Morphisms between complete Riemannian pseudogroups✩

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Abstract

We introduce the concept of morphism of pseudogroups generalizing the étalé morphisms of Haefliger. With our definition, any continuous foliated map induces a morphism between the corresponding holonomy pseudogroups. The main theorem states that any morphism between complete Riemannian pseudogroups is complete, has a closure and its maps are $C^\infty$ along the orbit closures. Here, completeness and closure are versions for morphisms of concepts introduced by Haefliger for pseudogroups. This result is applied to approximate foliated maps by smooth ones in the case of transversely complete Riemannian foliations, yielding the foliated homotopy invariance of their spectral sequence. This generalizes the topological invariance of their basic cohomology, shown by El Kacimi-Alaoui–Nicolau. A different proof of the spectral sequence invariance was also given by the second author.

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Contents

1. Introduction and main results ................................................................. 545
2. Morphisms of pseudogroups ................................................................. 548
3. Extending concepts to pseudogroups ..................................................... 550
4. Holonomy pseudogroups of foliated spaces ........................................... 551
5. Holonomy morphisms of foliated maps ............................................... 552
6. Integrable and foliated homotopies ....................................................... 554
7. Complete pseudogroups and complete morphisms .................................. 554
8. Singular foliated spaces ........................................................................ 555
9. Riemannian pseudogroups .................................................................... 557
10. The pseudogroup generated by the elements close to identity maps ........... 557
11. Locally homogeneous structure of the orbit closures ............................. 560

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1. Introduction and main results

We introduce the concept of morphism of pseudogroups generalizing the étalé morphisms of Haefliger [20]: it is a slight change of Haefliger’s definition so that they may contain, not only homeomorphisms, but also arbitrary continuous maps. With our definition, any continuous map between foliated spaces mapping leaves to leaves (a foliated map) induces a morphism between the corresponding holonomy pseudogroups; indeed, this defines a functor between the categories of foliated maps and morphisms of pseudogroups, called the holonomy functor. A morphism of pseudogroups can be interpreted as a morphism of S-atlas, defined by van Est [48], and is more general than a homomorphism of étalé groupoids: only transverse foliated maps induce homomorphisms of holonomy groupoids. Any pseudogroup morphism induces a continuous map between the corresponding orbit spaces.

Haefliger has also introduced other morphisms between topological groupoids [18]. Any Haefliger morphism between étalé groupoids induces a morphism of our type. Somehow, our morphisms extract the information about local maps from Haefliger morphisms, and forget about their additional structure. Since our main results will be about maps, our morphism seems to be more appropriate for our study.

For each topological space, we can consider the pseudogroup generated by its identity map. Then, each continuous map between topological spaces generates (in an obvious sense) a morphism between the corresponding pseudogroups. This assignment defines a canonical injective functor between the category of continuous maps and the category of morphisms. In this sense, many topological and geometric concepts can be generalized to pseudogroups. For instance, the concepts of homotopy and homotopy equivalence will be used for pseudogroups. The definition of
the fundamental group of a pseudogroup with a distinguished orbit was given by Haefliger [19] and van Est [48]. For a pseudogroup consisting of $C^\infty$ transformations, its de Rham cohomology is isomorphic to its invariant cohomology (the cohomology of the complex of differential forms that are invariant by the local transformations of the pseudogroup).

We will mainly consider morphisms between pseudogroups of local isometries of Riemannian manifolds, called Riemannian pseudogroups. In [20], Haefliger has introduced the condition of completeness for pseudogroups, and the closure of a complete Riemannian pseudogroup. Now, we give versions of these concepts for morphisms.

Recall that the orbit closures of a complete Riemannian pseudogroup are the leaves of a $C^\infty$ singular foliation [42]. Then a morphism between such pseudogroups is said to be of class $C^{0,\infty}$ when it consists of maps that are $C^\infty$ along the orbit closures, and moreover the corresponding leafwise derivatives of arbitrary order are continuous on the ambient manifold; this makes sense even for singular foliations! Our main result is the following.

**Theorem A.** Any morphism between complete Riemannian pseudogroups is complete, has a closure and is of class $C^{0,\infty}$.

The last part of Theorem A is a generalization of the well-known fact that any continuous homomorphism between Lie groups is $C^\infty$. Thus a version of last part of Theorem A for “measurable morphisms” seems to be possible (Problem 31.4).

The proof of Theorem A only involves basic techniques, but it is rather complicated and has a lot of steps. A simplified version of the proof, without details and only for the case of dense orbits, was given in [5]; it may be useful to understand the complete proof of this paper.

A Riemannian foliation is a $C^\infty$ foliation whose holonomy pseudogroup is Riemannian. These foliations can be locally described by Riemannian submersions for some Riemannian metric (a bundle-like metric) [41]. A Riemannian foliation is said to be transversely complete when, for some bundle-like metric, the geodesics orthogonal to the leaves are complete; this condition implies that its holonomy pseudogroup is complete [20, Example 1.2.1].

We will give examples of non-complete morphisms between complete pseudogroups (Examples 30.3 and 30.4), showing that the Riemannian condition cannot be removed in Theorem A. However, a weaker equicontinuity condition could be enough to get completeness and the existence of a closure (see [37, Appendix E] and [3]), although it is not enough to get smoothness along the orbit closures (Example 30.5).

Certain topology, called the strong adapted topology, is introduced on the set of foliated maps between transversely complete Riemannian foliations. Its definition takes into account the special structure of these foliations [37], but it equals the strong compact-open topology when the leaf closures are compact.

A homotopy consisting of foliated maps is called a foliated homotopy, and the corresponding concept of foliated homotopy equivalence can be defined too. The strong adapted topology has the following nice behavior with respect to foliated homotopies.

**Theorem B.** If two foliated maps between transversely complete Riemannian foliations are close enough with respect to the strong adapted topology, then there is a foliated homotopy between them. Moreover the foliated homotopy can be chosen to be proper if the foliated maps are proper.

The following result gives smooth approximations of continuous foliated maps in the case of transversely complete Riemannian foliations. It is easy to find counterexamples showing that it cannot be generalized to arbitrary $C^\infty$ foliations.

**Theorem C.** In the space of continuous foliated maps between two transversely complete Riemannian foliations with the strong adapted topology, the subset of $C^\infty$ foliated maps is dense.

In the proof of Theorem C, the transverse smoothness inside the leaf closures is granted by Theorem A for any foliated map. Then standard arguments are used to improve differentiability, firstly, along the leaves and, secondly, along the geodesics orthogonal to the leaf closures. Indeed, a relative version of Theorem C is proved, which, together with Theorem B, has the following consequences.
Corollary D. If two continuous foliated maps between transversely complete Riemannian foliations are foliatedly homotopic, then there is a $C^\infty$ foliated homotopy between them.

Corollary E. Any continuous foliated map between transversely complete Riemannian foliations is foliatedly homotopic to a $C^\infty$ foliated map.

The cohomological information of a $C^\infty$ manifold $M$ can be enlarged by the presence of a $C^\infty$ foliation $\mathcal{F}$. Precisely, $\mathcal{F}$ gives rise to the spectral sequence $(E_1 = E_1(\mathcal{F}, d_1))$ [31,43,28,2], which is the obvious generalization of the de Rham version of the Leray spectral sequence of a $C^\infty$ fiber bundle: it is induced by a filtration of the de Rham complex of $M$ locally defined like for fiber bundles. Several cohomologies usually associated to $C^\infty$ foliations are contained in this spectral sequence. For instance, the terms $E_1^{0,*}$ and $E_2^{0,*}$ are respectively called leafwise cohomology and basic cohomology.

Another version of the spectral sequence, $(E_{c,i}, d_i)$, can be defined by considering compactly supported differential forms. The term $E_{c,2}^{*,p}$ ($p = \dim \mathcal{F}$) is isomorphic to the transverse cohomology [17], also called Haefliger cohomology.

In general, $(E_i, d_i)$ is not a topological invariant (a counterexample is given in [27]). Nevertheless, by Corollaries D and E, any continuous foliated map between transversely complete Riemannian foliations induces a homomorphism between their spectral sequences, yielding the following spectral sequence invariance:

Corollary F. Any foliated homotopy equivalence between transversely complete Riemannian foliations induces an isomorphism on $E_i$ for $i \geq 2$.

In the case of dense leaves, there is a version of Corollary F for $E_1$.

Corollary F generalizes the topological invariance of the basic cohomology for Riemannian foliations on closed manifolds, obtained by El Kacimi-Alaoui–Nicolau [27].

The second author has given a different proof of Corollary F in [33]. He introduces an Alexander–Spanier version of the spectral sequence, which is a topological invariant by definition, and indeed is shown to be invariant by foliated homotopy equivalences. Then, for transversely complete Riemannian foliations, he shows that both spectral sequences are isomorphic from the second term on.

Versions of the above approximation and invariance results are also proved for proper foliated maps:

Theorem G. In the space of continuous foliated maps between two transversely complete Riemannian foliations with the strong adapted topology, the subset of proper continuous foliated maps is open, and therefore the subset of proper $C^\infty$ foliated maps is dense.

Corollary H. For transversely complete Riemannian foliations, if two proper continuous foliated maps are foliatedly homotopic, then there is a proper $C^\infty$ foliated homotopy between them.

Corollary I. For transversely complete Riemannian foliations, there is a proper foliated homotopy between any proper continuous foliated map and some proper $C^\infty$ foliated map.

Corollary J. Any proper foliated homotopy equivalence between transversely complete Riemannian foliations induces an isomorphism on $E_{c,i}$ for $i \geq 2$.

We also introduce a version of the strong compact-open topology on the set of morphisms between two pseudogroups. Then the above approximation and invariance results have the following versions for morphisms:

Theorem K. In the space of morphisms between two complete Riemannian pseudogroups with the strong compact-open topology, the subset of $C^\infty$ morphisms is dense.

Corollary L. If two morphisms between complete Riemannian pseudogroups are homotopic, then there is a $C^\infty$ homotopy between them.
Corollary M. Any morphism between complete Riemannian pseudogroups is homotopic to a $C^\infty$ morphism.

Corollary N. Any homotopy equivalence between complete Riemannian pseudogroups induces an isomorphism between their invariant cohomologies.

In Sections 2–15, besides introducing some new concepts, we recall the needed preliminaries about pseudogroups and foliated spaces. Special emphasis is put on the case of complete Riemannian pseudogroups and transversely complete Riemannian foliations. Some of the preliminaries are elaborated to fit our needs. Theorem A is proved in Sections 16 and 17. Sections 18–23 are devoted to the approximation results for foliated maps. The preliminaries on the spectral sequence are given in Section 24, and its invariance is shown in Section 25. Sections 26–29 are devoted to the pseudogroup versions of the approximation and invariance results. Finally, examples and open problems are given in the last two sections.

2. Morphisms of pseudogroups

Recall the following definitions from [20, Section 1.4]. A pseudogroup of local transformations of a topological space $T$ (or acting on $T$) is a collection $\mathcal{H}$ of homeomorphisms between open subsets of $T$ which contains the identity map $\text{id}_T$ of $T$, and is closed under the operations of composition (wherever defined), inversion, restriction to open sets and combination (defining homeomorphisms). Here, the composition $h_2 \circ h_1$ of two local transformations $h_1 : U \to V_i$, $i = 1, 2$, refers to the composite of the maps

$$h_1^{-1}(V_1 \cap U_2) \xrightarrow{h_1} V_1 \cap U_2 \xrightarrow{h_2} h_2(V_1 \cap U_2).$$

On the other hand, being closed by combination means that, if a family of maps in $\mathcal{H}$ can be combined to define a homeomorphism, then this combination is in $\mathcal{H}$. Sometimes it will be convenient to consider the elements of $\mathcal{H}$ as open embeddings with target space $T$. The pseudogroup $\mathcal{H}$ is said to be generated by a subset $S \subset \mathcal{H}$ if any map in $\mathcal{H}$ can be obtained from maps in $S$ involving the above operations. The restriction of $\mathcal{H}$ to a subspace $T_0 \subset T$ is the pseudogroup $\mathcal{H}|_{T_0}$ consisting of the homeomorphisms between open subsets of $T_0$ that can be locally extended to maps in $\mathcal{H}$. If $T_0$ is open in $T$, then $\mathcal{H}|_{T_0}$ consists of the maps in $\mathcal{H}$ whose domain and image is contained in $T_0$.

The orbit of a point $x \in T$ is the set $\mathcal{H}(x)$ of the images of $x$ by all maps in $\mathcal{H}$ whose domains contain $x$. The union of orbits that meet some subset $T_0 \subset T$ is called the saturation of $T_0$ and denoted by $\mathcal{H}(T_0)$; the set $T_0$ is said to be invariant or saturated if $\mathcal{T}_0 = \mathcal{H}(T_0)$. The quotient space of orbits is denoted by $\mathcal{H} \setminus T$. The pseudogroup $\mathcal{H}$ is said to be of class $C^\infty$ if $T$ is a $C^\infty$ manifold and $\mathcal{H}$ consists of $C^\infty$ maps.

Suppose that $T = \bigcup_i T_i$. Given a pseudogroup $\mathcal{H}_i$ acting on each $T_i$, there is a pseudogroup $\mathcal{H}$ acting on $T$ whose elements are the homeomorphisms between open subsets of $T$, $h : U \to V$, such that $h(U \cap T_i) = V \cap T_i$ and the restriction $h : U \cap T_i \to V \cap T_i$ is in $\mathcal{H}_i$ for all $i$. If the maps in each $\mathcal{H}_i$ can be locally extended to maps in $\mathcal{H}$, then $\mathcal{H}|_{T_i} = \mathcal{H}_i$ for all $i$, and $\mathcal{H}$ is called the combination of the pseudogroups $\mathcal{H}_i$. When every $T_i$ is open in $T$, then the combination of the pseudogroups $\mathcal{H}_i$ is defined just when $\mathcal{H}_i|_{T_i \cap T_j} = \mathcal{H}_j|_{T_j \cap T_i}$ for all $i$ and $j$.

Let $\mathcal{H}$ and $\mathcal{H}'$ be pseudogroups of local transformations of spaces $T$ and $T'$, respectively. According to [20, Section 1.4], an étalé morphism $\Phi : \mathcal{H} \to \mathcal{H}'$ is a maximal collection of homeomorphisms of open subsets of $T$ to open subsets of $T'$ satisfying the following conditions:

(i) If $\phi \in \Phi$, $h \in \mathcal{H}$ and $h' \in \mathcal{H}'$, then $h' \circ \phi \circ h \in \Phi$.
(ii) The domains of elements of $\Phi$ cover $T$.
(iii) If $\phi, \psi \in \Phi$, then $\psi \circ \phi^{-1} \in \mathcal{H}'$.

If moreover $\Phi^{-1} = \{\phi^{-1} \mid \phi \in \Phi\}$ is an étalé morphism, then $\Phi$ is called an equivalence of pseudogroups, and $\mathcal{H}$ is said to be equivalent to $\mathcal{H}'$. If $\Phi : \mathcal{H} \to \mathcal{H}'$ is an equivalence, then $\mathcal{H}'' = \mathcal{H} \cup \mathcal{H}' \cup \Phi \cup \Phi^{-1}$ is a pseudogroup on the topological sum $T'' = T \cup T'$ whose orbits cut $T$ and $T'$, and so that $\mathcal{H}'|_{T'} = \mathcal{H}$ and $\mathcal{H}''|_{T'} = \mathcal{H}'$. Reciprocally, for any pseudogroup $\mathcal{H}''$ on $T''$ satisfying these conditions, the family

$$\{\phi \in \mathcal{H}'' \mid \text{dom} \phi \subset T, \text{im} \phi \subset T'\}$$

is an equivalence $\mathcal{H} \to \mathcal{H}'$. 

We want to generalize étalé morphisms by involving arbitrary local maps, and thus the above condition (iii) must be weakened accordingly. Precisely, we introduce the following concept.

**Definition 2.1.** A morphism \( \Phi : \mathcal{H} \to \mathcal{H}' \) is a maximal collection of continuous maps of open subsets of \( T \) to \( T' \) satisfying the following properties:

(i) If \( \phi \in \Phi \), \( h \in \mathcal{H} \) and \( h' \in \mathcal{H}' \), then \( h' \circ \phi \circ h \in \Phi \).

(ii) The domains of elements of \( \Phi \) cover \( T \).

(iii) If \( \phi, \psi \in \Phi \) and \( x \in \text{dom} \phi \cap \text{dom} \psi \), then there is some \( h' \in \mathcal{H}' \) with \( \phi(x) \in \text{dom} h' \) and so that \( h' \circ \phi = \psi \) on some neighborhood of \( x \).

**Remarks 1.** In Definition 2.1, observe the following:

(a) Property (i) implies that \( \Phi \) is closed by restrictions to open sets.

(b) Properties (i) and (iii) imply that, if \( \phi, \psi \in \Phi \), \( h \in \mathcal{H} \) and \( x \in \text{dom} \phi \cap h^{-1}(\text{dom} \psi) \), then there is some \( h' \in \mathcal{H}' \) with \( \phi(x) \in \text{dom} h' \) and so that \( h' \circ \phi = \psi \circ h \) on some neighborhood of \( x \).

(c) \( \Phi \) induces a continuous map \( \Phi_{\text{orb}} : \mathcal{H} \setminus T \to \mathcal{H}' \setminus T' \) defined by

\[
\Phi_{\text{orb}}(O) = \bigcup_{\phi \in \Phi} \phi(O \cap \text{dom} \phi).
\]

The set of morphisms \( \mathcal{H} \to \mathcal{H}' \) will be denoted by \( C(\mathcal{H}, \mathcal{H}') \). A morphism \( \Phi : \mathcal{H} \to \mathcal{H}' \) is said to be of class \( C^\infty \) if \( \mathcal{H} \) and \( \mathcal{H}' \) are \( C^\infty \) pseudogroups and \( \Phi \) consists of \( C^\infty \) maps; the set of \( C^\infty \) morphisms \( \mathcal{H} \to \mathcal{H}' \) will be denoted by \( C^\infty(\mathcal{H}, \mathcal{H}') \). According to Remark 1(c), the mapping \( \Phi \mapsto \Phi_{\text{orb}} \) defines a canonical map \( C(\mathcal{H}, \mathcal{H}') \to C(\mathcal{H} \setminus T, \mathcal{H}' \setminus T') \) called the orbit map.

**Lemma 2.2.** Let \( \Phi \) be collection of continuous maps of open subsets of \( T \) to \( T' \) satisfying the properties (i)–(iii) of Definition 2.1. Then \( \Phi \) is a morphism \( \mathcal{H} \to \mathcal{H}' \) if and only if \( \Phi \) is closed under combination of maps.

**Proof.** Suppose that \( \Phi \) is a morphism \( \mathcal{H} \to \mathcal{H}' \). The family \( \Psi \) of all possible combinations of maps in \( \Phi \) contains \( \Phi \) and also satisfies properties (i)–(iii) of Definition 2.1. Hence \( \Phi = \Psi \) by the maximality of \( \Phi \).

Now, assume that \( \Phi \) is closed under combination of maps and let us show the maximality of Definition 2.1. Suppose \( \Phi \) is contained in another collection \( \Psi \) of continuous maps of open subsets of \( T \) to \( T' \) satisfying the properties (i), (ii) and (iii) of Definition 2.1. Take any \( \psi \in \Psi \). By property (ii) for \( \Phi \) and property (iii) for \( \Psi \), for every \( x \in \text{dom} \psi \) there is some \( \phi \in \Phi \) with \( x \in \text{dom} \phi \) and there is some \( h' \in \mathcal{H}' \) with \( \phi(x) \in \text{dom} h' \) and such that \( h' \circ \phi = \psi \) on some neighborhood of \( x \). But \( h' \circ \phi \in \Phi \) because \( \Phi \) satisfies property (i). Therefore every germ of \( \psi \) is the germ of some element of \( \Phi \), yielding that \( \psi \) is a combination of elements of \( \Phi \). Thus \( \psi \in \Phi \) as desired because \( \Phi \) is closed under combination of maps. \( \square \)

**Lemma 2.3.** Let \( \Phi_0 \) be a family of continuous maps of open subsets of \( T \) to \( T' \) satisfying the following properties:

(ii) The \( \mathcal{H} \)-saturation of the domains of maps in \( \Phi_0 \) cover \( T \).

(iii) There is a subset \( S \) of generators of \( \mathcal{H} \) such that, if \( \phi, \psi \in \Phi_0 \), \( h \in S \) and \( x \in \text{dom} \phi \cap h^{-1}(\text{dom} \psi) \), then there is some \( h' \in \mathcal{H}' \) with \( \phi(x) \in \text{dom} h' \) and so that \( h' \circ \phi = \psi \circ h \) on some neighborhood of \( x \).

Then there is a unique morphism \( \Phi : \mathcal{H} \to \mathcal{H}' \) containing \( \Phi_0 \).

**Proof.** Let \( \Phi \) be the family of the following types of maps of open subsets of \( T \) to \( T' \):

- All composites \( h' \circ \phi \circ h \) with \( \phi \in \Phi_0 \), \( h \in \mathcal{H} \) and \( h' \in \mathcal{H}' \), wherever defined.
- All possible combinations of composites of the above type.
This collection \( \Phi \) clearly satisfies properties (i), (ii) and (iii) of Definition 2.1. Moreover \( \Phi \) is closed under combination of maps, and thus it is a morphism by Lemma 2.2.

Finally, the uniqueness of \( \Phi \) follows because, if \( \Psi \) is another morphism \( \mathcal{H} \to \mathcal{H}' \) containing \( \Phi_0 \), then it also contains \( \Phi \) by property (i) for \( \Psi \) and its maximality, and thus \( \Phi = \Psi \) by the maximality of \( \Phi \). \( \square \)

In Lemma 2.3, it will be said that \( \Phi_0 \) generates \( \Phi \). Observe that morphisms consisting of local homeomorphisms are precisely those generated by étalé morphisms (by taking the appropriate target space).

The composition of two consecutive morphisms,

\[
\mathcal{H} \xrightarrow{\Phi} \mathcal{H}' \xrightarrow{\Psi} \mathcal{H}'',
\]

is the morphism \( \Psi \circ \Phi : \mathcal{H} \to \mathcal{H}'' \) generated by all composites of maps in \( \Phi \) with maps in \( \Psi \) (wherever defined). With this operation, the morphisms of pseudogroups form a category \( \text{PsGr} \), whose isomorphisms are the morphisms generated by equivalences of pseudogroups. For a pseudogroup \( \mathcal{H} \) acting on a space \( T \), the identity morphism \( \text{id}_\mathcal{H} \) of \( \text{PsGr} \) at \( \mathcal{H} \) is the morphism generated by \( \text{id} \) of \( \mathcal{H} \).

The restriction of a morphism \( \Phi : \mathcal{H} \to \mathcal{H}' \) to a subspace \( T_0 \subset T \) is the morphism \( \Phi|_{T_0} : \mathcal{H}|_{T_0} \to \mathcal{H}' \) consisting of all maps of open subsets of \( T_0 \) to \( T' \) that can be locally extended to maps in \( \Phi \). If \( T_0 \) is open in \( T \), then \( \Phi|_{T_0} \) consists of all maps in \( \Phi \) whose domain is contained in \( T_0 \). The inclusion map \( T_0 \hookrightarrow T \) generates a morphism \( \mathcal{H}|_{T_0} \to \mathcal{H} \), whose composition with \( \Phi \) is \( \Phi|_{T_0} \).

Suppose that \( T = \bigcup_i T_i \). Given a morphism \( \Phi_i : \mathcal{H}|_{T_i} \to \mathcal{H}' \) for each \( i \), let \( \Phi \) be the family of continuous maps \( \phi : U \to T' \), where \( U \) is an open subset of \( T \), such that \( \phi|_{U \cap T_i} \in \Phi_i \) for all \( i \). If \( \Phi \) is a morphism and the maps in each \( \Phi_i \) can be locally extended to maps in \( \Phi \), then \( \Phi|_{T_i} = \Phi_i \) for all \( i \), and \( \Phi \) is called the combination of the morphisms \( \Phi_i \). When every \( T_i \) is open in \( T \), then the combination of the morphisms \( \Phi_i \) is defined just when \( \Phi|_{T_i \cap T_j} = \Phi_i|_{T_i \cap T_j} \) for all \( i \) and \( j \).

The image of a morphism \( \Phi : \mathcal{H} \to \mathcal{H}' \) is the \( \mathcal{H}' \)-saturated set \( \text{im} \Phi = \bigcup_{\phi \in \Phi} \text{im} \phi \). If \( \mathcal{H} \) acts on a space \( T \) and \( T_0 \subset T \), define the direct image

\[
\Phi(T_0) = \text{im} \Phi|_{T_0} = \bigcup_{\phi \in \Phi} (T_0 \cap \text{dom} \phi),
\]

which is \( \mathcal{H}' \)-saturated; thus \( \Phi_{\text{orb}}(O) = \Phi(O) \) for any orbit \( O \) of \( \mathcal{H} \). It is said that \( \Phi \) is constant if \( \text{im} \Phi \) is one orbit (\( \Phi_{\text{orb}} \) is constant). If \( \mathcal{H}' \) acts on a space \( T' \) and \( T_0 \subset T' \), define the inverse image

\[
\Phi^{-1}(T'_0) = \bigcup_{\phi \in \Phi} \phi^{-1}(T_0 \cap \text{im} \phi),
\]

which is \( \mathcal{H} \)-saturated. If \( \text{im} \Phi \subset T'_0 \), then the restrictions \( \phi : \text{dom} \phi \to T'_0 \), for \( \phi \in \Phi \), form a morphism that may be denoted by \( \Phi : \mathcal{H} \to \mathcal{H}'|_{T'_0} \) and called the restriction of \( \Phi \) too.

The product of two pseudogroups, \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) acting on spaces \( T_1 \) and \( T_2 \), respectively, is the pseudogroup \( \mathcal{H}_1 \times \mathcal{H}_2 \) acting on \( T_1 \times T_2 \) generated by the products of maps in \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \). The product of two morphisms \( \Phi_i : \mathcal{H}_i \to \mathcal{H}'_i \), \( i = 1, 2 \), is the morphism \( \Phi_1 \times \Phi_2 : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}'_1 \times \mathcal{H}'_2 \) generated by the products of maps in \( \Phi_1 \) and \( \Phi_2 \). The pair of two morphisms \( \Phi_i : \mathcal{H} \to \mathcal{H}'_i \), \( i = 1, 2 \), is the morphism \( (\Phi_1, \Phi_2) : \mathcal{H} \to \mathcal{H}'_1 \times \mathcal{H}'_2 \) generated by the pairs \( (\phi_1, \phi_2) \), where \( \phi_1 \in \Phi_1 \) and \( \phi_2 \in \Phi_2 \) have the same domain.

3. Extending concepts to pseudogroups

Let \( \text{Top} \) denote the category of continuous maps between topological spaces. There is a canonical injective covariant functor \( \text{Top} \to \text{PsGr} \) which assigns the pseudogroup generated by \( \text{id}_T \) to each space \( T \), and assigns the morphism generated by \( f \) to each continuous map \( f \). We will consider \( \text{Top} \) as a subcategory of \( \text{PsGr} \) in this way; i.e., topological spaces and continuous maps will be also considered as pseudogroups and morphisms.

Let \( X \) be a space and \( \mathcal{H} \) a pseudogroup acting on another space \( T \). Any continuous map \( X \to T \) generates a morphism \( X \to \mathcal{H} \). Nevertheless, a continuous map \( f : T \to X \) generates a morphism \( \mathcal{H} \to X \) if and only if \( f \) is constant on the \( \mathcal{H} \)-orbits; in this case, the notation \( f : \mathcal{H} \to T' \) will be used for such a morphism, which consists of the restrictions of \( f \) to all open subsets of \( T \).
Some topological concepts can be extended to pseudogroups by using orbits and morphisms instead of points and continuous maps. For instance, a pseudogroup $\mathcal{H}$ is said to be path connected if, for any pair of orbits $O$ and $O'$ of $\mathcal{H}$, there is a morphism $\Phi : I \to \mathcal{H}$ with $\Phi(0) = O$ and $\Phi(1) = O'$, where $I = [0, 1]$ is considered as a pseudogroup in the above sense. If $\mathcal{H}$ is path connected, then its space of orbits is path connected. A homotopy between two morphisms $\Phi_0, \Phi_1 : \mathcal{H} \to \mathcal{H}'$ is a morphism $\Psi : \mathcal{H} \times I \to \mathcal{H}'$ such that $\Phi_0$ and $\Phi_1$ can be identified to the restrictions of $\Psi$ to $T \times \{0\} \equiv T$ and $T \times \{1\} \equiv T$. Since the restriction of $\mathcal{H} \times I$ to each slice $T \times \{t\}$ can be identified with $\mathcal{H}$, a homotopy $\Psi : \mathcal{H} \times I \to \mathcal{H}'$ can be considered as the family of its restrictions $\Psi_t = \Psi|_{T \times \{t\}} : \mathcal{H} \to \mathcal{H}'$. A morphism $\Phi : \mathcal{H} \to \mathcal{H}'$ is called a homotopy equivalence if there is a morphism $\Phi' : \mathcal{H}' \to \mathcal{H}$ such that $\Phi' \circ \Phi$ and $\Phi \circ \Phi'$ are homotopic to the identity morphisms $\text{id}_\mathcal{H}$ and $\text{id}_{\mathcal{H}'}$, respectively. If $\text{id}_\mathcal{H}$ is homotopic to a constant morphism, then $\mathcal{H}$ is said to be contractible.

Other typical topological concepts that can be obviously generalized to pseudogroups in this way are coverings, fiber bundles, the most usual types of (co)homologies, homotopy groups, LS category, $K$-theory and cobordism. In particular, coverings and fundamental groups of pseudogroups were defined in [19] and [48].

The differential calculus can be also extended to $C^\infty$ pseudogroups by using morphisms. For instance, we can define their de Rham cohomology (see Section 28) and study transversality of $C^\infty$ morphisms.

Finally, morphisms also allow the generalization of many typical concepts of differential geometry to pseudogroups. For example, for pseudogroups of local isometries of Riemannian manifolds (called Riemannian pseudogroups), the geodesics and geodesic segments can be defined as morphisms, and, with more generality, we can consider harmonic morphisms between Riemannian pseudogroups. The length of a geodesic segment of a Riemannian pseudogroup makes sense in this way. Moreover geodesic completeness can be considered for Riemannian pseudogroups.

### 4. Holonomy pseudogroups of foliated spaces

With the terminology of [20] and [21], a foliated structure $\mathcal{F}$ of dimension $n \in \mathbb{N}$ on a space $X$ can be described by a foliated cocycle, which is a collection $\{U_i, p_i\}$, where $\{U_i\}$ is an open cover of $X$ and each $p_i$ is a topological submersion of $U_i$ onto some space $T_i$ whose fibers are connected open subsets of $\mathbb{R}^n$ and such that the following compatibility condition is satisfied: for every $x \in U_i \cap U_j$, there is an open neighborhood $U_{i,j}^x$ of $x$ in $U_i \cap U_j$ and a homeomorphism $h_{i,j}^x : p_i(U_{i,j}^x) \to p_j(U_{i,j}^x)$ such that $p_j = h_{i,j}^x \circ p_i$ on $U_{i,j}^x$; the notation $T_{i,j}^x = p_i(U_{i,j}^x)$ will be used for the sake of simplicity. Two foliated cocycles determine the same foliated structure when their union is a foliated cocycle. Thus $\mathcal{F}$ is given as an equivalence class of foliated cocycles—the logical problems of this definition can be easily avoided, either by using the theory of universes of Grothendieck’s school, or by using only foliated cocycles with values in a given set of spaces (see e.g. [30]). The space $X$ endowed with $\mathcal{F}$ is called a foliated space.

Many interesting examples of foliated spaces are given, e.g., in [10]. Foliated structures and foliated spaces with boundary or corners can be defined similarly. Usually, foliated spaces are supposed to be Polish (completely metrizable and separable), and local compactness is also assumed often; recall that a space is locally compact and Polish if and only if it is locally compact and second countable [29, Theorem 5.3]. But these conditions are not needed for the purposes of this section and the next one. Also, it would be enough that the fibers of the submersions $p_i$ be arbitrary connected locally path connected spaces. When a foliated space is a manifold, then the common term foliation is used for its foliated structure.

Let us recall the generalization to foliated spaces of some common terminology for foliations. The domains of the submersions of the foliated cocycles are called simple open sets. An open covering of $X$ is called simple when it consists of simple open sets. The simple open sets form a base of the topology of $X$. The local quotient of a simple open set is its quotient space whose points are the plaques, which can be identified with the target space of the corresponding submersion. The fibers of those submersions of any foliated cocycle are called plaques. The plaques of all foliated cocycles form a base of a finer topology on $X$, called the leaf topology, whose connected components are called leaves. The leaf through each point $x$ is usually denoted by $L_x$. The quotient space of leaves will be denoted by $X/\mathcal{F}$. The images of the local sections of the submersions of foliated cocycles are called local transversals.

The condition on each $p_i$ to be a topological submersion means that, for each $x \in U_i$, there is a homeomorphism $\theta_i^x$ of an open neighborhood $U_i^x$ of $x$ to $p_i(U_i^x) \times B$, for some fixed ball $B$ of $\mathbb{R}^n$, so that $p_i$ corresponds to the first factor projection by $\theta_i^x$. Then $(U_i^x, \theta_i^x)$ is called a foliated chart. A collection of foliated charts whose domains cover $X$ is called a foliated atlas.
The foliated structure $\mathcal{F}$ can be also identified with its canonical foliated cocycle, which consists of the canonical projections of all simple open subsets of $X$ onto their local quotients. Other well-known descriptions of $\mathcal{F}$ can be given by using foliated atlases, or by using the partition of $X$ into the leaves, satisfying appropriate conditions.

A simple open set $V$ is said to be uniform in another simple open set $U$ when $V \subset U$ and every plaque of $V$ meets only one plaque in $U$. A simple open covering $\{U_i\}$ of $X$ is called regular if, whenever $U_i$ meets $U_j$, there is some simple open set of $X$ where both $U_i$ and $U_j$ are uniform; in particular, each plaque of $U_i$ meets at most one plaque of $U_j$, and thus we can take $U_{i,j}^x = U_i \cap U_j$ in the above compatibility condition. If $X$ is Polish and locally compact, then it has arbitrarily fine locally finite regular simple open coverings (use the arguments of [23,14] and [4]).

For a foliated cocycle $\{U_i, p_i\}$ of $\mathcal{F}$ with $p_i : U_i \to T_i$, the homeomorphisms $h_{i,j}^x$, given by its compatibility condition, generate a pseudogroup $\mathcal{H}$ acting on the topological sum $T = \bigsqcup_i T_i$, and the maps $p_i$ generate a morphism $\mathcal{P} : X \to \mathcal{H}$; these $\mathcal{H}$ and $\mathcal{P}$ are said to be induced by $\{U_i, p_i\}$. Let $\{U'_a, p'_a\}$ be another foliated cocycle of $\mathcal{F}$ with $p'_a : U'_a \to T'_a$, which induces a pseudogroup $\mathcal{H}'$ acting on $T' = \bigsqcup_a T'_a$ and a morphism $\mathcal{P}' : X \to \mathcal{H}'$. Then the foliated cocycle $\{U_i, p_i, U'_a, p'_a\}$ induces a pseudogroup $\mathcal{H}''$ acting on $T'' = T \sqcup T'$ whose orbits meet $T$ and $T'$, and so that $\mathcal{H}''|_T = \mathcal{H}$ and $\mathcal{H}''|_{T'} = \mathcal{H}'$. Therefore $\mathcal{H}''$ defines a canonical equivalence $\Phi_0 : \mathcal{H} \to \mathcal{H}'$, which generates a canonical isomorphism $\Phi : \mathcal{H} \to \mathcal{H}'$ satisfying $\Phi \circ \mathcal{P} = \mathcal{P}'$. The equivalence class of the pseudogroup induced by any foliated cocycle of $\mathcal{F}$ is usually called the holonomy pseudogroup. But the canonical foliated cocycle induces a canonical representative of this class, which will be also called holonomy pseudogroup and denoted by $\text{Hol}(\mathcal{F})$. By identifying the local quotient of each simple open set with a local transversal, we see that the holonomy pseudogroup can be given by sliding local transversals along the leaves; thus it represents the “transverse dynamics” of the foliated space.

Let us introduce the following terminology. Let $m, n \in \mathbb{N}$. For spaces $T$ and $T'$, and open subsets $U \subset T \times \mathbb{R}^m$ and $V \subset T' \times \mathbb{R}^n$, a foliated map $f : U \to V$ is said to be of class $C^{0,\infty}$ if, for each $y \in Y$, the mapping $z \mapsto \text{pr}_2 \circ f(y, z)$ is $C^\infty$ with partial derivatives of arbitrary order depending continuously on $y$, where $\text{pr}_2 : T' \times \mathbb{R}^n \to \mathbb{R}^n$ is the second factor projection. A $C^{0,\infty}$ structure on $\mathcal{F}$ is a maximal foliated atlas $\{U_i, \theta_i\}$ so that each composite $\theta_j \circ \theta_i^{-1}$ is $C^{0,\infty}$. When $X$ is endowed with a $C^{0,\infty}$ structure, it is called a $C^{0,\infty}$ foliated space.

The foliated structure $\mathcal{F}$ is said to be of class $C^{\infty,0}$ when $\text{Hol}(\mathcal{F})$ is $C^{\infty}$; in particular, $\mathcal{F}$ has to be a foliation.

If $X$ is a manifold, the topological submersions of any foliated cocycle of $\mathcal{F}$ have values in manifolds. If $X$ is a $C^\infty$ manifold and there is a foliated cocycle of $\mathcal{F}$ consisting of $C^\infty$ submersions, then $\mathcal{F}$ is called a $C^\infty$ foliation.

5. Holonomy morphisms of foliated maps

Let $X$ and $Y$ be foliated spaces with respective foliated structures $\mathcal{F}$ and $\mathcal{G}$. A foliated map $f : (X, \mathcal{F}) \to (Y, \mathcal{G})$, or simply $f : \mathcal{F} \to \mathcal{G}$, is a map $f : X \to Y$ which maps leaves of $\mathcal{F}$ to leaves of $\mathcal{G}$; thus $f$ induces a map $\widetilde{f} : X/\mathcal{F} \to Y/\mathcal{G}$, which is continuous if $f$ is continuous. The identity map $\text{id}_X$, considered as a foliated map $\mathcal{F} \to \mathcal{F}$, will be denoted by $\text{id}_\mathcal{F}$. The set of continuous foliated maps $\mathcal{F} \to \mathcal{G}$ will be denoted by $C(X, \mathcal{F}; Y, \mathcal{G})$, or simply $C(\mathcal{F}, \mathcal{G})$. Continuous foliated maps between foliated spaces form a category with the operation of composition, which will be denoted by Fol. The mapping $f \mapsto \widetilde{f}$ defines a functor Fol $\to$ Top.

When $\mathcal{F}$ and $\mathcal{G}$ are of class $C^{0,\infty}$, a foliated map $f : \mathcal{F} \to \mathcal{G}$ is said to be of class $C^{0,\infty}$ when, for all foliated charts $(U, \theta)$ and $(U', \theta')$ of the $C^{0,\infty}$ structures of $\mathcal{F}$ and $\mathcal{G}$ such that $f(U) \subset U'$, the composite $\theta' \circ f \circ \theta^{-1}$ is $C^{0,\infty}$. The set of $C^{0,\infty}$ foliated maps $\mathcal{F} \to \mathcal{G}$ will be denoted by $C^{0,\infty}(\mathcal{F}, \mathcal{G})$.

When $\mathcal{F}$ and $\mathcal{G}$ are $C^\infty$ foliations, the set of $C^\infty$ foliated maps $\mathcal{F} \to \mathcal{G}$ will be denoted by $C^\infty(\mathcal{F}, \mathcal{G})$.

Let $\{U_i, p_i\}$ and $\{V_a, p'_a\}$ be foliated cocycles of $\mathcal{F}$ and $\mathcal{G}$ with $p_i : U_i \to T_i$ and $p'_a : V_a \to T'_a$. For $x \in U_i \cap U_j$ and $y \in V_a \cap V_b$, the compatibility condition is satisfied with open sets $U_{i,j}^x$ and $V_{a,b}$, and homeomorphisms $h_{i,j}^x : T_{i,j}^x \to T_{j,i}^x$ and $h_{a,b}^y : T_{a,b}^y \to T_{b,a}^y$. Let $\mathcal{H}$ and $\mathcal{H}'$ be the pseudogroups induced by $\{U_i, p_i\}$ and $\{V_a, p'_a\}$, acting on $T = \bigsqcup_i T_i$ and $T' = \bigsqcup_a T'_a$, and let $\mathcal{P}, \mathcal{P}' : X \to \mathcal{H}, \mathcal{H}'$ be the corresponding morphisms.

For any fixed $x \in C(\mathcal{F}, \mathcal{G})$, we can choose the open sets $U_{i,j}^x$ and $V_{a,b}$ such that $f$ maps each fiber of $p_i$ in $U_{i,j}^x$ to a fiber of $p'_a$ on $V_{a,b}^{f(x)}$ for some mapping $i \mapsto a_i$. So there are continuous maps $\phi_{i,j}^x : T_{i,j}^x \to T_{a,b}^{f(x)}$ satisfying
\[
\phi_{i,j}^x \circ p_i = p'_a \circ f
\]
on $U_{i,j}^x$. Let $\Phi_0$ be the family of such maps $\phi_{i,j}^x$. 

5. Holonomy morphisms of foliated maps
Lemma 5.1. $\Phi_0$ generates a morphism $\Phi : \mathcal{H} \to \mathcal{H'}$ such that $\mathcal{P}' \circ f = \Phi \circ \mathcal{P}$.

**Proof.** This $\Phi_0$ obviously satisfies hypothesis (ii') of Lemma 2.3, and the hypothesis (iii') is given by the commutativity of the diagram

$$
\begin{array}{c}
T'_{i,j} \xrightarrow{\Phi'_{i,j}} T'_{a_i,a_j} \\
\downarrow h'_{i,j} \\
T_{i,j} \xrightarrow{\Phi_{i,j}} T_{a_i,a_j}
\end{array}
$$

for $x \in U_i \cap U_j$, which holds because

$$
\phi'_{i,j} \circ h_{i,j}^x \circ p_i = \phi_{i,j} \circ p_j = p_{a_j} \circ f = h_{a_i,a_j}^{f(x)} \circ p_{a_j} \circ f = h_{a_i,a_j}^{f(x)} \circ \phi_{i,j} \circ p_i
$$
on $U_{i,j}^x$ by (5.1). So $\Phi_0$ generates a morphism $\Phi : \mathcal{H} \to \mathcal{H'}$, and the equality $\mathcal{P}' \circ f = \Phi \circ \mathcal{P}$ also follows from (5.1). □

It will be said that $\Phi$ is induced by $f$, or that $f$ is a lift of $\Phi$. In particular, for the canonical representatives, we get a morphism $\text{Hol}(f) : \text{Hol}(\mathcal{F}) \to \text{Hol}(\mathcal{G})$, which is called the holonomy morphism induced by $f$; we may also say that $\Phi$ is a representative of $\text{Hol}(f)$. In this way, we get a covariant functor $\text{Hol} : \text{Fol} \to \text{PsGr}$, which is called the holonomy functor.

When $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{C}^{\infty,0}$ foliations, a foliated map $f : \mathcal{F} \to \mathcal{G}$ is said to be of class $\mathcal{C}^{\infty,0}$ if $\text{Hol}(f) : \text{Hol}(\mathcal{F}) \to \text{Hol}(\mathcal{G})$ is a $\mathcal{C}^{\infty}$ morphism. The set of $\mathcal{C}^{\infty,0}$ foliated maps $\mathcal{F} \to \mathcal{G}$ will be denoted by $\mathcal{C}^{\infty,0}(\mathcal{F}, \mathcal{G})$.

Any topological space $X$ can be considered as a foliated space whose leaves are its points; the notation $\text{Man}$ will be considered as an inclusion.

Let $X$ be a space and $\mathcal{F}$ a foliated structure on another space $Y$. Any continuous map $X \to Y$ is a foliated map $X_{pr} \to \mathcal{F}$. However, a continuous map $Y \to X$ is a foliated map $\mathcal{F} \to X$ if and only if it is constant on the leaves of $\mathcal{F}$. Any manifold $M$ (possibly with boundary) can be also considered as a foliated space (possibly with boundary) whose leaves are its connected components; this foliated space will be also denoted by $M$. Furthermore any map between manifolds is a foliated map in this sense. This defines an injective functor $\text{Man} \to \text{Fol}$, which will be also considered as an inclusion, where $\text{Man}$ denotes the category of continuous maps between manifolds. The composition of $\text{Man} \to \text{Fol}$ with the holonomy functor is the functor of projection to the discrete space of connected components.

Let $Y$ be a foliated space with foliated structure $\mathcal{G}$, and let $\mathcal{F}$ be a foliated structure on some subspace $X \subset Y$. If each leaf of $\mathcal{F}$ is a submanifold of some leaf of $\mathcal{G}$, then $\mathcal{F}$ is called a subfoliated structure of $\mathcal{G}$; in this case, the inclusion map $X \hookrightarrow Y$ is a foliated map of $\mathcal{F}$ to $\mathcal{G}$ denoted by $\mathcal{F} \hookrightarrow \mathcal{G}$.

For foliated spaces $X_1$ and $X_2$ with respective foliated structures $\mathcal{F}_1$ and $\mathcal{F}_2$, the product $\mathcal{F}_1 \times \mathcal{F}_2$ is the foliated structure on $X_1 \times X_2$ whose leaves are the products of leaves of $\mathcal{F}_1$ and leaves of $\mathcal{F}_2$. If $(U_i^1, p_i^1)$ and $(U_j^2, p_j^2)$ are foliated cocycles of $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively, then $(U_i^1 \times U_j^2, p_i^1 \times p_j^2)$ is a foliated cocycle of $\mathcal{F}_1 \times \mathcal{F}_2$, called the product of $(U_i^1, p_i^1)$ and $(U_j^2, p_j^2)$. The factor projections $\text{pr}_k : X_1 \times X_2 \to X_k$ are foliated maps $\text{pr}_k : \mathcal{F}_1 \times \mathcal{F}_2 \to \mathcal{F}_k$, $k = 1, 2$. Then

$$(\text{Hol}(\text{pr}_1), \text{Hol}(\text{pr}_2)) : \text{Hol}(\mathcal{F}_1 \times \mathcal{F}_2) \to \text{Hol}(\mathcal{F}_1) \times \text{Hol}(\mathcal{F}_2)$$
is an isomorphism. Indeed, $\text{Hol}(\mathcal{F}_1) \times \text{Hol}(\mathcal{F}_2)$ is induced by the product of the canonical foliated cocycles of $\mathcal{F}_1$ and $\mathcal{F}_2$, and $(\text{Hol}(\text{pr}_1), \text{Hol}(\text{pr}_2))$ is the canonical isomorphism. Observe that $\text{Hol}(\mathcal{F}_1) \times \text{Hol}(\mathcal{F}_2)$ is the restriction of $\text{Hol}(\mathcal{F}_1 \times \mathcal{F}_2)$ to an open set that cuts every orbit, and the corresponding inclusion map generates $\text{Hol}(\text{pr}_1), \text{Hol}(\text{pr}_2))^{-1}$.

The product $f_1 \times f_2$ of foliated maps $f_i : \mathcal{F}_i \to \mathcal{G}_i$, $i = 1, 2$, is a foliated map $\mathcal{F}_1 \times \mathcal{F}_2 \to \mathcal{G}_1 \times \mathcal{G}_2$. When $f_1$ and $f_2$ are continuous, the morphism $\text{Hol}(f_1) \times \text{Hol}(f_2)$ corresponds to $\text{Hol}(f_1 \times f_2)$ by the canonical isomorphisms. The pair $(f_1, f_2)$ of foliated maps $f_i : \mathcal{F} \to \mathcal{G}_i$, $i = 1, 2$, is a foliated map $\mathcal{F} \to \mathcal{G}_1 \times \mathcal{G}_2$. When $f_1$ and $f_2$ are continuous, the morphism $\text{Hol}(f_1), \text{Hol}(f_2))$ corresponds to $\text{Hol}(f_1, f_2)$ by the canonical equivalence.
If $M$ is a connected manifold (possibly with boundary or corners) considered as a foliation with one leaf, then $\text{Hol}(M)$ has only one orbit, and it is thus isomorphic to a singleton space $\{\ast\}$. More precisely, the holonomy morphism of the map $M \to \{\ast\}$ is an isomorphism $\text{Hol}(M) \to \text{Hol}(\{\ast\}) \equiv \{\ast\}$. Let $F$ be a foliated structure, and let $pr_1 : F \times M \to F$ and $pr_2 : F \times M \to M$ be the factor projections. Then the equivalence

$$(\text{Hol}(pr_1), \text{Hol}(pr_2)) : \text{Hol}(F \times M) \to \text{Hol}(F) \times \text{Hol}(M)$$

corresponds to $\text{Hol}(pr_1) : \text{Hol}(F \times M) \to \text{Hol}(F)$ by the isomorphism

$$\text{Hol}(F) \times \text{Hol}(M) \cong \text{Hol}(F) \times \{\ast\} \equiv \text{Hol}(F).$$

Given any point $y \in M$, let $t_y : F \to F \times M$ be the foliated map defined by $t_y(x) = (x, y)$. Since $pr_1 \circ t_y = \text{id}_F$, we get $\text{Hol}(t_y) = (\text{Hol}(pr_1))^{-1}$, which is independent of $y$.

6. Integrable and foliated homotopies

Let $X$ and $Y$ be foliated spaces with foliated structures $F$ and $G$. An integrable homotopy between continuous foliated maps $f, g : F \to G$ is a homotopy $H : X \times I \to Y$ between $f$ and $g$ which is a foliated map $F \times I \to G$; i.e., each homotopy curve $t \mapsto H(x, t)$ lies in a leaf of $G$. If there is an integrable homotopy between $f$ and $g$, then these maps are said to be integrably homotopic. A continuous foliated map $f : F \to G$ is called an integrable homotopy equivalence if there is a continuous foliated map $g : G \to F$ such that $g \circ f$ and $f \circ g$ are integrably homotopic to $\text{id}_F$ and $\text{id}_G$.

Proposition 6.1. Integrably homotopic foliated maps define the same holonomy morphism.

Proof. Let $H : F \times I \to G$ be an integrable homotopy between foliated maps $f, g : F \to G$, and let $t_i : F \to F \times I$, $i = 0, 1$, be the foliated maps defined by $t_i(x) = (x, i)$. Since $I$ is connected, we have $\text{Hol}(t_0) = \text{Hol}(t_1)$ (Section 5). Therefore

$$\text{Hol}(f) = \text{Hol}(H \circ t_0) = \text{Hol}(H) \circ \text{Hol}(t_0) = \text{Hol}(H) \circ \text{Hol}(t_1) = \text{Hol}(H \circ t_1) = \text{Hol}(g).$$

\[ \square \]

Corollary 6.2. With the above notation, if $f : F \to G$ is an integrable homotopy equivalence, then $\text{Hol}(f)$ is an isomorphism.

A foliated homotopy between continuous foliated maps $f, g : F \to G$ is a homotopy $H$ between $f$ and $g$ which is a foliated map $F \times I_{pt} \to G$; i.e., the homotopy $H$ consists of foliated maps $H_t = H(\cdot, t) : F \to G$. Any integrable homotopy is a foliated homotopy. Two foliated maps are said to be foliatedly homotopic if there is a foliated homotopy between them. A continuous foliated map $f : F \to G$ is called a foliated homotopy equivalence if there is a continuous foliated map $g : G \to F$ such that $g \circ f$ and $f \circ g$ are foliatedly homotopic to $\text{id}_F$ and $\text{id}_G$, respectively. Since $\text{Hol}(F \times I_{pt}) \cong \text{Hol}(F) \times I$ canonically, if $H : F \times I_{pt} \to G$ if a foliated homotopy between $f$ and $g$, then $\text{Hol}(H) : \text{Hol}(F \times I_{pt}) \to \text{Hol}(G)$ canonically defines a homotopy between $\text{Hol}(f)$ and $\text{Hol}(g)$. Therefore we get the following.

Proposition 6.3. If $f$ is a foliated homotopy equivalence, then $\text{Hol}(f)$ is a homotopy equivalence.

A continuous foliated map $f : F \to G$ is called a proper integrable homotopy equivalence if it is proper and there is a proper continuous foliated map $g : G \to F$ such that there are proper integrable homotopies between $g \circ f$ and $\text{id}_F$, and between $f \circ g$ and $\text{id}_G$. A proper foliated homotopy equivalence can be defined similarly by using proper foliated homotopies instead of proper integrable homotopies.

7. Complete pseudogroups and complete morphisms

For any map $h : T \to T'$ between topological spaces, let $\gamma(h, x)$ denote its germ at any $x \in T$. If $\mathcal{H}$ is a family of maps of open subsets of $T$ to open subsets of $T'$, let $\gamma(\mathcal{H})$ denote the space of all germs of maps in $\mathcal{H}$ with the étalé
topology. If $T$ is a manifold, then $\gamma(H)$ is a manifold of the same dimension. If $H$ is a pseudogroup acting on $T$, then $\gamma(H)$ is a topological groupoid with the operation induced by composition.

Recall from [20] that a pseudogroup $H$ acting on a space $T$ is said to be complete if, for all $x, y \in T$, there are open neighborhoods $U$ and $V$ of $x$ and $y$ such that, for any $h \in H$ and any $z \in U \cap \operatorname{dom} h$ with $h(z) \in V$, there is some $\tilde{h} \in H$ so that $U \subset \operatorname{dom} \tilde{h}$ and $\gamma(h, z) = \gamma(\tilde{h}, z)$; in this case, $(U, V)$ is called a completeness pair.

**Definition 7.1.** Let $H$ and $H'$ be pseudogroups acting on topological spaces $T$ and $T'$, respectively. A morphism $\Phi : H \rightarrow H'$ is said to be complete when, given $\phi, \psi \in H$, $x \in \operatorname{dom} \phi$ and $y \in \operatorname{dom} \psi$, there are open neighborhoods $U$ and $V$ of $x$ and $y$ in $\operatorname{dom} \phi$ and $\operatorname{dom} \psi$, respectively, such that, for all $h \in H$ and every $z \in U \cap \operatorname{dom} h$ with $h(z) \in V$, there is some $\tilde{h} \in H$ and some $h' \in H'$ so that $U \subset \operatorname{dom} \tilde{h}$, $\tilde{h}(U) \subset \operatorname{dom} \psi$, $\gamma(h, z) = \gamma(\tilde{h}, z)$, $\phi(U) \subset \operatorname{dom} h'$, and $h' \circ \phi = \psi \circ \tilde{h}$ on $U$. In this case, $(\phi, U; \psi, V)$ is called a completeness quadruple.

**Remark 1.** In Definition 7.1, if $H$ is complete, then $(U, V)$ can be chosen to be a completeness pair of $H$. Thus, in this case, to prove that $(\phi, U; \psi, V)$ is a completeness quadruple of $H$, it is enough to take elements $h \in H$ with $\operatorname{dom} h = U$.

Observe that a pseudogroup $H$ is complete if and only if the identity morphism $\operatorname{id}_H$ is complete.

8. **Singular foliated spaces**

By adapting a definition by [46], a singular foliated structure $F$ on a space $X$ can be described by a singular foliated cocycle, which is a collection $\{U_i, p_i\}$, where $\{U_i\}$ is an open cover of $X$ and each $p_i$ is a topological submersion of $U_i$ onto some space $T_i$ whose fibers are connected open subsets of some Euclidean space whose dimension depends on $i$, and such that the following properties are satisfied:

- **Compatibility condition:** any $x \in X$ is contained in some $U_i$ such that, if $x \in U_j$, then there is some open neighborhood $U_{j,i}$ of $x$ in $U_j \cap U_i$ and a topological submersion $h_{j,i} : p_j(U_{j,i}) \rightarrow p_i(U_{j,i})$ such that $p_i = h_{j,i} \circ p_j$ on $U_{j,i}$; in this case, the fiber $p_i^{-1}(p_i(x))$ is called a plaque of $\{U_i, p_i\}$ (or of $p_i$ or $U_i$).
- Some fiber of any $p_i$ is a plaque.

Two singular foliated cocycles determine the same singular foliated structure when their union is a singular foliated cocycle. The space $X$ endowed with $F$ is called a singular foliated space.

The plaques of all singular foliated cocycles defining $F$ form a base of a topology, called the leaf topology. The connected components of $X$ with the leaf topology are called the leaves of $F$. The quotient space of leaves will be denoted by $X/F$.

Given a foliated cocycle $\{U_i, p_i\}$ and a leaf $L$ of $F$, each intersection $L \cap U_i$ is a union of fibers of $p_i$; such fibers are open in $L$ just when they are plaques.

For $x$ is in a plaque of some $p_i$, then the image $\Sigma$ of any local section of $p_i$ containing $x$ is called a local transversal of $F$ through $x$. There is a unique singular foliated structure $F_{\Sigma}$ on $\Sigma$ such that, for $V = p_i^{-1}(p_i(\Sigma))$, the submersion $p_i$ defines a foliated map $p_i : F|_V \rightarrow F_{\Sigma}$, which restricts to submersions of the leaves of $F|_V$ to the leaves of $F_{\Sigma}$. Observe that $F_{\Sigma}$ is the foliated structure by points in the singular case. Notice also that, in the singular case, an open subset of a local transversal may not be a local transversal through any point.

Other concepts and notations of foliated structures can be directly generalized to the singular case: saturations, restrictions, products of singular foliated structures, foliated maps, integrable homotopies, foliated homotopies, etc. If $X$ is a manifold, then $F$ is called a singular foliation; in this case, the submersions of a foliated cocycle have values in manifolds. If $X$ is a $C^\infty$ manifold and there is some foliated cocycle consisting of $C^\infty$ submersions, then $F$ is called a $C^\infty$ singular foliation.

Singular foliated structures whose leaves are manifolds with boundary, or arbitrary connected locally path connected spaces, can be defined similarly.

Any foliated structure is a singular foliated structure; it may be said that these foliated structures are regular for emphasis. A $(C^\infty)$ singular foliated structure is a $(C^\infty)$ regular foliated structure if and only if all of its leaves have the same dimension.
By [40], for any continuous local action of a local Lie group on a separable metric space, the connected components of the orbits are the leaves of a singular foliated structure, which is regular just when the local action is locally free. In the case of $C^\infty$ local actions of local Lie groups on $C^\infty$ manifolds, we get $C^\infty$ singular foliations.

Let $\mathcal{F}$ and $\mathcal{G}$ be singular foliated structures. The set of continuous foliated maps $\mathcal{F} \to \mathcal{G}$ will be denoted by $C(\mathcal{F}, \mathcal{G})$. When $\mathcal{F}$ and $\mathcal{G}$ are $C^\infty$ singular foliations, the set of $C^\infty$ foliated maps $\mathcal{F} \to \mathcal{G}$ will be denoted by $C^\infty(\mathcal{F}, \mathcal{G})$.

The following notation will be used.

**Notation 1.** The tangent vector bundle of $C^\infty$ manifold $M$ will be denoted by $TM$, and the tangent space of $M$ at some point $x \in M$ will be denoted by $T_xM$. As usual, the Lie algebra of $C^\infty$ tangent vector fields on $M$ is denoted by $\mathfrak{X}(M)$. For any $C^\infty$ map between $C^\infty$ manifolds, $f : M \to N$, its tangent homomorphism $TM \to TN$ is denoted by $T_f$ or $f_*$.

By the main result of [46], a partition $\mathcal{F}$ of $M$ into $C^\infty$ immersed connected submanifolds is a $C^\infty$ singular foliation if and only if the evaluation map $\mathfrak{X}(\mathcal{F}) \to T_xM$ is surjective for each $x \in M$, where $\mathfrak{X}(\mathcal{F}) \subset \mathfrak{X}(M)$ is the Lie subalgebra of $C^\infty$ vector fields on $M$ tangent to the leaves, and $T_x\mathcal{F} \subset T_xM$ is the linear subspace of vectors tangent to the leaf through $x$.

Let $\mathcal{F}$ and $\mathcal{G}$ be $C^\infty$ singular foliations on $C^\infty$ manifolds $M$ and $N$, and let $f \in C(\mathcal{F}, \mathcal{G})$ with $C^\infty$ restrictions to the leaves of $\mathcal{F}$. Let also $\phi : U \to \mathbb{R}^m$ and $\psi : V \to \mathbb{R}^n$ be charts of $M$ and $N$ such that $f(U) \subset V$. For any positive integer $r$, let $X_1, \ldots, X_r \in \mathfrak{X}(\mathcal{F})$. Since the restriction of $f$ to the leaves of $\mathcal{F}$ is $C^\infty$, the order $r$ derivative

\[
D^r(\psi \circ f \circ \phi^{-1})(x)(\phi_*(X_1(x)), \ldots, \phi_*(X_r(x)))
\]

is well defined for all $x \in \phi(U)$.

**Definition 8.1.** The map $f$ will be said to be of class $C^{0,k}$ if, for all $r \leq k$ and all possible charts $\phi$ and $\psi$ as above, the derivatives of the type (8.1) depend continuously on $x$. The map $f$ will be said to be of class $C^{0,\infty}$ if it is of class $C^{0,k}$ for all $k$. The set of $C^{0,\infty}$ maps $\mathcal{F} \to \mathcal{G}$ will be denoted by $C^{0,\infty}(\mathcal{F}, \mathcal{G})$.

This definition generalizes to the $C^\infty$ singular foliations the concept of $C^{0,\infty}$ foliated map between $C^{0,\infty}$ foliated spaces (Section 5).

**Lemma 8.2.** Consider foliated maps

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{f} & \mathcal{F}' \\
\xrightarrow{g} & & \xrightarrow{g'} \mathcal{F}''
\end{array}
\]

between singular $C^\infty$ foliations. Suppose the composite $g \circ f$ is $C^{0,\infty}$, and $f$ is a $C^\infty$ surjective submersion such that, for all $Y \in \mathfrak{X}(\mathcal{F}')$, there is some $X \in \mathfrak{X}(\mathcal{F})$ with $f_*(X) = Y$. Then $g$ is $C^{0,\infty}$.

**Proof.** Observe that the hypothesis on $f$ imply that it is also a surjective submersion as a map from leaves to leaves. So the restriction of $g$ to the leaves is of class $C^\infty$ because the restriction of $g \circ f$ to the leaves is of class $C^\infty$.

Let $M$, $M'$ and $M''$ be the ambient manifolds of $\mathcal{F}$, $\mathcal{F}'$ and $\mathcal{F}''$, and let $\psi : V \to \mathbb{R}^n$ and $\chi : W \to \mathbb{R}^p$ be charts of $M'$ and $M''$ such that $g(V) \subset W$. Since $f$ is a $C^\infty$ surjective submersion, we can assume $V = f(U)$ for some chart $\phi : U \to \mathbb{R}^m$ of $M$. For any positive integer $r$, given $Y_1, \ldots, Y_r \in \mathfrak{X}(\mathcal{F}')$, we know the existence of some $X_1, \ldots, X_r \in \mathfrak{X}(\mathcal{F})$ such that $f_*(X_i) = Y_i$, $i = 1, \ldots, r$. Then, if $y = \psi \circ f \circ \phi^{-1}(x)$ for $x \in \phi(U)$ and $y \in \psi(V)$, we have

\[
D^r(\chi \circ g \circ \psi^{-1})(y)(\psi_*(Y_1(y)), \ldots, \psi_*(Y_r(y))) = D^r(\chi \circ g \circ f \phi^{-1})(x)(\phi_*(X_1(x)), \ldots, \phi_*(X_r(x))).
\]

But the right-hand side of (8.2) depends continuously on $x \in \phi(U)$ because $g \circ f$ is of class $C^{0,\infty}$. Then the left-hand side of (8.2) depends continuously on $y \in \psi(V)$ because $\psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V)$ is a surjective submersion, and thus a quotient map.

**Remark 2.** Since being $C^{0,\infty}$ is a local property, in Lemma 19.9, it is enough to require that the lifting property of $f$ is satisfied locally.
9. Riemannian pseudogroups

A pseudogroup \( \mathcal{H} \) of local isometries of an \( n \)-dimensional Riemannian manifold \( T \) will be called a *Riemannian pseudogroup* on \( T \). The corresponding groupoid of germs \( \gamma(\mathcal{H}) \) is Hausdorff because a local isometry with connected domain is the identity if it is the identity on some nontrivial open subset (*quasi-analyticity*). Let \( J^1(T) \) denote the topological groupoid of 1-jets of local diffeomorphisms of \( T \), and \( j^1 : \gamma(\mathcal{H}) \to J^1(T) \) be defined by mapping each germ to its 1-jet. \( J^1(T) \) is a manifold of dimension \( n^2 + 2n \) and \( j^1 \) is a continuous homomorphism. Moreover \( j^1 \) is injective because germs of local isometries are determined by their 1-jets.

For any open subset \( U \subseteq T \), let \( \mathcal{H}_U = \{ h \in \mathcal{H} \mid \text{dom } h = U \} \) endowed with the topology of uniform convergence.

**Lemma 9.1.** If \( U \) is a connected relatively compact subset of \( T \) and \( x \in U \), then the map

\[
U \times \mathcal{H}_U \to J^1(T), \quad (x, f) \mapsto j^1(\gamma(f, x)),
\]

is an embedding.

**Proof.** This follows because, by the conditions on \( U \), any \( f \in \mathcal{H}_U \) is well known to be continuously determined by its 1-jet at any fixed point. \( \square \)

From now on in this section, assume that \( \mathcal{H} \) is complete.

**Theorem 9.2.** (*Haefliger [20, Proposition 3.1].*) With the above notation and conditions, there is a unique Riemannian pseudogroup \( \overline{\mathcal{H}} \) on \( T \) such that \( j^1(\gamma(\mathcal{H})) = j^1(\gamma(\overline{\mathcal{H}})) \). Moreover \( \overline{\mathcal{H}} \) is complete, its orbits are the closures of the \( \mathcal{H} \)-orbits, and \( \overline{\mathcal{H}} \setminus T \) is Hausdorff.

The pseudogroup \( \overline{\mathcal{H}} \) of Theorem 9.2 is called the *closure* of \( \mathcal{H} \), and \( \mathcal{H} \) is said to be *closed* if \( \mathcal{H} = \overline{\mathcal{H}} \).

From Theorem 9.2, we get \( \overline{\mathcal{H}}(x) = \overline{\mathcal{H}}(y) \) for all \( x \in T \) and all \( y \in \overline{\mathcal{H}}(x) \). It follows that any \( \mathcal{H} \)-saturated open subset \( U \subseteq T \) is \( \overline{\mathcal{H}} \)-saturated too; indeed, by the above observation, \( U \) cuts all \( \mathcal{H} \)-orbits in \( \overline{\mathcal{H}}(x) \) for any \( x \in U \).

Let \( \mathcal{H}' \) be another complete Riemannian pseudogroup on a Riemannian manifold \( T \). For any morphism \( \Phi : \mathcal{H} \to \mathcal{H}' \), the orbit map \( \Phi_{\text{orb}} : \mathcal{H} \setminus T \to \mathcal{H}' \setminus T' \) induces a continuous map \( \Phi_{\text{orb}} : \overline{\mathcal{H}} \setminus T \to \overline{\mathcal{H}'} \setminus T' \) called the *orbit closure map*.

The following is a generalization of the theorem of Myers–Steenrod.

**Theorem 9.3.** (*Salem [42].*) With the above notation and conditions, suppose that \( \mathcal{H} \) is closed. For each point \( x \in T \), there is an open neighborhood \( U \) of \( x \) and a finite dimensional Lie algebra \( \mathfrak{S}(U) \) of Killing vector fields over \( U \) such that, for each relatively compact open set \( V \) with \( \overline{V} \subseteq U \), the elements of \( \mathcal{H} \) with domain \( V \) and close enough to the identity are the maps of the form \( \exp \xi \) for \( \xi \in \mathfrak{S}(U) \) small enough.

In Theorem 9.3, the notation \( \exp t \xi \) is used for the local uniparametric group of diffeomorphisms defined by a \( C^\infty \) vector field \( \xi \).

The elements of the Lie algebra \( \mathfrak{S}(U) \) of Theorem 9.3 are the sections on \( U \) of a locally constant sheaf \( \mathfrak{S} \) of Lie algebras of germs of vector fields over \( T \), upon which \( \mathcal{H} \) acts by automorphisms [20, Section 3.4]. Such a \( \mathfrak{S} \) is called the *sheaf of infinitesimal transformations* of \( \mathcal{H} \), and its typical stack, denoted by \( \mathfrak{g} \), is called the *structural Lie algebra* of \( \mathcal{H} \). When \( \mathcal{H} \) is not closed, the same terminology is used for the sheaf and Lie algebra associated to \( \overline{\mathcal{H}} \).

10. The pseudogroup generated by the elements close to identity maps

Let \( \mathcal{H} \) be a closed complete Riemannian pseudogroup on a Riemannian manifold \( T \), and let \( \mathfrak{S} \) denote its sheaf of infinitesimal transformations. Then let \( \mathcal{H}_0 \) denote the Riemannian pseudogroup on \( T \) generated by the maps \( \exp \xi \), where \( \xi \) is any local section of \( \mathfrak{S} \). Using combination of maps and the formula \( \exp \xi = (\exp \xi/N)^N \), \( N \in \mathbb{N} \), it follows that \( \mathcal{H}_0 \) is also generated by the maps \( \exp \xi \) with \( \xi \in \mathfrak{S}(U) \) small enough, and \( U \) small enough as well. So \( \mathcal{H}_0 \subseteq \mathcal{H} \); in fact, according to Theorem 9.3, \( \mathcal{H}_0 \) is also generated by the elements of \( \mathcal{H} \) that are close enough to the identity map in their domains. The main goal of this section is to prove the following.
Theorem 10.1. With the above notation and conditions, the Riemannian pseudogroup \( \mathcal{H}_0 \) is complete and closed, and its orbits are the connected components of the orbits of \( \mathcal{H} \).

The following notation will be used.

Notation 2. Let \( G \) be a groupoid. For units \( x \) and \( y \) of \( G \), let \( G_x \) (respectively, \( G^y \)) denote the subset of elements of \( G \) with source (respectively, target) \( x \), and let \( G^x_y = G_x \cap G^y \). If \( X, Y \) are subsets of the space of units of \( G \), let \( G_X = \bigcup_{x \in X} G_x \), \( G^Y = \bigcup_{y \in Y} G^y \), and \( G^X_T = G_X \cap G^T \). Sometimes, the unit \( x \) will be also denoted by \( 1_x \).

For \( x, y \in T \), the subspace \( J^1(\gamma(\mathcal{H}))^x \) is a closed subspace of \( J^1(T)^x \) because \( \mathcal{H} \) is closed. Then, \( J^1(T)^x \) can be considered as the space of linear maps \( T_x T \rightarrow T_x T \), and because the linear maps in \( J^1(\gamma(\mathcal{H}))^x \) are orthogonal, it follows that \( J^1(\gamma(\mathcal{H}))^x \) is compact.

Lemma 10.2. We have:

(i) For all \( x, y \in T \), \( J^1(\gamma(\mathcal{H}))^x \) is closed in \( J^1(\gamma(\mathcal{H}))^y \), and thus \( J^1(\gamma(\mathcal{H}))^x \) is also compact.

(ii) \( J^1(\gamma(\mathcal{H})) \) is open in \( J^1(\gamma(\mathcal{H})) \).

Proof. To prove property (i), let \( h_m \) be a sequence of maps in \( \mathcal{H}_0 \) with \( x \in \text{dom} \ h_m \) and \( h_m(x) = y \) for all \( m \). Suppose the sequence \( J^1(\gamma(h_m, x)) \) is convergent in \( J^1(T) \). Since \( J^1(\gamma(\mathcal{H}))^x \) is closed in \( J^1(T) \), there is some \( h \in \mathcal{H} \) such that \( x \in \text{dom} \ h \), \( h(x) = y \) and \( J^1(\gamma(h_m, x)) \rightarrow J^1(\gamma(h, x)) \) as \( m \rightarrow \infty \).

By the completeness of \( \mathcal{H} \), we can assume that there are open neighborhoods \( U \) and \( V \) of \( x \) and \( y \), and that there is a sequence \( g_m \in \mathcal{H} \) such that \( h \) and every \( g_m \) are isometries \( U \rightarrow V \), and \( \gamma(g_m, x) = \gamma(h_m, x) \) for all \( m \). By Lemma 9.1 and since \( J^1(\gamma(h_m, x)) \) converges to \( J^1(\gamma(h, x)) \), it follows that \( g_m \rightarrow h \) uniformly on \( U \), and thus \( g_m^{-1} \circ h \rightarrow \text{id}_U \) uniformly. So \( g_m^{-1} \circ h \in \mathcal{H}_0 \) for \( m \) large enough. Hence

\[
J^1(\gamma(h,x)) = J^1(\gamma(g_m,x)) \cdot J^1(\gamma(g_m^{-1} \circ h,x)) = J^1(\gamma(h_m,x)) \cdot J^1(\gamma(g_m^{-1} \circ h,x)),
\]

which is in

\[
J^1(\gamma(\mathcal{H}))^x \cdot J^1(\gamma(\mathcal{H}))^x \subset J^1(\gamma(\mathcal{H}))^x
\]

for \( m \) large enough, as desired.

To prove property (ii), take any \( h \in \mathcal{H}_0 \) and any \( x \in \text{dom} \ h \). We have to show that, for \( g \in \mathcal{H} \) and \( y \in \text{dom} \ g \), if \( J^1(\gamma(g,y)) \) is close enough to \( J^1(\gamma(h,x)) \), then \( J^1(\gamma(g,y)) \in J^1(\gamma(\mathcal{H})) \); i.e., the restriction of \( g \) to some open neighborhood of \( y \) is in \( \mathcal{H}_0 \).

Let \( (U, V) \) be a completeness pair of \( \mathcal{H} \) with \( x \in U \) and \( h(x) \in V \). Let \( W \) be a relatively compact connected open neighborhood of \( x \) with \( \overline{W} \subset \text{dom} \ h \cap U \) and \( h(\overline{W}) \subset V \). Since \( J^1(\gamma(g,y)) \) is as close as desired to \( J^1(\gamma(h,x)) \), we can suppose that \( y \in W \) and \( \text{dom} \ g = U \). Then \( g \) is uniformly close to \( h \) on \( W \) by Lemma 9.1, and thus \( g \circ h^{-1} \) is close to the identity map on \( h(W) \), yielding \( g \circ h^{-1} \in \mathcal{H}_0 \) by Theorem 9.3. Now we have

\[
J^1(\gamma(g,y)) = J^1(\gamma(g \circ h^{-1}, h(y))) \cdot J^1(\gamma(h,y)),
\]

where both \( J^1(\gamma(g \circ h^{-1}, h(y))) \) and \( J^1(\gamma(h,y)) \) are in \( J^1(\gamma(\mathcal{H})) \). Thus \( J^1(\gamma(g,y)) \) is also in \( J^1(\gamma(\mathcal{H})) \) as desired. \( \Box \)

Corollary 10.3. If \( U \) and \( V \) are relatively compact open subsets of \( T \), then \( J^1(\gamma(\mathcal{H}))^U \) is a relatively compact open subset of \( J^1(\gamma(\mathcal{H})) \).

Proof. We know that \( J^1(\gamma(\mathcal{H}))^U \) is open in \( J^1(\gamma(\mathcal{H})) \) by Lemma 10.2(ii).

When \( P \) and \( Q \) run in the family of open neighborhoods of points \( x, y \in T \), the subsets \( J^1(\gamma(\mathcal{H}))^P \) form a base of open neighborhoods of \( J^1(\gamma(\mathcal{H}))^x \) in \( J^1(\gamma(\mathcal{H})) \). Moreover \( J^1(\gamma(\mathcal{H}))^x \) is compact by Lemma 10.2(ii). Also, \( J^1(\gamma(\mathcal{H})) \) is locally compact because it is open in the closed subset \( J^1(\gamma(\mathcal{H})) \) of the Hausdorff manifold \( J^1(T) \). So \( J^1(\gamma(\mathcal{H}))^P \) is relatively compact in \( J^1(\gamma(\mathcal{H})) \) for \( P \) and \( Q \) small enough.
Now, for each $x \in \overline{U}$ and $y \in \overline{V}$, choose corresponding open neighborhoods $P_{x,y}$ and $Q_{x,y}$ such that $j^1(\gamma(\mathcal{H}_0))_{P_{x,y}}^Q$ is relatively compact in $j^1(\gamma(\mathcal{H}_0))_{P_{x,y}}$. For each $x \in \overline{U}$, the compact subspace $\overline{V}$ can be covered by a finite number of open sets $Q_{x,y}$, which are denoted by $Q_{x,j} = Q_{x,y_j}$, $j = 1, \ldots, \ell_x$. For $P_{x,j} = P_{x,y_j}$, the open subsets
\[ P_x = P_{x,1} \cap \cdots \cap P_{x,\ell_x}, \quad x \in \overline{U}, \]
cover the compact subspace $\overline{U}$, and thus there is a finite subcovering consisting of sets $P_l = P_{x_l}$, $i = 1, \ldots, k$. Let $Q_{l,j} = Q_{x_{i,j}}$, $i = 1, \ldots, k$, $j = 1, \ldots, \ell_i$. Then
\[ j^1(\gamma(\mathcal{H}_0))_{U}^Q \subset \bigcup_{i=1}^k \bigcup_{j=1}^\ell_i j^1(\gamma(\mathcal{H}_0))_{P_{l,j}}^Q, \]
where each subset of the right-hand side is relatively compact in $j^1(\gamma(\mathcal{H}_0))$, and thus $j^1(\gamma(\mathcal{H}_0))_{U}^Q$ is relatively compact. \( \square \)

**Corollary 10.4.** The orbits of $\mathcal{H}_0$ are the connected components of the orbits of $\mathcal{H}$.

**Proof.** First we prove that the orbits of $\mathcal{H}_0$ are connected. Let $h \in \mathcal{H}_0$ and $x \in \text{dom} \ h$. Then
\[ h = \exp \xi_k \circ \cdots \circ \exp \xi_1 \]
on some neighborhood $W$ of $x$, where each $\xi_i$ is a local section of $\mathcal{G}$. Arguing by induction on $k$, it is enough to prove that $h(x)$ is in the same connected component of $\mathcal{H}_0(x)$ as
\[ h_0(x) = \exp \xi_{k-1} \circ \cdots \circ \exp \xi_1(x). \]
But the composite
\[ h_t = \exp \xi_k \circ \exp \xi_{k-1} \circ \cdots \circ \exp \xi_1 \in \mathcal{H}_0 \]
is also defined on $W$ for all $t \in I$. So $t \mapsto h_t(x)$, $t \in I$, is a curve in $\mathcal{H}_0(x)$ joining $h_0(x)$ and $h(x)$.

Second, we prove that $\mathcal{H}_0(x)$ is open in $\mathcal{H}(x)$ for each $x \in T$. By Lemma 10.2(ii), the subset $j^1(\gamma(\mathcal{H}_0))_{x}$ is open in $j^1(\gamma(\mathcal{H}))_{x}$. But it is easy to see that the target projection $\beta : j^1(\gamma(\mathcal{H}))_{x} \to \mathcal{H}(x)$ is open. Therefore $\mathcal{H}_0(x) = \beta(j^1(\gamma(\mathcal{H}_0))_{x})$ is open in $\mathcal{H}(x)$.

The result now follows because each orbit of $\mathcal{H}$ is a disjoint union of orbits of $\mathcal{H}_0$ since $\mathcal{H}_0 \subset \mathcal{H}$. \( \square \)

**Corollary 10.5.** The pseudogroup of local isometries $\mathcal{H}_0$ is closed.

**Proof.** Since $\mathcal{H}$ is closed, it is enough to prove that $j^1(\gamma(\mathcal{H}_0))$ is closed in $j^1(\gamma(\mathcal{H}))$. Let $\sigma_m$ be a sequence in $j^1(\gamma(\mathcal{H}))$ converging to some $\sigma \in j^1(\gamma(\mathcal{H}))$; we have to prove that $\sigma$ is in $j^1(\gamma(\mathcal{H}))$. Let $x$ and $y$ be the source and target of $\sigma$, and let $U$ and $V$ be relatively compact neighborhoods of $x$ and $y$. Then $j^1(\gamma(\mathcal{H}))_{U}^V$ is an open neighborhood of $\sigma \in j^1(\gamma(\mathcal{H}))$, and thus we can assume $\sigma_m \in j^1(\gamma(\mathcal{H}))_{U}^V$ for all $m$; so $\sigma_m$ in $j^1(\gamma(\mathcal{H}))_{U}^V$. Hence $\sigma$ is in the closure of $j^1(\gamma(\mathcal{H}))_{U}^V$ in $j^1(\gamma(\mathcal{H}))$. But the closure of $j^1(\gamma(\mathcal{H}))_{U}^V$ in $j^1(\gamma(\mathcal{H}))$ equals its closure in $j^1(\gamma(\mathcal{H}))$ because $j^1(\gamma(\mathcal{H}))$ is Hausdorff and $j^1(\gamma(\mathcal{H}))_{U}^V$ is relatively compact in $j^1(\gamma(\mathcal{H}))$ (Corollary 10.3). Therefore $\sigma \in j^1(\gamma(\mathcal{H}_0))$. \( \square \)

**Lemma 10.6.** Let $\sigma \in j^1(\gamma(\mathcal{H}_0))_{x}$ for some $x \in T$. Then there is some open neighborhood $V$ of $x$ such that any $\tau \in j^1(\gamma(\mathcal{H}_0))$ close enough to $\sigma$ is the 1-jet of the germ of some element of $\mathcal{H}_0$ with domain $V$.

**Proof.** Let $\sigma = j^1(\gamma(h,x))$ for some $h \in \mathcal{H}_0$ and let $P = \text{dom} \ h$. We can suppose $P \subset U$ for some open set $U$ of those given by Theorem 9.3; i.e., for every relatively compact open set $W$ with $W \subset U$, the elements of $\mathcal{H}_0$ with domain $W$ and close enough to the identity are those of the form $\exp \xi$ for $\xi \in \mathcal{G}(U)$ small enough. Then take as $V$ any open neighborhood of $x$ with $\overline{V} \subset P$.

Let $\tau = j^1(\gamma(g,y))$ for some $g \in \mathcal{H}_0$ and $y \in \text{dom} \ g = W$. Assuming that $W$ is connected, by Lemma 9.1, if $\tau$ is close enough to $\sigma$, we can suppose that $W \subset V$, $g(W) \subset h(V)$, and $g$ is as close to $h$ on $W$ as desired. Thus the
composite $h^{-1} \circ g$ is defined in $W$, and can be made close enough to the identity to be of the form $\exp \xi$ for some small $\xi \in \mathfrak{g}(U)$ by Theorem 9.3. But if $h^{-1} \circ g$ is close enough to the identity on $W$, then $\xi$ is so small that $\exp \xi$ is defined on $V$ and $\exp \xi(V) \subset P$ because $\overline{V} \subset P$ and $\overline{P} \subset U$. Thus $h \circ \exp \xi \in \mathcal{H}_0$ is an extension of $g$ defined on the whole of $V$. □

**Corollary 10.7.** $\mathcal{H}_0$ is complete.

**Proof.** Let $x, y \in T$. Since the orbits of $\mathcal{H}_0$ are closed by Corollary 10.4, if $x$ and $y$ are not in the same orbit of $\mathcal{H}_0$, then there are open neighborhoods $U$ and $V$ of $x$ and $y$ so that no orbit of $\mathcal{H}_0$ intersects both $U$ and $V$. Such a pair $(U, V)$ obviously satisfies the completeness condition of $\mathcal{H}_0$. Therefore we can assume $x$ and $y$ are in the same orbit of $\mathcal{H}_0$; so $j^1(\gamma(\mathcal{H}_0))_x \neq 0$. Now, by Lemma 10.6, each $\sigma \in j^1(\gamma(\mathcal{H}_0))_x$ has an open neighborhood $\Theta_\sigma$ in $j^1(\gamma(\mathcal{H}_0))$ such that, for some neighborhood $U_\sigma$ of $x$, every element of $\Theta_\sigma$ is the 1-jet of some map in $\mathcal{H}_0$ defined on the whole of $U_\sigma$. Because $j^1(\gamma(\mathcal{H}_0))_x$ is compact (Lemma 10.2(i)), it can be covered by a finite number of open sets $\Theta_i = \Theta_{\sigma_i}, i = 1, \ldots, k$, and let $U_i = U_{\sigma_i}$. Since the open sets $j^1(\gamma(\mathcal{H}_0))_x^V$ of $j^1(\gamma(\mathcal{H}_0))_x$ form a neighborhood base of the compact set $j^1(\gamma(\mathcal{H}_0))_x$ when $U$ and $V$ run over the open neighborhoods of $x$ and $y$, there is such a pair $(U, V)$ so that

$$j^1(\gamma(\mathcal{H}_0))_x^V \subset \Theta_1 \cup \cdots \cup \Theta_k.$$ 

Moreover we can assume $U \subset U_1 \cap \cdots \cap U_k$. Then it is easy to check that $(U, V)$ satisfies the completeness condition of $\mathcal{H}_0$; indeed, if $h \in \mathcal{H}_0$ and $z \in U \cap \text{dom} \ h$, then $j^1(\gamma(h, z)) \in j^1(\gamma(\mathcal{H}_0))_x^V$, and thus $j^1(\gamma(h, z)) \in \Theta_i$ for some $i$, yielding that $\gamma(h, z)$ is the germ at $z$ of some $g \in \mathcal{H}_0$ with $\text{dom} \ g = U_i \supset U$. □

Theorem 10.1 is the combination of Corollaries 10.4, 10.5 and 10.7.

The following is a version of Theorem 9.3 in terms of $\mathcal{H}_0$ and local actions of local Lie groups.

**Theorem 10.8.** Let $\mathcal{H}$ be a closed complete Riemannian pseudogroup on a Riemannian manifold $T$. Then $\mathcal{H}_0$ is generated by an effective isometric local action on $T$ of a local Lie group $G$ whose Lie algebra is the structural Lie algebra $\mathfrak{g}$ of $\mathcal{H}$.

**Proof.** According to Theorem 9.3, the sheaf $\mathfrak{g}$ defines an infinitesimal action of $\mathfrak{g}$ on each open set $U$ considered in its statement. By [8, Corollary 1, p. 184, Chapters 2 and 3], this infinitesimal action on each $U$ is induced by a local action of a local Lie group with Lie algebra $\mathfrak{g}$. These local actions can be glued together by [8, Proposition 11, p. 182, Chapters 2 and 3], defining the required local action on $T$, which is effective and isometric by the definition of $\mathfrak{g}$. □

Now suppose that $\mathcal{H}$ is a (possibly non-closed) complete Riemannian pseudogroup on a Riemannian manifold $T$. By Theorem 10.8, the pseudogroup $\mathcal{H}_0$ is generated by an effective isometric local action of a local Lie group $G$ on $T$. Let $\Lambda$ be the subset of $g \in G$ whose local action on $T$ defines an element of $\mathcal{H}$. It easily follows that such a $\Lambda$ is a dense local subgroup of $G$, and the restriction of the local action of $G$ to $\Lambda$ induces the pseudogroup $\mathcal{H}_0 = \mathcal{H} \cap \mathcal{H}_0$. We get that $\mathcal{H}_0$ is complete and $\mathcal{H}_0 = \mathcal{H}_0$.

By Theorems 10.1 and 10.8, or directly from Theorem 9.3, the connected components of the orbits closures of $\mathcal{H}$ are the leaves of a $C^\infty$ singular foliation. It follows that any $\mathcal{H}$-saturated open subset of $T$ is $\mathcal{H}$-saturated too.

**11. Locally homogeneous structure of the orbit closures**

Let $\mathcal{H}$ be a complete Riemannian pseudogroup on a Riemannian manifold $T$. To describe locally the orbit closures of $\mathcal{H}$, we can assume that $\mathcal{H}$ is closed by Theorem 9.2. Even with this assumption, there may not be any pseudogroup in the equivalence class of $\mathcal{H}$ induced by the action of a Lie group [20] (the orbits would be homogeneous spaces). But, since $\mathcal{H}_0$ is generated by the local action of a local Lie group (Theorem 10.8), and since the orbits of $\mathcal{H}_0$ are the connected components of the orbits of $\mathcal{H}$ (Theorem 10.1), the orbits of $\mathcal{H}$ have certain “local homogeneous structure” that will be described in this section.

Let $G$ be the local Lie group with an effective isometric left local action on $T$ that induces $\mathcal{H}_0$. The identity element of $G$ will be denoted by $e$. Such a local action will be denoted by $\mu : \Omega \to T$, where $\Omega$ is an open neighborhood
of \([e] \times T\) in \(G \times T\). If \(\mathfrak{g}\) denotes the Lie algebra of right invariant tangent vector fields on \(G\), we have a corresponding infinitesimal action of \(\mathfrak{g}\) on \(T\).

Fix \(x \in T\). Let \(\mathfrak{k} \subset \mathfrak{g}\) be the Lie subalgebra of elements of \(\mathfrak{g}\) whose infinitesimal transformations of \(T\) vanish at \(x\), and let \(\mathcal{K}\) be the distribution on \(G\) defined by the left translates of \(\mathfrak{k}\); i.e., \(\mathcal{K}_x = L_{g_x} \mathfrak{k}\) for each \(g \in G\). Such \(\mathcal{K}\) is completely integrable because \(\mathfrak{k}\) is a subalgebra, and thus defines a foliation on \(G\) that will be also denoted by \(\mathcal{K}\); this is the left foliation on \(G\) induced by the Lie subalgebra \(\mathfrak{k} \subset \mathfrak{g}\) according to the terminology of [8, Chapter III, § 4.1, pp. 166–167].

Let \(\Omega_x = \{g \in G \mid (g, x) \in \Omega\}\), which is open in \(G\), and let \(\mu_x : \Omega_x \to T\) be defined by \(\mu_x(g) = \mu(g, x)\). Since \(\mu\) is locally transitive on \(\mathcal{H}_0(x)\), the restriction \(\mu_x : \Omega_x \to \mathcal{H}_0(x)\) is a \(C^\infty\) submersion; in particular, it is open. Thus the connected components of the fibers of \(\mu_x\) are the leaves of a foliation, which is easily seen to be equal to the restriction of \(\mathcal{K}\) to \(\Omega_x\).

**Lemma 11.1.** If \(x_n \to x\) in \(\mathcal{H}(x)\), then there is an open neighborhood \(U\) of \(x\) in \(T\) and a sequence \(h_n \in \mathcal{H}_U\) such that \(h_n(x) = x_n\) and \(h_n \to \text{id}_U\) uniformly.

**Proof.** This holds because \(\mu_x : \Omega_x \to \mathcal{H}_0(x)\) is open. \(\square\)

Let \(V\) be an open neighborhood of \(e\) in \(\Omega_x\) which is simple with respect to \(\mathcal{K}\). Then \(\mu_x(V)\) is open in \(\mathcal{H}_0(x)\) because \(\mu_x : \Omega_x \to \mathcal{H}_0(x)\) is open. Let \(Q\) be the corresponding local quotient of \(V\), which is a manifold with a unique \(C^\infty\) structure so that the canonical projection \(V \to Q\) is a \(C^\infty\) submersion. Since \(\mu_x : \Omega_x \to \mathcal{H}_0(x)\) is a \(C^\infty\) submersion, it induces a local diffeomorphism \(\bar{\mu}_x : Q \to \mathcal{H}_0(x)\). We can choose \(V\) cutting each fiber of \(\mu_x\) at most in one plaque, which means that \(\bar{\mu}_x : Q \to \mu_x(V)\) is a bijection, and thus a diffeomorphism.

**12. Version of Molino’s theory for pseudogroups**

The following is the obvious adaptation to pseudogroups of Molino’s description of the so-called Riemannian foliations [37]. The proofs from [37] can be easily adapted to this setting too.

A \(C^\infty\) pseudogroup acting on a \(C^\infty\) manifold \(T\) is said to be parallelizable if its maps preserve same parallelism of \(T\); such a pseudogroup becomes Riemannian by declaring this parallelism to be orthonormal. Suppose that moreover \(\mathcal{H}\) is complete. Then the following properties hold:

- \(\mathcal{H}/T\) is a manifold with a unique \(C^\infty\) structure so that the canonical projection \(\pi : T \to \mathcal{H}/T\) is a \(C^\infty\) submersion.
- Each \(\mathcal{H}\)-orbit closure \(F\) has an open neighborhood \(U\) in \(\mathcal{H}/T\) such that \(\mathcal{H}|_{\pi^{-1}(U)} \cong \mathcal{H}|_F \times U\). \hfill (12.1)

Now, let \(\mathcal{H}\) be a complete Riemannian pseudogroup on a Riemannian manifold \(T\). The tangent homomorphisms \(\mathcal{T}h\) of maps \(h \in \mathcal{H}\) generate a pseudogroup \(\mathcal{T} \mathcal{H}\) on the tangent bundle \(TT\), and the bundle projection \(\pi_T : TT \to T\) generates a morphism \(\Pi_T : \mathcal{T} \mathcal{H} \to \mathcal{H}\). The pseudogroup \(\mathcal{T} \mathcal{H}\) is complete and Riemannian with respect to the Sasaki metric on \(TT\).

Let \(O(T)\) denote the \(O(n)\)-principal bundle of orthonormal frames of \(TT\), \(n = \text{dim } T\). For any \(h \in \mathcal{H}\), let \(O(h) : O(\text{dom } h) \to O(\text{im } h)\) be the map defined by \(O(h)(f_1, \ldots, f_n) = (T h(f_1), \ldots, T h(f_n))\).

The maps \(O(h)\) generate a pseudogroup \(O(\mathcal{H})\) acting on \(O(T)\), and the bundle projection \(\pi_O : O(T) \to T\) generates a homomorphism \(\Pi_O : O(\mathcal{H}) \to \mathcal{H}\). The pseudogroup \(O(\mathcal{H})\) is parallelizable and complete, obtaining the following properties:

- \(W = \overline{O(\mathcal{H}) \setminus O(T)}\) is a manifold with a unique \(C^\infty\) structure so that the canonical projection \(\pi : O(T) \to \overline{O(\mathcal{H}) \setminus O(T)}\) is a \(C^\infty\) submersion.
- The action of \(O(n)\) on \(O(T)\) induces a \(C^\infty\) right action of \(O(n)\) on \(W\). The canonical projection \(W \to W / O(n)\) will be denoted by \(\bar{\pi}\).
The map
\[
\tilde{H} \setminus T \to W/O(n), \quad F \mapsto \tilde{\pi} \circ \pi(\pi_0^{-1}(F)),
\]  
(12.2)
is a homeomorphism, whose inverse is given by \( E \mapsto \pi_0((\tilde{\pi} \circ \pi)^{-1}(E)). \)

The homeomorphism (12.2) induces a bijection \( C^\infty(W/O(n)) \to C^\infty(\tilde{H} \setminus T), \) where \( C^\infty(W/O(n)) \) and \( C^\infty(\tilde{H} \setminus T) \) are the sets of real valued functions on \( W/O(n) \) and \( \tilde{H} \setminus T \) inducing \( C^\infty \) functions on \( W \) and \( T \) via \( \tilde{\pi} \) and the canonical projection \( T \to \tilde{H} \setminus T, \) respectively.

As a first consequence of the above properties, it follows that each locally finite open covering of \( \tilde{H} \setminus T \) has a subordinated partition of unity consisting of functions in \( C^\infty(\tilde{H} \setminus T). \)

Consider the nested sequence
\[
\emptyset = T_{-1} \subset T_0 \subset \cdots \subset T_n = T,
\]
where each \( T_\ell \) is the union of all orbit closures of \( \mathcal{H} \) with dimension \( \leq \ell. \) Since \( \tilde{H} \setminus T \) is homeomorphic to \( W/O(n), \) it follows that every \( T_\ell \) is closed in \( T, \) each \( T_\ell \setminus T_{\ell-1} \) is open and dense in \( T_\ell, \) and \( T_\ell \setminus T_{\ell-1} \) is an \( \overline{H} \)-invariant \( C^\infty \) submanifold of \( T. \) Thus the orbit closures define a \( C^\infty \) regular foliation on \( T_\ell \setminus T_{\ell-1} \) because all of them have the same dimension.

Recall the following definitions from e.g.\([39].\) The order of a family of subsets, not all empty, of some set is the largest integer \( k \) for which there is a subfamily with \( k + 1 \) elements whose intersection is non-empty, or is \( \infty \) if there is no such largest integer. A family of empty sets is declared to have order \( -1. \) Then the covering dimension \( \dim X \) of a space \( X \) is the least integer \( k \) such that every finite open covering of \( X \) has an open refinement of order not exceeding \( k \) or is \( \infty \) if there is no such integer. By \([39, \text{Theorem } 4.3],\) if \( X \) is normal, then we can use locally finite open coverings instead of finite ones in the definition of covering dimension; thus we can use arbitrary open coverings when \( X \) is normal and paracompact.

For any \( C^\infty \) action of a compact Lie group \( G \) on a \( C^\infty \) manifold \( M, \) consider the isotropy type stratification of \( M/G \) \([44]: \) two orbits have the same isotropy type (and thus belong to the same stratum) when they have the same conjugacy classes of isotropy groups in \( G. \) Any orbit type stratum is a manifold, and is open in its closure, which consists of strata of lower dimension \([44].\) Then, by applying Proposition 1.5, Theorem 2.5 and Corollary 5.8 of Chapter 3 of \([39]\) to the inclusion \( S \subset S \) for each isotropy type stratum \( S, \) we get by induction on \( \dim S \) that \( \dim M/G \) equals the top codimension of the orbits. Thus, using the above homeomorphism between \( \tilde{H} \setminus T \) and \( W/O(n), \) we get that \( \dim \tilde{H} \setminus T \) equals the top codimension of the orbit closures; in particular, it is finite.

13. Description around the orbit closures

In this section, we describe of complete Riemannian pseudogroups around the orbit closures. This is an adaptation of the description of transversely complete Riemannian foliations around the leaf closures given by Molino \([36].\) A more refined description around the orbit closures, with a complete set of invariants, was given by Haefliger \([20].\)

**Notation 3.** Let \( M \) be a metric space with distance function \( d. \) For each \( x \in M \) and \( r > 0, \) we use the standard notation \( B(x, r) \) for the open \( r \)-ball in \( M \) centered at \( x. \) For any subset \( S \subset M, \) Pen\((F, r)\) denotes the \( r \)-penumbra of \( S \) in \( M, \) whose definition is
\[
\text{Pen}(F, r) = \{ y \in M \mid d(y, K) < r \} = \bigcup_{x \in M} B(x, r).
\]

Let \( \mathcal{H} \) be a complete Riemannian pseudogroup on a Riemannian manifold \( T. \) With the notation of the above section, the exponential map is defined on some open neighborhood \( \tilde{\Omega} \) of the zero section of \( TT; \) let \( \tilde{\Omega}' = T\mathcal{H}(\tilde{\Omega}). \) By elementary properties, \( \exp : \tilde{\Omega} \to T \) generates a morphism \( \text{Exp} : T\mathcal{H}(\tilde{\Omega}) \to \mathcal{H}. \) Let \( \Omega = (\pi, \exp)(\tilde{\Omega}) \) and \( \Omega' = (\mathcal{H} \times \mathcal{H})(\tilde{\Omega}). \) Observe that, if two orbit closures \( F_1 \) and \( F_2 \) are close enough in \( \tilde{H} \setminus T, \) then \( F_1 \times F_2 \subset \tilde{\Omega}'. \) We can choose \( \tilde{\Omega} \) so that \((\pi, \exp) : \tilde{\Omega} \to \Omega \) is a diffeomorphism, and thus \((\Pi, \text{Exp}) : T\mathcal{H}(\tilde{\Omega}) \to (\mathcal{H} \times \mathcal{H})(\tilde{\Omega}) \) is an isomorphism. Moreover we can assume that \( t \cdot \tilde{\Omega} \subset \tilde{\Omega} \) for all \( t \in I. \) Hence the mapping \((v, t) \mapsto t \cdot v\) generates a homotopy \( \tilde{\Psi} : T\mathcal{H}(\tilde{\Omega}) \times I \to T\mathcal{H}(\tilde{\Omega}), \) which defines a homotopy \( \Psi : (\mathcal{H} \times \mathcal{H})(\tilde{\Omega}) \times I \to (\mathcal{H} \times \mathcal{H})(\tilde{\Omega}) \) via \((\Pi, \text{Exp}) : T\mathcal{H}(\tilde{\Omega}) \to (\mathcal{H} \times \mathcal{H})(\tilde{\Omega}). \) Observe that \( \Psi_0 \) is generated by mapping \((x, y) \mapsto (x, x), \) and \( \Psi_1 \) is the identity morphism at \((\mathcal{H} \times \mathcal{H})(\tilde{\Omega}). \)
For any \( x \in T \), let \( F = \overline{H}(x) \), which is \( H \)-invariant. The normal subbundle \( TF^\perp \subset TT \) is invariant by \( TH \). For \( \epsilon > 0 \), besides the \( \epsilon \)-penumbra \( \text{Pen}(F, \epsilon) \) of \( F \) in \( M \), consider the \( \epsilon \)-penumbra \( \overline{\text{Pen}}(F, \epsilon) \) of the zero section in \( TF^\perp \). There is some \( \epsilon > 0 \) so that \( \overline{\text{Pen}}(F, \epsilon) \subset \overline{\Omega}^1 \) and the restriction

\[
\exp: \overline{\Omega} \cap \text{Pen}(F, \epsilon) \to \exp(\overline{\Omega}) \cap \text{Pen}(F, \epsilon)
\]

is a diffeomorphism. Let \( H_{F, \epsilon} \) and \( TH_{F, \epsilon} \) denote the restrictions of \( H \) and \( TH \) to \( \text{Pen}(F, \epsilon) \) and \( \overline{\text{Pen}}(F, \epsilon) \), respectively. Then the restriction \( \exp:H_{F, \epsilon} \to H_{F, \epsilon} \) is an isomorphism generated by (13.1). We have \( \overline{\Psi}(\text{Pen}(F, \epsilon) \times I) \subset \overline{\text{Pen}}(F, \epsilon) \), and thus \( \overline{\Psi} \) induces a homotopy \( \Psi_{F, \epsilon}: H_{F, \epsilon} \times I \to H_{F, \epsilon} \) via \( \exp:H_{F, \epsilon} \to H_{F, \epsilon} \). Notice that \( \text{im}\Psi_{F, \epsilon, 0} \subset F \), \( \Psi_{F, \epsilon, 1} \) is the identity morphism at \( H_{F, \epsilon} \), and, for each \( x \in \text{Pen}(F, \epsilon) \), the restriction of \( \Psi_{F, \epsilon} \) to \( \{x\} \times I = I \) is a geodesic segment of \( H \) whose length is \(< d(x, F) \) (in the sense of Section 3). Let \( f = (f_1, \ldots, f_n) \in O(T)|F \). Suppose that \( f \) is adapted to \( F \) in the sense that \( f_1, \ldots, f_r \in TF \) and \( f_{r+1}, \ldots, f_n \in TF^\perp \), where \( r = \dim F \). The orbit closure \( P = \overline{O(H)(f)} \) is a principal bundle over \( F \) with structure Lie group \( H \subset O(n) \). Since \( P \) consists of frames adapted to \( F \), it follows that \( H \subset O(r) \times O(n') \), where \( n' = n - r \). Let \( H' \) denote the projection of \( H \) to \( O(n') \). The restriction of \( O(H) \) to \( P \) will be denoted by \( H_P \), and the bundle projection \( P \to F \) generates a morphism \( H_P \to H_F \).

On the other hand, let \( O(T F^\perp) \) denote the \( (n') \)-principal bundle over \( F \) of orthonormal frames of \( TF^\perp \). For each \( h \in H \), let

\[
O^1_F(h) : O(T (F \cap \text{dom} h)^\perp) \to O(T (F \cap \text{im} h)^\perp)
\]

be the map defined by

\[
O^1_F(h)(e_1, \ldots, e_n) = (Th(e_1), \ldots, Th(e_n)).
\]

The maps \( O^1_F(h) \) generate a pseudogroup \( O^1_F(H) \) on \( O(T F^\perp) \), and the bundle projection \( O(T F^\perp) \to F \) generates a morphism \( O^1_F(H) \to H_F \). There is a canonical map \( P \to O(T F^\perp) \) defined by forgetting the first \( r \) components of each frame. Its image is an \( H' \)-principal bundle over \( F \) denoted by \( P' \), which is invariant by \( O^1_F(H) \). The restriction of \( O^1_F(H) \) to \( P' \) will be denoted by \( H_{P'} \); observe that \( H_{P'} \) has dense orbits. Moreover the canonical projection \( P \to P' \) generates a morphism \( H_P \to H_{P'} \).

Let \( B_\epsilon \) denote the open ball in \( \mathbb{R}^{n'} \) of radius \( \epsilon \) and centered at the origin. Consider the diagonal action of \( H' \) on \( P' \times B_\epsilon \), and let \( P' \times_{H'} B_\epsilon \) denote the corresponding quotient space. As usual, we get a canonical identity

\[
\overline{\text{Pen}}(F, \epsilon) = P' \times_{H'} B_\epsilon.
\]

Furthermore \( H_{P'} \times B_\epsilon \) induces a pseudogroup \( H_{P'} \times_{H'} B_\epsilon \) acting on the quotient \( P' \times_{H'} B_\epsilon \). Then (13.2) generates an identity

\[
TH_{F, \epsilon} = H_{P'} \times_{H'} B_\epsilon,
\]

yielding that

\[
H_{F, \epsilon} \cong H_{P'} \times_{H'} B_\epsilon
\]

via the isomorphism \( \exp: TH_{F, \epsilon} \to H_{F, \epsilon} \), which generalizes (12.1) when \( U \) is diffeomorphic to an open ball in \( \mathbb{R}^{n'} \).

14. Riemannian foliations

A \( C^\infty \) foliation \( \mathcal{F} \) on a manifold \( M \) is said to be Riemannian when its holonomy pseudogroup is Riemannian for some metric. In this case, there is a foliated cocycle of \( \mathcal{F} \) consisting of Riemannian submersions for some Riemannian metric on \( M \), which is called a bundle-like metric \([41,37]\); thus the geodesics orthogonal to the leaves are projected to geodesics by the maps of such foliated cocycle. A characteristic property of bundle-like metrics is that, if any geodesic is orthogonal to the leaves at some point, then it remains orthogonal to the leaves at every point \([41,37]\); these geodesics are called horizontal. This condition can be considered for a \( C^\infty \) singular foliation too, obtaining the definition of singular Riemannian foliation \([37]\). A Riemannian foliation \( \mathcal{F} \) on a manifold \( M \) is said to be transversely complete when the horizontal geodesics are complete for some bundle-like metric (a transversely complete bundle-like metric); in this case, \( \text{Hol}(\mathcal{F}) \) is complete \([37]\) (this is a direct consequence of Lemma 15.2 below, which is included here for other purposes).
For instance, for any isometric local action of a local Lie group on a Riemannian manifold, the connected components of the orbits are the leaves of a singular Riemannian foliation. Therefore, by Theorems 10.1 and 10.8, or by the version of Molino’s theory for pseudogroups (Section 12), the connected components of the orbit closures of a complete Riemannian pseudogroup are the leaves of a singular Riemannian foliation [20]. It follows that the leaf closures of any transversely complete Riemannian foliation $\mathcal{F}$ on a manifold $M$ are the leaves of a singular Riemannian foliation $\mathcal{F}$, which is also a consequence of Molino’s theory [37]. Like for complete Riemannian pseudogroups, it follows that any $\mathcal{F}$-saturated open set of $M$ is $\mathcal{F}$-saturated too.

For any transversely complete Riemannian foliation $\mathcal{F}$ on a manifold $M$, the following properties follow from the corresponding ones for pseudogroups (Sections 12 and 13), or from Molino’s theory [37]:

- The space $M/\mathcal{F}$ is homeomorphic to a space of orbit closures of an action of a compact Lie group. Let $C^\infty(M/\mathcal{F})$ be the set of functions $M/\mathcal{F} \rightarrow \mathbb{R}$ whose composite with the canonical projection $M \rightarrow M/\mathcal{F}$ is a $C^\infty$ function on $M$. For every locally finite open covering of $M/\mathcal{F}$, there is a subdivided partition of unity consisting of functions in $C^\infty(M/\mathcal{F})$.
- Let $p = \dim \mathcal{F}$ and $q = \codim \mathcal{F}$. Consider the nested sequence

$$\emptyset = M_{-1} \subset M_0 \subset \cdots \subset M_q = M,$$

where each $M_\ell$ is the union of all leaf closures of $\mathcal{F}$ with dimension $\leq p + \ell$. Every $M_\ell$ is closed in $M$. Each $M_\ell \setminus M_{\ell-1}$ is open and dense in $M_\ell$, and is an $\mathcal{F}$-saturated $C^\infty$ submanifold of $T$. The restriction of $\mathcal{F}$ to $M_\ell \setminus M_{\ell-1}$ is a regular Riemannian foliation because all leaf closures have the same dimension on this submanifold.
- The covering dimension of $M/\mathcal{F}$ equals the top codimension of the leaf closures, which is thus finite.
- There is a description of $\mathcal{F}$ around each leaf closure that corresponds to (13.3). A finer description around the leaf closures was also given by Haefliger [20].

Let $\mathcal{F}_T$ be the foliation on $TM$ defined by the following condition. For each $x \in M$ and $v \in T_x \mathcal{F}$, take some simple open neighborhood $U$ of $x$, and let $P$ be the plaque of $U$ that contains $x$. Then a neighborhood of $v$ in the corresponding leaf of $\mathcal{F}_T$ is given by all tangent vectors of the form $X(y)$, where $y \in P$, and $X$ runs in the set of infinitesimal transformations of $\mathcal{F}|_U$ so that $X(x) = v$. The projection of the leaves of $\mathcal{F}_T$ to the normal bundle $\nu \mathcal{F} = TM/T \mathcal{F}$ are the leaves of a foliation $\nu \mathcal{F}$, which is the horizontal lift of $\mathcal{F}$ with respect to the partial Bott connection. Observe that the bundle projections $\pi_T: TM \rightarrow M$ and $\nu_T: \nu \mathcal{F} \rightarrow M$ are foliated maps $\mathcal{F}_T \rightarrow \mathcal{F}$ and $\nu \mathcal{F} \rightarrow M$, respectively. Moreover $\pi_T$ restricts to covering maps from the leaves of $\mathcal{F}_T$ to the leaves of $\mathcal{F}$.

Via the canonical identity $\nu \mathcal{F} \equiv T\mathcal{F}^\perp$, $\mathcal{F}_T$ corresponds to a foliation $\mathcal{F}$ on $T\mathcal{F}^\perp$. Then $\pi_T$ restricts to a foliated map $\mathcal{F}_T \rightarrow \mathcal{F}$, whose restrictions to the leaves are covering maps.

By the above definition of $\mathcal{F}_T$, the mapping $(v, t) \mapsto t \cdot v$ defines a $C^\infty$ foliated map $H_T: \mathcal{F}_T \times \mathbb{R}_{pt} \rightarrow \mathcal{F}_T$, which induces a $C^\infty$ foliated map $H_\nu: \nu \mathcal{F} \times \mathbb{R} \rightarrow \nu \mathcal{F}$. Obviously, $H_T(T\mathcal{F}^\perp \times \mathbb{R}) \subset T\mathcal{F}^\perp$, and the restriction $H_T: T\mathcal{F}^\perp \times \mathbb{R} \rightarrow T\mathcal{F}^\perp$ is foliated maps $\mathcal{F}_T \rightarrow \mathcal{F}$ and $\nu \mathcal{F} \rightarrow \mathcal{F}$, respectively. Moreover $\pi_T$ restricts to covering maps from the leaves of $\mathcal{F}_T$ to the leaves of $\mathcal{F}$.

Since the metric is bundle-like and transversely complete, the exponential map $\exp$ of $M$ is defined on the whole of $T\mathcal{F}^\perp$ and is a foliated map $\mathcal{F} \rightarrow \mathcal{F}$. Moreover there is an $\mathcal{F}$-saturated open neighborhood $\mathcal{O}$ of the zero section of $T\mathcal{F}^\perp$ such that the map $(\pi_T, \exp): \mathcal{O} \rightarrow M \times M$ is a $C^\infty$ embedding, and $t \cdot \mathcal{O} \subset \mathcal{O}$ for all $t \in I$. Then $\Omega = (\pi, \exp)(\mathcal{O})$ is a regular $C^\infty$ submanifold of $M \times M$, and $(\pi_T, \exp): \mathcal{O} \rightarrow (\mathcal{F} \times \mathcal{F})|_\mathcal{O}$ is a foliated diffeomorphism. So $\tilde{H}$ induces via $(\pi, \exp)$ a $C^\infty$ foliated homotopy $H: (\mathcal{F} \times \mathcal{F})|_\mathcal{O} \times \mathbb{R}_{pt} \rightarrow (\mathcal{F} \times \mathcal{F})|_\mathcal{O}$. Observe that $H_0(x, y) = (x, x)$ for all $(x, y) \in \Omega$, and $H_1 = \text{id}_\mathcal{O}$.

Let $F$ be a leaf closure of $\mathcal{F}$. For $\epsilon > 0$, consider the $\epsilon$-penumbra $\text{Pen}(F, \epsilon) \subset M$, and the $\epsilon$-penumbra $\widetilde{\text{Pen}}(F, \epsilon)$ of the zero section in $T\mathcal{F}^\perp$. Let $\mathcal{F}_{F, \epsilon} = \mathcal{F}|_{\text{Pen}(F, \epsilon)}$ and $\widetilde{\mathcal{F}}_{F, \epsilon} = \widetilde{\mathcal{F}}|_{\widetilde{\text{Pen}}(F, \epsilon)}$. Then $\tilde{H}$ restricts to a foliated map $\mathcal{F}_{F, \epsilon} \rightarrow \mathcal{F}$. If $\epsilon$ is small enough, then $\text{Pen}(F, \epsilon) \subset \mathcal{O}$ and $H_\mathcal{O}: \mathcal{O} \rightarrow \mathcal{F}_{F, \epsilon}$ is a foliated $C^\infty$ diffeomorphism. On the other hand, we have $H(\text{Pen}(F, \epsilon) \times I) \subset \text{Pen}(F, \epsilon)$, and thus $\tilde{H}$ induces a $C^\infty$ foliated homotopy $H_{F, \epsilon}: \mathcal{F}_{F, \epsilon} \times I \rightarrow \mathcal{F}_{F, \epsilon}$ via $\exp: \mathcal{F}_{F, \epsilon} \rightarrow \mathcal{F}_{F, \epsilon}$, and the mapping $t \mapsto H_{F, \epsilon}(x, t)$ is a horizontal geodesic segment of length $< d(x, F)$ for each $x \in \text{Pen}(F, \epsilon)$. 


15. Existence of complete bundle-like metrics

Let $\mathcal{F}$ be a transversely complete Riemannian foliation on a manifold $M$, and let $g$ be a transversely complete bundle-like metric on $M$. The goal of this section is to prove the following result, which is an adaptation of a theorem of [38] to Riemannian foliations.

**Proposition 15.1.** There exists a complete bundle-like metric $g'$ on $M$ such that:

(i) $g$ and $g'$ define the same orthogonal complement $T\mathcal{F}^\perp$ of $T\mathcal{F}$;
(ii) $g$ and $g'$ have the same restriction to $T\mathcal{F}^\perp$; and
(iii) $g'$ is a conformal change of $g$ on the leaves.

Let $d_{\mathcal{F}}$ denote the distance function of the leaves. For each $x \in M$ and $r > 0$, let $B_{\mathcal{F}}(x, r)$ denote the open $r$-ball in the leaf $L_x$. Also, let $\tilde{B}(x, r)$ be the open $r$-ball in $T_{\mathcal{F}}^\perp$ centered at the origin, and let $B_{\perp}(x, r) = \exp(\tilde{B}(x, r))$. Observe that, if $r$ is small enough, then $B_{\perp}(x, r)$ is a $C^\infty$ local transversal of $\mathcal{F}$ through $x$. For $S \subset M$, let

$$\widetilde{\text{Pen}}(S, r) = \bigcup_{x \in S} \tilde{B}(x, r),$$

$$\text{Pen}_{\perp}(S, r) = \bigcup_{x \in S} B_{\perp}(x, r) = \exp(\widetilde{\text{Pen}}(S, r)).$$

**Lemma 15.2.** For each compact subset $K \subset M$, there is some $s > 0$ such that:

(i) $\exp : \tilde{B}(x, s) \rightarrow M$ is a $C^\infty$ embedding transverse to $\mathcal{F}$ for all $x$ in the $\mathcal{F}$-saturation $K'$ of $K$; and
(ii) for any leaf $L$ that meets $K$ and any curve $\gamma : I \rightarrow L$, there is a continuous map $h_\gamma : I \times B_{\perp}(\gamma(0), s) \rightarrow M$, which is (piecewise) $C^\infty$ if $\gamma$ is (piecewise) $C^\infty$, and such that

$$h_\gamma(t) = B_{\perp}(\gamma(t), s)$$

for all $t \in I$, and

$$h_\gamma(0, z) = z, \quad h_\gamma(I \times \{z\}) \subset L_z$$

for every $z \in B_{\perp}(x, s)$.

**Proof.** With the notation of Section 14, the map $\exp : \tilde{\Omega} \cap T_{\mathcal{F}}^\perp \rightarrow M$ is a $C^\infty$ embedding transverse to $\mathcal{F}$ for each $x \in M$. On the other hand, by the compactness of $K$, there is some $s > 0$ such that $\text{Pen}(K, s) \subset \tilde{\Omega}$. Since $\tilde{\Omega}$ is $\mathcal{F}$-saturated and $\text{Pen}(K', s)$ is the $\mathcal{F}$-saturation of $\text{Pen}(K, s)$, it follows that $\text{Pen}(K', s) \subset \tilde{\Omega}$, yielding part (i).

For each $v \in T_{\gamma(0)}\mathcal{F}^\perp$, let $L_v$ be the leaf of $\mathcal{F}$ through $v$. Since $\pi_{\mathcal{F}}$ restricts to covering maps from the leaves of $\mathcal{F}$ to the leaves of $\mathcal{F}$, there is a unique curve $\tilde{\gamma}_v : I \rightarrow L_v$ such that $\pi_{\mathcal{F}} \circ \tilde{\gamma}_v = \gamma$. This $\tilde{\gamma}_v$ is (piecewise) $C^\infty$, and has a $C^\infty$ dependence on $v$. Then part (ii) is satisfied with the map $h_\gamma : I \times B_{\perp}(x, s) \rightarrow M$ defined by $h_\gamma(x, z) = \exp(\tilde{\gamma}_v(t))$, where $v$ is the unique point in $\tilde{B}(\gamma(0), s)$ satisfying $\exp(v) = z$.  

**Lemma 15.3.** For any $S \subset M$ and $s > 0$, $\text{Pen}_{\perp}(S, s)$ is compact if and only if $\tilde{S}$ is compact.

**Proof.** The “only if” part is obvious because $S \subset \text{Pen}_{\perp}(S, s)$.

If $\tilde{S}$ is compact, then $\text{Pen}(S, s)$ has compact closure in $T\mathcal{F}^\perp$. Since the domain of its exponential map contains $T\mathcal{F}^\perp$ because $g$ is transversely complete, it follows that $\text{Pen}_{\perp}(S, s) = \exp(\text{Pen}(S, s))$ has compact closure in $M$.  

For $x \in M$ and $r, s > 0$, let

$$\Pi(x, r, s) = \text{Pen}_{\perp}(B_{\mathcal{F}}(x, r), s).$$

Observe that

$$\Pi(x, r, s) \subset B(x, r + s).$$

(15.1)
Lemma 15.4. If the leaves are complete Riemannian submanifolds, then $g$ is complete.

Proof. If the leaves are complete, then $\bar{\Pi}(x, r, s)$ is compact for all $x \in M$ and $r, s > 0$ by Lemma 15.3. Hence $\bar{B}(x, r)$ is compact for all $x \in M$ and $r > 0$ by (15.1), and thus $g$ is complete. \qed

Let $\tau : M \to (0, \infty]$ be the function defined by

$$\tau(x) = \sup\{r > 0 \mid B_F(x, r) \text{ is compact}\}.$$  

For any leaf $L$ of $\mathcal{F}$ and all $x, y \in L$, we easily get

$$|\tau(x) - \tau(y)| < d_F(x, y).$$  \hspace{1cm} (15.2)

Lemma 15.5. $\tau$ is continuous.

Proof. The continuity of $\tau$ on the leaves follows from (15.2), but the statement asserts the continuity of $\tau$ on $M$.

Fix some $x \in M$, and let $K$ be a compact neighborhood of $x$. Take some $s > 0$ satisfying the statement of Lemma 15.2 with this $K$. Let $x_n$ be a sequence in $K$ converging to $x$. By the above observation, we can assume that $x_n \in B_F(x, s)$ for all $n$.

For $0 < r' < r < \tau(x)$ and $n$ large enough, we have $B_F(x_n, r') \subset \Pi(x, r, s)$ by Lemma 15.2, and thus $\bar{B}_F(x_n, r')$ is compact by Lemma 15.3. It follows that

$$\tau(x) \leq \liminf_{n \to \infty} \tau(x_n).$$

For $0 < r < r'$ and $n$ large enough, we get $B_F(x, r) \subset \Pi(x_n, r', s)$ by Lemma 15.2. Hence $\bar{B}_F(x, r)$ is compact if $\bar{B}_F(x_n, r')$ is compact by Lemma 15.3. This yields

$$\tau(x) \geq \limsup_{n \to \infty} \tau(x_n),$$

and the result follows. \qed

Observe that $\tau^{-1}(\infty)$ is a saturated set: it is the union of complete leaves. Moreover it is closed by Lemma 15.5, but it may not be open. Thus there may be points $x$ and $y$ in the same connected component of $M$ such that $\tau(x) = \infty$ and $\tau(y) < \infty$. Therefore the argument of [38] has to be slightly modified to finish the proof of Proposition 15.1. We proceed as follows.

Proof of Proposition 15.1. Setting $\frac{1}{\infty} = 0$ as usual, the mapping $x \mapsto \max\{\frac{1}{\tau(x)}, 1\}$ is continuous by Lemma 15.5. So there is a $C^\infty$ function $\omega : M \to \mathbb{R}$ such that $\omega(x) > \max\{\frac{1}{\tau(x)}, 1\}$ for all $x \in M$. Let $g'$ be the Riemannian tensor on $M$ satisfying the properties (i)–(iii) of Proposition 15.1, whose restriction to the leaves is equal to $\omega^2 g$. For each $x \in M$ and every $r > 0$, let $B'_F(x, r)$ be the open $g'$-ball in $L_x$ of center $x$ and radius $r$.

Claim 1. For all $x \in M$, we have

$$B'_F(x, \frac{1}{3}) \subset \begin{cases} B_F(x, \frac{\tau(x)}{2}) & \text{if } \tau(x) < \infty, \\ B_F(x, \frac{1}{2}) & \text{if } \tau(x) = \infty. \end{cases}$$

By Claim 1, $B'_F(x, \frac{1}{3})$ is compact for all $x \in M$. Hence $g'$ has complete restrictions to the leaves, and Proposition 15.1 follows by Lemma 15.4.

Let us prove Claim 1. For any leaf $L$ and any $C^\infty$ curve $c : I \to L$ joining points $x$ and $y$, let $\ell$ and $\ell'$ its lengths defined by $g$ and $g'$, respectively. By the mean value theorem, we have

$$\ell' = \int_0^1 \omega(c(t)) \|\gamma'(t)\| \, dt$$
\[
\omega(c(t_0)) = \int_0^1 \|y'(t)\| \, dt
\]
for some \( t_0 \in I \).

Assume first that \( r(x) < \infty \), and thus \( r(z) < \infty \) for all \( z \in L \). Let \( d'_F \) denote the \( g' \)-distance function on the leaves.

Suppose that \( d_F(x, y) \geq r(x)^2 \). Then

\[
\ell' > \frac{\ell}{r(x) + \ell} \geq \frac{1}{3}
\]
by (15.4) and since \( \ell \geq \frac{r(x)}{2} \). Hence \( d'_F(x, y) \geq \frac{1}{3} \), showing Claim 1 in this case.

Suppose now that \( r(x) = \infty \). So \( r(L) = \infty \) and \( \omega(L) = 1 \). Therefore \( \ell' = \ell \) by (15.3), yielding \( d'_F(x, y) = d_F(x, y) \).

We get \( B'_F(x, \frac{1}{3}) = B_F(x, \frac{1}{3}) \), which completes the proof of Claim 1.

Let us use the term horizontal metric for the Carnot–Caratheodory metric \( d_H : M \times M \to [0, \infty] \) induced by the polarization \( T F^\perp \subset TM \) (see e.g. [15]), which is defined as follows. A horizontal curve in \( M \) is a piecewise \( C^\infty \) curve orthogonal to the leaves of \( F \). Then \( d_H(x, y) = \infty \) if there is no horizontal curve between \( x \) and \( y \), and, otherwise, \( d_H(x, y) \) is the infimum of the lengths of horizontal curves between \( x \) and \( y \). For \( x \in M \) and \( r > 0 \), the horizontal ball of radius \( r > 0 \) and center \( x \) is the set

\[
B_H(x, r) = \{y \in N \mid d_H(x, y) < r\}.
\]

For \( K \subset M \), the horizontal penumbra of radius \( r \) around \( K \) is the set

\[
\text{Pen}_H(K, r) = \bigcup_{x \in K} B_H(x, r).
\]

**Proposition 15.6.** If \( K \) is compact, then \( \overline{\text{Pen}_H(K, r)} \) is compact.

**Proof.** By Proposition 15.1, there is a complete bundle-like metric on \( M \) defining the same horizontal penumbras as \( g \). So we can assume that \( g \) is complete. Then \( \text{Pen}(K, r) \) is compact, and the result follows because \( \overline{\text{Pen}_H(K, r)} \subset \text{Pen}(K, r) \).

16. Morphisms between complete Riemannian pseudogroups

Let \( \mathcal{H} \) and \( \mathcal{H}' \) be complete Riemannian pseudogroups on Riemannian manifolds \( T \) and \( T' \), respectively, and let \( F \) and \( F' \) be the corresponding possibly singular Riemannian foliations defined by their orbits closures. For any morphism \( \Phi : \mathcal{H} \to \mathcal{H}' \), every \( \phi : U \to T' \) in \( \Phi \) is a foliated map \( F|_U \to F' \). The morphism \( \Phi \) is said to be of class \( C^{0,\infty} \) if all of its elements are \( C^{0,\infty} \) as foliation maps in the above sense. The set of all \( C^{0,\infty} \) morphisms \( \mathcal{H} \to \mathcal{H}' \) will be denoted by \( C^{0,\infty}(\mathcal{H}, \mathcal{H}') \).

Now, our main result (Theorem A) can be stated as follows.

**Theorem 16.1.** With the above notation and conditions, any morphism \( \Phi : \mathcal{H} \to \mathcal{H}' \) satisfies the following properties:
(i) $\Phi$ is complete.
(ii) $\Phi$ generates a morphism $\overline{\Phi} : \overline{\mathcal{H}} \to \overline{\mathcal{H}}$.
(iii) $\Phi$ is of class $C^{0,\infty}$.

The morphism $\overline{\Phi}$ of Theorem 16.1(ii) will be called the closure of $\Phi$.

**Remarks 2.** In Theorem 16.1, observe the following:

(a) By property (iii), $\Phi$ is $C^\infty$ if $\mathcal{H}'$ has dense orbits.
(b) With the notation of Section 9 and Theorem 16.1, observe that $\Phi_{\text{orb}} = \overline{\Phi}_{\text{orb}} : \overline{\mathcal{H}} \setminus T \to \overline{\mathcal{H}} \setminus T'$.

Theorem 16.1 will follow from the following proposition, whose large proof is given in the following section.

**Proposition 16.2.** Let $\mathcal{H}$, $\mathcal{H}'$ and $\Phi$ be as in the statement of Theorem 16.1. For each $\phi \in \Phi$ and any $x \in \text{dom} \phi$, there is an open neighborhood $U$ of $x$ in $\text{dom} \phi$ satisfying the following properties:

(i) $(U, U)$ is a completeness pair of $\overline{\mathcal{H}}$.
(ii) There exists a neighborhood $O$ of $\text{id}_U$ in $\overline{\mathcal{H}}_U$ such that, if $h \in O$, then $h(U) \subset \text{dom} \phi$ and there is some $h' \in \overline{\mathcal{H}}$ with $\phi(U) \subset \text{dom} h'$ and so that $h' \circ \phi = \phi \circ h$ on the whole of $U$.
(iii) If $h'_1, h'_2 \in \overline{\mathcal{H}}$ satisfy $\phi(U) \subset \text{dom} h'_1 \cap \text{dom} h'_2$ and $h'_1 \circ \phi = h'_2 \circ \phi$ on some neighborhood of $x$, then $h'_1 \circ \phi = h'_2 \circ \phi$ on the whole of $U$.
(iv) The map $\phi$ is $C^{0,\infty}$ on $U$.

The rest of this section will be devoted to prove that Theorem 16.1 follows from Proposition 16.2. To begin with, Theorem 16.1(iii) is the same condition as Proposition 16.2(iv).

To prove Theorem 16.1(ii), by Lemma 2.3 it is enough to prove that, given $\phi, \psi \in \Phi$, $h \in \overline{\mathcal{H}}$ and $x \in \text{dom} \phi \cap \text{dom} h$ with $h(x) \in \text{dom} \psi$, there is some $h' \in \overline{\mathcal{H}}'$ with $\phi(x) \in \text{dom} h'$ and such that $h' \circ \phi = \psi \circ h$ on some neighborhood of $x$. To prove the existence of such an $h'$, we take the neighborhood $U$ of $y = h(x)$ and the neighborhood $O$ of $\text{id}_U$ in $\overline{\mathcal{H}}_U$ given by Proposition 16.2 for $\psi$ and $y$; moreover we can assume that $U$ is connected and relatively compact. Let

$$
\sigma = j^1(\gamma(h, x)) \in j^1(\gamma(\overline{\mathcal{H}}))_x = j^1(\gamma(\mathcal{H}))_x.
$$

Then $\sigma$ can be approximated as much as desired by elements $r \in j^1(\gamma(h, x))$, which are of the form $\tau = j^1(\gamma(f, x))$ for $f \in \mathcal{H}$ with $x \in \text{dom} f$. Thus $\sigma \cdot \tau^{-1}$ is as close as desired to $\gamma(h, x) \in j^1(\overline{\mathcal{H}})$, where $z = f(x)$ can be assumed to be in $U$. Because $O$ is a neighborhood of $\text{id}_U$ in $\overline{\mathcal{H}}_U$, and since $U$ is connected and relatively compact, it follows from Lemma 9.1 that, for $\tau$ close enough to $\sigma$, there is some $g \in O$ such that $j^1(\gamma(g, z)) = \sigma \cdot \tau^{-1}$. Therefore $g = h \circ f^{-1}$ on some neighborhood of $z$ because both maps are local isometries. Now, since $\Phi$ is a morphism $\mathcal{H} \to \mathcal{H}'$, there is some $f' \in \mathcal{H}'$ so that $\phi(x) \in \text{dom} f'$ and $f' \circ \phi = \psi \circ f$ on some neighborhood of $x$. On the other hand, from Proposition 16.2, we get $g(U) \subset \text{dom} \psi$ and the existence of some $g' \in \overline{\mathcal{H}}'$ such that $\psi(U) \subset \text{dom} g'$ and $g' \circ \psi = \psi \circ g$ on $U$. It follows that

$$
\psi \circ h = \psi \circ g \circ f = g' \circ \psi \circ f = g' \circ f' \circ \phi
$$

on some neighborhood of $x$, which is the desired property with $h' = g' \circ f' \in \overline{\mathcal{H}}'$. Thus $\Phi$ generates a morphism $\overline{\Phi} : \overline{\mathcal{H}} \to \overline{\mathcal{H}}$.

**Lemma 16.3.** The morphism $\overline{\Phi}$ is complete.

Before proving Lemma 16.3, we show how it implies property (i) of Theorem 16.1 (the completeness of $\Phi$). Fix $\phi, \psi \in \Phi$, $x \in \text{dom} \phi$ and $y \in \text{dom} \psi$. Let $U$ and $V$ be neighborhoods of $x$ and $y$ such that $(\phi, U; \psi, V)$ is a completeness quadruple of $\overline{\Phi}$. We can assume $U$ and $V$ are so small that $\phi(U) \subset U'$ and $\psi(V) \subset V'$ for some completeness pair $(U', V')$ of $\overline{\mathcal{H}}$. We can also suppose that $U$ satisfies the properties of Proposition 16.2 with respect to $\phi$ and $x$. With these assumptions, we are going to show that $(\phi, U; \psi, V)$ is also a completeness quadruple of $\Phi$. 
Take \( h \in \mathcal{H}_U \) with \( h(U) \cap V \neq \emptyset \). Since \((\phi, U; \psi, V)\) is a completeness quadruple of \( \Phi \), we have \( h(U) \subset \text{dom} \psi \), and moreover there is some \( h' \in \overline{\Phi} \) satisfying \( \phi(U) \subset \text{dom} h' \) and \( h' \circ \phi = \psi \circ h \) on \( U \). On the other hand, because \( h \in \mathcal{H} \) and \( \Phi \) is a morphism \( \mathcal{H} \to \mathcal{H}' \), there is some \( h' \in \mathcal{H}' \) with \( \phi(x) \subset \text{dom} h' \) and such that \( h' \circ \phi = \psi \circ h \) on some neighborhood of \( x \). We can suppose that \( \text{dom} h' = U' \supset \phi(U) \) because \((U', V')\) is a completeness pair of \( \Phi \).

So \( h' \circ \phi = \bar{h}' \circ \phi \) on some neighborhood of \( x \), and thus also on \( U \) by Proposition 16.2(iii). Therefore \( h' \circ \phi = \psi \circ h \) on \( U \) as desired.

**Proof of Lemma 16.3.** Let \( \phi, \psi \in \Phi \), \( x \in \text{dom} \phi \) and \( y \in \text{dom} \psi \). Choose relatively compact open neighborhoods \( P \) and \( Q \) of \( x \) and \( y \) in \( T \) and \( T' \), respectively, satisfying the following properties:

1. There are relatively compact connected open neighborhoods \( P_1 \) and \( Q_1 \) of \( P \) and \( Q \) in \( \text{dom} \phi \) and \( \text{dom} \psi \), respectively, so that \((P_1, Q_1)\) is a completeness pair of \( \Phi \).
2. There is an open neighborhoods \( O \) of \( \text{id}_U \) in \( \overline{\Phi}_P \) such that, for all \( h \in O \), we have \( h(P) \subset \text{dom} \phi \), and moreover there is some \( h' \in \overline{\Phi} \) with \( \phi(P) \subset \text{dom} h' \) and \( h' \circ \phi = \phi \circ h \) on \( P \).
3. The space \( j^1(\gamma(\overline{\Phi}))_{\overline{P}} \) is compact.

Here, it is obvious that \( P \) and \( Q \) can be chosen so that property (A) holds. Property (B) can be assumed by Proposition 16.2. Finally, property (C) can be assumed because \( j^1(\gamma(\overline{\Phi}))_{\overline{P}} \) is locally compact, \( j^1(\gamma(\overline{\Phi}))_{\overline{P}} \) is compact, and the family of sets \( j^1(\gamma(\overline{\Phi}))_{\overline{P}} \) is a base of open neighborhoods of \( \gamma(\overline{\Phi})_{\overline{P}} \) in \( j^1(\gamma(\overline{\Phi}))_{\overline{P}} \) when \( P \) and \( Q \) run over the open neighborhoods of \( x \) and \( y \).

**Claim 2.** If some \( h \in \overline{\Phi}_{P_1} \) satisfies \( h(P) \cap \overline{Q} \neq \emptyset \), then there is some neighborhood \( U_h \) of \( x \) in \( P \) and some neighborhood \( G_h \) of \( h \) in \( \overline{\Phi}_{P_1} \) such that, for all \( f \in G_h \), there exists some \( f' \in \overline{\Phi} \) with \( \phi(U_h) \subset \text{dom} f' \) and \( f' \circ \phi = \psi \circ f \) on \( U_h \).

Fix \( h \in \overline{\Phi}_{P_1} \) with \( h(P) \cap \overline{Q} \neq \emptyset \) to prove Claim 2. Because \( \overline{\Phi} \) is a morphism, there is some \( h' \in \overline{\Phi} \) with \( \phi(x) \subset \text{dom} h' \) and so that \( \psi \circ h = h' \circ \phi \) on some neighborhood \( W_h \) of \( x \) in \( P \). Choose \( U_h \) such that \( \overline{U_h} \subset W_h \), and choose the neighborhood \( G_h \) of \( h \) in \( \overline{\Phi}_{P_1} \) so small that \( f(P) \subset h(P_1) \) and \( g = h^{-1} \circ f \) in \( Q \) for all \( f \in G_h \).

Then, by (B), there is some \( g' \in \overline{\Phi} \) with \( \phi(P) \subset \text{dom} g' \) and \( g' \circ \phi = \phi \circ g \) on \( P \). Hence \( f' = h' \circ g' \in \overline{\Phi} \) satisfies \( \phi(U_h) \subset \text{dom} f' \) and we have

\[
f' \circ \phi = h' \circ g' \circ \phi = h' \circ \phi \circ g = \psi \circ h \circ g = \psi \circ f
\]

on \( U_h \), which completes the proof of Claim 2.

Now, to finish the proof of Lemma 16.3, consider the subspace

\[
\mathcal{F} = \{(h, z) \in \overline{\Phi}_{P_1} \times \overline{P} \mid h(z) \in \overline{Q} \} \subset \overline{\Phi}_{P_1} \times \overline{P}.
\]

Since \( P_1 \) is connected and relatively compact, the map

\[
\mathcal{F} \to j^1(\gamma(\overline{\Phi}))_{\overline{P}}, \quad (h, z) \mapsto j^1(\gamma(h, z)),
\]

is a homeomorphism by Lemma 9.1. Thus \( \mathcal{F} \) is compact by (C). When \( h \) runs over the elements of \( \overline{\Phi}_{P_1} \) satisfying \( h(P) \cap \overline{Q} \neq \emptyset \), the sets

\[
\mathcal{F}_h = \mathcal{F} \cap (G_h \times \overline{P}) \neq \emptyset,
\]

form an open covering of \( \mathcal{F} \), where the sets \( G_h \) are given by Claim 2. Then there is a finite number of elements \( h_1, \ldots, h_n \in \overline{\Phi}_{P_1} \) with \( h_i(P) \cap \overline{Q} \neq \emptyset \) and so that the corresponding sets \( \mathcal{F}_i = \mathcal{F}_{h_i} \) cover \( \mathcal{F} \). Let also \( G_i = G_{h_i} \) and \( U_i = U_{h_i} \), according to Claim 2. Now let \( U \) be any connected open neighborhood of \( x \) in \( U_1 \cap \cdots \cap U_n \), and let \( V \) be any open neighborhood of \( y \) with \( V \subset Q \). With this choice of \( U \) and \( V \), we are going to prove that \((\phi, U; \psi, V)\) is a completeness quadruple of \( \Phi \).

Take any \( f \in \overline{\Phi}_U \) with \( f(U) \cap V \neq \emptyset \). Because \( U \) is connected and \((P_1, Q_1)\) is a completeness pair of \( \Phi \), there is some extension \( \tilde{f} \in \overline{\Phi}_{P_1} \) of \( f \). We have \( \tilde{f}(P) \cap \overline{Q} \supset f(U) \cap V \neq \emptyset \), and thus there is some \( z \in \overline{P} \) such that \((\tilde{f}, z) \in \mathcal{F} \). So \((\tilde{f}, z) \in \mathcal{F}_i \) for some \( i \in \{1, \ldots, n\} \); in particular, \( \tilde{f} \in G_i \). By Claim 2, there is some \( f' \in \overline{\Phi} \) with \( \phi(U_i) \subset \text{dom} f' \) and \( f' \circ \phi = \psi \circ \tilde{f} \) on \( U_i \). Therefore \( f' \circ \phi = \psi \circ f \) on \( U \) as desired. \( \square \)
17. Proof of Proposition 16.2

Let \( \mathcal{H}, \mathcal{H}' \) and \( \Phi \) be as in the statement of Proposition 16.2, and fix \( \phi \in \Phi \). To begin with, choose relatively compact connected open subsets, \( U_0 \subset T \) and \( U'_0 \subset T' \), such that:

(A) \( \overline{U}_0 \subset \text{dom} \phi \) and \( \phi(\overline{U}_0) \subset U'_0 \subset \text{im} \phi \).
(B) \( (U_0, U_0) \) and \( (U'_0, U'_0) \) are completeness pairs of \( \overline{H} \) and \( \overline{H}' \), respectively.

Since \( \overline{U}_0 \) is compact and \( \phi \) continuous, we have \( \phi(\overline{U}_0) = \overline{\phi(U_0)} \) and the restriction \( \phi : \overline{U}_0 \rightarrow \phi(\overline{U}_0) \) is a quotient map.

We shall use the following notation. Let \( d \) denote the distance function on both \( T \) and \( T' \). For \( y \in T \), \( y' \in T' \) and \( R > 0 \), let \( B(y, R) \) and \( B(y', R) \) be the open balls in \( T \) and \( T' \) of radius \( R \) centered at \( y \) and \( y' \), respectively. Let \( \tilde{B}(y', R) \) be the open ball in \( T' \) of radius \( R \) centered at the origin. Finally, let \( \exp_{y'} \) denote the exponential map from some neighborhood of the origin in \( T' \) to \( T' \).

Now, consider connected open subsets, \( U_1 \subset U_0 \) and \( U'_1 \subset U'_0 \), and choose \( R, R' > 0 \) such that the following properties hold:

(C) \( \phi(U_1) \subset U'_1 \).
(D) \( \text{diam}(\overline{U}_1) < R, d(\overline{U}_1, T \setminus U_0) > R; \) thus \( \overline{U}_1 \subset U_0 \).
(E) \( d(U'_1, T' \setminus U'_0) > R' \); thus \( U'_1 \subset U'_0 \).
(F) \( \phi(\tilde{B}(y, \overline{R})) \subset B(\phi(y), R') \) for all \( y \in \overline{U}_1 \).
(G) The map \( \exp_{y'} : \tilde{B}(y', R') \rightarrow B(y', R') \) is defined and is a diffeomorphism for all \( y' \in \overline{U}'_1 \).
(H) \( U_1 \cap \overline{H}(y) \) is connected for all \( y \in U_1 \); i.e., the orbits of \( \overline{H} \) have connected intersections with \( U_1 \).
(I) \( U_1 \cap h_1(U_1) \cap h_2(U_1) \cap \overline{H}(y) \neq \emptyset \) for all \( y \in U_1 \) and all \( h_1, h_2 \in \overline{H}_{U_0} \) close enough to \( \text{id}_{U_0} \).

Observe that, for any \( x \in \text{dom} \phi \), we can choose neighborhoods \( U_i \) and \( U'_i \), \( i = 0, 1 \), of \( x \) and \( \phi(x) \) satisfying properties (A)–(I). Indeed, properties (A)–(E) can be obviously assumed; property (F) can be assumed because \( \overline{U}_0 \) is a compact subset of \( \text{dom} \phi \); property (G) can be assumed because \( \overline{U}'_1 \) is compact; and finally, properties (H) and (I) can be assumed by the description of a neighborhood of an orbit closure (Section 13).

For \( y' \in \overline{U}'_1 \), let \( \log_{y'} : B(y', R') \rightarrow \tilde{B}(y', R') \) denote the inverse of \( \exp_{y'} : \tilde{B}(y', R') \rightarrow B(y', R') \). Fix \( 0 < r \leq R \) and \( y \in \overline{U}_1 \). Let \( y' = \phi(y) \), which is in \( \overline{U}'_1 \) by (C). Then the subset \( \log_{y'}(\phi(B(y, r))) \subset T' \) is well defined by (F) and (G), and let \( E(y, r) \) denote the linear span of \( \log_{y'}(\phi(B(y, r))) \) in \( T' \). Finally let

\[
E(y) = \bigcup_{0 < r \leq R} E(y, r),
\]

which is a linear subspace of \( T' \).

Lemma 17.1. For all \( y \in \overline{U}_1 \) there is some \( r \in (0, R] \) such that \( E(y) = E(y, r) \).

Proof. This is an easy consequence of the finite dimension of \( T' \).

Lemma 17.2. Let \( r \in (0, R] \), \( y \in \overline{U}_1 \) and \( h'_1, h'_2 \in \overline{H}_{U'_0} \). Then \( h'_1 \circ \phi = h'_2 \circ \phi \) on \( B(y, r) \) if and only if \( h'_{1*} = h'_{2*} \) on \( E(y, r) \).

Proof. Let \( y' = \phi(y) \) and \( y'_i = h'_i(y') \), \( i = 1, 2 \). Then the diagrams

\[
\begin{array}{ccc}
\tilde{B}(y', R') & \xrightarrow{h'_i} & \tilde{B}(y'_i, R')V \\
\downarrow \exp_{y'} & & \downarrow \exp_{y'_i} \\
B(y', R') & \xrightarrow{h'_i} & B(y'_i, R')
\end{array}
\] (17.1)
are well defined and commutative. We have \( h'_1 \circ \phi = h'_2 \circ \phi \) on \( B(y, r) \) if and only if \( h'_1 = h'_2 \) on \( B(y, r) \), which is equivalent to \( h'_{1*} = h'_{2*} \) on \( \log_{\gamma'} \phi (B(y, r)) \) by the commutativity of (17.1), which in turn is obviously equivalent to \( h'_{1*} = h'_{2*} \) on \( E(y, r) \).  

**Corollary 17.3.** Let \( y \in U_1 \) and \( h'_1, h'_2 \in \overline{H}_{U_0}' \). Then \( h'_1 \circ \phi = h'_2 \circ \phi \) on some neighborhood of \( y \) if and only if \( h'_{1*} = h'_{2*} \) on \( E(y) \).

**Proof.** This is a direct consequence of Lemmas 17.1 and 17.2.  

**Lemma 17.4.** Let \( h \in \mathcal{H}_{U_0}', h' \in \mathcal{H}_{U_0}', y \in U_1 \) and \( y' = \phi(y) \). If \( \phi \circ h = h' \circ \phi \) around \( y \), then the tangent map \( h'_* : T_yT' \to T_{h'(y')}T' \) restricts to an isomorphism \( h'_* : E(y) \xrightarrow{\approx} E(h(y)) \).

**Proof.** We have \( \phi \circ h = h' \circ \phi \) on \( B(y, r) \) for some \( r \in (0, R] \) by hypothesis. Then, because \( \phi : \overline{U_0} \to \phi(\overline{U_0}) \) is a quotient map and since \( h : B(y, s) \to B(h(y), s) \) is a homeomorphism for all \( s \in (0, R] \), it follows that

\[
h' : \phi(B(y, s)) \to \phi(B(h(y), s))
\]

is a homeomorphism if \( s \leq r \). So, for \( s \leq r \),

\[
h'_* : \log_{\gamma'} \theta(B(y, s)) \to \log_{\gamma' h(y)} \theta(B(h(y), s))
\]

is a homeomorphism as well because the diagram

\[
\begin{array}{ccc}
\tilde{B}(y', R') & \xrightarrow{h'_*} & \tilde{B}(h'(y'), R')V \\
\exp \downarrow & & \exp_{h'(y')} \downarrow \\
B(y', R') & \xrightarrow{h'} & B(h'(y'), R')
\end{array}
\]

is well defined and commutative. Therefore \( h'_* : T_yT' \to T_{h'(y')}T' \) restricts to an isomorphism \( h'_* : E(y, s) \xrightarrow{\approx} E(h(y), s) \) for all \( s \leq r \), and the result follows.  

For \( X \subset U_1 \), let \( E(X) = \bigcup_{y \in X} E(y) \). The following is some kind of semicontinuity for the spaces \( E(y) \).

**Lemma 17.5.** Let \( y \in U_1 \) and let \( V \) be an open subset of \( U_1 \). If \( z \in \overline{V \cap \mathcal{H}(y)} \), then

\[
E(z) \subset \bigcap_w E(V \cap W \cap \mathcal{H}(y)),
\]

where \( W \) runs over the open neighborhoods of \( z \) in \( T \).

**Proof.** For \( z' = \phi(z) \), let \( \Sigma \subset j^1(y(\overline{V}))' \) denote the subset whose elements are limits in \( j^1(y(\overline{V}))' \) of sequences \( j^1(y(h'_n, z')) \), where \( h'_n \in \mathcal{H}_{U_0}' \) satisfies \( h'_n \circ \phi = \phi \circ h_n \) around \( z \) for some sequence \( h_n \to \text{id}_{U_0} \) in \( \mathcal{H}_{U_0} \) with \( h_n(z) \in V \) for all \( n \). If \( z \notin V \), then we cannot take \( h_n = \text{id}_{U_0} \) and \( h'_n = \text{id}_{U_0} \) for all \( n \); thus, a priori, it is not clear that \( 1_{z'} \in \Sigma \).

**Claim 3.** \( \Sigma \neq \emptyset \).

Let us prove this assertion. Because \( z \in \overline{V \cap \mathcal{H}(y)} \), there is some sequence \( h_n \in \mathcal{H} \) with \( z \in \text{dom } h_n \) and \( h_n(z) \to z \). Since \( (U_0, U_0) \) is a completeness pair of \( \overline{H} \), we can assume \( h_n \in \mathcal{H}_{U_0} \) for all \( n \). Furthermore we can suppose \( h_n \to \text{id}_{U_0} \) in \( \mathcal{H}_{U_0} \) by Lemma 11.1. Since \( \Phi \) is a morphism \( \mathcal{H} \to \mathcal{H}' \) and \( (U_0', U_0') \) is a completeness pair of \( \overline{H} \), there is a sequence \( h'_n \in \mathcal{H}_{U_0}' \) satisfying \( \Phi \circ h_n = h'_n \circ \phi \) around \( z \) for each \( n \). The sequence \( j^1(y(h'_n, z')) \) approaches the compact subspace \( j^1(y(\overline{V}))' \), and thus some subsequence is convergent to some \( \sigma \in j^1(y(\overline{V}))' \). It follows that \( \sigma \in \Sigma \), concluding the proof of Claim 3.
Claim 4. If \( \sigma, \tau \in \Sigma \), then \( \sigma \cdot \tau \in \Sigma \).

Let us prove Claim 4. By the definition of \( \Sigma \), there are sequences \( h_m, g_n \in \mathcal{H}_{U_0} \) and \( h'_m, g'_n \in \mathcal{H}'_{U_0} \) such that:

- \( j^1(\gamma(h'_m, z')) \to \sigma, \ j^1(\gamma(g'_n, z')) \to \tau \);
- \( h'_m \circ \phi = \phi \circ h_m \) and \( g'_n \circ \phi = \phi \circ g_n \) around \( z \);
- \( h_m(z), g_n(z) \in V \); and
- \( h_m \to \text{id}_{U_0} \) and \( g_n \to \text{id}_{U_0} \) in \( \mathcal{H}_{U_0} \).

For each \( m \in \mathbb{Z}_+ \), let \( P_m \) be a neighborhood of \( z \) where \( h'_m \circ \phi = \phi \circ h_m \) and \( h_m(P_m) \subset V \). Since \( g_n \to \text{id}_{U_0} \), for each \( m \) there is some \( n_m \in \mathbb{Z}_+ \) such that \( g_{n_m}(z) \in P_m \). Moreover we can take \( n_m \uparrow \infty \) as \( m \to \infty \). Then there is some neighborhood \( Q_m \) of \( z \) so that \( g_{n_m}(Q_m) \subset P_m \) and \( \phi \circ g_{n_m} = g'_{n_m} \circ \phi \) on \( Q_m \). Because \((U_0, U_0')\) and \((U'_0, U'_0')\) are completeness pairs of \( \mathcal{H} \) and \( \mathcal{H}' \), respectively, there are maps \( f_m \in \mathcal{H}_{U_0} \) and \( f'_m \in \mathcal{H}'_{U_0} \) that are equal to \( h_m \circ g_{n_m} \) and \( h'_m \circ g'_n \) on some neighborhoods \( W_m \) and \( W'_m \) of \( z \) and \( z' \), respectively. We can assume \( \phi(W_m) \subset W'_m \). Hence

\[
\lim_{m} f_m(z) = h_m \circ g_{n_m}(z) \in h_m \circ g_{n_m}(Q_m) \subset h_m(P_m) \subset V.
\]

Moreover

\[
\lim_{m} j^1(\gamma(f_m, z)) = \lim_{m} \left( j^1(\gamma(h_m, g_{n_m}(z))) \cdot j^1(\gamma(g_{n_m}, z)) \right) = \lim_{m} j^1(\gamma(h_m, g_{n_m}(z))) \cdot \lim_{m} j^1(\gamma(g_{n_m}, z)) = 1_z,
\]

\[
\lim_{m} j^1(\gamma(f'_m, z')) = \lim_{m} \left( j^1(\gamma(h'_m, g'_m(z'))) \cdot j^1(\gamma(g'_m, z')) \right) = \lim_{m} j^1(\gamma(h'_m, g'_m(z'))) \cdot \lim_{m} j^1(\gamma(g'_m, z')) = \sigma \cdot \tau,
\]

by the properties of the maps \( h_m, g_n, h'_m \) and \( g'_n \), because \( n_m \uparrow \infty \), and since

\[
g'_{n_m}(z') = \phi \circ g_{n_m}(z) \to \phi(z) = z'.
\]

So \( f_m \to \text{id}_{U_0} \) in \( \mathcal{H}_{U_0} \) by Lemma 9.1. It follows that \( \sigma \cdot \tau \) satisfies the conditions to be in \( \Sigma \) as desired.

Claim 5. \( \Sigma \) is closed, and thus compact.

To prove Claim 5, take any sequence \( \sigma_m \in \Sigma \) converging to some element \( \tau \in j^1(\gamma(\mathcal{H}')) \). By the definition of \( \Sigma \), for each \( m \), there are sequences \( h_{m,n} \in \mathcal{H}_{U_0} \) and \( h'_{m,n} \in \mathcal{H}'_{U_0} \) such that:

- \( j^1(\gamma(h'_{m,n}, z')) \to \sigma_m \);
- \( h'_{m,n} \circ \phi = \phi \circ h_{m,n} \) around \( z \);
- \( h_{m,n}(z) \in V \);
- \( h_{m,n} \to \text{id}_{U_0} \) in \( \mathcal{H}_{U_0} \) as \( n \to \infty \).

A standard diagonal convergence argument easily yields the existence of a sequence \( n_m \uparrow \infty \) such that \( h_{m,n_m} \to \text{id}_{U_0} \) in \( \mathcal{H}_{U_0} \) and \( j^1(\gamma(h'_{m,n_m}, z')) \to \tau \). Therefore \( \tau \) satisfies the conditions to be in \( \Sigma \) as desired.

The following property is well known.

Claim 6. Let \( G \) be a first countable compact topological group. For each \( g \in G \), the subset

\[
K = \left\{ g^n \mid n \in \mathbb{Z}_+ \right\} \subset G
\]

is a subgroup.
A proof of Claim 6 is included for completeness. First consider the closed subset

\[ L = \bigcap_{m=1}^{\infty} \{ g^n \mid n \geq m \} \subset K. \]

We have \( L \neq \emptyset \) because \( G \) is compact. Furthermore \( L \) is a subgroup; in fact, if \( \sigma, \tau \in L \), then there are sequences of positive integers, \( m_k, n_k \uparrow \infty \), such that \( \sigma = \lim_k g^{m_k} \) and \( \tau = \lim_k g^{n_k} \). We can also suppose \( m_k - n_k \uparrow \infty \), yielding \( \sigma \cdot \tau^{-1} = \lim_k g^{m_k-n_k} \in L \).

On the other hand, we clearly have \( gL \subset L \). Thus \( g \in L \), yielding \( K = L \) because \( L \) is a closed subgroup of \( G \). Hence \( K \) is a subgroup as desired.

**Claim 7.** \( 1_{\Sigma} \in \Sigma \).

To prove this, we know the existence of some \( \sigma \in \Sigma \) by Claim 3. Since \( j^1(\gamma(H))_{\Sigma'} \) is a compact Lie group, the subset

\[ K = \{ \sigma^n \mid n \in \mathbb{Z}_+ \} \subset j^1(\gamma(H))_{\Sigma'} \]

is a subgroup by Claim 6. Moreover \( K \subset \Sigma \) by Claims 4 and 5. Therefore \( 1_{\Sigma} \in \Sigma \) as desired.

Now the proof of Lemma 17.5 can be completed as follows. By Claim 7, there are sequences \( h_n \) and \( h'_n \) in \( \mathcal{H} \mathcal{U}_0 \) and \( \mathcal{H}' \mathcal{U}_0' \) such that:

- \( j^1(\gamma(h'_n, z')) \to 1_{\Sigma} ; \)
- \( h'_n \circ \phi = \phi \circ h_n \) around \( z ; \)
- \( h_n(z) \in V ; \)
- \( h_n \to \text{id}_{U_0} \) in \( \mathcal{H} \mathcal{U}_0 \).

By Lemma 17.4, every tangent homomorphism \( h'_{n\Phi} : T_zT' \to \tau_{h'_n(z)}T' \) restricts to an isomorphism \( h'_{n\Phi} : E(z) \cong E(h_n(z)) \). So, as \( W \) runs over the neighborhoods of \( z \) in \( T \), we get

\[ \bigcap_{W} E(V \cap W \cap \mathcal{H}(y)) \supset \bigcap_{m \geq m} \bigcup_{n \geq m} E(h_n(z)) = \bigcap_{m \geq m} \bigcup_{n \geq m} h'_{n\Phi}(E(z)). \] (17.2)

On the other hand, for each \( u \in T_zT' \), we have \( h'_{n\Phi}(u) \to u \) in \( TT' \) because \( j^1(\gamma(h'_n, z')) \to 1_{\Sigma} \), yielding

\[ E(z) \subset \bigcap_{m \geq m} \bigcup_{n \geq m} h'_{n\Phi}(E(z)). \] (17.3)

The result now follows from (17.2) and (17.3). \( \square \)

From now on in this section, let \( X_y = U_1 \cap \overline{\mathcal{H}}(y) \) and \( X'_y = \phi(X_y) \) for each \( y \in U_1 \).

**Corollary 17.6.** Let \( y \in U_1 \), \( h \in \mathcal{H} \mathcal{U}_0 \) and \( h' \in \mathcal{H}' \mathcal{U}_0' \) such that \( h(U_1) \subset U_0 \). If \( \phi \circ h = h' \circ \phi \) on some neighborhood of \( y \) in \( T \), then \( \phi \circ h = h' \circ \phi \) on some neighborhood of \( X_y \) in \( T \).

**Proof.** Let

\[ A = \{ z \in U_1 \mid \phi \circ h = h' \circ \phi \text{ on some neighborhood of } z \text{ in } T \}. \]

Clearly, \( A \) is an open subset of \( U_1 \) and contains \( y \). So, since \( X_y \) is connected (property (H)), it is enough to prove that \( A \cap \overline{\mathcal{H}}(y) \) is closed in \( X_y \).

Let \( z \) be a point in the closure of \( A \cap \overline{\mathcal{H}}(y) \) in \( X_y \); i.e.,

\[ z \in U_1 \cap A \cap \overline{\mathcal{H}}(y) = U_1 \cap \overline{A \cap \mathcal{H}}(y). \]

We have to prove that \( z \in A \).
Because \( \Phi \) is a morphism \( \mathcal{H} \rightarrow \mathcal{H}' \) and \((U'_0, U'_0')\) a completeness pair of \( \overline{\mathcal{H}}' \), there is some \( h'' \in \mathcal{H}'_{U_0} \) such that \( \phi \circ h = h'' \circ \phi \) on some open neighborhood \( P \) of \( z \). So \( z \in V \cap \mathcal{H}(y) \), where \( V = P \cap A \). Furthermore \( h' \circ \phi = \phi \circ h = h'' \circ \phi \) in \( V \), yielding \( h'_x \circ h'' = h'' \circ \phi \) on \( E(V) \) by Corollary 17.3. Hence, by the continuity of \( h'_x \) and \( h''_x \), we get \( h'_x = h''_x \) on \( W \cap V \), where \( W \) runs over the neighborhoods of \( z \). It follows that \( h'_x = h''_x \) on \( E(z) \) by Lemma 17.5, and thus \( h' \circ \phi = h'' \circ \phi \) around \( z \) by Corollary 17.3. Therefore \( h' \circ \phi = \phi \circ h \) around \( z \), yielding \( z \in A \) as desired. \( \square \)

**Corollary 17.7.** Let \( y \in U_1, h \in \mathcal{H}_{U_0} \) and \( h' \in \mathcal{H}'_{U_0'} \) such that \( h(U_1) \subset U_0 \). If \( \phi \circ h = h' \circ \phi \) on some neighborhood of \( y \) in \( X_y \), then \( \phi \circ h = h' \circ \phi \) on the whole of \( X_y \).

**Proof.** Let \( \mathcal{G} \) denote the restriction of \( \mathcal{H} \) to \( \overline{\mathcal{H}}(y) \), and let \( \Psi : \mathcal{G} \rightarrow \mathcal{H}' \) denote the restriction of \( \Phi \). With respect to \( \Psi \), the open sets \( U_i \cap \overline{\mathcal{H}}(y) \) and \( U'_{i'} \), \( i = 0, 1 \), clearly satisfy properties (A)–(H). Thus the result is a direct consequence of Corollary 17.6. \( \square \)

**Corollary 17.8.** There are neighborhoods \( \mathcal{P}_0 \) and \( \mathcal{Q}_0 \) of \( \text{id}_{U_0} \) and \( \text{id}_{U'_0} \) in \( \overline{\mathcal{H}}_{U_0} \) and \( \overline{\mathcal{H}}'_{U'_0} \), respectively, such that all \( h \in \mathcal{P}_0 \) and \( h' \in \mathcal{Q}_0 \) satisfy \( U_1 \subset \text{dom} \ h^{-1}, \ U'_1 \subset \text{dom} \ h'^{-1} \), and moreover, if \( \phi \circ h = h' \circ \phi \) on \( X_y \), then \( \phi \circ h^{-1} = h'^{-1} \circ \phi \) on \( X_y \).

**Proof.** Let \( \mathcal{P}_0 \) be the set of \( h \in \overline{\mathcal{H}}_{U_0} \) satisfying \( h(U_1) \subset U_0, U_1 \subset \text{h}(U_0) \) and \( h(X_y) \cap X_y \neq \emptyset \) for all \( y \in U_1 \). Let \( \mathcal{Q}_0 \) be the set of \( h' \in \overline{\mathcal{H}}'_{U'_0} \) satisfying \( U'_1 \subset h'(U_0) \). By property (I), the sets \( \mathcal{P}_0 \) and \( \mathcal{Q}_0 \) are neighborhoods of \( \text{id}_{U_0} \) and \( \text{id}_{U'_0} \), respectively.

Take \( h \in \mathcal{P}_0 \) and \( h' \in \mathcal{Q}_0 \). We clearly have \( U_1 \subset \text{dom} \ h^{-1} \) and \( U'_1 \subset \text{dom} \ h'^{-1} \). Now, suppose that \( \phi \circ h = h' \circ \phi \) on \( X_y \) for some \( y \in U_1 \); in particular, the following diagram is commutative:

\[
\begin{array}{ccc}
X_y \cap h^{-1}(X_y) & \xrightarrow{h} & h(X_y) \cap X_y \\
\downarrow \phi & & \downarrow \phi \\
\phi(X_y \cap h^{-1}(X_y)) & \xrightarrow{h'} & \phi(h(X_y) \cap X_y).
\end{array}
\]

So \( \phi \circ h^{-1} = h'^{-1} \circ \phi \) on \( h(X_y) \cap X_y \). Furthermore, since \((U_0, U_0)\) and \((U'_0, U'_0)\) are completeness pairs of \( \overline{\mathcal{H}} \) and \( \overline{\mathcal{H}}' \), and since \( U_1 \) and \( U'_1 \) are connected, there is some \( f \in \overline{\mathcal{H}}_{U_0} \) and some \( f' \in \overline{\mathcal{H}}'_{U'_0} \) that are equal to \( h^{-1} \) and \( h'^{-1} \) on \( U_1 \) and \( U'_1 \), respectively. Thus \( \phi \circ f = f' \circ \phi \) on \( h(X_y) \cap X_y \). Moreover \( h(X_y) \cap X_y \neq \emptyset \) and \( f(U'_1) = h^{-1}(U_1) \subset U_0 \) because \( h \in \mathcal{P}_0 \). Therefore

\[ \phi \circ h^{-1} = \phi \circ f = f' \circ \phi = h'^{-1} \circ \phi \]

on \( X_y \) by Corollary 17.7. \( \square \)

**Corollary 17.9.** There are neighborhoods \( \mathcal{P}_1 \) and \( \mathcal{Q}_1 \) of \( \text{id}_{U_0} \) and \( \text{id}_{U'_0} \) in \( \overline{\mathcal{H}}_{U_0} \) and \( \overline{\mathcal{H}}'_{U'_0} \), respectively, such that all \( h_1, h_2 \in \mathcal{P}_1 \) and all \( h'_1, h'_2 \in \mathcal{Q}_1 \) satisfy \( U_1 \subset \text{dom} \ (h_1 \circ h_2), U'_1 \subset \text{dom} \ (h'_1 \circ h'_2) \), and moreover, if \( \phi \circ h_1 = h'_1 \circ \phi \) on \( X_y \) for some \( y \in U_1, i = 1, 2 \), then \( \phi \circ h_1 \circ h_2 = h'_1 \circ h'_2 \circ \phi \) on \( X_y \).

**Proof.** By property (I), there is a neighborhood \( \mathcal{P}_0 \) of \( \text{id}_{U_0} \) in \( \overline{\mathcal{H}}_{U_0} \) such that \( X_y \cap h_1(X_y) \cap h_2(X_y) \neq \emptyset \) for all \( h_1, h_2 \in \mathcal{P}_0 \) and all \( y \in U_1 \). Take an open set \( V \) in \( T \) such that \( U'_1 \subset V \) and \( V \subset U_0 \). Then the family \( \mathcal{P}_1 \) of maps \( h \in \mathcal{P}_0 \) satisfying \( h(U'_1) \subset V \) and \( h(V) \subset U_0 \) is a neighborhood of \( \text{id}_{U_0} \) in \( \overline{\mathcal{H}}_{U_0} \). Let \( \mathcal{Q}_1 \subset \overline{\mathcal{H}}'_{U'_0} \) be the open subset of maps \( h' \in \overline{\mathcal{H}}'_{U'_0} \) satisfying \( h'(U'_1) \subset U'_0 \).

Take \( h_1, h_2 \in \mathcal{P}_1 \) and \( h'_1, h'_2 \in \mathcal{Q}_1 \). We clearly have \( U_1 \subset \text{dom} \ (h_1 \circ h_2) \) and \( U'_1 \subset \text{dom} \ (h'_1 \circ h'_2) \). Now, suppose that \( \phi \circ h_1 = h'_1 \circ \phi \) on \( X_y \) for some \( y \in U_1, i = 1, 2 \). So, for

\[ Y = X_y \cap h_1(X_y) \cap h_2(X_y), \]
the following diagram is commutative:

\[
\begin{array}{ccc}
\phi(\mathbf{h}^{-1}_1(Y)) & \mathbf{h}^{-1}_2 & \mathbf{h}^{-1}_1(Y) \\
\downarrow & \downarrow & \downarrow \\
\phi(\mathbf{h}^{-1}_1(Y)) & \mathbf{h}^{-1}_1(Y) & Y \\
\end{array}
\]

Hence \(\phi \circ h_1 \circ h_2 = h'_1 \circ h'_2 \circ \phi\) on \(Y\). Furthermore, since \((U_0, U_0)\) and \((U'_0, U'_0)\) are completeness pairs of \(\overline{H}\) and \(\overline{F}'\), and since \(U_1\) and \(U'_1\) are connected, there is some \(f \in \overline{H} U_0\) which equals \(h_1 \circ h_2\) on \(U_1\), and there is some \(f' \in \overline{F}' U'_0\) which equals \(h'_1 \circ h'_2\) on \(U'_1\). Thus \(\phi \circ f = f' \circ \phi\) on \(h^{-1}_2 \circ h^{-1}_1(Y)\). Moreover \(Y \neq \emptyset\) and \(f(U_1) = h_1 \circ h_2(U_1) \subset h_1(V) \subset U_0\) because \(h_1, h_2 \in \mathcal{P}_1\). Therefore

\[\phi \circ h_1 \circ h_2 = \phi \circ f = f' \circ \phi = h'_1 \circ h'_2 \circ \phi\]

on \(X_y\) by Corollary 17.7.

**Corollary 17.10.** For any \(y \in U_1\) and any neighborhood \(Q\) of \(id U'_0\) in \(\overline{F}' U'_0\), there is some neighborhood \(\mathcal{P}_y\) of \(id U_0\) in \(\overline{H} U_0\) such that, for all \(h \in \mathcal{P}_y\), there exists some \(h' \in Q\) such that \(\phi \circ h = h' \circ \phi\) on \(X_y\).

**Proof.** Fix any \(y \in U_1\) and let \(y' = \phi(y)\).

**Claim 8.** If \(h \in \overline{H} U_0\) satisfies \(h(U_1) \subset U_0\), then there is some \(h' \in \overline{F}' U'_0\) such that \(\phi \circ h = h' \circ \phi\) on \(X_y\).

Let us prove this statement. By the definition of \(\overline{H}\), by Lemma 9.1, and since \((U_0, U_0)\) is a completeness pair of \(\overline{H}\), we get the existence of a sequence \(h_n \in \mathcal{H} U_0\) that converges to \(h\) in \(\overline{H} U_0\). We can suppose that \(h_n(y) \in U_0\) for all \(n\). Thus there is another sequence \(h'_n \in \mathcal{H} U'_0\) so that \(\phi \circ h_n = h'_n \circ \phi\) around \(y\) because \(\Phi\) is a morphism \(\mathcal{H} \rightarrow \mathcal{H}'\) and \((U'_0, U'_0)\) is a completeness pair of \(\overline{F}'\). So \(\phi \circ h_n = h'_n \circ \phi\) on \(X_y\) by Corollary 17.7. On the other hand, the sequence \(j^1(\gamma(h'_n, y')) \in j^1(\gamma(\overline{F}'))\)' approaches the compact subset \(j^1(\gamma(\overline{F}'))\)'\(y''\), where \(y'' = \phi(y)\) and \(y'' = \phi \circ h(y)\). Therefore \(j^1(\gamma(h'_n, y'))\) has some convergent subsequence; in fact, we can suppose that \(j^1(\gamma(h'_n, y'))\) is itself convergent. Its limit is of the form \(j^1(\gamma(h', y'))\) for some \(h' \in \overline{F}' U'_0\) because \((U'_0, U'_0)\) is a completeness pair of \(\overline{F}'\). Moreover \(h'_n \rightarrow h'\) in \(\overline{F}' U'_0\) by Lemma 9.1. So, from \(\phi \circ h_n = h'_n \circ \phi\) on \(X_y\), it follows that \(\phi \circ h = h' \circ \phi\) on \(X_y\) because \(\phi\) is continuous, yielding the proof of Claim 8.

To finish the proof of this lemma, given a sequence \(h_n \in \overline{H} U_0\) such that \(h_n \rightarrow id U_0\) uniformly, we have to show that there is a subsequence \(h_{nm}\) and a corresponding sequence \(h''_{nm} \in \overline{F}' U'_0\) so that \(h''_{nm} \rightarrow id U'_0\) and \(\phi \circ h_{nm} = h''_{nm} \circ \phi\) on \(X_y\) for all \(m\). We can assume that \(h_n(U_1) \subset U_0\) for all \(n\). Then, by Claim 8, there is a sequence \(h'_n \in \overline{F}' U'_0\) such that \(\phi \circ h_n = h'_n \circ \phi\) on \(X_y\). We have \(h'_n \circ \phi(y) = \phi \circ h_n(y) \rightarrow y'\), and thus \(\sigma_n = j^1(\gamma(h'_n, y'))\) approaches the compact space \(j^1(\gamma(\overline{F}'))\)'\(y''\). Hence some subsequence \(\sigma_{nm}\) is convergent to some \(\tau \in j^1(\gamma(\overline{F}'))\)'\(y''\). Because \((U'_0, U'_0)\) is a completeness pair of \(\overline{F}'\), there is some \(f' \in \overline{F}' U'_0\) such that \(\tau = j^1(\gamma(f', y'))\). Furthermore \(h''_{nm} \rightarrow f'\) on \(U'_0\) by Lemma 9.1, and thus \(f' \circ \phi = \phi\) on \(X_y\) because \(\phi\) is continuous, yielding that \(f'\) is the identity map on \(X'_y\). Therefore there is some neighborhood of \(X'_y\), where the composite \(h''_{nm} \circ f'^{-1}\) is defined for all \(m\), and we have

\[
j^1(\gamma(h''_{nm} \circ f'^{-1}, y')) = \sigma_{nm} \circ \tau^{-1} \rightarrow 1\).
\]

Again, since \((U'_0, U'_0)\) is a completeness pair of \(\overline{F}'\), there is some \(h''_{nm} \in \overline{F}' U'_0\) such that \(\sigma_n \circ \tau^{-1} = j^1(\gamma(h''_{nm}, y'))\). Then \(h''_{nm} = h''_{nm} \circ f'^{-1}\) on some neighborhood of \(X'_y\) because this space is connected by property (H). Therefore

\[
h''_{nm} \circ \phi = h''_{nm} \circ f'^{-1} \circ \phi = h''_{nm} \circ \phi = \phi \circ h_{nm}
\]

on \(X_y\) because \(f'\) is equal to the identity on \(X'_y\). Moreover \(h''_{nm} \rightarrow id\) on \(U'_0\) by (17.4) and Lemma 9.1. \(\square\)
Corollary 17.11. There is some neighborhood $\mathcal{P}_2$ of $\text{id}_{U_0}$ in $\overline{\mathcal{H}}_{U_0}$ such that, for any $y \in U_1$ and any neighborhood $\mathcal{P}$ of $\text{id}_{U_0}$ in $\overline{\mathcal{H}}_{U_0}$, there is some neighborhood $\mathcal{Q}_y$ of $\text{id}_{U'}$ in $\overline{\mathcal{H}}_{U'0}$ such that, if $h \in \mathcal{P}_2$ and $h' \in \mathcal{Q}_y$ satisfy $\phi \circ h = h' \circ \phi$ on $X_y$, then there is some $f \in \mathcal{P}$ such that $\phi \circ f = h' \circ \phi$ on $X_y$.

Proof. Let $\mathcal{P}_0$ be the neighborhood of $\text{id}_{U_0}$ in $\overline{\mathcal{H}}_{U_0}$ given by Corollary 17.8, and choose a neighborhood $\mathcal{P}_2$ of $\text{id}_{U_0}$ whose closure is contained in $\mathcal{P}_0$.

For each $y \in U_1$, we have to prove that, if some sequences $h_n \in \mathcal{P}_2$ and $h'_n \in \overline{\mathcal{H}}_{U_0}$ satisfy $\phi \circ h_n = h'_n \circ \phi$ on $X_y$ and $h'_n \to \text{id}_{U_0}$, then there is some subsequence $h'_{nm}$ and a sequence $f_m \in \overline{\mathcal{H}}_{U_0}$ such that $h'_{nm} \circ \phi = \phi \circ f_m$ on $X_y$ and $f_m \to \text{id}_{U_0}$.

Since $h_n \in \mathcal{P}_1$, the sequence $j^1(\gamma(h_n, y))$ approaches the compact set $j^1(\gamma(\overline{\mathcal{H}}_j)y)$, yielding that some subsequence $h'_{nm}$ is convergent to some $h$ in $\overline{\mathcal{H}}_{U_0}$ by Lemma 9.1. Since $\phi$ is continuous, we get $\phi \circ h = \phi \circ h'$ on $X_y$, and moreover $h \in \overline{\mathcal{P}_2} \subset \mathcal{P}_0$. Therefore $U_1 \subset \text{dom } h^{-1}$ and $\phi = \phi \circ h^{-1}$ on $X_y$ by Corollary 17.8.

Since $h'_{nm} \to h$ in $\overline{\mathcal{H}}_{U_0}$ and $\overline{U_1} \subset U_0$, it follows that the composite $h^{-1} \circ h_{nm}$ is defined on $U_1$ for $m$ large enough. Thus, because $U_1$ is connected and $(U_0, U_0)$ is a completeness pair of $\mathcal{H}$, it follows that there is a unique $f_m \in \overline{\mathcal{H}}_{U_0}$ which is equal to $h^{-1} \circ h_{nm}$ on $U_1$. From $h^{-1} \circ h_{nm} |_{U_1} \to \text{id}_{U_1}$ in $\overline{\mathcal{H}}_{U_1}$, we get $f_m \to \text{id}_{U_0}$ in $\overline{\mathcal{H}}_{U_0}$ by Lemma 9.1. Furthermore

$$\phi \circ f_m = \phi \circ h^{-1} \circ h_{nm} = \phi \circ h_{nm} = h'_{nm} \circ \phi$$

on $X_y$ for $m$ large enough. □

According to Theorem 10.8, there are local Lie groups $G$ and $G'$ with local actions on $T$ and $T'$ generating the pseudogroups $\mathcal{H}_{U_0}$ and $\mathcal{H}_{U'}$, respectively. The identity elements of $G$ and $G'$ will be denoted by $e$ and $e'$. Since $U_0$ and $U'$ are relatively compact, we can suppose that $G$ and $G'$ are small enough around $e$ and $e'$ so that the local action of all of their elements is defined on the whole of $U_0$ and $U'$, respectively. For $g \in G$ and $g' \in G'$, we shall also denote by $g$ and $g'$ the corresponding elements in $\mathcal{H}_{U_0}$ and $\mathcal{H}_{U'}$ defined by the local actions. Again, we can assume that $G$ and $G'$ are so small that their elements, considered as maps in $\mathcal{H}_{U_0}$ and $\mathcal{H}_{U'}$, belong to the neighborhoods $\mathcal{P}_i$ and $\mathcal{Q}_j$ given by Corollaries 17.8, 17.9 and 17.11, $i = 0, 1, 2, \ j = 0, 1$. Further assumptions on $G$ and $G'$ will be made when needed.

Fix any $y \in U_1$. Let $G_y$ be the set of the elements $g \in G$ such that there is some $g' \in G'$ so that $g' \circ \phi = \phi \circ g$ on $X_y$.

Lemma 17.12. $G_y$ is an open local subgroup of $G$.

Proof. The symmetry of $G_y$ follows from Corollary 17.8. Also, $G_y$ is a neighborhood of $e$ in $G$ by Corollary 17.10. So it only remains to prove that $G_y$ is open in $G$.

Take any $g \in G_y$, and choose some $g' \in G'$ such that $g' \circ \phi = \phi \circ g$ on $X_y$. Let $Q$ be a neighborhood of $e'$ in $G'$ so small that, for all $h' \in Q$, the product $g'h'$ is defined in $G'$. By Corollary 17.10, there is some neighborhood $P$ of $e$ in $G$ so small that:

- for all $h \in P$, there is some $h' \in Q$ such that $h' \circ \phi = \phi \circ h$ on $X_y$; in particular, $P \subset G_y$; and
- the product $gh$ is well defined in $G$ for all $h \in P$.

Then the neighborhood $gP$ of $g$ in $G$ is contained in $G_y$ by Corollary 17.9. □

Let $G'_y$ be the set of all $g' \in G'$ satisfying $g' \circ \phi = \phi \circ g$ on $X_y$ for some $g \in G_y$.

Lemma 17.13. $G'_y$ is a local Lie subgroup of $G'$.

Proof. $G'_y$ clearly contains the identity element $e'$ of $G'$. Moreover $G'_y$ is symmetric by Corollary 17.8.
Now take any \( g_0' \in G' \) and any \( g_0 \in G \) satisfying \( g_0' \circ \phi = \phi \circ g_0 \) on \( X_y \). Let \( P_0 \) and \( P \) be compact neighborhoods of \( g_0 \) and \( e \) in the open subset \( G_y \) of \( G \) such that the products \( gh \) and \( hg \) are defined in \( G \) for all \( g \in P_0 \) and all \( h \in P \). Then, by Corollary 17.11, there are compact neighborhoods \( Q_0 \) and \( Q \) of \( g_0' \) and \( e' \) in \( G' \), respectively, such that:

- for all \( g' \in G'_y \cap Q_0 \) and all \( h' \in G'_y \cap Q \), there exists some \( g \in G_y \cap P_0 \) and some \( h \in G_y \cap P \) so that \( g' \circ \phi = \phi \circ g \) and \( h' \circ \phi = \phi \circ h \) on \( X_y \);
- the products \( g'h' \) and \( h'g' \) are defined in \( G' \) for all \( g' \in Q_0 \) and all \( h' \in Q \).

Take \( g' \in G'_y \cap Q_0 \) and \( h' \in G'_y \cap Q \), and choose \( g \in G_y \cap P_0 \) and \( h \in G_y \cap P \) satisfying \( g' \circ \phi = \phi \circ g \) and \( h' \circ \phi = \phi \circ h \) on \( X_y \). Then, by Corollary 17.9, we get \( g' \circ h' \circ \phi = \phi \circ g \circ h \) and \( h' \circ g' \circ \phi = \phi \circ h \circ g \) on \( X_y \). So both \( g'h' \) and \( h'g' \) belong to \( G'_y \), and we get that \( G_y' \) is a local subgroup of \( G' \).

Finally, consider a sequence \( h'_n \in G'_y \cap Q \) converging to some element \( h' \in Q \). Then there is another sequence \( h_n \in P \) such that \( h'_n \circ \phi = \phi \circ h_n \) on \( X_y \). Since \( P \) is compact, there is a subsequence \( h'_m \circ \phi = \phi \circ h_m \) on \( X_y \), and it follows that \( h' \circ \phi = \phi \circ h \) on \( X_y \). So \( h' \in G'_y \cap Q \), obtaining that \( G'_y \cap Q \) is closed in \( Q \), and thus compact. Hence both \( g_0' \in (G'_y \cap Q) \) and \( (G'_y \cap Q)g'_0 \) are compact neighborhoods of \( g_0' \) in \( G'_y \). Therefore \( G_y' \) is a locally compact local subgroup of \( G' \), and thus a local Lie subgroup of \( G' \) by [8, Théorème 2, p. 227].

**Lemma 17.14.** \( X_y' = \phi(X_y) \) is an open subset of \( G'_y \)-orbit of \( y' = \phi(y) \) on \( T' \), and thus \( X_y' \) is a \( C^\infty \) submanifold of \( T' \).

**Proof.** Since \( G_y' \) is an open neighborhood of \( e \) in \( G \), we get that \( \mathcal{F}_0(y) \) is the orbit of the local action of \( G_y \) on \( T \) that contains \( y \). Take any \( z \in X_y \). Because \( X_y \) is a connected open subset of \( \mathcal{F}_0(y) \), there are \( g_1, \ldots, g_k \in G_y \) such that \( z = g_1 \ldots g_k y \) and \( g_i g_{i+1} \ldots g_k y \in X_y \) for \( i = 1, \ldots, k \). By the definition of \( G_y' \), there are \( g'_1, \ldots, g'_k \in G'_y \) such that \( g'_i \circ \phi = \phi \circ g_i \) on \( X_y \). Hence \( \phi(z) = g'_1 \ldots g'_k y' \) is in the \( G'_y \)-orbit of \( y' \). Therefore the whole of \( X_y' \) is contained in the \( G'_y \)-orbit of \( y' \) because \( z \) is arbitrary.

On the other hand, for all \( g' \in G'_y' \), we know the existence of some \( g \in G_y \) such that \( g' \circ \phi = \phi \circ g \) on \( X_y \); in particular, \( g' \circ \phi = \phi (g y) \). Furthermore, by Corollary 17.11, the above \( g \) can be chosen to approach \( e \) in \( G_y \) as \( g' \) approaches \( e' \) in \( G'_y \). So \( g y \in X_y \) if \( g' \) is close enough to \( e' \) in \( G'_y \), yielding \( g' \circ \phi = \phi(g y) \in X_y' \). Therefore \( X_y' \) contains a nontrivial open subset of the \( G'_y \)-orbit of \( y' \). It follows that \( X_y' \) is open in that orbit as desired.

We can assume that \( G' \) is an open local subgroup of a local Lie group \( \tilde{G} \) such that the product \( g'h' \) is defined in \( \tilde{G} \) for all \( g', h' \in G' \), and the local action of \( G' \) on \( T' \) can be extended to a local action of \( \tilde{G} \) on \( T' \). For the given point \( y \in U_1 \), two elements of \( G'_y \) will be said to be equivalent if, as elements of \( \mathcal{F}_0(y) \), they have the same restriction to \( X_y' \).

The corresponding quotient space will be denoted by \( G'_y / \tilde{K}_y \).

**Lemma 17.15.** The local Lie group structure of \( G'_y \) canonically induces a local Lie group structure on \( G''_y \), and the local action of \( G'_y \) on \( X_y' \) canonically induces an effective isometric local action of \( G''_y \) on \( X_y' \).

**Proof.** Let \( \tilde{K}_y \subset \tilde{G} \) be the closed normal local Lie subgroup of elements in \( \tilde{G} \) that fix all points of \( X_y' \). Right translations obviously define a local action of \( \tilde{K}_y \) on \( G'_y \), whose orbit space will be denoted by \( G'_y / \tilde{K}_y \). Since \( \tilde{K}_y \) is normal in \( \tilde{G} \), the local group structure of \( G'_y \) induces a local group structure on \( G'_y / \tilde{K}_y \), and the local action of \( G'_y \) on \( X_y' \) induces a local action of \( G'_y / \tilde{K}_y \) on \( X_y' \).

The pseudogroup on \( G'_y \) generated by the local action of \( \tilde{K}_y \) is parallelizable: it preserves the parallelism defined by any frame of right invariant vector fields. This pseudogroup is closed because \( \tilde{K}_y \) is closed in \( \tilde{G} \). Moreover it is obviously complete. So \( G'_y / \tilde{K}_y \) is a manifold and the quotient map \( G'_y \to G'_y / \tilde{K}_y \) is a \( C^\infty \) submersion according to Section 12. Therefore the result easily follows from the following assertion.

**Claim 9.** We have \( G'_y / \tilde{K}_y = G''_y \) as quotient spaces.
If the product \( g' a' \) is defined in \( G'_y \) for some \( g' \in G'_y \) and some \( a' \in \tilde{K}_y \), then \( g' \) and \( g' a' \) have the same restriction to \( X'_y \), and thus they are equivalent. Hence, to prove Claim 9, it only remains to show that two equivalent elements \( g', h' \in G'_y \) are in the same \( \tilde{K}_y \)-orbit. But for such \( g' \) and \( h' \), the inverse \( g'^{-1} \) is also an element of \( G'_y \), and thus the product \( a' = g'^{-1} h' \) is defined in \( \tilde{G}' \). Moreover \( a' \) fixes every point of \( X'_y \) because \( g' \) and \( h' \) are equivalent; i.e., \( a' \in \tilde{K}_y \). So \( g' \) and \( h' = g' a' \) are in the same \( \tilde{K}_y \)-orbit as desired. \( \square \)

Let \( g, g', g'' \) and \( g''_y \) be the Lie algebras of right invariant vector fields on \( G, G', G'_y \) and \( G''_y \), respectively; we take Lie algebras of right invariant vector fields since we consider left local actions. By Lemma 17.14, the elements of \( g'_y \) are just the elements in \( g' \) whose infinitesimal action is tangent to \( X'_y \). If \( \mathfrak{t}_y \subset g'_y \) denotes the normal Lie subalgebra of elements whose infinitesimal actions vanish on \( X'_y \), then there is an induced infinitesimal action of \( g'_y/\mathfrak{t}_y \) on \( X'_y \).

With the notation of the proof of Lemma 17.15, let \( K_y = \tilde{K}_y \cap G' \), which is an open local subgroup of \( \tilde{K}_y \), and a closed normal local Lie subgroup of \( G'_y \). Then the Lie algebra of \( K_y \) is \( \mathfrak{t}_y \). Moreover, by Lemma 17.15, we have \( g''_y \equiv g'/\mathfrak{t}_y \), and the above infinitesimal action of \( g'_y/\mathfrak{t}_y \) on \( X'_y \) can be identified with the infinitesimal action induced by the local action of \( G''_y \) on \( X'_y \).

**Lemma 17.16.** There is a homomorphism of local Lie groups, \( F_y : G_y \to G''_y \), such that the restriction \( \phi : X_y \to X'_y \) is \( F_y \)-equivariant.

**Proof.** For each \( g \in G_y \), there is some \( g' \in G'_y \) so that \( g' \circ \phi = \phi \circ g \) on \( X_y \). By the definition of \( G''_y \), any other element of \( G'_y \) satisfying the same property defines the same element in \( G''_y \); thus the notation \( F_y(g) \in G''_y \) makes sense for the class represented by \( g' \). This defines a map \( F_y : G_y \to G''_y \), which is a homomorphism of local groups by Corollary 17.9. Moreover \( \phi : X_y \to X'_y \) is \( F_y \)-equivariant, clearly. On the other hand, Corollary 17.10 implies that \( F_y \) is also continuous, and thus analytic too [8, Theorem 1, p. 225]. Therefore \( F_y \) is a homomorphism of local Lie groups. \( \square \)

If \( A \in g \) and \( A' \in g''_y \), the notation \( \tilde{A} \in \mathfrak{X}(T) \) and \( \tilde{B} \in \mathfrak{X}(X'_y) \) will be used for the corresponding infinitesimal actions.

**Corollary 17.17.** The map \( \phi : X_y \to X'_y \) is \( C^\infty \) and satisfies the following property: if \( A \in g, A' = F_y(A) \in g''_y, z \in X_y \) and \( z' = \phi(z) \in X'_y \), then
\[
\phi_*(\tilde{A}_z) = \tilde{A}'_{z'}.
\] (17.5)

**Proof.** Fix \( z \in X_y \) and let \( z' = \phi(z) \). According to Section 11, there are symmetric open neighborhoods \( R \) and \( S \) of \( e \) and \( e'' \) in \( G_y \) and \( G''_y \), and there are open neighborhoods \( P \) and \( Q \) of \( z \) and \( z' \) in \( X_y \) and \( X'_y \), such that the maps \( R \to P \) and \( S \to Q \), given by \( a \mapsto a z \) and \( a' \mapsto a' z' \), are well defined \( C^\infty \) surjective submersions. We can assume \( \phi(P) \subset Q \) and \( F_y(R) \subset S \). The \( F_y \)-equivariance of \( \phi : X_y \to X'_y \) implies the commutativity of the following diagram:
\[
\begin{array}{ccc}
R & \xrightarrow{F_y} & S \\
\downarrow & & \downarrow \\
P & \xrightarrow{\phi} & Q.
\end{array}
\] (17.6)

So \( \phi : P \to Q \) is \( C^\infty \), and thus \( \phi : X_y \to X'_y \) is \( C^\infty \) since \( z \) is arbitrary. Finally, equality (17.5) is a direct consequence of the commutativity of (17.6). \( \square \)

Now, we can suppose that \( \exp : B \to G \) and \( \exp : B' \to G' \) are diffeomorphisms for some balls \( B \) and \( B' \) in \( g \) and \( g' \), respectively, centered at the origin. Then, for \( -1 \leq r \leq 1 \), multiplication by \( r \) defines maps \( B \to B \) and \( B' \to B' \), which correspond via the exponential map to maps \( G \to G \) and \( G' \to G' \), which will be respectively denoted by \( g \mapsto g^r \) and \( g' \mapsto g'^r \).

**Corollary 17.18.** If \( g' \circ \phi = \phi \circ g \) on \( X_y \) for \( g \in G \) and \( g' \in G' \), then \( g'^r \circ \phi = \phi \circ g^r \) on \( X_y \) for \( -1 \leq r \leq 1 \). In particular, we have \( g^r \in G_y \) and \( g'^r \in G'_y \) for all \( g \in G_y \) and all \( g' \in G'_y \), and thus \( G_y \) and \( G'_y \) are connected.
Proof. The condition \( g' \circ \phi = \phi \circ g \) on \( X_y \) means that \( g \in G_y \), \( g' \in G'_y \) and \( F_y(g) = \bar{g} \), where \( \bar{g} \in G''_y \) denotes the class represented by \( g' \). Take \( A \in B \) and \( A' \in B' \) such that \( g = \exp A \) and \( \bar{g} = \exp A' \). Then the local actions of \( g' \) and \( \bar{g}' \) on \( X_y \) and \( X'_y \) are the corresponding uniparametric local groups of diffeomorphisms \( \exp r\bar{A} \) and \( \exp rA' \).

Therefore it is enough to show that \( \phi_x(\bar{A}) = \bar{A}' \) on \( X'_y \). But we have

\[
\exp F_y(A) = F_y(\exp(A)) = F_y(g) = \bar{g}' = \exp(A').
\]

So \( F_y(A) = A' \), and the result follows by Corollary 17.17. \( \Box \)

Now, fix an arbitrary point \( x \in \text{dom} \phi \) and let \( x' = \phi(x) \). Take neighborhoods \( U_1 \) and \( U'_1 \) of \( x \) and \( x' \) satisfying properties (A)–(I). Define \( \rho : G \to [0, R] \) in the following way. Given \( g \in G \), if there is some \( g' \in G' \) with \( \phi \circ g = g' \circ \phi \) around \( x \), then let \( \rho(g) \) be the supremum of \( r \in (0, R] \) such that \( \phi \circ g = g' \circ \phi \) on \( B(x, r) \) for some \( g' \in G' \); otherwise let \( \rho(g) = 0 \). The following is a key result.

Lemma 17.19. There is some \( C > 0 \) and some open local subgroup \( H \subset G \) such that \( \rho(g) \geq C \) for all \( g \in H \).

Proof. Let \( C_1 = d(x, T \smallsetminus U_1) > 0 \).

Claim 10. We have

\[
\rho(g_1g_2) \geq \min\{C_1, \rho(g_1), \rho(g_2)\}
\]

for all \( g_1, g_2 \in G \).

To prove this assertion, we can assume \( \rho(g_i) > 0, i = 1, 2 \). Set

\[
0 < s < \min\{C_1, \rho(g_1), \rho(g_2)\}.
\]

Then, by Corollary 17.7, there are \( g'_1, g'_2 \in G' \) such that \( g'_i \circ \phi = \phi \circ g_i \) on a tube in \( U_1 \) around \( X_x \) of radius \( s \).

Therefore \( (g'_1g'_2) \circ \phi = \phi \circ (g_1g_2) \) on a tube in \( U_1 \) around \( X_x \) of radius \( s \) by Corollary 17.9, and Claim 10 follows.

Claim 11. We have

\[
\rho(g') \geq \min\{\rho(g), C_1\}
\]

for \( -1 \leq r \leq 1 \) and \( g \in G \).

We can assume \( \rho(g) > 0 \) to prove this claim, and set \( 0 < s < \min\{\rho(g), C_1\} \). Then, by Corollary 17.7, there is some \( g' \in G' \) such that \( g' \circ \phi = \phi \circ g' \) on the tube in \( U_1 \) around \( X_x \) of radius \( s \). So \( g'' \circ \phi = \phi \circ g'' \) on such a tube for \( -1 \leq r \leq 1 \) by Corollary 17.18, and Claim 11 follows.

Let \( \Lambda \subset G \) be the dense local subgroup whose local action on \( T \) generates \( \mathcal{H}_0 \). Since \( \Phi \) is a morphism \( \mathcal{H} \to \mathcal{H}' \) and \( (U'_0, U'_{0}) \) is a completeness pair of \( \mathcal{H}' \), for all \( g \in \Lambda \), there is some \( h' \in \mathcal{H}'_{U'_{0}} \) so that \( h' \circ \phi = \phi \circ g \) around \( x \). But this does not directly imply \( \rho(g) > 0 \) because \( h' \) may not be defined by the action of some element of \( G' \); recall that the local action of \( G' \) on \( T' \) generates \( \mathcal{H}'_{T'_{0}} \), which may be different from \( \mathcal{H}'_{T_{0}} \). So the following assertion is not completely obvious, where the notation \( g^{(0,1]} = \{g' \mid r \in (0, 1]\} \) is used for any \( g \in G \).

Claim 12. For all \( g \in \Lambda \) and any neighborhood \( U \) of \( g^{(0,1]} \) in \( G \), there is some \( a \in \Lambda \cap U \) with \( \rho(a) > 0 \).

Fix any \( g \in \Lambda \) and any neighborhood \( U \) of \( g^{(0,1]} \) in \( G \). Take some \( h' \in \mathcal{H}'_{U'_{0}} \) such that \( h' \circ \phi = \phi \circ g \) on some neighborhood \( O \) of \( x \) in \( U_1 \). It is easy to see that there are sequences, \( g_n \in \Lambda \) and \( h' \in \mathcal{H}'_{U'_{0}} \), so that:

(a) \( h'_n \circ \phi = \phi \circ g_n \) on some neighborhood \( O_n \) of \( x \) in \( O \);
(b) \( g_n \to g \);
(c) \( g_n(O_n) \subset g_m(O_m) \) if \( m < n \);
(d) \( g_m^{-1}g_n \in U \) if \( m < n \).
Recall the notation $x' = \phi(x)$, and let $y = g(x)$ and $y' = \phi(y)$. Let also $x'_n = h'_n(x') = \phi \circ g_n(x)$. Since $\phi$ is continuous, from (a) and (b) we get that $j^1(\gamma(h'_n, x'))$ approaches the compact subspace $j^1(\gamma(T'))_{x'}$ of $J^1(T')$. So some subsequence of $j^1(\gamma(h'_n, x'))$ is convergent in $j^1(\gamma(T'))_{x'}$. Therefore there are $m$ and $n$ large enough, say $m < n$, so that
\[
j^1(\gamma(h'_m, x')) = j^1(\gamma(h'_m, x'))^{-1} \cdot j^1(\gamma(h'_n, x'))
\]
is as close to $1_{x'}$ as desired. In particular, we can assume $h'_m^{-1} \circ h'_n$ is defined by the local action of some $a'_{m,n} \in G'$ by Lemma 9.1 and Theorem 10.8. On the other hand, from (c) we get a commutative diagram
\[
\begin{array}{c}
O_n \xrightarrow{g_n} g_m(O_m) \xrightarrow{g_n^{-1}} O_m \\
\phi(O_n) \xrightarrow{h'_n} \phi \circ g_m(O_m) \xrightarrow{h'_m^{-1}} \phi(O_m).
\end{array}
\]
So $a'_{m,n} \circ \phi = \phi \circ g_m^{-1} g_n$ on $O_n$, yielding $\rho(g_m^{-1} g_n) > 0$, and Claim 12 follows by (d).

The proof of Lemma 17.19 can be completed as follows. Since $\Lambda$ is dense in $G$, there are elements $g_1, \ldots, g_k$ in $\Lambda$ which are the image by the exponential map of the elements of some base of $\mathfrak{g}$. It is easy to see that there is a neighborhood $U_i$ of each $g_i^{[0,1]}$ such that any choice of elements $a_i \in U_i, i = 1, \ldots, k$, is the image by the exponential map of some base of $\mathfrak{g}$. So all elements in some open local subgroup $H \subset G$ can be written as products of the form $a_1^{r_1} \cdots a_k^{r_k}$ with $-1 \leq r_i \leq 1$. Moreover, by Claim 12 such elements $a_i$ can be chosen so that $\rho(a_i) > 0$. Then the result easily follows from Claims 10 and 11. □

Let us finish the proof of Proposition 16.2. For $\phi$ and $x$ as above, by Lemma 17.19, there is an open local subgroup $H \subset G$ and some open connected neighborhood $U$ of $x$ in $U_1$ such that, for all $h \in H$, there is some $h' \in G'$ so that $h' \circ \phi = \phi \circ h$ on $U$. Indeed, we can assume $X_y \subset U$ if $y \in U$ by Corollary 17.7. So Proposition 16.2(iii) follows with such a $U$ and taking as $\mathcal{O}$ the neighborhood of $\text{id}_U \in \mathcal{H}_U$ defined by the local action of $H$ on $U$. Because $U \subset U_0$, this $U$ also satisfies Proposition 16.2(iii); i.e., $(U, U)$ is a completeness pair of $\mathcal{H}$. We can choose such a $U$ satisfying Proposition 16.2(iii) by Lemmas 17.1 and 17.2.

Only Proposition 16.2(iv) remains to be proved; that is, it remains to show that $\phi$ is $C^{0,\infty}$ on $U$. This will take some more work, but the main ideas were already used; it is only needed a slight sharpening of the arguments by using the existence of $H$ and $U$.

Let $H'$ be the set of elements $h' \in G'$ such that $h' \circ \phi = \phi \circ h$ on $U$ for some $h \in H$. Observe that $H'$ contains $e'$.

**Lemma 17.20.** For any neighborhood $Q$ of $e'$ in $H'$, there is some neighborhood $P$ of $e$ in $H$ such that, for all $g \in P$, there exists some $g' \in Q$ such that $\phi \circ g = g' \circ \phi$ on $U$.

**Proof.** Given a sequence $g_n \in H$ such that $g_n \to e$, we have to show that there is a subsequence $g_m$ and a corresponding sequence $h'_m \in H'$ so that $h'_m \to e'$ and $\phi \circ g_m = h'_m \circ \phi$ on $U$ for all $m$. We know that there is a sequence $g'_n \in H'$ such that $\phi \circ g_n = g'_n \circ \phi$ on $U$. Then $h'_n \circ \phi = \phi \circ g_n \to \phi$ on $U$, and thus $\sigma_n = j^1(\gamma(g'_n, x'))$ approaches the compact space $j^1(\gamma(T'))_{x'}$ (recall that $x' = \phi(x)$). Hence some subsequence $\sigma_{n_m}$ is convergent to some $\tau \in j^1(\gamma(T'))_{x'}$. Because $(U'_0, U'_0)$ is a completeness pair of $\mathcal{H}$, there is some $f' \in \mathcal{H}_{U'_0}$ such that $\tau = j^1(\gamma(f', x'))$. Furthermore $g'_{n_m} \to f'$ on $U'_0$ by Lemma 9.1. So $f' \circ \phi = \phi \circ f'$ on $U$, and thus $f'$ is the identity on $\phi(U)$. Hence there is some neighborhood of $\phi(U)$ where the composite $g'_{n_m} \circ f'^{-1}$ is defined for all $m$, and we have
\[
j^1(\gamma(g'_{n_m} \circ f'^{-1}, x')) = \sigma_{n_m} \cdot \tau^{-1} \to 1_{x'}.
\]
Again, since $(U'_0, U'_0)$ is a completeness pair of $\mathcal{H}$ and $U$ is connected, there is some $h'_m \in \mathcal{H}_{U'_0}$ such that $\sigma_n \cdot \tau^{-1} = j^1(\gamma(h'_m, x'))$. Then $h'_m = g'_{n_m} \circ f'^{-1}$ on some neighborhood of $\phi(U)$. Therefore
\[
h'_m \circ \phi = g'_{n_m} \circ f'^{-1} \circ \phi = g'_{n_m} \circ \phi = \phi \circ g_{n_m}
\]
on $U$ because $f'$ equals the identity on $\phi(U)$. Moreover $h'_m \to \text{id}_{U'_0}$ by (17.7) and Lemma 9.1; thus the maps $h'_m$ can be considered as elements of $H'$ for $m$ large enough. □
Lemma 17.21. For any neighborhood $P$ of $e$ in $G$, there is some neighborhood $Q$ of $e'$ in $G'$ such that for all $g' \in H' \cap Q$ there is some $\phi_0$ in $H \cap P$ such that $\phi \circ f = h' \circ \phi_0$ on $U$.

Proof. We have to prove that, if some sequences $g_n \in H$ and $g'_n \in H'$ satisfy $\phi \circ g_n = g'_n \circ \phi$ on $U$ and $g_n \to e'$, then there is some subsequence $g'_{n_m}$ and a sequence $f_m \in H$ such that $g'_{n_m} \circ \phi = \phi \circ f_m$ on $U$ and $f_m \to e$.

The sequence $j^1(\gamma(g_n, x))$ approaches the compact set $j^1(\gamma(\overline{H}))_x^U$. So some subsequence $g_{n_m}$ converges to $h$ in $\overline{HU}_0$ by Lemma 9.1. Since $\phi$ is continuous, we get $\phi \circ h = \phi$ on $U$, and moreover $h \in \overline{P}_2 \subset P_0$. Therefore $U_1 \subset \text{dom } h^{-1}$ and $\phi = \phi \circ h^{-1}$ on $U$ by Corollary 17.8.

Since $g_{n_m} \to h$ in $\overline{HU}_0$ and $U_1 \subset U_0$, the composite $h^{-1} \circ h_{n_m}$ is defined on $U_1$ for $m$ large enough. Thus, because $U_1$ is connected and $(U_0, U_0)$ is a completeness pair of $\overline{H}$, there is a unique $f_m \in \overline{HU}_0$ which equals $h^{-1} \circ h_{n_m}$ on $U_1$. From the convergence of $h^{-1} \circ h_{n_m}$ to the identity map on $U_1$, we get $f_m \to \text{id}_{U_0}$ in $\overline{HU}_0$ by Lemma 9.1. Furthermore $\phi \circ f_m = \phi \circ h^{-1} \circ g_{n_m} = \phi \circ g_{n_m} = g'_{n_m} \circ \phi$ on $U$ for $m$ large enough. Finally, since $f_m \to \text{id}_{U_0}$, the maps $f_m$ can be considered as elements of $G$, and thus of $H$, for $m$ large enough. \qed

Lemma 17.22. $H'$ is a local Lie subgroup of $G'$.

Proof. $H'$ contains the identity element, and is symmetric by Corollary 17.8.

Now take any $g'_0 \in H'$ and any $g_0 \in H$ satisfying $g'_0 \circ \phi = \phi \circ g_0$ on $U$. Let $P_0$ and $P$ be compact neighborhoods of $g_0$ and $e$ in $H$, respectively, such that the products $gh$ and $hg$ are defined in $G$ for all $g \in P_0$ and all $h \in P$. Then, by Lemma 17.21, there are compact neighborhoods $Q_0$ and $Q$ of $g'_0$ and $e'$ in $G'$, respectively, such that:

- for all $g' \in H' \cap Q_0$ and all $h' \in H' \cap Q$, there exists some $g \in P_0$ and some $h \in P$ so that $g \circ \phi = \phi \circ g$ and $h \circ \phi = \phi \circ h$ on $U$;
- the products $g' h'$ and $h' g'$ are defined in $G'$ for all $g' \in Q_0$ and all $h' \in Q$.

Take $g' \in H' \cap Q_0$ and $h' \in H' \cap Q$, and choose $g \in P_0$ and $h \in P$ satisfying $g' \circ \phi = \phi \circ g$ and $h' \circ \phi = \phi \circ h$ on $U$. Then, by Corollary 17.9, we get $(g' h') \circ \phi = \phi \circ (gh)$ and $(h' g') \circ \phi = \phi \circ (hg)$ on $U$. So both $g' h'$ and $h' g'$ belong to $H'$, and we get that $H'$ is a local subgroup of $G'$.

Finally, consider a sequence $h'_{n} \in H' \cap Q$ converging to some element $h' \in Q$. Then there is another sequence $h_n \in P$ such that $h'_n \circ \phi = \phi \circ h_n$ on $U$. Since $P$ is compact, there is a subsequence $h_{n_m}$ converging to some element $h \in P$. On the other hand, since $\phi$ is continuous, from $h'_{n_m} \circ \phi = \phi \circ h_{n_m}$ on $U$, it follows that $h' \circ \phi = \phi \circ h$ on $U$. So $h' \in H' \cap Q$, obtaining that $H' \cap Q$ is closed in $Q$, and thus compact. Hence both $g'_0(H' \cap Q)$ and $(H' \cap Q)g'_0$ are compact neighborhoods of $g'_0$ in $H'$. Therefore $H'$ is a locally compact local subgroup of $G'$, and thus a local Lie subgroup of $G'$ by [8, Théorème 2, p. 227]. \qed

Lemma 17.23. For each $y \in U$, $X'_y$ is open in the $H'$-orbit of $y' = \phi(y)$ on $T'$.

Proof. Since $H$ is an open neighborhood of $e$ in $G$, we get that $\overline{H}_0(y)$ is the orbit of the local action of $H$ on $T$ that contains $y$. Take any $z \in X_y$. Then, because $X_y$ is a connected open subset of $\overline{H}_0(y)$, there are $g_1, \ldots, g_k \in H$ such that $z = g_1 \ldots g_k y$ and $g_1 g_{i+1} \ldots g_k y \in X_y$ for $i = 1, \ldots, k$. By definition of $H$, there are $g'_1, \ldots, g'_k \in H'$ such that $g'_i \circ \phi = \phi \circ g_i$ on $U$. Hence $\phi(z) = g'_1 \ldots g'_k y'$ is in the $H'$-orbit of $y'$. Therefore the whole $X'_y$ is contained in the $H'$-orbit of $y'$ because $z$ is arbitrary. Now the result follows from Lemma 17.14. \qed

Two elements of $H'$ will be said to be equivalent if they have the same action on $\phi(U)$. The corresponding quotient space will be denoted by $H''$.

Lemma 17.24. The local Lie group structure of $H'$ canonically induces a local Lie group structure on $H''$, and the local action of $H'$ on $\phi(U)$ canonically induces an effective isometric local action of $H''$ on some $G'$-submanifold $U''$ of $U'$ that contains $\phi(U)$.
Proof. Let $\tilde{L} \subset \tilde{G}'$ be the closed normal local Lie subgroup of elements that fix the points of $\phi(U)$. Let $U''$ be the space of points in $U'_1$ that are fixed by $\tilde{L}$. Since the local action of $\tilde{L}$ is isometric, $U''$ is a $C^\infty$ submanifold of $U_1$. This follows because the local action of $\tilde{K}$ on $T'$ can be linearized around each $y' \in U''$ via the exponential map, and the fixed points of this linearized local action is a linear subspace of $T_\gamma T'$; thus normal coordinates of $T'$ around points of $U''$ restrict to local coordinates on $U''$; and the changes of such local coordinates on $U''$ are $C^\infty$, obviously. Furthermore $U''$ is a $G'$-subset of $U'_1$; if $y' \in U''$, $g' \in G'$ and $a' \in \tilde{L}$, there is some $b' \in \tilde{L}$ such that $a'g' = g'b'$ because $\tilde{L}$ is normal in $\tilde{G}'$, yielding $a'g'y' = g'b'y' = g'y'$, and thus $g'y' \in U''$.

Now, as in the proof of Lemma 17.15, the local Lie group structure of $H'$ induces a local Lie group structure on $H''$, and the local action of $H'$ on $U''$ induces a local action of $H''$ on $U''$. □

With the notation of the proofs of Lemmas 17.15 and 17.24, observe that
\[ H' \subset \bigcap_{y \in U} G'_y, \quad \tilde{L} \subset \bigcap_{y \in U} \tilde{K}_y. \]

So there is a canonical homomorphism $H'' \to G''_y$ for each $y \in U$. The identity element of $H''$ will be also denoted by $e''$.

**Lemma 17.25.** There is a homomorphism of local Lie groups, $F : H \to H''$, such that the diagram
\[
\begin{array}{ccc}
H & \xrightarrow{F} & H'' \\
\downarrow & & \downarrow \\
G_y & \xrightarrow{F} & G''_y
\end{array}
\]  
(17.8)
is commutative for all $y \in U$.

**Proof.** For each $h \in H$, there is some $h' \in H'$ so that $g' \circ \phi = \phi \circ g$ on $U$. By the definition of $H''$, any other element in $H'$ satisfying the same property represents the same element in $H''$; thus the notation $F(h) \in H''$ makes sense for the element represented by $h'$. This defines a map $F : H \to H''$, which is a homomorphism of local groups by Corollary 17.9. On the other hand, Lemma 17.20 implies that $F$ is also continuous, and thus it is a homomorphism of local Lie groups [8, Theorem 1, p. 225]. The commutativity of (17.8) is easy to check. □

**Corollary 17.26.** The restriction $\phi : U \to U''$ is $F$-equivariant.

**Proof.** This follows from Lemma 17.16 and the commutativity of (17.8). □

Let $\mu : \Omega \to U$ and $\mu'' : \Omega'' \to U''$ denote the local actions of $H$ on $U$ and of $H''$ on $U''$, where $\Omega \subset H \times U$ and $\Omega'' \subset H' \times U''$ are open neighborhoods of $\{e\} \times U$ and $\{e''\} \times U''$. The equivariance of $\phi$ means that the following diagram is commutative:
\[
\begin{array}{ccc}
\Omega & \xrightarrow{F \times \phi} & \Omega'' \\
\downarrow & & \downarrow \\
U & \xrightarrow{\phi} & U''.
\end{array}
\]  
(17.9)

Let $\mathcal{G}$ and $\mathcal{G}''$ be the foliations on $\Omega$ and $\Omega''$ whose leaves are the intersections of $\Omega$ and $\Omega''$ with the corresponding product slices $H \times \{y\}$ and $H'' \times \{y''\}$, for $y \in U$ and $y'' \in U''$. Also, the $\mathcal{F}$-orbits $X_y$ in $U$ are the leaves of a singular Riemannian foliation $\mathcal{F}$ on $U$, and the $\mathcal{F}''$-orbits define a singular Riemannian foliation $\mathcal{F}''$ on $U''$ since $U''$ is a $G'$-submanifold of $U'_1$ (Lemma 17.24). With respect to these singular Riemannian foliations, (17.9) consists of foliated maps. Moreover the top map of (17.9) is $C^{0,\infty}$, obviously, and the vertical maps $\mu$ and $\mu''$ of (17.9) are $C^\infty$ surjective submersions. So the composite
\[
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{\mu} & \mathcal{F} & \xrightarrow{\phi} & \mathcal{F}''
\end{array}
\]
is $C^{0,\infty}$ by the commutativity of (17.9). Therefore the fact that $\phi$ is $C^{0,\infty}$ follows from Lemma 19.9 since $\mu$ satisfies the following key property.

Lemma 17.27. For each $V \in \mathcal{X}(F)$, there is some $W \in \mathcal{X}(G)$ so that $\mu_*(W) = V$.

Proof. Since $H$ is an open local group of $G$, the Lie algebra of right invariant vector fields on $H$ can be also identified with $\mathfrak{g}$, and for each $A \in \mathfrak{g}$, let $\tilde{A} \in \mathcal{X}(F)$ be the corresponding infinitesimal action. Since infinitesimal actions of elements of $\mathfrak{g}$ generate $\mathcal{X}(F)$ as $C^\infty(U)$-module, it is enough to check the result when $V = \tilde{A}$ for some $A \in \mathfrak{g}$. In this case, it is easy to check that the result holds with $W = (A, 0) \in \mathcal{X}(G)$, where we use the canonical injection of $\mathcal{X}(H) \oplus \mathcal{X}(U)$ into $\mathcal{X}(H \times U)$. □

18. The strong plaquewise topology

In this section, we give a version for continuous foliated maps of the strong and weak compact-open topologies. Roughly speaking, two foliated maps will be close with respect to this topology when, on foliated charts, they have the same representations with transverse coordinates, and close representations with plaquewise coordinates.

Let $X$ and $Y$ be foliated spaces with foliated structures $F$ and $G$. Fix the following data:

- Any foliated map $f : F \to G$.
- Any locally finite collection $U = \{U_i\}$ of simple open sets of $X$.
- A family $V = \{V_i\}$ of simple open sets of $Y$, with the same index set. Let $p_i : V_i \to T_i$ be the canonical projection to the local quotient.
- A family $K = \{K_i\}$ of compact subsets of $X$, with the same index set, such that $K_i \subset U_i$ and $f(K_i) \subset V_i$ for all $i$.

Let $\mathcal{N}(f, U, V, K)$ be the set of foliated maps $g : F \to G$ such that $g(K_i) \subset V_i$ and $p_i \circ g(x) = p_i \circ f(x)$ for each $i$ and every $x \in K_i$. Such sets $\mathcal{N}(f, U, V, K)$ form a base of a topology on $C(F, G)$, called the strong plaquewise topology, and the corresponding space is denoted by $C_{SP}(F, G)$. A weak version of this topology can be defined by taking finite families; it can be called the weak plaquewise topology, and the corresponding space can be denoted by $C_{WP}(F, G)$. Both of these topologies are equal if $X$ is compact; in this case, we can use the term plaquewise topology and the notation $C_P(F, G)$.

From now on in this section, assume that $X$ is locally compact and Polish.

Theorem 18.1. With the above notation and conditions, if two continuous foliated maps $f, g : F \to G$ are close enough in $C_{SP}(F, G)$, then there is an integrable homotopy between them. If moreover $f$ and $g$ are proper, then there is a proper integrable homotopy between them.

The proof of Theorem 18.1 uses the following lemma.

Lemma 18.2. Let $f, g : F \to G$ be continuous foliated maps. Suppose that, for each $x \in M$, both points $f(x)$ and $g(x)$ belong to the same plaque of some simple open set of $Y$ depending on $x$. Then there is an integrable homotopy between $f$ and $g$. If moreover $f$ and $g$ are proper, then there is a proper integrable homotopy between them.

Proof. It easily follows from the hypotheses that we can get the following:

- A locally finite family $\{U_i\}$ of open subsets of $X$.
- A covering of $X$ by compact subset, $\{K_i\}$, with the same index set and such that $K_i \subset U_i$ for all $i$.
- For each $i$, a foliated chart $\theta_i : V_i \to T_i \times B$ of $G$ such that $f(x)$ and $g(x)$ lie in the same plaque of $V_i$ for each $x \in U_i$. Here, $B$ is an open ball of $\mathbb{R}^n$, where $n = \dim G$. Let $p_i : V_i \to T_i$ and $q_i : V_i \to B$ denote the composites of $\theta_i$ with the factor projections of $T_i \times B$.

Since $\{U_i\}$ is locally finite, its index set is countable; say, either $\mathbb{Z}_+$, or $\{1, \ldots, k\}$ for some $k \in \mathbb{Z}_+$. Set $f_0 = f$. 

Claim 13. For each $i$, there is a continuous foliated map $f_i : \mathcal{F} \rightarrow \mathcal{G}$ such that:

- $f_i(U_j) \subset V_j$ for all $j$;
- $f_i(x)$ and $f_{i-1}(x)$ lie in the same plaque of $V_i$ for all $i$ and $x \in U_i$;
- $f_i(x) = g(x)$ for all $i$ and all $x \in K_1 \cup \cdots \cup K_i$;
- $f_i = f_{i-1}$ on some neighborhood of $M \setminus U_i$ for all $i$;
- there is an integrable homotopy $H_i : \mathcal{F} \times [0,1] \rightarrow \mathcal{G}$ between $f_{i-1}$ and $f_i$ for all $i$; and
- if $f_{i-1}$ is proper, then $f_i$ and $H_i$ are proper.

The result follows from Claim 13 in the following way. Take a partition $0 = t_1 < \cdots < t_k < t_{k+1} = 1$ of $I$ when $i$ runs in $\{1, \ldots, k\}$, and take any sequence $t_i \uparrow 1$ with $t_0 = 0$ when $i$ runs in $\mathbb{Z}_+$. In any case, let $\rho_i : [t_i, t_{i+1}] \rightarrow I$ be any orientation preserving homeomorphism for each $i$. Then it is easy to check that an integrable homotopy $H : \mathcal{F} \times I \rightarrow \mathcal{G}$ between $f$ and $g$ can be defined by

$$H(x,t) = \begin{cases} H_i(x, \rho_i(t)) & \text{if } t \in [t_i, t_{i+1}] \text{ for some } i, \\ g(x) & \text{if } t = 1. \end{cases}$$

Moreover $H$ is proper if $f$ and $g$ are proper.

It remains to prove Claim 13, which is done by induction on $i$. We have already set $f_0 = f$. Now suppose that $f_{i-1}$ is given for some $i > 0$. Let $\lambda_i : X \rightarrow I$ be a continuous map such that $\text{supp} \lambda_i \subset U_i$ and $\lambda_i(x) = 1$ for all $x \in K_i$. Then it is easy to show that the statement of Claim 13 holds with the following definitions of $f_i$ and $H_i$. Let $H_i(x,t) = f_{i-1}(x)$ for $x \in X \setminus U_i$,

$$H_i(x,t) = \theta_i^{-1}\left(p_i \circ g(x), \left(1 - t \lambda_i(x)\right) \cdot q_i \circ f_{i-1}(x) + t \lambda_i(x) \cdot q_i \circ g(x)\right)$$

for $x \in U_i$, and $f_i(x) = H_i(x,1)$ for any $x \in M$. □

Proof of Theorem 18.1. Consider a basic neighborhood of $f$ in $C_{\text{sp}}(\mathcal{F}, \mathcal{G})$ of the form $\mathcal{N}(f, U, \mathcal{V}, \mathcal{K})$ defined above. We can suppose that $\mathcal{K}$ covers $X$. Then, by Lemma 18.2, there is an integrable homotopy between $f$ and any map in $\mathcal{N}(f, U, \mathcal{V}, \mathcal{K})$. □

Corollary 18.3. If two foliated maps $f, g : \mathcal{F} \rightarrow \mathcal{G}$ are close enough in $C_{\text{sp}}(\mathcal{F}, \mathcal{G})$, then $\text{Hol}(f) = \text{Hol}(g)$.

Proof. This follows from Proposition 6.1 and Theorem 18.1. □

Now, assume that $Y$ is locally compact and Polish too, and let $\text{Prop}(\mathcal{F}, \mathcal{G})$ denote the set of proper continuous foliated maps $\mathcal{F} \rightarrow \mathcal{G}$.

Theorem 18.4. With the above notation and conditions, $\text{Prop}(\mathcal{F}, \mathcal{G})$ is open in $C_{\text{sp}}(\mathcal{F}, \mathcal{G})$.

Proof. Given $f \in \text{Prop}(\mathcal{F}, \mathcal{G})$, take a neighborhood $\mathcal{N} = \mathcal{N}(f, U, \mathcal{V}, \mathcal{K})$ of $f$ in $C_{\text{sp}}(\mathcal{F}, \mathcal{G})$ as above.

Claim 14. We can choose $U$, $\mathcal{V}$ and $\mathcal{K}$ such that $\mathcal{K}$ covers $X$ and $\mathcal{V}$ is locally finite.

To prove this assertion, let $\Lambda$ be the set of indices $i$. We can assume that $\mathcal{K}$ covers $X$ because $X$ is locally compact and Polish. On the other hand, since $Y$ is locally compact and Polish, it has a locally finite open covering $\mathcal{W}$. Then, for each $i \in \Lambda$, the set

$$\mathcal{W}_i = \{W \in \mathcal{W} \mid W \cap f(K_i) \neq \emptyset\}$$

is finite, and there is an expression

$$K_i = \bigcup_{W \in \mathcal{W}_i} K_i, W,$$
where each $K_{i,W}$ is compact and so that $f(K_{i,W}) \subset W$. For each $i \in \Lambda$ and $W \in \mathcal{W}_i$, there is a finite covering $\mathcal{V}'_{i,W}$ of $f(K_{i,W})$ by simple open sets which are uniform in $U_i$. Then, for each $i$ and $W \in \mathcal{W}_i$, there is another expression

$$K_{i,W} = \bigcup_{V' \in \mathcal{V}'_{i,W}} K_{i,W,V'},$$

where each $K_{i,W,V'}$ is compact and so that $f(K_{i,W,V'}) \subset V'$. Now let

$$\Lambda' = \{(i, W, V') \mid i \in \Lambda, \ W \in \mathcal{W}_i, \ V' \in \mathcal{V}'_{i,W}\},$$

and consider the indexed sets

$$U' = \{U'_{i,W,V'}\}, \quad \mathcal{V}' = \{V'_{i,W,V'}\}, \quad \mathcal{K}' = \{K'_{i,W,V'}\},$$

with $(i, W, V') \in \Lambda'$, where

$$U'_{i,W,V'} = U_i, \quad V'_{i,W,V'} = V', \quad K'_{i,W,V'} = K_{i,W}.$$

Then $U'$, $\mathcal{V}'$ and $\mathcal{K}'$ satisfy the conditions of Claim 14, and $\mathcal{N}(f, U, \mathcal{V}, K')$ is defined because each $V' \in \mathcal{V}'_{i,W}$ is uniform in $V_i$.

When the properties of Claim 14 are satisfied, for any compact subset $R \subset Y$, there is a finite subfamily $\Lambda_R \subset \Lambda$ such that

$$i \in \Lambda \setminus \Lambda_R \implies R \cap V_i = \emptyset.$$

On the other hand, by the conditions of Claim 14 and the definition of $\mathcal{N}$, for any $g \in \mathcal{N}$ and $i \in \Lambda$, we have:

- $g(X) \subset \bigcup_{i \in \Lambda} V_i$; and
- there is some finite subset $\Lambda_{g,i} \subset \Lambda$ such that $g^{-1}(V_i) \subset \bigcup_{j \in \Lambda_{g,i}} K_j$.

Therefore

$$g^{-1}(R) \subset \bigcup_{i \in \Lambda_R} g^{-1}(V_i) \subset \bigcup_{i \in \Lambda_R} \bigcup_{j \in \Lambda_i} K_j,$$

and thus $g^{-1}(R)$ is compact. So $\mathcal{N} \subset \text{Prop}(\mathcal{F}, \mathcal{G})$, and the result follows. \qed

19. $C^{0,\infty}$ approximations of foliated maps

19.1. The general case

With the notation of the above section, suppose that $\mathcal{F}$ and $\mathcal{G}$ are $C^{0,\infty}$ (Section 4), and assume that $X$ is locally compact and Polish.

**Theorem 19.1.** With the above notation and conditions, suppose that some $f \in C(\mathcal{F}, \mathcal{G})$ is of class $C^{0,\infty}$ on some neighborhood of a closed subset $E \subset X$. Then any neighborhood of $f$ in $\mathcal{C}_{\text{SP}}(\mathcal{F}, \mathcal{G})$ contains some $g \in C^{0,\infty}(\mathcal{F}, \mathcal{G})$ such that $g = f$ on some neighborhood of $E$. In particular, $C^{0,\infty}(\mathcal{F}, \mathcal{G})$ is dense in $\mathcal{C}_{\text{SP}}(\mathcal{F}, \mathcal{G})$.

The first step to prove Theorem 19.1 is the following local result. Let $T$ and $T'$ be topological spaces, and let $L$ and $L'$ be connected open subsets of Euclidean spaces; assume that $T$ is locally compact and Polish. Let $\mathcal{F}$ and $\mathcal{G}$ be the $C^\infty$ foliations on $X = T \times L$ and $Y = T' \times L'$ with leaves $\{x\} \times L$ and $\{x\}' \times L'$ for $x \in T$ and $x' \in T'$, respectively. For any $u \in C(\mathcal{F}, \mathcal{G})$, write $u(x, y) = (\tilde{u}(x), \check{u}(x, y))$ for all $(x, y) \in X$. Obviously, the foliated map $u$ is of class $C^{0,\infty}$ if and only if the mapping $y \mapsto \tilde{u}(x, y)$ is $C^\infty$ for each $x \in T$ and its partial derivatives of arbitrary order depend continuously on $x$.

**Lemma 19.2.** Let $Q \subset X$ be a closed subset and $W \subset X$ an open subset. If some $u \in C(\mathcal{F}, \mathcal{G})$ is of class $C^{0,\infty}$ on some neighborhood of $Q \setminus W$, then any neighborhood of $u$ in $\mathcal{C}_{\text{SP}}(\mathcal{F}, \mathcal{G})$ contains some map $v$ which is $C^{0,\infty}$ on some neighborhood of $Q$, and which equals $u$ on $X \setminus W$. 

Proof. Fix any neighborhood $N$ of $u$ in $C_{sp}(F, G)$. Since $X$ is locally compact and Polish, if $N$ is small enough, we can assume that $\bar{v} = \bar{u}$ for all $v \in N$. So the strong compact-open topology and strong plaqueswise topology have the same restrictions to $N$, and the corresponding space will be denoted by $N_S$. Observe also that the second factor projection, $pr_2 : Y \to L'$, induces a homeomorphism

$$pr_2 : N_S \to C_S(X, L'), \quad v \mapsto pr_2 \circ v = \bar{v}.$$  

Therefore, by the relative approximation theorem [24, pp. 48–49], there is some $h \in pr_2(N)$ which is $C^{0, \infty}$ on some neighborhood of $Q$, and which equals $\bar{u}$ on $M \setminus W$. Then the result follows with the map $v \in N$ satisfying $\bar{v} = h$. 

Proof of Theorem 19.1. This is similar to the proof of the usual approximation theorem [24, p. 49]. Fix the following:

- A locally finite open covering $\{U_i\}$ of $X$ by relatively compact domains of foliated charts of $F$.
- A covering $\{K_i\}$ of $X$ by compact subsets, with the same index set, satisfying $K_i \subset U_i$ for all $i$.
- An open subset $V_i \subset M$ for each $i$ so that $K_i \subset V_i$ and $\overline{V_i} \subset U_i$.
- For each $i$, a simple open set $U'_i$ of $Y$ such that $f(U'_i) \subset U'_i$. Let $p_i : U'_i \to T'_i$ denote the canonical projection to the local quotient.
- An open neighborhood $O$ of $E$ so that $f$ is $C^{0, \infty}$ around $\overline{O}$.
- Any neighborhood $M$ of $f$ in $C_{sp}(F, G)$ satisfying $g(U'_i) \subset U'_i$ for all $i$ and all $g \in M$; this property holds when $M$ is small enough.

Since the open covering $\{U_i\}$ of $X$ is locally finite, and $X$ is locally compact and Polish, the above index set is countable; say, either $\mathbb{N}$, or $\{0, \ldots, k\}$ for some $k \in \mathbb{N}$.

Claim 15. For each $i$, there is some $g_i \in M$ which is $C^{0, \infty}$ on some neighborhood of $K_0 \cup \cdots \cup K_i$, which equals $f$ on some neighborhood of $\overline{O}$, and which equals $g_{i-1}$ on some neighborhood of $X \setminus U_i$ if $i > 0$.

The result follows from this assertion because a foliation map $g \in M$ is well defined by $g(x) = g_{i_x}(x)$ for each $x \in M$, where

$$i_x = \max\{i \mid x \in U_i\}.$$  

Such a $g$ is $C^{0, \infty}$ because the sets $K_i$ cover $M$, and $g$ is obviously equal to $f$ on $O$.

To prove Claim 15, each $g_i$ is defined by induction on $i$. To simplify the construction of this sequence, we can assume $U_0 = K_0 = \emptyset$, and take $g_0 = f$. Now let $i > 0$, and assume that $g_{i-1}$ is defined. Let $F_i = F_{\mid U_i}$ and $G_i = G_{\mid U'_i}$, and let $u_{i-1}$ denote the restriction $g_{i-1} : F_i \to G_i$. Let $\mathcal{Y}_i$ denote the subspace of $C_{sp}(F_i, G_i)$ whose elements are the foliated maps $v \in C(F_i, G_i)$ satisfying

$$p_i \circ v = p_i \circ u_{i-1}, \quad v|_{U_i \setminus V_i} = u_{i-1}|_{U_i \setminus V_i}.$$  

Let $H_i : \mathcal{Y}_i \to C_{sp}(F_i, G_i)$ be the extension map given by

$$H_i(v) = \begin{cases} v & \text{on } U_i, \\ g_{i-1} & \text{on } X \setminus U_i. \end{cases}$$  

It is easy to prove that $H_i$ is continuous, and thus there is some neighborhood $N_i$ of $u_{i-1}$ in $\mathcal{Y}_i$ such that $H_i(N_i) \subset M$. Let $N'_i$ be a neighborhood of $u_{i-1}$ in $C_{sp}(F_i, G_i)$ with $N'_i = \mathcal{Y}_i \cap N_i$. Then, by Lemma 19.9, there is some $v_i \in N'_i$ which is $C^{0, \infty}$ on some neighborhood of $K_i$, and which equals $u_{i-1}$ on some neighborhood of $U_i \setminus W_i$, where $W_i = V_i \cap (U_i \setminus \overline{O})$. It follows that $v_i \in \mathcal{Y}_i$, and Claim 15 holds with $g_i = H_i(v_i)$. 

Corollary 19.3. If there is an integrable homotopy between two $C^{0, \infty}$ foliated maps $F \to G$, then there also exists a $C^{0, \infty}$ integrable homotopy between them.
Proof. Let $H$ be an integrable homotopy between two maps $f, g \in C^{0,\infty}(\mathcal{F}, \mathcal{G})$. Then let $\overline{H} : \mathcal{F} \times \mathbb{R} \to \mathcal{G}$ be the foliated map defined by

$$\overline{H}(x, t) = \begin{cases} H(x, t) & \text{if } t \in I, \\ f(x) & \text{if } t \leq 0, \\ g(x) & \text{if } t \geq 1. \end{cases}$$

This $\overline{H}$ is $C^{0,\infty}$ on $X \times (\mathbb{R} \setminus I)$. Then, by Theorem 19.1, there is a $C^{0,\infty}$ foliated map $F : \mathcal{F} \times \mathbb{R} \to \mathcal{G}$ which equals $H$ on $X \times (-\infty, -1]$ and $X \times [2, \infty)$. Thus the foliated map $F' : \mathcal{F} \times I \to \mathcal{G}$, defined by $F'(x, t) = F(x, 3t - 1)$, is a $C^{0,\infty}$ integrable homotopy between $f$ and $g$. \qed

Corollary 19.4. Any map in $C(\mathcal{F}, \mathcal{G})$ is integrably homotopic to a map in $C^{0,\infty}(\mathcal{F}, \mathcal{G})$.

Proof. This is a direct consequence of Theorem 18.1 and Corollary 19.3. \qed

19.2. The case of proper maps

Now, assume that $Y$ is locally compact and Polish too. When Prop$(\mathcal{F}, \mathcal{G})$ is endowed with the restriction of the strong plaquewise topology, it will be denoted by Prop$_{SP}$(\mathcal{F}, \mathcal{G}). Let $\text{Prop}^{0,\infty}(\mathcal{F}, \mathcal{G}) = \text{Prop}(\mathcal{F}, \mathcal{G}) \cap C^{0,\infty}(\mathcal{F}, \mathcal{G})$.

Theorem 19.5. With the above notation and conditions, in Theorem 19.1, if $f$ is proper, then $g$ can be chosen to be proper too. In particular, $\text{Prop}^{0,\infty}(\mathcal{F}, \mathcal{G})$ is dense in Prop$_{SP}(\mathcal{F}, \mathcal{G})$.

Proof. This follows from Theorems 18.4 and 19.1. \qed

Corollary 19.6. If there is a proper integrable homotopy between two proper $C^{0,\infty}$ foliated maps $\mathcal{F} \to \mathcal{G}$, then there also exists a proper $C^{0,\infty}$ integrable homotopy between them.

Proof. This can be proved like Corollary 19.3, by using Theorem 19.5 instead of Theorem 19.1. \qed

Corollary 19.7. There is a proper integrable homotopy between any proper continuous foliated map $\mathcal{F} \to \mathcal{G}$ and some proper $C^{0,\infty}$ foliated map.

Proof. This is a direct consequence of Theorem 18.1 and Corollary 19.6. \qed

19.3. The case of $C^{\infty,0}$ foliated maps

If $\mathcal{F}$ and $\mathcal{G}$ are $C^{\infty,0}$ foliations, the notation $C^{\infty,0}_{\text{SP}}(\mathcal{F}, \mathcal{G})$ is used when the set $C^{\infty,0}(\mathcal{F}, \mathcal{G})$ is endowed with the restriction of the strong plaquewise topology. If moreover $\mathcal{F}$ and $\mathcal{G}$ are $C^{\infty}$ foliations, the regularity of the approximation in Theorem 19.1 can be improved as follows.

Theorem 19.8. With the above notation and conditions, suppose that some $f \in C^{\infty,0}(\mathcal{F}, \mathcal{G})$ is of class $C^{\infty}$ on some neighborhood of a closed subset $E \subset M$. Then any neighborhood of $f$ in $C_{\text{SP}}(\mathcal{F}, \mathcal{G})$ contains some $g \in C^{\infty}(\mathcal{F}, \mathcal{G})$ such that $g = f$ on some neighborhood of $E$. In particular, $C^{\infty}(\mathcal{F}, \mathcal{G})$ is dense in $C^{\infty,0}_{\text{SP}}(\mathcal{F}, \mathcal{G})$.

Theorem 19.8 follows by adapting the proof of Theorem 19.1 and using the following version of Lemma 19.9.

Lemma 19.9. With the notation of Lemma 19.2, suppose that $T$ and $T'$ are open sets in Euclidean spaces. Let $Q \subset X$ be a closed subset and $W \subset X$ an open subset. If some $u \in C^{\infty,0}(\mathcal{F}, \mathcal{G})$ is of class $C^{\infty}$ on some neighborhood of $Q \setminus W$, then any neighborhood of $u$ in $C_{\text{SP}}(\mathcal{F}, \mathcal{G})$ contains some map $v$ which is $C^{\infty}$ on some neighborhood of $Q$, and which equals $u$ on $X \setminus W$. 

Proof. The condition that \( u \) is \( C^{\infty,0} \) means that \( \bar{u} \) is \( C^\infty \). On the other hand, the proof of the approximation theorem [24, Chapter 2, Theorem 2.6] can be easily adapted to get a relative approximation theorem. Then, with the notation of the proof of Lemma 19.2, we can assume that \( h \) is \( C^\infty \), and thus \( v \) is \( C^\infty \). □

Corollary 19.10. With the above notation and conditions, let \( f \) and \( g \) be \( C^{\infty,0} \) foliated maps \( F \to G \). If there is an integrable homotopy between \( f \) and \( g \), then there also exists a \( C^\infty \) integrable homotopy between \( f \) and \( g \).

Proof. This follows by using Theorem 19.8 instead of Theorem 19.1 in the proof Corollary 19.3. □

Corollary 19.11. Any map in \( C^{\infty,0}(F, G) \) is integrably homotopic to a map in \( C^\infty(F, G) \).

Proof. This is a direct consequence of Theorem 18.1 and Corollary 19.10. □

19.4. The case of proper \( C^{\infty,0} \) foliated maps

In Section 19.3, suppose that \( Y \) is also locally compact and Polish. Let \( \text{Prop}^\infty_0(F, G) \) denote the set \( \text{Prop}^\infty_0(F, G) \) endowed with the restriction of the strong plaquewise topology, and let

\[
\text{Prop}^\infty(F, G) = \text{Prop}(F, G) \cap C^\infty(F, G).
\]

Theorem 19.12. With the above notation and conditions, in Theorem 19.8, if \( f \) is proper, then \( g \) can be chosen to be proper too. In particular, \( \text{Prop}^\infty(F, G) \) is dense in \( \text{Prop}^\infty_0(F, G) \).

Proof. This follows from Theorems 18.4 and 19.8. □

Corollary 19.13. If there is a proper integrable homotopy between two maps in \( \text{Prop}^\infty(F, G) \), then there also exists a proper \( C^\infty \) integrable homotopy between them.

Proof. This follows by using Theorem 19.12 instead of Theorem 19.8 in the proof Corollary 19.10. □

Corollary 19.14. There is a proper integrable homotopy between any proper \( C^{\infty,0} \) foliated map \( F \to G \) and some proper \( C^\infty \) foliated map.

Proof. This is a direct consequence of Theorem 18.1 and Corollary 19.13. □

19.5. The case of Riemannian foliations with dense leaves

Assume that \( F \) and \( G \) are transversely complete Riemannian foliations, and that the leaves of \( G \) are dense. Then \( C(F, G) = C^{\infty,0}(F, G) \) by Theorem 16.1(iii). So Theorem 19.8 and Corollaries 19.10 and 19.11 have the following consequences.

Theorem 19.15. With the above notation and conditions, suppose that some \( f \in C(F, G) \) is of class \( C^\infty \) on some neighborhood of a closed subset \( E \subset M \). Then any neighborhood of \( f \) in \( C_{SP}(F, G) \) contains some \( g \in C^\infty(F, G) \) such that \( g = f \) on some neighborhood of \( E \). In particular, \( C^\infty(F, G) \) is dense in \( C_{SP}(F, G) \).

Corollary 19.16. If there is an integrable homotopy between two continuous foliated maps \( F \to G \), then there also exists a \( C^\infty \) integrable homotopy between them.

Corollary 19.17. Any continuous foliated map \( F \to G \) is integrably homotopic to a \( C^\infty \) foliated map.
19.6. The case of proper foliated maps between Riemannian foliations with dense leaves

Theorem 19.12, and Corollaries 19.13 and 23.6 have the following consequences.

Theorem 19.18. In Theorem 19.15, if \( f \) is proper, then \( g \) can be chosen to be proper too. In particular, \( \text{Prop}^\infty(\mathcal{F}, \mathcal{G}) \) is dense in \( \text{Prop}_\text{SP}(\mathcal{F}, \mathcal{G}) \).

Corollary 19.19. If there is a proper integrable homotopy between two maps in \( \text{Prop}(\mathcal{F}, \mathcal{G}) \), then there also exists a proper \( C^\infty \) integrable homotopy between them.

Corollary 19.20. There is a proper integrable homotopy between each proper continuous foliated map \( \mathcal{F} \to \mathcal{G} \) and some proper \( C^\infty \) foliated map.

20. The strong horizontal topology

In this section, we introduce a second version of the strong and weak compact-open topologies. Now, we consider the set of foliated maps between two transversely complete Riemannian foliations, and the topology is closely related with the horizontal metric.

Let \( \mathcal{F} \) and \( \mathcal{G} \) be transversely complete Riemannian foliations on manifolds \( M \) and \( N \), and fix any transversely complete bundle-like metric on \( N \). Consider the following data:

- Any \( f \in C(\mathcal{F}, \mathcal{G}) \).
- Any locally finite family \( Q = \{Q_a\} \) of saturated closed subsets of \( M \) with compact projection to \( M/\mathcal{F} \).
- A family \( \mathcal{E} = \{\epsilon_a\} \) of positive real numbers, with the same index set as \( Q \).

Let \( \mathcal{M}(f, Q, \mathcal{E}) \) be the set of continuous foliated maps \( g : \mathcal{F} \to \mathcal{G} \) such that there is some sequence \( (f_0, f_1, \ldots) \) in \( C(\mathcal{F}, \mathcal{G}) \) satisfying the following properties:

- \( f_0 = f \);
- \( f_k = g \) on each \( Q_a \) for all but finitely many \( k \in \mathbb{N} \); and
- for each \( x \in M \) and \( k \in \mathbb{N} \), there is some horizontal geodesic arc \( c_{x,k} \) between \( f_k(x) \) and \( f_{k+1}(x) \) so that

\[
x \in Q_a \implies \sum_{k=0}^{\infty} \text{length}(c_{x,k}) < \epsilon_a.
\]

Suppose that \( \epsilon_a \geq \epsilon'_a + \epsilon''_a \) with \( \epsilon'_a, \epsilon''_a > 0 \), and let \( \mathcal{E}' = \{\epsilon'_a\} \) and \( \mathcal{E}'' = \{\epsilon''_a\} \). Then

\[
g \in \mathcal{M}(f, Q, \mathcal{E}') \implies \mathcal{M}(g, Q, \mathcal{E}'') \subset \mathcal{M}(f, Q, \mathcal{E}). \tag{20.1}
\]

It follows from (20.1) that the family of such sets \( \mathcal{M}(f, Q, \mathcal{E}) \) form a base of a topology, called the strong horizontal topology, and the corresponding space is denoted by \( C_{\text{SH}}(\mathcal{F}, \mathcal{G}) \). A weak version of this topology can be defined by considering finite families; it can be called the weak horizontal topology, and the notation \( C_{\text{WH}}(\mathcal{F}, \mathcal{G}) \) can be used for the corresponding space. Both of these topologies are equal when \( M/\mathcal{F} \) is compact; in this case, the term horizontal topology and the notation \( C_H(\mathcal{F}, \mathcal{G}) \) can be used.

Proposition 20.1. With the above notation and conditions, if two continuous foliated maps \( f, g : \mathcal{F} \to \mathcal{G} \) are close enough in \( C_{\text{SH}}(\mathcal{F}, \mathcal{G}) \), then there exists a foliated homotopy \( G : \mathcal{F} \times I_{\mathbb{R}} \to \mathcal{G} \) between them. If moreover \( f \) and \( g \) are proper, then \( G \) can also be chosen to be proper.

Proof. Consider an open set \( \mathcal{M} = \mathcal{M}(f, Q, \mathcal{E}) \) as above, which is a neighborhood of \( f \) in \( C_{\text{SH}}(\mathcal{F}, \mathcal{G}) \). For each \( g \in \mathcal{M} \), take the maps \( f_k \) given by the definition of \( \mathcal{M} \). Suppose that \( Q \) covers \( M \). Then, according to Section 14, there is a \( C^\infty \) regular submanifold \( \Omega \subset N \times N \) and a foliated homotopy \( H : (\mathcal{G} \times \mathcal{G})|_\Omega \times I_{\mathbb{R}} \to (\mathcal{G} \times \mathcal{G})|_\Omega \) such that:
• if the numbers $\epsilon_a$ are small enough, then $(f_k, f_{k+1})(M) \subset \Omega$; and
• $H_0(x, y) = (x, x)$ for all $(x, y) \in \Omega$, and $H_1 = \text{id}_\Omega$.

Therefore, when the numbers $\epsilon_a$ are small enough, a foliated homotopy $G_k$ between each $f_k$ and $f_{k+1}$ is given by the composite

$$\mathcal{F} \times I_{pt}^{(f_k, f_{k+1}) \times \text{id}_I} (G \times G)|_{\Omega} \times I_{pt}^{H} (G \times G)|_{\Omega} \xrightarrow{\text{pr}_2} G,$$

where $\text{pr}_2 : G \times G \to G$ is the second factor projection. Observe that $G_k(x, t) = f_k(x)$ if $f_k(x) = f_{k+1}(x)$.

Now, fix any increasing sequence $0 = t_0 < t_1 < \cdots \uparrow 1$, and let $\rho_k : [t_k, t_{k+1}] \to I$ be any orientation preserving homeomorphism for each $k \in \mathbb{N}$. Then a foliated homotopy $G : \mathcal{F} \times I_{pt} \to G$ between $f$ and $g$ can be defined by

$$G(x, t) = \begin{cases} G_k(x, \rho_k(t)) & \text{if } t_k \leq t \leq t_{k+1} \text{ for some } k \in \mathbb{N}, \\ g(x) & \text{if } t = 1. \end{cases}$$

Assume that moreover $\epsilon = \max \mathcal{E} < \infty$. Observe that the curve $t \mapsto H(x, t)$ is horizontal for each $x \in M$. Hence, if $\text{pr}_1 : N \times I \to N$ denotes the first factor projection, we have

$$f \circ \text{pr}_1(G^{-1}(R)) \subset \text{Pen}_H(R, \epsilon)$$

for any $R \subset N$, yielding

$$G^{-1}(R) \subset f^{-1}(\text{Pen}_H(R, \epsilon)) \times I.$$ 

Therefore, by Proposition 15.6, $G$ is proper if $f$ is proper. $\square$

**Proposition 20.2.** With the above notation and conditions, $\text{Prop}(\mathcal{F}, G)$ is open in $C_{\text{SH}}(\mathcal{F}, G)$.

**Proof.** Let $f \in \text{Prop}(\mathcal{F}, G)$. Choose $Q$ and $\mathcal{E}$ as above such that $Q$ covers $M$ and $\epsilon = \max \mathcal{E} < \infty$. Let $\mathcal{M} = \mathcal{M}(f, Q, \mathcal{E})$, which is a neighborhood of $f$ in $C_{\text{SH}}(\mathcal{F}, G)$. For any $g \in \mathcal{M}$ and $R \subset N$, we have

$$f(g^{-1}(R)) \subset \text{Pen}_H(R, \epsilon)$$

because $Q$ covers $M$, yielding

$$g^{-1}(R) \subset f^{-1}(\text{Pen}_H(R, \epsilon)).$$

So $\mathcal{M} \subset \text{Prop}(\mathcal{F}, G)$ by Proposition 15.6, and the result follows. $\square$

**21. $C^{\infty,0}$ approximations of foliated maps**

Approximation of foliated maps by $C^{\infty,0}$ ones is impossible in general (Example 30.6). But, in the case of transversely complete Riemannian foliations, such a $C^{\infty,0}$ approximation is a consequence of Theorem 16.1 and the structure of neighborhoods of orbit closures. The strong horizontal topology is appropriate to get this approximation.

**Theorem 21.1.** With the notation and conditions of Section 20, suppose that some $f \in C(\mathcal{F}, G)$ is $C^{\infty,0}$ on some neighborhood of a closed saturated subset $E \subset M$. Then any neighborhood of $f$ in $C_{\text{SH}}(\mathcal{F}, G)$ contains some $g \in C^{\infty,0}(\mathcal{F}, G)$ such that $g = f$ on some neighborhood of $E$. In particular, $C^{\infty,0}(\mathcal{F}, G)$ is dense in $C_{\text{SH}}(\mathcal{F}, G)$.

**Proof.** Let $\mu = \dim N / \overline{G} \in \mathbb{N}$. If $\mu = 0$, then $N / \overline{G}$ is totally disconnected [39, Chapter 3, Proposition 1.3], and thus the leaf closures of $\overline{G}$ are open in $N$. It follows that $f$ itself is $C^{\infty,0}$ by Theorem 16.1(iii). Therefore we can assume $\mu > 0$.

Let $\mathcal{M} = \mathcal{M}(f, Q, \mathcal{E})$ be a neighborhood of $f$ of the type described in Section 20, with $Q = \{Q_a\}$ and $\mathcal{E} = \{\epsilon_a\}$.

Let $p = \dim \overline{G}$ and $q = \text{codim} \overline{G}$. Consider the nested sequence

$$\emptyset = N_{-1} \subset N_0 \subset \cdots \subset N_q = N,$$

where each $N_{\ell}$ is the union of all leaf closures of $\overline{G}$ with dimension $\leq p + \ell$. According to Section 14, every $N_{\ell}$ is closed in $N$, each $N_{\ell} \setminus N_{\ell-1}$ is open and dense in $N_{\ell}$, $N_{\ell} \setminus N_{\ell-1}$ is a $\overline{G}$-saturated $C^\infty$ submanifold of $N$, and the restriction of $\overline{G}$ to $N_{\ell} \setminus N_{\ell-1}$ is a regular Riemannian foliation. Let $\mathcal{E}' = \{|\epsilon_a'|\}$ with $\epsilon_a' = \epsilon_a/(q+1)$, and set $f_{-1} = f$. 

Claim 16. For each $\ell \in \{0, \ldots, q\}$, there is some $f_\ell \in \mathcal{M}(f_{\ell-1}, Q, \mathcal{E}')$ which is $C^{\infty, 0}$ on some neighborhood of $f_{\ell-1}^{-1}(N_\ell)$, and such that $f_\ell = f_{\ell-1}$ on some neighborhood of $E$.

By (20.1), the result follows from Claim 16 with $g = f_q$.

Claim 16 is proved by induction on $\ell \in \{-1, 0, \ldots, q\}$. We have already defined $f_{-1} = f$, which is not required to satisfy any condition. Now, suppose that $f_{\ell-1}$ is already constructed for some $\ell \in \{0, \ldots, q\}$. Let $U$ be an open neighborhood of $E \cup f^{-1}(N_{\ell-1})$ where $f_{\ell-1}$ is $C^{\infty, 0}$. We can assume that $U$ is $\mathcal{F}$-saturated according to Section 14. Let $U_0$ be another $\mathcal{F}$-saturated open neighborhood of $E \cup f^{-1}(N_{\ell-1})$ whose closure is contained in $U$. Consider the notation of Section 14 applied to $\mathcal{G}$ and $N$.

Claim 17. For each $m \in \mathbb{Z}_+$, there are saturated open subsets $V_m, W_m \subset M \setminus \overline{U}_0$, a leaf closure $F_m$ of $\mathcal{G}$ in $N_\ell \setminus N_{\ell-1}$, and some $\delta_m > 0$ such that:

(i) $\{V_m\}$ is locally finite and of order $\leq \mu$;
(ii) $\{W_m\}$ covers $f_{\ell-1}^{-1}(N_\ell \setminus N_{\ell-1}) \setminus U$;
(iii) $W_m \subset V_m$;
(iv) $\text{Pen}(F_m, \delta_m) \subset \overline{\Omega}$;
(v) $\exp : \mathcal{F}_{F_m, \delta_m} \rightarrow \mathcal{F}_{F_m, \delta_m}$ is a foliated diffeomorphism;
(vi) $f_{\ell-1}(V_m) \subset \text{Pen}(F_m, 2^{-\mu}\delta_m)$; and
(vii) $\delta_m \leq \bar{\epsilon}_m / \mu$, where

\[ \bar{\epsilon}_m = \min \{\epsilon'_a \mid V_m \cap Q_a \neq \emptyset\}. \]

Since $\mathcal{G}$ is a regular Riemannian foliation on $N_\ell \setminus N_{\ell-1}$, there is a covering $\{O_\alpha\}$ of $N_\ell \setminus N_{\ell-1}$ by saturated open subsets, and a collection $\{\gamma_\alpha\}$ of positive real numbers such that $\text{Pen}(F, \gamma_\alpha) \subset \overline{\Omega}$ and $\exp : \mathcal{F}_{F, \gamma_\alpha} \rightarrow \mathcal{F}_{F, \gamma_\alpha}$ is a foliated diffeomorphism for each index $\alpha$ and every leaf closure $F$ of $\mathcal{G}$ in $O_\alpha$. Let $\{V_k'\}$ (k $\in \mathbb{Z}_+$) be a locally finite family of saturated open subsets of $M \setminus U_0$ that covers the closed set $f_{\ell-1}^{-1}(N_\ell) \setminus U$, and such that $V_k' \cap (N_\ell \setminus N_{\ell-1}) \subset f_{\ell-1}^{-1}(O_{\alpha_k})$ for some mapping $k \mapsto \alpha_k$. Then

\[ \bar{\epsilon}'_k = \min \{\epsilon'_a \mid V_k' \cap Q_a \neq \emptyset\} > 0 \]

for all $k$, and let $\delta'_k = \min(\gamma_{\alpha_k}, \bar{\epsilon}'_k / \mu)$. When $k \in \mathbb{Z}_+$ and $F$ runs in the family of leaf closures of $\mathcal{G}$ in $O_{\alpha_k}$, the family $\{f_{\ell-1}^{-1}(\text{Pen}(F, 2^{-\mu}\delta'_k))\}$ is another covering of $f_{\ell-1}^{-1}(N_\ell) \setminus U$ by saturated open subsets of $M \setminus U_0$. From [39, Chapter 3, Theorem 4.3], it follows that there is a common locally finite refinement $\{V_m\}$ (m $\in \mathbb{Z}_+$) of the coverings $\{V_k'\}$ and $\{f_{\ell-1}^{-1}(\text{Pen}(F, 2^{-\mu}\delta'_k))\}$ by saturated open subsets of $M \setminus U_0$, and whose order is $\leq \mu$. Thus there are mappings $m \mapsto k_m$ and $m \mapsto F_m$, where $F_m$ is a leaf closure of $\mathcal{G}$ in $O_{k_m}$, so that

\[ V_m \subset V_{k_m}, \quad f_{\ell-1}(V_m) \subset \text{Pen}(F_m, 2^{-\mu}\delta'_{k_m}). \] (21.1)

Thus the sets $V_m$ satisfy Claim 17(i), and Claim 17(vi) follows from the second part of (21.1) with $\delta_m = \delta'_{k_m}$. By the first part of (21.1), we get \(\bar{\epsilon}'_{k_m} \leq \bar{\epsilon}_m\) with the definition of \(\bar{\epsilon}_m\) given in Claim 17(vii). So

\[ \delta_m \leq \min(\gamma_{\alpha_{k_m}}, \bar{\epsilon}_m / \mu), \]

yielding the conditions (iv), (v) and (vii) of Claim 17. A shrinking of the covering $\{V_m\}$ gives the family $\{W_m\}$ satisfying conditions (ii) and (iii) of Claim 17, finishing the proof of this assertion.

According to Section 14, from Claim 17, we get the following for each $m \in \mathbb{Z}_+$:

- There is a $C^{\infty}$ function $\lambda_m : M \rightarrow I$, constant on every leaf of $\mathcal{F}$, such that $\lambda_m \equiv 1$ on some saturated open neighborhood $O_m$ of $M \setminus V_m$ and $\lambda_m \equiv 0$ on $W_m$.
- There is a $C^{\infty}$ foliated homotopy $H_m : \mathcal{G}_{F_m, \delta_m} \times I_{pt} \rightarrow \mathcal{G}_{F_m, \delta_m}$ such that:
  - $\text{im} \ H_{m, 0} \subset F_m$;
  - $H_{m, 1} = \text{id}_{\mathcal{G}_{F_m, \delta_m}}$; and
  - the mapping $t \mapsto H_m(x', t)$ is a horizontal geodesic segment of length $< d(x', F_m)$ for each $x' \in \text{Pen}(F_m, \delta_m)$, where $d$ denotes the distance function of $N$. 

Let $\mu_0 = 0$ and, for $k \in \mathbb{Z}_+$, let $\mu_k$ be the order of the family $\{V_1, \ldots, V_k\}$; thus $\mu_0 \leq \mu_1 \leq \cdots$ and $\mu_k \leq \mu$ for all $k \in \mathbb{N}$.

**Claim 18.** For each $k \in \mathbb{N}$, there is some $f_{\ell,k} \in C(\mathcal{F}, \mathcal{G})$ such that:

(i) $f_{\ell,k} = f_{\ell,1}$ on $M \setminus V_k$;
(ii) $f_{\ell,k}$ is $C^{\infty,0}$ on $W_1 \cup \cdots \cup W_k \cup U$ for $k > 0$;
(iii) $f_{\ell,k}(V_m) \subset \text{Pen}(F_m, 2^\mu_k - \mu \delta_m)$ for all $k$ and $m$; and
(iv) for each $x \in V_m \cap V_k$, there is a horizontal geodesic segment $c_{x,k}$ between $f_{\ell,k-1}(x)$ and $f_{\ell,k}(x)$ whose length is $< 2^{\mu_k-1} - \mu \delta_m$.

First, set $f_{\ell,0} = f_{\ell,1}$, which satisfies the conditions of Claim 18 because $\mu_0 = 0$. Now assume that $f_{\ell,k-1}$ is defined for some $k > 0$ and satisfies the conditions of Claim 18. Then let $f_{\ell,k}$ be the combination of the restriction $f_{\ell,k-1} : \mathcal{F}|_\Omega_k \to \mathcal{G}$ and the composite

$$\mathcal{F}|_{V_k} \xrightarrow{(f_{\ell,k-1}, \lambda_k)} \mathcal{G}_{F_k, \delta_k} \times I_{pt} \xrightarrow{H_k} \mathcal{G}_{F_k, \delta_k} \hookrightarrow \mathcal{G}.$$ 

Observe that $f_{\ell,k}$ is well defined because $f_{\ell,k-1}$ satisfies Claim 18(iii). Moreover $f_{\ell,k}$ is a continuous foliated map $\mathcal{F} \to \mathcal{G}$ because $f_{\ell,k-1}$ and $H_k$ are continuous, and $\lambda_k$ is constant on the leaves.

We have $f_{\ell,k} = f_{\ell,k-1}$ on $M \setminus V_k$, yielding Claim 18(i) for $f_{\ell,k}$ because $f_{\ell,k-1}$ satisfies this condition.

On the one hand, $f_{\ell,k}$ is $C^{\infty,0}$ on any open set where $f_{\ell,k-1}$ is $C^{\infty,0}$ because $H_k$ and $\lambda_k$ are $C^{\infty}$. On the other hand, $f_{\ell,k}(W_k) \subset F_k$, and thus $f_{\ell,k}$ is $C^{\infty,0}$ on $W_k$ by Theorem 16.1(iii). Therefore $f_{\ell,k}$ satisfies Claim 18(ii) since so does $f_{\ell,k-1}$.

Take any $m \in \mathbb{Z}_+$ and any $x \in V_m$. If $x \in V_m \setminus V_k$, then $f_{\ell,k}(x) = f_{\ell,k-1}(x)$, yielding

$$d(f_{\ell,k}(x), F_m) = d(f_{\ell,k-1}(x), F_m) < 2^{\mu_k-1} - \mu \delta_m \leq 2^{\mu_k-1} - \mu \delta_m.$$

If $x \in V_m \cap V_k$, then $\mu_k = \mu_k-1 + 1$, and the mapping

$$t \mapsto H_k(f_{\ell,k-1}(x), 1 - t + t\lambda_k(x)), \quad 0 \leq t \leq 1,$$

is a horizontal geodesic segment $c_{x,k}$ between $f_{\ell,k-1}(x)$ and $f_{\ell,k}(x)$ whose length is $< 2^{\mu_k-1} - \mu \delta_m$ since $f_{\ell,k-1}$ satisfies Claim 18(iii). Then Claim 18(iv) follows, and moreover

$$d(f_{\ell,k}(x), F_m) \leq d(f_{\ell,k}(x), f_{\ell,k-1}(x)) + d(f_{\ell,k-1}(x), F_m) \leq 2^{\mu_k-1} - \mu \delta_m + 2^{\mu_k-1} - \mu \delta_m = 2^{\mu_k-1} - \mu \delta_m$$

because Claim 18(iii) holds for $f_{\ell,k-1}$. Therefore $f_{\ell,k}$ also satisfies Claim 18(iii), completing the proof of Claim 18.

Now, let $f_\ell : M \to N$ be the map defined by $f_\ell(x) = f_{\ell,k}(x)$, where

$$k_x = \begin{cases} \max\{k \in \mathbb{Z}_+ \mid x \in V_k\} & \text{if } x \in \bigcup_m V_m, \\ 0 & \text{if } x \notin \bigcup_m V_m. \end{cases}$$

Observe that $f_\ell = f_{\ell,k}$ on some saturated neighborhood of each point if $k$ is large enough because $\{V_m\}$ is of finite order. So the following properties hold:

- $f_{\ell} = f_{\ell,-1}$ on $U_0$ by Claim 18(i) since $U_0 \cap \bigcup_m V_m = \emptyset$;
- $f_\ell$ is a continuous foliated map $\mathcal{F} \to \mathcal{G}$ because each $f_{\ell,k}$ is a continuous foliated map; and
- $f_{\ell}$ is $C^{\infty,0}$ on the neighborhood $U \cup \bigcup_m W_m$ of $f_{\ell,-1}(N_\ell)$ by Claim 18(ii).

Take any $x \in M$. Consider the horizontal geodesic segments $c_{x,k}$ given by Claim 18(iv) if $x \in V_k$, and let $c_{x,k}$ be the constant geodesic segment at $f_{\ell,k-1}(x)$ if $x \in M \setminus V_k$. Then

$$\sum_{k=1}^{\infty} \text{length}(c_{x,k}) = 0.$$
if \( x \notin \bigcup_m V_m \), whilst, by Claim 17(vii),
\[
\sum_{k=1}^{\infty} \text{length}(c_{x,k}) < \sum_{x \in V_k} 2^{\mu_k-1-\mu} \delta_m \leq \mu \delta_m \leq \bar{\epsilon}_m
\]
if \( x \in V_m \) for some \( m \). So
\[
x \in Q_a \implies \sum_{k=1}^{\infty} \text{length}(c_{x,k}) < \epsilon'_a,
\]
which means that \( f_{\ell} \in \mathcal{M}(f_{\ell-1}, Q, \mathcal{E}') \), and Claim 16 follows.

22. The strong adapted topology

With the notation and conditions of the above section, the intersection of the strong plaquewise topology and the strong horizontal topology on \( C(F, G) \) is called the strong adapted topology, and the corresponding space is denoted by \( C_{SA}(F, G) \). The weak adapted topology can be also defined as the intersection of the weak plaquewise topology and the weak horizontal topology, and the corresponding space can be denoted by \( C_{WA}(F, G) \). Both of these topologies are equal when \( M \) is compact; in this case, the term adapted topology and the notation \( C_A(F, G) \) can be used. If \( F \) has compact leaf closures, then the strong adapted topology equals the strong compact-open topology on \( C(F, G) \).

Then, with the notation of Section 20, let
\[
\mathcal{M}(f, Q, \mathcal{E}, N) = \bigcup_{g \in N} \mathcal{M}(g, Q, \mathcal{E}).
\]
The following result is easily verified.

Lemma 22.1. The sets \( \mathcal{M}(f, Q, \mathcal{E}, N) \) form a base of the strong adapted topology.

Theorem 22.2. With the above notation and conditions, if two foliated maps are close enough in \( C_{SA}(F, G) \), then there exists a foliated homotopy between them.

Proof. This follows from Theorem 18.1, Proposition 20.1 and Lemma 22.1.

Corollary 22.3. If two foliated maps \( f, g : F \to G \) are close enough in \( C_{SA}(F, G) \), then \( \text{Hol}(f) \) is homotopic to \( \text{Hol}(g) \).

Proof. This is a consequence of Proposition 6.3 and Theorem 22.2.

Theorem 22.4. With the above notation and conditions, \( \text{Prop}(F, G) \) is open in \( C_{SA}(F, G) \).

Proof. This is a consequence of Theorem 18.4 and Proposition 20.2.

23. \( C^\infty \) approximations of foliated maps

23.1. The general case

Combining the \( C^{0,\infty} \) and \( C^{\infty,0} \) approximations, we get \( C^\infty \) approximation of foliated maps for transversely complete Riemannian foliations.

Theorem 23.1. With the notation and conditions of Section 22, suppose that some \( f \in C(F, G) \) is of class \( C^\infty \) on some neighborhood of a closed saturated subset \( E \subset M \). Then any neighborhood of \( f \) in \( C_{SA}(F, G) \) contains some \( g \in C^\infty(F, G) \) such that \( g = f \) on some neighborhood of \( E \). In particular, \( C^\infty(F, G) \) is dense in \( C_{SA}(F, G) \).
Proof. This is a consequence of Theorems 19.1 and 21.1.

Corollary 23.2. If there is a foliated homotopy between two $C^\infty$ foliated maps $\mathcal{F} \to \mathcal{G}$, then there also exists a $C^\infty$ foliated homotopy between them.

Proof. This follows with the arguments of Corollary 19.3, by using foliated homotopies instead of integrable homotopies, and using Theorem 23.1 instead of Theorem 19.1.

Corollary 23.3. Any foliated map $\mathcal{F} \to \mathcal{G}$ is foliatedly homotopic to a $C^\infty$ foliated map.

Proof. This is a direct consequence of Theorems 22.2 and 23.1.

23.2. The case of proper foliated maps

The notation $\text{Prop}_{SA}(\mathcal{F}, \mathcal{G})$ will be used when the set $\text{Prop}(\mathcal{F}, \mathcal{G})$ is endowed with the restriction of the strong adapted topology.

Theorem 23.4. In Theorem 23.1, if $f$ is proper, then $g$ can be chosen to be proper too. In particular, $\text{Prop}^\infty(\mathcal{F}, \mathcal{G})$ is dense in $\text{Prop}_{SA}^\infty(\mathcal{F}, \mathcal{G})$.

Proof. This follows from Theorems 22.4 and 23.1.

Corollary 23.5. If there is a proper foliated homotopy between two maps in $\text{Prop}^\infty(\mathcal{F}, \mathcal{G})$, then there also exists a proper $C^\infty$ foliated homotopy between them.

Proof. This can be proved with the arguments of Corollary 19.3, by using proper foliated homotopies instead of integrable homotopies, and using Theorem 23.4 instead of Theorem 19.1.

Corollary 23.6. There is a proper foliated homotopy between any proper continuous foliated map $\mathcal{F} \to \mathcal{G}$ and some proper $C^\infty$ foliated map.

Proof. This is a direct consequence of Theorems 22.2 and 23.4.

24. The spectral sequence of a $C^\infty$ foliation

Let $\mathcal{F}$ be a $C^\infty$ foliation of dimension $p$ and codimension $q$ on a $C^\infty$ manifold $M$. Consider the decreasing filtration of the de Rham differential algebra $(\Omega = \Omega(M), d)$ by the differential ideals

$$\Omega = F^0 \Omega \supset F^1 \Omega \supset \cdots \supset F^q \Omega \supset F^{q+1} \Omega = 0,$$

where and $r$-form is in $F^k \Omega$ if it vanishes when $r-k+1$ vectors are tangent to the leaves; intuitively, this means that its “transverse degree” is $\geq k$. The induced spectral sequence $(E_i = E_i(\mathcal{F}), d_i)$ is a differentiable invariant of $\mathcal{F}$ [31,43,28,2] (see also [34] for the general theory of spectral sequences).

A compactly supported version $(E_{c,i} = E_{c,i}(\mathcal{F}), d_i)$ can be also defined by restricting the above filtration to the subcomplex $\Omega_c \subset \Omega$ of compactly supported differential forms.

Let $\mathcal{G}$ be another $C^\infty$ foliation on a manifold $N$. For each $f \in C^\infty(\mathcal{F}, \mathcal{G})$, the corresponding homomorphism $f^* : \Omega(N) \to \Omega(M)$ preserves the filtrations and induces a spectral sequence homomorphism $E_i(f) : (E_i(\mathcal{G}), d_i) \to (E_i(\mathcal{F}), d_i)$.

If $f$ is a proper map, then $f^*$ restricts to a homomorphism $\Omega_c(N) \to \Omega_c(M)$, inducing a spectral sequence homomorphism $E_{c,i}(f) : (E_{c,i}(\mathcal{G}), d_i) \to (E_{c,i}(\mathcal{F}), d_i)$.

Proposition 24.1. For $C^\infty$ foliated maps $f, g : \mathcal{F} \to \mathcal{G}$, we have the following:
(i) If there is a $C^\infty$ integrable homotopy between $f$ and $g$, then $E_i(f) = E_i(g)$ for $i \geq 1$.
(ii) If there is a $C^\infty$ foliated homotopy between $f$ and $g$, then $E_i(f) = E_i(g)$ for $i \geq 2$.

**Proof.** This follows easily from the following observations with a general spectral sequence argument. The operator on differential forms induced by a $C^\infty$ integrable homotopy, as defined, e.g., in [7], preserves the filtration, whilst, for a $C^\infty$ foliated homotopy, the corresponding operator decreases the filtration degree at most by one.

The following result has a similar proof.

**Proposition 24.2.** For $C^\infty$ proper foliated maps $f, g : \mathcal{F} \to \mathcal{G}$, we have the following:

(i) If there is a $C^\infty$ proper integrable homotopy between $f$ and $g$, then $E_{c,i}(f) = E_{c,i}(g)$ for $i \geq 1$.
(ii) If there is a $C^\infty$ proper foliated homotopy between $f$ and $g$, then $E_{c,i}(f) = E_{c,i}(g)$ for $i \geq 2$.

### 25. Invariance of the spectral sequence

Let $\mathcal{F}$ and $\mathcal{G}$ be transversely complete Riemannian foliations.

#### 25.1. The general case

According to Theorem 23.1, any $f \in C(\mathcal{F}, \mathcal{G})$ is foliatedly homotopic to some $g \in C^\infty(\mathcal{F}, \mathcal{G})$. Moreover, by Corollary 23.2 and Proposition 24.1(ii), the homomorphism $E_i(g)$ is independent of the choice of $g$ for $i \geq 2$, and can be denoted by $E_i(f)$.

**Theorem 25.1.** With the above notation and conditions, for $f, g \in C(\mathcal{F}, \mathcal{G})$, we have the following:

(i) If $f$ and $g$ are foliatedly homotopic, then $E_i(f) = E_i(g)$ for $i \geq 2$.
(ii) If $f$ is a foliated homotopy equivalence, then $E_i(f)$ is an isomorphism for $i \geq 2$.

**Proof.** This is a consequence of Corollary 23.2 and Proposition 24.1(ii).

#### 25.2. The case of proper foliated maps

According to Corollary 23.6, there is a proper foliated homotopy between each $f \in \text{Prop}(\mathcal{F}, \mathcal{G})$ and some $g \in C^\infty(\mathcal{F}, \mathcal{G})$. Moreover, by Corollary 23.5 and Proposition 24.2(ii), the homomorphism $E_{c,i}(g)$ is independent of the choice of $g$ for $i \geq 2$, and can be denoted by $E_{c,i}(f)$.

**Theorem 25.2.** With the above notation and conditions, for $f, g \in \text{Prop}(\mathcal{F}, \mathcal{G})$, we have the following:

(i) If there is a proper foliated homotopy between $f$ and $g$, then $E_{c,i}(f) = E_{c,i}(g)$ for $i \geq 2$.
(ii) If $f$ is a proper foliated homotopy equivalence, then $E_{c,i}(f)$ is an isomorphism for $i \geq 2$.

**Proof.** This is a consequence of Corollary 23.5 and Proposition 24.2(ii).

#### 25.3. The case with dense leaves

Now assume that the leaves of $\mathcal{G}$ are dense. According to Corollary 19.17, any $f \in C(\mathcal{F}, \mathcal{G})$ is integrably homotopic to some $g \in C^\infty(\mathcal{F}, \mathcal{G})$. Moreover, by Corollary 19.16 and Proposition 24.1(i), the homomorphism $E_1(g)$ is also independent of the choice of $g$ and can be denoted by $E_1(f)$.

**Theorem 25.3.** With the above notation and conditions, for $f, g \in C(\mathcal{F}, \mathcal{G})$, we have the following:
25.4. The case of proper foliated maps and dense leaves

In this case, another description of these topologies can be given as follows. Fix an \( (\Phi, Q) \) such that \( \Phi(Q_a) \subset U'_a \) for all \( a \). All such sets \( M(Q, U') \) form a base of open sets of \( C_S(\mathcal{H}, \mathcal{H}') \). If we consider only finite families, we get a base of \( C_{c-o}(\mathcal{H}, \mathcal{H}') \).

Observe that \( C_{c-o}(\mathcal{H}, \mathcal{H}') \) and \( C_S(\mathcal{H}, \mathcal{H}') \) may not be Hausdorff; for instance, they have the trivial topology when \( \mathcal{H}' \) has dense orbits.

From now on, assume that \( \mathcal{H} \) and \( \mathcal{H}' \) are complete Riemannian pseudogroups. Then the orbit closure map \( C_S(\mathcal{H}, \mathcal{H}') \to C_S(\mathcal{H} \setminus T, \mathcal{H} \setminus T') \) is an identification. A similar property holds for the compact-open topologies. In this case, another description of these topologies can be given as follows. Fix an \( \mathcal{H}' \)-invariant metric on \( T' \), and let \( d \) be the corresponding distance function. Consider the following data:

- Any \( \Phi \in C(\mathcal{H}, \mathcal{H}') \).
- Any locally finite family \( Q = \{ Q_a \} \) of saturated closed subsets of \( T \) with compact projection to \( \mathcal{H} \setminus T \).
- A family \( E = \{ \epsilon_a \} \) of positive real numbers, with the same index set as \( Q \).

Let \( M(\Phi, Q, E) \) be the set of morphisms \( \Psi : \mathcal{H} \to \mathcal{H}' \) such that \( d(\Phi(F), \Psi(F)) < \epsilon_a \) for all index \( a \) and every orbit closure \( F \) in \( Q_a \). Notice that this condition means that, between \( \Phi(F) \) and \( \Psi(F) \), there is some geodesic segment on \( \mathcal{H}' \) of length \( < \epsilon_a \) (in the sense of Section 3). Then the sets \( M(\Phi, Q, E) \) form a base of open sets of \( C_S(\mathcal{H}, \mathcal{H}') \).

A base of \( C_{c-o}(\mathcal{H}, \mathcal{H}') \) is given by considering only finite families.
Proposition 26.1. Let \( F \) and \( G \) be transversely complete Riemannian foliations, where \( G \) is endowed with a bundle-like metric. Then the holonomy functor defines continuous maps

\[
C_{SA}(F, G) \rightarrow C_S(\text{Hol}(F), \text{Hol}(G)), \quad C_{WA}(F, G) \rightarrow C_{\text{WA}}(\text{Hol}(F), \text{Hol}(G)).
\]

Proof. This follows easily from the observation that each horizontal geodesic segment of the bundle-like metric of \( G \) project to geodesic segments of \( \text{Hol}(G) \) with the same length. \( \square \)

Theorem 26.2. With the above notation and conditions, if two morphisms are close enough in \( C_S(H, H') \), then there exists a homotopy between them.

Proof. According to Section 13, there is an open neighborhood \( \Omega' \) of the diagonal in \( T' \times T' \), which is invariant by \( H' \times H' \), and there is a homotopy \( \Psi : (H' \times H')|_{\Omega'} \times I \rightarrow (H' \times H')|_{\Omega'} \) such that \( \Psi_0 \) is generated by the map \( (x, y) \mapsto (x, x) \), and \( \Psi_1 \) is the identity morphism at \( (H' \times H')|_{\Omega'} \). If two morphisms \( \Phi_1 \) and \( \Phi_2 \) are close enough in \( C_S(H, H') \), then \( \Phi_1(F) \times \Phi_2(F) \subset \Omega' \) for all \( H \)-orbit closure \( F \). Then a homotopy between \( \Phi_1 \) and \( \Phi_2 \) is given by the composite

\[
(1, I) \rightarrow \Phi_1(\Phi_2) \rightarrow \Phi_2(\Phi_1) \rightarrow \Phi_1 \times \Phi_2 \rightarrow \Phi_2 \times \Phi_1 \rightarrow (H' \times H')|_{\Omega'} \rightarrow H',
\]

where the right-hand side morphism is generated by the restriction \( \Omega' \rightarrow T' \) of second factor projection \( T' \times T' \rightarrow T' \). \( \square \)

Corollary 26.3. If the orbits of \( H' \) are dense, then all morphisms \( H \rightarrow H' \) are homotopic to each other.

27. \( C^\infty \) approximations of morphisms

The following is a pseudogroup version of Theorem 23.1.

Theorem 27.1. Let \( H \) and \( H' \) be complete Riemannian pseudogroups. Suppose that some \( \Phi \in C(H, H') \) is \( C^\infty \) on some neighborhood of a closed saturated subset \( E \subset T \). Then any neighborhood of \( \Phi \) in \( C_S(H, H') \) contains some \( \Psi \in C^\infty(H, H') \) such that \( \Phi = \Psi \) on some neighborhood of \( E \). In particular, \( C^\infty(H, H') \) is dense in \( C_S(H, H') \).

Proof. This result can be proved with an easy adaptation to pseudogroups of the arguments of Theorem 21.1. Instead of observations from Section 14, we have to use the corresponding observations from Sections 12 and 13. Moreover we have to use the morphism versions of some operations with maps (see Section 2); e.g., the combination of morphisms is used in the needed version of Claim 18. \( \square \)

Like in the foliated setting, Theorem 27.1 has the following consequences.

Corollary 27.2. If there is a homotopy between two \( C^\infty \) morphisms \( H \rightarrow H' \), then there also exists a \( C^\infty \) homotopy between them.

Corollary 27.3. Any morphism \( H \rightarrow H' \) is homotopic to a \( C^\infty \) morphism.

28. Cohomologies of \( C^\infty \) pseudogroups

Let \( H \) be a \( C^\infty \) pseudogroup acting on a \( C^\infty \) manifold \( T \). The invariant complex of \( H \) is the subcomplex \( \Omega_{\text{inv}}(H) \subset \Omega(T) \) consisting of differential forms that are \( H \)-invariant; i.e., those forms \( \alpha \) satisfying \( h^* (\alpha_{\lim}) = \alpha_{\text{dom}} \) for all \( h \in H \). The corresponding cohomology is called the invariant cohomology and denoted by \( H_{\text{inv}}(H) \).

A \( C^\infty \) morphism \( \Phi : H \rightarrow H' \) induces a homomorphism of complexes \( \Phi^* : \Omega_{\text{inv}}(H') \rightarrow \Omega_{\text{inv}}(H) \), which is well defined by the condition \( (\Phi^* \alpha)|_{\text{dom} \Phi} = \phi^* \alpha \) for \( \alpha \in \Omega_{\text{inv}}(H') \) and \( \phi \in \Phi \). This assignment \( \Phi \mapsto \Phi^* \) is functorial; thus isomorphic pseudogroups have isomorphic invariant complexes. We get an induced homomorphism \( \Phi^* : H_{\text{inv}}(H') \rightarrow H_{\text{inv}}(H) \).
Let $\mathcal{F}$ be a $C^\infty$ foliation on a $C^\infty$ manifold $M$, and $\mathcal{P} : M \to \text{Hol}(\mathcal{F})$ the corresponding canonical morphism. Then $\mathcal{P}^* : \Omega_{\text{inv}}(\text{Hol}(\mathcal{F})) \to \Omega(M)$ restricts to an isomorphism $\mathcal{P}^* : \Omega_{\text{inv}}(\text{Hol}(\mathcal{F})) \to E^0_1(\mathcal{F})$, yielding the well-known isomorphism $H_{\text{inv}}(\text{Hol}(\mathcal{F})) \cong E^0_2(\mathcal{F})$.

**Proposition 28.1.** If there is a $C^\infty$ homotopy between two $C^\infty$ morphisms $\Phi_0, \Phi_1 : \mathcal{H} \to \mathcal{H}'$, then $\Phi^*_0 = \Phi^*_1 : H_{\text{inv}}(\mathcal{H}') \to H_{\text{inv}}(\mathcal{H})$.

**Proof.** This is a direct adaptation of the proof of the corresponding result for manifolds [7] by using morphisms instead of maps. $\Box$

In the sense of Section 3, we can also consider the de Rham cohomology $H_{\text{dR}}(\mathcal{H})$ of a $C^\infty$ pseudogroup $\mathcal{H}$ acting on some $C^\infty$ manifold $T$, which is precisely defined as follows. Let $\bigwedge T^h$ be the pseudogroup on $\bigwedge T^*$ generated by the local transformations of the form $\bigwedge T^h : \bigwedge T(\text{dom} h)^* \to \bigwedge T(h)^*$ with $h \in \mathcal{H}$; notice that $h^* h = \bigwedge T^h \circ h \circ T^h$ for all $h \in \mathcal{H}$. For each degree $r$, the restriction of $\bigwedge T^h$ to $\bigwedge^r T^h$ will be denoted by $\bigwedge^r T^h$. The fiber bundle projection $\pi : \bigwedge T^h \to T$ generates a morphism $\Pi : \bigwedge T^h \to \mathcal{H}$ since $\pi \circ \bigwedge T^h = h^{-1} \circ h$ for all $h \in \mathcal{H}$. A morphism $\Theta : \mathcal{H} \to \bigwedge T^h$ is called a differential form on $\mathcal{H}$ if it is a section of $\Pi$ in the sense that $\Pi \circ \Theta = \text{id}_{\mathcal{H}}$; it is said that $\Theta$ is of degree $r$ if $\im \Theta \subset \bigwedge^r T^h$. Any differential form $\Theta$ on $\mathcal{H}$ is generated by differential forms on open subsets of $T$: if the domain of some $\theta \in \Theta$ is small enough, then $h = \pi \circ \Theta \in \mathcal{H}$ and $\Theta \circ h^{-1} \in \bigwedge (\text{dom} h) \cap \Theta$. The exterior derivative of a $C^\infty$ differential form $\Theta$ on $\mathcal{H}$ is the $C^\infty$ differential form $d\Theta$ on $\mathcal{H}$ generated by the exterior derivatives $d\theta$ of those $\theta \in \Theta$ that are differential forms on open subsets of $T$. The vector space $\Omega(\mathcal{H})$ of $C^\infty$ differential forms on $\mathcal{H}$ becomes a complex endowed with the exterior derivative $d$ and the grading defined as above. Then $H_{\text{dR}}(\mathcal{H})$ is the cohomology of $(\Omega(\mathcal{H}), d)$. There is a canonical isomorphism $\Omega_{\text{inv}}(\mathcal{H}) \cong \Omega_{\text{dR}}(\mathcal{H})$, which assigns to each $\alpha \in \Omega_{\text{inv}}(\mathcal{H})$ the morphism generated by $\alpha$. Thus $H_{\text{inv}}(\mathcal{H}) \cong H_{\text{dR}}(\mathcal{H})$, obtaining a known simpler expression for the de Rham cohomology of $\mathcal{H}$.

Another complex associated to $\mathcal{H}$ was introduced by A. Haefliger as follows [17]. If some $\alpha \in \Omega_r(T)$ is supported in the image of some $h \in \mathcal{H}$, let $h^* h \alpha \in \Omega_r(T)$ denote the extension by zero of $h^* h \alpha$. The Haefliger complex $\Omega_{\text{Ha}}(\mathcal{H}, d)$ of $\mathcal{H}$ is the quotient complex $\Omega_r(T, d)$ over the subcomplex generated by forms of the type $\alpha - h^* \alpha$, where $h \in \mathcal{H}$ and $\alpha \in \Omega_r(T)$ with $\text{supp} \alpha \subset \text{dom} h$. The corresponding cohomology is called the Haefliger cohomology and denoted by $H_{\text{Ha}}(\mathcal{H})$. The Haefliger cohomology is not Hausdorff in general with the topology induced by the $C^\infty$ topology on $\Omega_r(T)$ [17]; so it makes sense to consider its maximal Hausdorff quotient, which is called the reduced Haefliger cohomology and denoted by $\overline{H}_{\text{Ha}}(\mathcal{H})$.

Exterior product and integration defines a pairing

$$\Omega^r_{\text{inv}}(\mathcal{H}) \otimes \Omega^{n-r}_{\text{Ha}}(\mathcal{H}) \to \mathbb{R}$$

for each degree $r$, where $n = \dim T$. It induces a pairing

$$H^r_{\text{inv}}(\mathcal{H}) \otimes \overline{H}^{n-r}_{\text{Ha}}(\mathcal{H}) \to \mathbb{R},$$

which degenerates in general; nevertheless, the pseudogroup version of the arguments of [1] and [32] show that this pairing is non-degenerate for complete Riemannian pseudogroups.

Any étalé morphism $\Phi : \mathcal{H} \to \mathcal{H}'$ functorially induces a continuous homomorphism $\Phi_* : \Omega_{\text{Ha}}(\mathcal{H}) \to \Omega_{\text{Ha}}(\mathcal{H}')$ [17, Section 1.2]; thus isomorphic pseudogroups have isomorphic Haefliger complexes.

For any $C^\infty$ foliation $\mathcal{F}$, $\Omega_{\text{Ha}}(\text{Hol}(\mathcal{F}))$ and $H_{\text{Ha}}(\text{Hol}(\mathcal{F}))$ are called the transverse complex and transverse cohomology of $\mathcal{F}$, and there is a homomorphism $\int_\mathcal{F} : \Omega_*(\mathcal{F}) \to \Omega_{\text{Ha}}(\text{Hol}(\mathcal{F}))$, called integration along the leaves, which induces an isomorphism $\int_\mathcal{F} : E^p_{c,1}(\mathcal{F}) \to \Omega_{\text{Ha}}(\text{Hol}(\mathcal{F}))$ ($p = \dim \mathcal{F}$) [17, Section 3], yielding $E^p_{c,2}(\mathcal{F}) \cong H_{\text{Ha}}(\text{Hol}(\mathcal{F}))$.

A morphism $\Phi : \mathcal{H} \to \mathcal{H}'$ is called a $C^\infty$ submersion when its maps are $C^\infty$ submersions. In this case, $\Phi$ induces a continuous open homomorphism $\Phi_* : \Omega_{\text{Ha}}(\mathcal{H}) \to \Omega_{\text{Ha}}(\mathcal{H}')$ in the following way. For each $\alpha \in \Omega_r(T)$, there are some $\alpha_1, \ldots, \alpha_k \in \Omega_r(T)$ and some $\phi_1, \ldots, \phi_k \in \Phi$ such that $\alpha = \alpha_1 + \cdots + \alpha_k$ and $\text{supp} \alpha \subset \text{dom} \phi_i$ for all $i \in \{1, \ldots, k\}$. Since each $\phi_i$ is a submersion, the integration along the fibers $\int_{\phi_i} \alpha_i \in \Omega_r(\text{im} \phi_i)$ is defined, whose extension by zero to $T'$ is also denoted by $\int_{\phi_i} \alpha_i \in \Omega_r(T')$.

With the obvious generalization of the arguments of [17, Theorem 3.1], it follows that the class of $\int_{\phi_i} \alpha_1 + \cdots + \int_{\phi_k} \alpha_k$ in $\Omega_{\text{Ha}}(\mathcal{H}')$ is independent of the choices of $\alpha_1, \ldots, \alpha_k$ and $\phi_1, \ldots, \phi_k$. 
Then, if \( \zeta \in \Omega_{H_a}(H) \) is the element represented by \( \alpha \), define \( \Phi_\alpha(\zeta) \in \Omega_{H_b}(H') \) to be the element represented by \( \int_{\phi_1} \alpha_1 + \cdots + \int_{\phi_k} \alpha_k \). The arguments of [17, Theorem 3.1] also show that \( \Phi_\alpha \) is continuous and open. The induced continuous homomorphism \( H_{a}(H) \to H_{b}(H') \) is also denoted by \( \Phi_\alpha \), or by \( H_{a}(H) \). This defines a covariant functor from PsGr to the category of continuous homomorphisms between topological vector spaces.

As a particular example, for any \( C^\infty \) foliation \( F \) on a \( C^\infty \) manifold \( M \), Haefliger’s integration along the fibers, \( \int_F : \Omega_c(M) \to \Omega_{H_a}(\text{Hol}(F)) \), is the homomorphism \( P_\alpha \) induced by the canonical morphism \( P : M \to \text{Hol}(F) \).

### 29. Invariance of the invariant cohomology

The following is a version for pseudogroups of Theorem 25.1.

**Theorem 29.1.** Let \( \Phi, \Psi : H \to H' \) be morphisms between complete Riemannian pseudogroups. We have the following:

(i) If \( \Phi \) and \( \Psi \) are homotopic, then \( \Phi^* = \Psi^* : H_{\text{inv}}(H') \to H_{\text{inv}}(H) \).

(ii) If \( \Phi \) is a homotopy equivalence, then \( \Phi^* : H_{\text{inv}}(H') \to H_{\text{inv}}(H) \) is an isomorphism.

**Proof.** This follows from Corollaries 27.2 and 27.3, and Proposition 28.1. \( \square \)

### 30. Examples

**Example 30.1.** Theorem 16.1(i) supplies a large class of complete morphisms. Another source of complete morphisms is the following: any pseudogroup generated by a group action is complete, and any equivariant map generates a complete pseudogroup equivalences. As pointed out in [20, Section 1.3], the pseudogroup of all local isometries of a Riemannian manifold is not complete in general; for instance, it is not complete in the case of a sphere with a metric which is flat on some part with non-empty interior, and has positive scalar curvature elsewhere. The identity morphism of any of the above non-complete pseudogroups is not complete. Thus, for Riemannian pseudogroups, the relation between completeness and geodesic completeness seems to be weak.

**Example 30.2.** For \( \lambda > 1 \), the mapping \( x \mapsto \lambda x \) generates a complete pseudogroup \( H \), whose restriction to \( U = (-1, 1) \) is not complete. This \( H \) is equivalent to \( H|_U \) since \( U \) cuts every \( H \)-orbit. So completeness is not invariant by pseudogroup equivalences. As pointed out in [20, Section 1.3], the pseudogroup of all local isometries of a Riemannian manifold is not complete in general; for instance, it is not complete in the case of a sphere with a metric which is flat on some part with non-empty interior, and has positive scalar curvature elsewhere. The identity morphism of any of the above non-complete pseudogroups is not complete. Thus, for Riemannian pseudogroups, the relation between completeness and geodesic completeness seems to be weak.

**Example 30.3.** A pseudogroup \( H \) acting on a space \( T \) is called **globally complete** when, for all \( h \in H \) and any \( x \in \text{dom } h \), there is some \( \tilde{h} \in H \) such that \( \text{dom } \tilde{h} = T \) and \( \gamma(\tilde{h}, x) = \gamma(h, x) \); i.e., \( (T, T) \) is a completeness pair for \( H \).

Let \( H' \) be another pseudogroup acting on a space \( T' \), and let \( \Phi : H \to H' \) be a morphism. It is said that \( \Phi \) is **globally complete** when there is some \( \phi \in \Phi \) such that \( \text{dom } \phi = T \), and moreover, for all \( h \in H \) and every \( z \in \text{dom } h \), there is some \( \tilde{h} \in H \) and some \( h' \in H' \) so that \( \text{dom } \tilde{h} = T \) and \( \gamma(\tilde{h}, x) = \gamma(h, x) \), \( \text{im } \phi \subset \text{dom } h' \), and \( h' \circ \phi = \phi \circ \tilde{h} \). This case, \( H \) is globally complete, \( \Phi \) is generated by \( \phi \), and \( (\phi, T; \phi, T) \) is a completeness quadruple of \( \Phi \).

Now, consider a family of pseudogroups \( H_i \), where each \( H_i \) acts on a space \( T_i \). As a straightforward generalization of the finite product of pseudogroups (Section 2), we can define the **product pseudogroups** \( \prod_i H_i \) acting on \( \prod_i T_i \). It is easy to check that \( \prod_i H_i \) is complete if and only if each \( H_i \) is complete, and all but finitely many pseudogroups \( H_i \) are globally complete.

Consider another family of pseudogroups \( H'_i \), with the same index set, and a family of morphisms \( \Phi_i : H_i \to H'_i \). As a straightforward generalization of the finite product of morphisms (Section 2), we can define the **product morphism** \( \prod_i \Phi_i : \prod_i H_i \to \prod_i H'_i \). This \( \prod_i \Phi_i \) is complete if and only if each \( \Phi_i \) is complete, and all but finitely many of the morphisms \( \Phi_i \) are globally complete.

This allows the construction of concrete examples of non-complete morphisms between complete pseudogroups, which shows that the Riemannian condition cannot be removed in Theorem 16.1(i). For instance, let \( H \) be the globally complete pseudogroup acting on the line \( \mathbb{R} \) generated by all homeomorphisms \( \mathbb{R} \to \mathbb{R} \), and let \( H' \) be the globally complete pseudogroup acting on circle \( S^1 \) generated by all homeomorphisms \( S^1 \to S^1 \). The universal covering map \( \mathbb{R} \to S^1 \) generates a complete morphism \( \Phi : H \to H' \), which is not globally complete. Then, with \( T_i = \mathbb{R} \), \( T'_i = S^1 \),
This case, it will be said that \( \mathcal{H}_i = \mathcal{H} \) and \( \mathcal{H}'_i = \mathcal{H}' \) for all \( i \in \mathbb{N} \), the pseudogroups \( \prod_i \mathcal{H}_i \) and \( \prod_i \mathcal{H}'_i \) are complete, but the morphism \( \prod_i \Phi_i : \prod_i \mathcal{H}_i \to \prod_i \mathcal{H}'_i \) is not complete.

**Example 30.4.** This is another example of a non-complete morphism between complete pseudogroups, which furthermore is \( C^\infty \). Let \( \mathcal{H} \) be the \( C^\infty \) pseudogroup acting on \( \mathbb{R}^2 \) generated by the group of diffeomorphisms \( h : \mathbb{R}^2 \to \mathbb{R}^2 \) such that there is some \( \epsilon > 0 \) and some \( c \in \mathbb{R} \) so that \( h(x, y) = (x + c, y) \) for all \( (x, y) \in \mathbb{R} \times (-\epsilon, \epsilon) \). Let \( \mathcal{H}' \) be the \( C^\infty \) pseudogroup acting on \( \mathbb{R}^2 \) generated by the group of diffeomorphisms \( \mathbb{R}^2 \to \mathbb{R}^2 \) that fix the origin and have the same germ at the origin as a rotation. Let \( \Phi : \mathbb{R}^2 \to \mathbb{R}^2 \) be the \( C^\infty \) map defined by \( \Phi(x, y) = (y \cos x, y \sin x) \). This \( \Phi \) generates a \( C^\infty \) morphism \( \Phi : \mathcal{H} \to \mathcal{H}' \). It is easy to prove that \( \mathcal{H} \) and \( \mathcal{H}' \) are complete, while \( \Phi \) is not complete: its completeness condition fails around the origin.

**Example 30.5.** Consider Arnold’s example of a diffeomorphism \( h : S^1 \to S^1 \) which is topologically conjugated but not \( C^1 \) conjugated to a rotation \( h' : S^1 \to S^1 \) [6]. By suspension, we get examples of homeomorphic \( C^\infty \) foliations with non-isomorphic basic cohomologies [27]; so these foliations cannot be diffeomorphic. Thus, if \( \mathcal{H} \) and \( \mathcal{H}' \) denote the pseudogroups generated by \( h \) and \( h' \), respectively, the isomorphism \( \Phi : \mathcal{H} \to \mathcal{H}' \) generated by any fixed homeomorphism \( \phi : S^1 \to S^1 \) satisfying \( \phi \circ h = h' \circ \phi \) is not \( C^\infty \). This does not contradict Theorem 16.1(iii) and Theorem 27.1 because \( \mathcal{H} \) is not Riemannian. However \( \mathcal{H} \) is equicontinuous [37, Appendix E, Section 5], [3], showing that Theorem 16.1(iii) cannot be generalized to equicontinuous pseudogroups.

**Example 30.6.** Let \( \mathcal{H} \) be the pseudogroup on \( \mathbb{R} \) generated by the map \( h \) defined by \( h(x) = \lambda x \) for some \( \lambda > 1 \). The map \( \phi \) defined by \( \phi(x) = |x| \) satisfies \( \phi \circ h = h \circ \phi \). So \( \phi \) generates a morphism \( \Phi : \mathcal{H} \to \mathcal{H} \). It is easy to see that \( \Phi \) cannot be approximated by any \( C^\infty \) morphism in \( C_5(\mathcal{H}, \mathcal{H}) \). By suspension, we get a foliated map between \( C^\infty \) foliations with compact space of leaves that cannot be strongly approximated by \( C^\infty \) foliated maps. This shows that Theorems 21.1, 23.1 and 27.1 cannot be generalized to arbitrary \( C^\infty \) foliations.

**Example 30.7.** Let \( \mathcal{H} \) and \( \mathcal{H}' \) be pseudogroups acting on spaces \( T \) and \( T' \), respectively. A morphism \( \Phi : \mathcal{H} \to \mathcal{H}' \) is said to be simply complete if all \( x \in T \) and \( x' \in T' \) have respective open neighborhoods \( U \) and \( U' \) such that, for any \( \phi \in \Phi \) and every \( y \in U \cap \text{dom } \phi \) with \( \phi(y) \in U' \), there is some \( \phi' \in \Phi \) such that \( U \subset \text{dom } \phi \) and \( \gamma(\phi', \phi) = \phi(y) \); in this case, it will be said that \( (U, U') \) is a simple completeness pair of \( \Phi \). Observe that \( \mathcal{H} \) is complete if and only if the identity morphism \( \text{id}_\mathcal{H} \) simply complete; thus this is another version of completeness for morphisms different from Definition 7.1. If \( T \) and \( T' \) are locally connected, and \( \Phi \) is quasi-analytic and simply complete, then \( \Phi \) is complete. Any morphism generated by a globally defined map is simply complete. The following result shows that the reciprocal statement is also true under certain topological conditions.

**Proposition 30.8.** If \( \Phi \) is simply complete, \( T \) is simply connected and \( T' \) is compact, then \( \Phi \) is generated by a single map \( T \to T' \).

**Proof.** Since \( \Phi \) is simply complete, for each \( x \in T \), there is a family of simple completeness pairs of \( \Phi \), \( \{(U_i, U'_i)\} \), such that each \( U_i \) contains \( x \) and \( \{U'_i\} \) covers \( T' \). Because \( T' \) is compact, \( \{U'_i\} \) admits a finite subcovering \( \{U'_{i_1}, \ldots, U'_{i_n}\} \). Then \( U = U_{i_1} \cap \cdots \cap U_{i_n} \) is an open neighborhood of \( x \) and satisfies the following condition: for any \( \phi \in \Phi \) and every \( y \in U \cap \text{dom } \phi \), there is some \( \tilde{\phi} \in \Phi \) such that \( U \subset \text{dom } \tilde{\phi} \) and \( \gamma(\tilde{\phi}, \phi) = \phi(y) \). This means that the source map \( \gamma(\Phi) \to T \) is a covering map, which admits a global section \( \theta : T \to \gamma(\mathcal{H}) \) because \( T \) is simply connected. Then \( \Phi \) is generated by the composite of \( \theta \) with the target projection \( \gamma(\Phi) \to T' \).

**Example 30.9 (Haefliger).** In [18], Haefliger has introduced another type of morphisms between topological groupoids inspired by the concept of Morita equivalence. Haefliger morphisms between étalé groupoids can be considered as another type of morphisms between pseudogroups. Each Haefliger morphism induces a morphism of our type in an obvious way. If a morphism of our type consists of open maps, then it is induced by a unique Haefliger morphism. For instance, the pseudogroup \( \mathcal{H} \) generated by a rotation of order \( n \) of the plane is contractible in our sense, whilst there are \( n \) Haefliger morphisms of the circle to \( \mathcal{H} \) which are not homotopic to each other.
31. Open problems

Problem 31.1. Develop the algebraic topology, differential topology and differential geometry of pseudogroups in the sense of Section 3. To begin with, we may ask whether results like the Hurewicz isomorphism theorem [45] or the Van Kampen theorem [22] can be generalized to pseudogroups. Or what about any version of the Hopf–Rinow theorem for Riemannian pseudogroups? This question can be also asked for Haefliger morphisms, of course.

Problem 31.2. For morphisms of pseudogroups, is there a functorial way to induce morphisms between the corresponding $C^*$-algebras and their $K$-theory? Haefliger morphisms could be more appropriate for these relations.

Problem 31.3. Let $\mathcal{F}$ and $\mathcal{G}$ be foliated structures, $f: \mathcal{F} \to \mathcal{G}$ a continuous foliated map, and $\Psi: \text{Hol}(\mathcal{F}) \times I \to \text{Hol}(\mathcal{G})$ a homotopy such that $\Psi_0 \equiv \text{Hol}(f)$. What conditions are needed for the existence of a lift $H: \mathcal{F} \times I_p \to \mathcal{G}$ of $\Psi$ so that $H_0 = f$? The needed conditions must be very restrictive (see [35] for the solution of a related problem). It is easy to produce counterexamples to the existence of such a lift by using vanishing cycles.

Problem 31.4. In the case of pseudogroups generated by the left translations on Lie groups, Theorem 16.1(iii) asserts that any continuous homomorphism between Lie groups is $C^\infty$. But indeed it is well known that any measurable homomorphism between Lie groups is $C^\infty$. Then, according to the generalization of the theorem of Myers–Steenrod for complete Riemannian pseudogroups given in [42], there should be a version of Theorem 16.1(iii) for “measurable morphisms”.

Problem 31.5. Theorem 16.1(iii) cannot be generalized to equicontinuous pseudogroups (Example 30.5), but what about its assertions (i) and (ii)? Observe that Theorem 16.1(ii) would make sense for equicontinuous pseudogroups because they also have a closure under mild topological conditions [3,47].

Problem 31.6. Is there any version of completeness for Haefliger morphisms? And a version of Theorem 16.1?

Problem 31.7. Is it possible to adapt the convolution technique [24] to improve differentiability of foliated maps between transversely complete Riemannian foliations? This is clearly possible in the case of transversely parallelizable foliations.

Problem 31.8. The spectral sequence can be given by the differential sheaf of local basic differential forms [33], which leads to the following question. For transversely complete Riemannian foliations, is it possible to prove the homotopy invariance of the spectral sequence directly from Theorem 16.1(iii) with a sheaf theoretic argument?

Problem 31.9. We may ask whether a “proper” morphism $\Phi: \mathcal{H} \to \mathcal{H}'$ induces a homomorphism $\Phi^*: H_{Ha}(\mathcal{H}') \to H_{Ha}(\mathcal{H})$ in a functorial way. The commutativity of the diagram

$$
\begin{array}{ccc}
E^{\bullet,p}_{c,1}(\mathcal{G}) & \xrightarrow{E^{\bullet,p}_{c,1}(f)} & E^{\bullet,p}_{c,1}(\mathcal{F})V \\
\downarrow f_G & & \downarrow f_\mathcal{F} \\
\Omega_{Ha}(\text{Hol}(\mathcal{G})) & \xrightarrow{\text{Hol}(f)^*} & \Omega_{Ha}(\text{Hol}(\mathcal{F}))
\end{array}
$$

must be required for any proper $C^\infty$ foliated map $f: \mathcal{F} \to \mathcal{G}$ between $C^\infty$ foliations of dimension $p$. If this can be made, then there must be a version of Theorem 25.2 for the Haefliger cohomology.

Of course, first, the right definition of proper morphism must be given! We may say that a morphism $\Phi$ is proper if $\Phi_{\text{orb}}$ is proper, but this does not seem to be enough. With a good definition of proper morphism, the canonical injective functor $\text{Top} \to \text{PsGr}$ and the holonomy functor $\text{Fol} \to \text{PsGr}$ must assign proper morphisms to proper maps. Thus this condition may have some relation with Haefliger’s definition of compact generation for pseudogroups [19,21].

Problem 31.10. A pseudomonoid can be defined like a pseudogroup with arbitrary continuous maps and without using inversion. Orbits, morphisms and equivalences can be generalized to pseudomonoids. But, by the lack of inversion, it
also makes sense to consider $\alpha$- and $\omega$-orbits, referring to the backwards and forwards direction. We similarly have $\alpha$- and $\omega$-morphisms, and $\alpha$- and $\omega$-equivalences. For instance, for any foliated space, all maps between local quotients induced by inclusions of simple open sets form a pseudomonoid. Its $\alpha$-class gives the holonomy pseudogroup of all open subsets. This may be useful to deal with invariants like the transverse LS category $\text{cat}_{h}F$ of a foliated structure $F$, whose definition involves non-saturated open sets [13]. For instance, to know how far it is from being a transverse invariant.

**Problem 31.11.** The saturated transverse LS category $\text{cat}_{h}^{S}F$ of a foliated structure $F$ is defined like $\text{cat}_{h}F$, but using only saturated open sets. It is not a transverse invariant either, but it is closer to being so. The “transverse invariant LS category” of $F$ is the LS category $\text{cat Hol}(F)$ of its holonomy pseudogroup, defined in the sense of Section 3. We easily get

$$\text{cat Hol}(F) \leq \text{cat}_{h}^{S}F,$$

which may be a strict inequality by the possible non-existence of lifts of homotopies on holonomy pseudogroups (Problem 31.3). We propose the study of $\text{cat Hol}(F)$. For instance, it should be equal to the “transverse category” of the classifying foliated space of $F$ [16,9,18]. Moreover the known results for $\text{cat}_{h}F$ or $\text{cat}_{h}^{S}F$ should have a version for $\text{cat Hol}(F)$ (see [13,12,11,25,26]).

**Problem 31.12.** Give a good definition of $C^{0,\infty}$ singular foliated spaces.

**Problem 31.13.** (Based on an idea of José Luis Arraut.) The concept of holonomy groupoid can be extended to the singular case as follows. Let $F$ be a singular foliated structure on a space $X$, let $c : I \to F$ be a foliated curve between points $x$ and $y$, and let $\Sigma_{x}$ and $\Sigma_{y}$ be local transversals of $F$ through $x$ and $y$, respectively. There is some open neighborhood $\Delta$ of $x$ in $T_{x}$ and some continuous foliated map $H : \Delta_{pt} \times I \to F$ such that:

- $H(x, \cdot) = c$;
- each map $H(\cdot, t)$ is an embedding whose image is a local transversal of $F$ through $c(t)$;
- $H(\cdot, 0)$ is the inclusion map $\Delta \hookrightarrow X$; and
- $H(\Delta \times \{1\})$ is an open subset of $\Sigma_{y}$.

Then $h = H(\cdot, 1) : \Delta \to \Sigma_{y}$ is an open embedding, but, in the singular case, the germ $\gamma(h, x)$ may depend on the choice of $H$. To avoid this dependence, we introduce an equivalence relation on the set of such germs as follows. First, observe that $h$ is a foliated map $F_{\Delta} \to F_{\Sigma_{y}}$ (see Section 8). Then, given another continuous foliated map $H' : \Delta'_{pt} \times I \to F$ satisfying the same properties as $H$, for $h' = H'(\cdot, 1) : \Delta' \to \Sigma_{y}$, let us say that $\gamma(h, x)$ is equivalent to $\gamma(h', x)$ if there is some open neighborhood $\Delta_{0}$ of $x$ in $\Delta \cap \Delta'$ and some integrable homotopy $G : F_{\Delta_{0}} \times I \to F_{\Sigma_{y}}$ between $h$ and $h'$ such that each $G(\cdot, t)$ is an open embedding. This defines an equivalence relation, and it is easy to check that the equivalence class of $\gamma(h, x)$ depends only on $c$ (once $\Sigma_{x}$ and $\Sigma_{y}$ are given), and can be called the holonomy defined by $c$. On the set of foliated curves $I \to F$, define an equivalence relation by declaring that two curves are equivalent when they have the same end points and define the same holonomy for any choice of local transversals. The quotient set $\mathfrak{G} = \mathfrak{G}(F)$ becomes a topological groupoid with the operation induced by the path product and the topology induced by the compact-open topology; this $\mathfrak{G}$ can be called the holonomy groupoid, and can be identified to the usual holonomy groupoid when $F$ is regular.

Like in the regular case, another version of the holonomy groupoid can be also given by a singular foliated cocycle (Section 8), by adapting the steps of the above construction of $\mathfrak{G}$.

It is a natural question whether the important role played by the holonomy groupoid of foliated spaces can be extended to the singular case. As a first approach, we may ask for a singular version of the Reeb’s local stability theorem.

**Problem 31.14.** Is it possible generalize our approximation and invariance results to the case of singular Riemannian foliations? Such generalizations should be consequences of a singular version of Theorem 16.1.
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References