} defined in eq. (5.9) for the Schrödinger solutions discussed earlier.

For Schrödinger solutions in $D = 5$ we consider the vielbeins as given in eq. (5.18) and compute the spin connections explicitly by solving the torsion free condition,

[7] The general Lifshitz metric has $g_{tt} = -L_{\rm Lif}^2/r^{\frac{2(D-2)(z-1)}{D-2-\theta}+2}$ and $g_{rr} = L_{\rm Lif}^2 r^{\frac{2\theta}{D-2-\theta}-2}$ where θ is the hyper-scaling violating exponent [26–28]. These solutions for non-zero θ do not exist in Lovelock theories.

$$\omega^{12} = -\omega^{21} = \frac{z}{r^z} dt - r^{z-2} d\xi, \quad \omega^{13} = -\omega^{31} = -\frac{z-1}{r} dr,$$
$$\omega^{23} = -\omega^{32} = -\frac{z-1}{r^z} dt + r^{z-2} d\xi, \quad \omega^{24} = -\omega^{42} = \frac{1}{r} dx, \quad \omega^{25} = -\omega^{52} = \frac{1}{r} dy. \tag{A.11}$$

The curvature 2-forms, R^{ab}, can be calculated and the non-vanishing components are

$$\begin{aligned}
&R^{12} = L_{\text{sch}}^2 \left[\frac{z^2 + (z-1)^2}{r^{z+1}} dt \wedge dr + r^{z-3} dr \wedge d\xi \right], \ R^{13} = \frac{L_{\text{sch}}^2}{r^2} dt \wedge d\xi, \\
&R^{23} = -L_{\text{sch}}^2 \left[\frac{2z(z-1)}{r^{z+1}} dt \wedge dr + r^{z-3} dr \wedge d\xi \right], \ R^{24} = -\frac{L_{\text{sch}}^2}{r^2} dr \wedge dx, \\
&R^{14} = L_{\text{sch}}^2 \left[\frac{z}{r^{z+1}} dt \wedge dx - r^{z-3} d\xi \wedge dx \right], \ R^{25} = -\frac{L_{\text{sch}}^2}{r^2} dr \wedge dy, \\
&R^{15} = L_{\text{sch}}^2 \left[\frac{z}{r^{z+1}} dt \wedge dy - r^{z-3} d\xi \wedge dy \right], \ R^{35} = L_{\text{sch}}^2 \left[\frac{z-1}{r^{z+1}} dt \wedge dy - r^{z-3} d\xi \wedge dy \right], \\
&R^{34} = L_{\text{sch}}^2 \left[\frac{z-1}{r^{z+1}} dt \wedge dx - r^{z-3} d\xi \wedge dx \right], \ R^{45} = -\frac{L_{\text{sch}}^2}{r^2} dx_2 \wedge dx_3.
\end{aligned} \tag{A.12}$$

Finally we compute the AdS curvature F^{AB} defined in eq. (5.9), note $A, B = 1, \cdots, 6$ and $a, b = 1, \cdots, 5$

$$\begin{aligned}
&F^{12} = L_{\text{sch}}^2 \frac{2z(z-1)}{r^{1+z}} dt \wedge dr, \ F^{13} = -L_{\text{sch}}^2 \frac{z(z-2)}{r^2} dt \wedge d\xi, \ F^{14} = L_{\text{sch}}^2 \frac{z-1}{r^{1+z}} dt \wedge dx, \\
&F^{15} = L_{\text{sch}}^2 \frac{z-1}{r^{1+z}} dt \wedge dy, \ F^{34} = L_{\text{sch}}^2 \frac{z-1}{r^{1+z}} dt \wedge dx, \ F^{35} = L_{\text{sch}}^2 \frac{z-1}{r^{1+z}} dt \wedge dy, \\
&F^{23} = -L_{\text{sch}}^2 \frac{2z(z-1)}{r^{1+z}} dt \wedge dr.
\end{aligned} \tag{A.13}$$

All the other components, such as, F^{24}, F^{25}, F^{45}, $F^{a6} = T^a$, $F^{6b} = -T^b$, are evaluated to be zero.

References

[1] D.T. Son, Toward an AdS/cold atoms correspondence: a geometric realization of the Schrodinger symmetry, Phys. Rev. D 78 (2008) 046003, arXiv:0804.3972 [hep-th].

[2] Koushik Balasubramanian, John McGreevy, Gravity duals for non-relativistic CFTs, Phys. Rev. Lett. 101 (2008) 061601, arXiv:0804.4053 [hep-th].

[3] Allan Adams, Alexander Maloney, Aninda Sinha, Samuel E. Vazquez, 1/N effects in non-relativistic gauge-gravity duality, J. High Energy Phys. 03 (2009) 097, arXiv:0812.0166 [hep-th].

[4] Xián O. Camanho, Naresh Dadhich, Alfred Molina, Pure Lovelock Kasner metrics, Class. Quantum Gravity 32 (17) (2015) 175016, arXiv:1505.06675 [gr-qc].

[5] Alberto Nicolis, Riccardo Rattazzi, Enrico Trincherini, The Galileon as a local modification of gravity, Phys. Rev. D 79 (2009) 064036, http://dx.doi.org/10.1103/PhysRevD.79.064036, arXiv:0811.2197 [hep-th].

[6] C. Deffayet, Gilles Esposito-Farese, A. Vikman, Covariant Galileon, Phys. Rev. D 79 (2009) 084003, http://dx.doi.org/10.1103/PhysRevD.79.084003, arXiv:0901.1314 [hep-th].

[7] C. Deffayet, S. Deser, G. Esposito-Farese, Generalized Galileons: all scalar models whose curved background extensions maintain second-order field equations and stress-tensors, Phys. Rev. D 80 (2009) 064015, http://dx.doi.org/10.1103/PhysRevD.80.064015, arXiv:0906.1967 [gr-qc].
[8] C. Deffayet, Xian Gao, D.A. Steer, G. Zahariade, From k-essence to generalised Galileons, Phys. Rev. D 84 (2011) 064039, http://dx.doi.org/10.1103/PhysRevD.84.064039, arXiv:1103.3260 [hep-th].
[9] C. Charmousis, B. Gouteraux, E. Kiritsis, Higher-derivative scalar-vector-tensor theories: black holes, Galileons, singularity cloaking and holography, J. High Energy Phys. 09 (2012) 011, http://dx.doi.org/10.1007/JHEP09(2012)011, arXiv:1206.1499 [hep-th].
[10] G.R. Dvali, Gregory Gabadadze, Massimo Porrati, 4-D gravity on a brane in 5-D Minkowski space, Phys. Lett. B 485 (2000) 208–214, http://dx.doi.org/10.1016/S03702693(00)00669-9, arXiv:hep-th/0005016.
[11] Georgios Kofinas, Rodrigo Olea, Universal regularization prescription for Lovelock AdS gravity, J. High Energy Phys. 11 (2007) 069, arXiv:0708.0782 [hep-th].
[12] Dileep P. Jatkar, Georgios Kofinas, Olivera Miskovic, Rodrigo Olea, Conformal mass in Einstein–Gauss–Bonnet AdS gravity, Phys. Rev. D 91 (10) (2015) 105030, arXiv:1501.06861 [hep-th].
[13] Gustavo Dotti, Julio Oliva, Ricardo Troncoso, Exact solutions for the Einstein–Gauss–Bonnet theory in five dimensions: black holes, wormholes and spacetime horns, Phys. Rev. D 76 (2007) 064038, http://dx.doi.org/10.1103/PhysRevD.76.064038, arXiv:0706.1830 [hep-th].
[14] M.H. Dehghani, Robert B. Mann, Lovelock–Lifshitz black holes, J. High Energy Phys. 07 (2010) 019, arXiv:1004.4397 [hep-th].
[15] Shamit Kachru, Xiao Liu, Michael Mulligan, Gravity duals of Lifshitz-like fixed points, Phys. Rev. D 78 (2008) 106005, arXiv:0808.1725 [hep-th].
[16] Marika Taylor, Non-relativistic holography, arXiv:0812.0530 [hep-th], 2008.
[17] Jan de Boer, Manuela Kulaxizi, Andrei Parnachev, Holographic Lovelock gravities and black holes, J. High Energy Phys. 06 (2010) 008, arXiv:0912.1877 [hep-th].
[18] Xian O. Camanho, Jose D. Edelstein, Causality in AdS/CFT and Lovelock theory, J. High Energy Phys. 06 (2010) 099, arXiv:0912.1944 [hep-th].
[19] Xian O. Camanho, Jose D. Edelstein, Miguel F. Paulos, Lovelock theories, holography and the fate of the viscosity bound, J. High Energy Phys. 05 (2011) 127, arXiv:1010.1682 [hep-th].
[20] Sachin Jain, Nilay Kundu, Kallol Sen, Aninda Sinha, Sandip P. Trivedi, A strongly coupled anisotropic fluid from dilaton driven holography, J. High Energy Phys. 01 (2015) 005, arXiv:1406.4874 [hep-th].
[21] Edward Witten, $(2+1)$-dimensional gravity as an exactly soluble system, Nucl. Phys. B 311 (1988) 46.
[22] Jorge Zanelli, Lecture notes on Chern–Simons (super-)gravities, in: Proceedings, 7th Mexican Workshop on Particles and Fields (MWPF 1999), 2005 (second edition, February 2008), arXiv:hep-th/0502193.
[23] Juan Crisostomo, Ricardo Troncoso, Jorge Zanelli, Black hole scan, Phys. Rev. D 62 (2000) 084013, arXiv:hep-th/0003271.
[24] Jorge Zanelli, Chern–Simons forms in gravitation theories, Class. Quantum Gravity 29 (2012) 133001, arXiv:1208.3353 [hep-th].
[25] Ricardo Troncoso, Jorge Zanelli, Higher dimensional gravity, propagating torsion and AdS gauge invariance, Class. Quantum Gravity 17 (2000) 4451–4466, arXiv:hepth/9907109.
[26] Christos Charmousis, Blaise Gouteraux, Bom Soo Kim, Elias Kiritsis, Rene Meyer, Effective holographic theories for low-temperature condensed matter systems, J. High Energy Phys. 11 (2010) 151, arXiv:1005.4690 [hep-th].
[27] Norihiro Iizuka, Nilay Kundu, Prithvi Narayan, Sandip P. Trivedi, Holographic Fermi and non-Fermi liquids with transitions in dilaton gravity, J. High Energy Phys. 01 (2012) 094, arXiv:1105.1162 [hep-th].
[28] Liza Huijse, Subir Sachdev, Brian Swingle, Hidden Fermi surfaces in compressible states of gauge-gravity duality, Phys. Rev. B 85 (2012) 035121, arXiv:1112.0573 [cond-mat.str-el].
[29] Tian-Yi Yu, Rui-Hong Yue, Asymptotic Lifshitz-like solution in Lovelock gravity $(n = 3)$, Commun. Theor. Phys. 56 (2011) 281–288.
[30] Yan Liu, Spatially homogeneous Lifshitz black holes in five dimensional higher derivative gravity, J. High Energy Phys. 06 (2012) 024, arXiv:1202.1748 [hep-th].