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Non-abelian differentiable gerbes

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Abstract

We study non-abelian differentiable gerbes over stacks using the theory of Lie groupoids. More precisely, we develop the theory of connections on Lie groupoid G-extensions, which we call "connections on gerbes", and study the induced connections on various associated bundles. We also prove analogues of the Bianchi identities. In particular, we develop a cohomology theory which measures the existence of connections and curvings for G-gerbes over stacks. We also introduce G-central extensions of groupoids, generalizing the standard groupoid S^1-central extensions. As an example, we apply our theory to study the differential geometry of G-gerbes over a manifold.

Keywords: Non-abelian gerbes; Lie groupoid extensions; Connections; Horizontal cohomology; Non-abelian cocycles

Contents

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1. Introduction

The general theory of gerbes over a site [1] was developed by Giraud [26]. Its main motivation is the study of non-abelian cohomology theory. Indeed, Giraud's theory shows, in particular, that S^1-bound gerbes over a manifold are in one–one correspondence with the third integer-coefficients cohomology group of the manifold. Due to this fact, gerbes are often referred to, in the literature, as geometric realizations of degree 3 integer cohomology classes. Recently, string theory [24,57,58] fostered the study of the differential geometry of gerbes. The differential geometry of S^1-bound gerbes over manifolds—Murray named them bundle gerbes [42]—was investigated by many authors among whom Brylinski [17], Murray [42], Chatterjee [19] and Hitchin [29]. A theory of connections, curvings and 3-curvatures was developed, which was used to define the characteristic classes, called Dixmier Douday classes, of bundle gerbes [42].

However, the differential geometry of the more general non-abelian gerbes is a subtler question. For bundle gerbes, a connection is defined to be a connection 1-form for the usual principal S^1-bundle satisfying some additional property [29,42]. However, as non-abelian G-gerbes can no longer be considered principal G-bundles, the obvious generalization will not work, and it is not clear what a connection on a non-abelian gerbe should be.

Breen and Messing were the first to study the differential geometry of non-abelian gerbes. Their important approach relies on the use of Kocks' synthetic geometry [33] as a tool to study the differential geometry of non-abelian gerbes from an algebraic geometry perspective. They introduced the concepts of connections, curvings and 3-curvatures and obtained various identities between them. Later on, Aschieri, Cantini and Jurčo investigated a non-abelian analogue of bundle gerbes from a more physical point of view [2]. Yet another interesting approach pioneered by Baez and Schreiber [3–5,49] is to use higher gauge theory [6,23,27,47]. As was shown by Breen [10,11], a G-gerbe is equivalent to a 2-group bundle in the sense of Dedecker [22], where the 2-group is the one corresponding to the crossed module $G \to \mathrm{Aut}(G)$ (see [25] for a geometrical construction). Hence, a "connection" on a G-gerbe should correspond to a "connection" on the 2-group principal bundle. The latter is the viewpoint adopted by Baez and Schreiber [4,5]. See also [36] for the case of bundle gerbes.

The aim of this paper is to propose an approach based on Lie groupoid extensions to study geometry of non-abelian differentiable gerbes. It is based on the theory of differentiable stacks developed in [7–9] (see also [43]). Roughly speaking, differentiable stacks are Lie groupoids *up to Morita equivalence*. Any Lie groupoid Γ defines a differentiable stack: the differentiable stack $\mathfrak{X}_\Gamma$ of Γ-torsors. Two differentiable stacks $\mathfrak{X}_\Gamma$ and $\mathfrak{X}'_\Gamma$ are isomorphic if, and only if, the Lie groupoids Γ and Γ' are Morita equivalent. In a certain sense, Lie groupoids are like "local charts" on a differentiable stack. The relation between differentiable gerbes and Lie groupoids is described by the following table, whose right-hand side lists the main topics investigated in this paper.

STACK LANGUAGE	GROUPOID LANGUAGE
differentiable stacks	Morita equivalent Lie groupoids
gerbes over stacks	Morita equivalent groupoid extensions
G-gerbes over stacks	Morita equivalent groupoid G-extensions
G-bound gerbes over stacks	Morita equiv. groupoid G-extensions with trivial band

Our approach has the advantage of avoiding abstract stacks by working directly with Lie groupoids, which are much more down-to-earth. Moreover it allows us to use the usual techniques of differential geometry and formulate our results in a global fashion without resorting to local charts. Such a viewpoint has already been taken by a few authors. For example, by Murray [42] and Murray and Stevenson [44] in their study of bundle gerbes and in the proof of their theorem concerning stable equivalence classes. For equivariant bundle gerbes, see [24,41,54]. In [7,8] and [18], S^1-bound gerbes over differentiable stacks were studied in terms of S^1-central extensions of Lie groupoids, and the characteristic classes of gerbes, i.e. the Dixmier–Douady classes, were introduced in terms of connection-like data. This perspective is also central in a series of work of Moerdijk [45,46]; in particular, [45] explores the deep connection between the classification of extensions of regular Lie groupoids, Giraud's non-abelian cohomology [26] and subsequent work of Breen [12].

In this paper, we develop a theory of connections on groupoid extensions, which we call "connections on gerbes." A connection on a groupoid extension $X_1 \to Y_1 \rightrightarrows M$ is an Ehresmann connection on the fiber bundle $\phi : X_1 \to Y_1$, which is compatible with the groupoid structure on $X_1 \rightrightarrows M$. More precisely, a connection on a Lie groupoid extension $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ is a horizontal distribution H on $X_1 \xrightarrow{\phi} Y_1$ which is also a subgroupoid of the tangent groupoid

$TX_1 \rightrightarrows TM$. For S^1-central extensions, such a connection is automatically a connection for the principal S^1-bundle $X_1 \xrightarrow{\phi} Y_1$, i.e. it is invariant under the action of S^1. This is in agreement with the definition of connections in [7,8,56], which coincides with the one used by Brylinski (who uses the name "connective structures") [17], Murray [42] and Hitchin [29]. However, in general, although $\phi : X_1 \to Y_1$ is a bi-principal $\mathcal{K}$–$\mathcal{K}$-bundle, where $\mathcal{K}$ is the kernel of ϕ, the horizontal distribution defining a groupoid extension connection is not invariant with respect to either $\mathcal{K}$-action of the principal bundle. We also introduce the notion of curvings and 3-curvatures for G-extensions. Finally, a cohomology theory is developed that measures the obstructions to the existence of connections and curvings.

We now describe the contents of this paper in more details.

Section 2 recalls some basic definitions and notions central to this paper. These include groupoid extensions, Morita equivalence of Lie groupoid extensions, and generalized morphisms of Lie groupoid extensions. Given a groupoid extension $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$, let $\mathcal{K} = \ker \phi$ be its kernel. The outer action is a canonical groupoid morphism from $Y_1 \rightrightarrows M$ to $\mathrm{Out}(\mathcal{K}, \mathcal{K}) \rightrightarrows M$, which is essential to the definition of the band of a groupoid extension.

Section 3 is devoted to the study of G-gerbes, or more precisely, groupoid G-extensions. The notion of band is introduced, which is a principal $\mathrm{Out}(G)$-bundle (or an $\mathrm{Out}(G)$-torsor) over the groupoid $Y_1 \rightrightarrows M$. Groupoid G-extensions with trivial band—they are called G-bound extensions in the paper—are emphasized and related to the so-called groupoid G-central extensions, a natural generalization of groupoid S^1-central extensions [7,8,56]. A one–one correspondence between groupoid G-central extensions and $Z(G)$-central extensions is proved. As a special case, we consider G-gerbes over a manifold, i.e. groupoid G-extensions of a Čech groupoid, where the non-abelian Čech 2-cocycle conditions as in [14] arise naturally. In particular, we derive, by a direct argument, the following non-trivial theorem of Giraud: the isomorphism classes of G-bound gerbes over a manifold N are in one–one correspondence with $H^2(N, Z(G))$.

Section 4 is devoted to the study of connections on groupoid extensions. As usual, we give three equivalent definitions: horizontal distributions, parallel transportations, and connection 1-forms. We then discuss the induced connections on the bundle of groups $\mathcal{K} \to M$ and its corresponding bundle of Lie algebras $\mathfrak{K} \to M$. We also prove analogues of the Bianchi identities.

In Section 5, we prove that a connection on a groupoid extension induces a canonical connection on its band. We discuss the relations between the curvatures of various induced connections, and in particular we derive an important formula which expresses the curvature on the band in terms of the curvature on the group bundle $\mathcal{K} \to M$.

In the last section, we develop a cohomology theory, called horizontal cohomology in the paper, which captures the obstruction to the existence of connections and curvings. In a certain sense, this is a generalization of smooth Deligne cohomology [17] to the non-abelian context. This cohomology is shown to be invariant under Morita equivalence, and hence depends only on the underlying gerbe. We then compute explicitly the horizontal cohomology groups for G-gerbes over a manifold. As a consequence we show that connections always exist on any G-gerbe over a manifold. Finally we introduce flat G-gerbes and prove that in the case of central extensions they are in one–one correspondence with flat $Z(G)$-gerbes.

After the present paper was completed, we learned that non-abelian gerbes were also investigated by Stevenson [53] from a different viewpoint. The relation between our approach and Breen and Messing's [14] was recently explored in [13]. An explicit correspondence between G-gerbes and $[G \to \mathrm{Aut}(G)]$-bundles is constructed geometrically in [25]. Non-abelian gerbes have also appeared in many recent works in deformation quantization [15,16,21,30–32,34,48, 59,60]. We hope that our theory will be useful in understanding the geometry underlying these works.

2. Differentiable gerbes as groupoid extensions

2.1. Lie groupoid extensions

The purpose of this section is to set up the basic notions of differentiable gerbes in terms of Lie groupoid extensions. Some materials here might be standard for experts [10,45].

Definition 2.1. A Lie **groupoid extension** is a morphism of Lie groupoids

$$\begin{tikzcd} X_1 \arrow[r, "\phi"] \arrow[d, shift left] \arrow[d, shift right] & Y_1 \arrow[d, shift left] \arrow[d, shift right] \\ M \arrow[r, equal] & M \end{tikzcd}$$

where ϕ is a fibration.

For short, we denote a groupoid extension simply by $X_\bullet \xrightarrow{\phi} Y_\bullet$ or $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$.

The **kernel** of a Lie groupoid extension $X_\bullet \xrightarrow{\phi} Y_\bullet$ is the preimage $\mathcal{K}$ (or $\ker\phi$) in X_1 of the unit space of $Y_\bullet$. It is obviously a subgroupoid of $X_\bullet$ on which the source and target maps coincide making it into a bundle of groups over M. Alternatively, we can define a Lie groupoid extension as a short exact sequence of Lie groupoids

$$\begin{tikzcd} \mathcal{K} \arrow[r] \arrow[d, shift right, swap, "\pi"] \arrow[d, shift left, "\pi"] & X_1 \arrow[r, "\phi"] \arrow[d, shift right, swap, "s"] \arrow[d, shift left, "t"] & Y_1 \arrow[d, shift right, swap, "s"] \arrow[d, shift left, "t"] \\ M \arrow[r, equal] & M \arrow[r, equal] & M \end{tikzcd} \tag{1}$$

where $\mathcal{K} \rightrightarrows M$ is a Lie groupoid whose target coincides with the source, i.e. a bundle of Lie groups.

There are a few important features that deserve to be pointed out. First $\mathcal{K} \rightrightarrows M$ is a subgroupoid of $X_1 \rightrightarrows M$. Therefore we can multiply an element of X_1 by a composable element of $\mathcal{K}$ from both the left and the right. These two actions commute and the quotient space by any of these actions is Y_1. Thus $X_1 \to Y_1$ is a $\mathcal{K}$–$\mathcal{K}$ principal bibundle.

Second, the groupoid $X_1 \rightrightarrows M$ acts on $\mathcal{K} \to M$ by conjugation. More precisely, any $x \in X_1$ induces a group isomorphism

$$\mathbf{Ad}_x : \mathcal{K}_{\mathbf{t}(x)} \to \mathcal{K}_{\mathbf{s}(x)}; \quad g \mapsto x \cdot g \cdot x^{-1},$$

where the multiplication on the right-hand side stands for the product on X_1.

2.2. Morita equivalence of Lie groupoid extensions

Let $\Gamma_1 \rightrightarrows \Gamma_0$ be a Lie groupoid, and $J : P_0 \to \Gamma_0$ a surjective submersion. Let P_1 denote the fibered product $P_0 \times_{J,\Gamma_0,\mathbf{s}} \Gamma_1 \times_{\mathbf{t},\Gamma_0,J} P_0$. Then $P_1 \rightrightarrows P_0$ has a natural structure of Lie groupoid with structure maps $s : P_1 \to P_0 : (p,x,q) \mapsto p$, $t : P_1 \to P_0 : (p,x,q) \mapsto q$, and $m : P_2 \to P_1 : ((p,x,q),(q,y,r)) \mapsto (p,xy,r)$. This is called the **pullback groupoid** of $\Gamma_1 \rightrightarrows \Gamma_0$ through J and is denoted by $\Gamma_1[P_0] \rightrightarrows P_0$. Recall that a morphism of Lie groupoids J from $P_1 \rightrightarrows P_0$ to $\Gamma_1 \rightrightarrows \Gamma_0$ is said to be a **Morita morphism** if $J : P_0 \to \Gamma_0$ is a surjective submersion and $P_1 \rightrightarrows P_0$ is isomorphic to the pullback groupoid of $\Gamma_1 \rightrightarrows \Gamma_0$ through J. Two Lie groupoids $\Gamma_1 \rightrightarrows \Gamma_0$ and $\Delta_1 \rightrightarrows \Delta_0$ are said to be **Morita equivalent** if there exists a third Lie groupoid $P_1 \rightrightarrows P_0$ together with a Morita morphism from $P_\bullet$ to $\Gamma_\bullet$ and a Morita morphism from $P_\bullet$ to $\Delta_\bullet$ [45]. Equivalently, two Lie groupoids $\Gamma_1 \rightrightarrows \Gamma_0$ and $\Delta_1 \rightrightarrows \Delta_0$ are Morita equivalent if there exists a manifold P_0, two surjective submersions $P_0 \xrightarrow{f} \Gamma_0$, $P_0 \xrightarrow{g} \Delta_0$ and an isomorphism of groupoids between $\Gamma_1[P_0] \rightrightarrows P_0$ and $\Gamma_1[P_0] \rightrightarrows P_0$.

Note that there is a 1–1 correspondence between Morita equivalence classes of Lie groupoids and (equivalence classes of) differentiable stacks [7,8].

Now we are ready to introduce the definition of Morita equivalence of groupoid extensions.

Definition 2.2. A **Morita morphism** f from a Lie groupoid extension $X_1' \xrightarrow{\phi} Y_1' \rightrightarrows M'$ to another extension $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ consists of Morita morphisms

$$\begin{array}{ccc} X_1' & \xrightarrow{f} & X_1 \\ \downdownarrows & & \downdownarrows \\ M' & \xrightarrow[f]{} & M \end{array} \quad \text{and} \quad \begin{array}{ccc} Y_1' & \xrightarrow{f} & Y_1 \\ \downdownarrows & & \downdownarrows \\ M' & \xrightarrow[f]{} & M \end{array}$$

such that the diagram

$$\begin{array}{ccc} X_1' & \xrightarrow{f} & X_1 \\ \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} \\ Y_1' & \xrightarrow{f} & Y_1 \\ \downdownarrows & & \downdownarrows \\ M' & \xrightarrow[f]{} & M \end{array} \tag{2}$$

commutes.

It is simple to see that any Morita morphism of Lie groupoids

$$\begin{array}{ccc} X_1' & \xrightarrow{f} & X_1 \\ \downarrow\downarrow & & \downarrow\downarrow \\ M' & \longrightarrow & M \end{array}$$

such that

$$f(\ker\phi') = \ker\phi \tag{3}$$

induces a Morita morphism between the groupoid extensions $X'_\bullet \xrightarrow{\phi'} Y'_\bullet$ and $X_\bullet \xrightarrow{\phi} Y_\bullet$. The converse is also true.

Definition 2.3. Two Lie groupoid extensions $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ and $X'_1 \xrightarrow{\phi'} Y'_1 \rightrightarrows M'$ are said to be **Morita equivalent** if there exists a third extension $X''_1 \xrightarrow{\phi''} Y''_1 \rightrightarrows M''$ together with a Morita morphism from $X''_\bullet \xrightarrow{\phi''} Y''_\bullet$ to $X_\bullet \xrightarrow{\phi} Y_\bullet$ and a Morita morphism from $X''_\bullet \xrightarrow{\phi''} Y''_\bullet$ to $X'_\bullet \xrightarrow{\phi'} Y'_\bullet$.

Equivalently, two Lie groupoid extensions $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ and $X'_1 \xrightarrow{\phi'} Y'_1 \rightrightarrows M'$ are Morita equivalent if there exist a manifold P, two surjective submersions $P \xrightarrow{\pi} M$, $P \xrightarrow{\pi'} M'$ and an isomorphism of Lie groupoid extensions between $X_1[P] \xrightarrow{\phi} Y_1[P] \rightrightarrows P$ and $X'_1[P] \xrightarrow{\phi'} Y'_1[P] \rightrightarrows P$.

One easily checks that this yields an equivalence relation on Lie groupoid extensions. Morita equivalence can also be defined in terms of bitorsors [28].

Recall that a Lie groupoid $\Gamma_1 \rightrightarrows \Gamma_0$ is said to **act on a manifold** P (from the left) if there exists a map $J : P \to \Gamma_0$, called the momentum map, and an action map $\Gamma_1 \times_{\mathbf{t},\Gamma_0,J} P \to P : (x, p) \mapsto x \cdot p$ such that $x_1 \cdot (x_2 \cdot p) = (x_1x_2) \cdot p$ and $J(p) \cdot p = p$, for all $(x_1, x_2, p) \in \Gamma_1 \times_{\mathbf{t},\Gamma_0,\mathbf{s}} \Gamma_1 \times_{\mathbf{t},\Gamma_0,J} P$. Let $\Gamma_1 \rightrightarrows \Gamma_0$ be a Lie groupoid and M a manifold. A (left) **$\Gamma_\bullet$-torsor over** M is a manifold P together with a surjective submersion $\pi : P \to M$ called structure map and a left action of $\Gamma_\bullet$ on P such that the action is free and the quotient space $\Gamma_1\backslash P$ is diffeomorphic to M. In other words, for any pair of elements p_1, p_2 in P satisfying $\pi(p_1) = \pi(p_2)$, there exists a unique solution $x \in \Gamma_1$ to the equation $x \cdot p_1 = p_2$. A **$\Gamma_\bullet$-$\Delta_\bullet$-bitorsor** is a manifold P together with two smooth maps $f : P \to \Gamma_0$ and $g : P \to \Delta_0$, and commuting actions of $\Gamma_\bullet$ from the left and of $\Delta_\bullet$ from the right on P, with momentum maps f and g respectively, such that P is at the same time a left-$\Gamma_\bullet$-torsor over Δ_0 (via g) and a right $\Delta_\bullet$-torsor over Γ_0 (via f).

Proposition 2.4. *Let $\Gamma_\bullet$ and $\Delta_\bullet$ be two Lie groupoids. There exists a $\Gamma_\bullet$-$\Delta_\bullet$-bitorsor if and only if $\Gamma_\bullet$ and $\Delta_\bullet$ are Morita equivalent.*

Proof. This is a standard result. We will sketch a proof here, which is needed later on.

$\boxed{\Rightarrow}$ Choose a bitorsor P_0. Let P_1 be the fibered product $\Gamma_1 \times_{\mathbf{t},\Gamma_0} P_0 \times_{\Delta_0,\mathbf{s}} \Delta_1$. There is a natural groupoid structure on $P_1 \rightrightarrows P_0$. The source and target maps are defined by

$$s(\gamma, p, \delta) = p \quad \text{and} \quad \mathbf{t}(\gamma, p, \delta) = \gamma \cdot p \cdot \delta. \tag{4}$$

Hence a pair $(\gamma_1, p_1, \delta_1), (\gamma_2, p_2, \delta_2)$ of elements of P_1 is composable if and only if $\gamma_1 \cdot p_1 \cdot \delta_1 = p_2$. The product is defined by

$$(\gamma_1, p_1, \delta_1) \cdot (\gamma_2, p_2, \delta_2) = (\gamma_2\gamma_1, p_1, \delta_1\delta_2). \tag{5}$$

One checks that the maps

$$\Gamma_1 \leftarrow P_1 \rightarrow \Delta_1,$$
$$\gamma^{-1} \leftarrow\!\shortmid (\gamma, p, \delta) \mapsto \delta$$

induce Morita morphisms

$$\begin{tikzcd} \Gamma_1 \arrow[d, shift left] \arrow[d, shift right] & P_1 \arrow[l] \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \Delta_1 \arrow[d, shift left] \arrow[d, shift right] \\ \Gamma_0 & P_0 \arrow[l] \arrow[r] & \Delta_0. \end{tikzcd}$$

Indeed, the maps

$$P_1 \rightarrow \Delta_1[P_0] : (\gamma, p, \delta) \mapsto (p, \delta, \gamma p \delta)$$

and

$$P_1 \rightarrow \Gamma_1[P_0] : (\gamma, p, \delta) \mapsto \big(p, \gamma^{-1}, \gamma p \delta\big)$$

induce the required Lie groupoid isomorphisms between $P_1 \rightrightarrows P_0$ and the pullbacks of $\Delta_\bullet$ and $\Gamma_\bullet$ over P_0.

$\boxed{\Leftarrow}$ This follows from the following two facts.

- If $\Gamma_\bullet \rightarrow \Delta_\bullet$ is a Morita morphism, then $P = \Gamma_0 \times_{\Delta_0,\mathbf{s}} \Delta_1$ is naturally a $\Gamma_\bullet$-$\Delta_\bullet$-bitorsor.
- If P is a $\Gamma_\bullet$-$\Delta_\bullet$-bitorsor and Q is a $\Delta_\bullet$-$E_\bullet$-bitorsor, then $(P \times_{\Delta_0} Q)/\Delta_1$ is a $\Gamma_\bullet$-$E_\bullet$-bitorsor. □

Proposition 2.5. *Let $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ and $X_1' \xrightarrow{\phi'} Y_1' \rightrightarrows M'$ be two Lie groupoid extensions with kernels $\mathcal{K}$ and $\mathcal{K}'$ respectively. Then $X_\bullet \xrightarrow{\phi} Y_\bullet$ and $X_\bullet' \xrightarrow{\phi'} Y_\bullet'$ are* ***Morita equivalent*** *if and only if there exists an $X_\bullet$-$X_\bullet'$-bitorsor B such that the orbits of the induced actions of $\mathcal{K}$ and $\mathcal{K}'$ on B coincide and the corresponding orbit space is a manifold.*

A bitorsor as in the above proposition will be called an **extension $\phi_\bullet$-$\phi_\bullet'$-bitorsor**. We refer the reader to [8,51] for the discussion on Morita equivalence of S^1-central extensions.

The proof of this proposition will be postponed to the next section.

Remark 2.6. There is a 1–1 correspondence between Morita equivalence classes of Lie groupoid extensions and (equivalence classes of) differentiable gerbes over stacks.

2.3. Generalized morphisms of Lie groupoid extensions

The material here is parallel to Section 2.1 in [51] and follows the approach in [28], to which we refer the reader for details.

Definition 2.7. A **strict morphism of groupoid extensions** is a commutative diagram as below, where the horizontal arrows are groupoid homomorphisms:

$$\begin{tikzcd} X_1' \arrow[r, "f"] \arrow[d, "\phi'"] & X_1 \arrow[d, "\phi"] \\ Y_1' \arrow[r, "f"] \arrow[d, shift left] \arrow[d, shift right] & Y_1 \arrow[d, shift left] \arrow[d, shift right] \\ M' \arrow[r, "f"'] & M \end{tikzcd} \tag{6}$$

and, for any $m' \in M'$, the restriction of $f : X_1' \to X_1$ to a map $\ker\phi'|_{m'} \to \ker\phi|_{f(m')}$ is an isomorphism.

In particular, Morita morphisms of groupoid extensions are strict homomorphisms of groupoid extensions.

Definition 2.8. Let $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ and $X_1' \xrightarrow{\phi'} Y_1' \rightrightarrows M'$ be two Lie groupoid extensions with kernels $\mathcal{K}$ and $\mathcal{K}'$ respectively. A **generalized morphism of groupoid extensions** from $X_1' \xrightarrow{\phi'} Y_1' \rightrightarrows M'$ to $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ is an $X_\bullet'$-$X_\bullet$ bimodule $M' \xleftarrow{f} B \xrightarrow{g} M$ (i.e. B is an $X_\bullet'$-left space and $X_\bullet$-right space and the $X_\bullet'$ and $X_\bullet$ actions commute) satisfying

1. B is an $X_\bullet$-torsor, and
2. the induced actions of $\mathcal{K}$ and $\mathcal{K}'$ on B are free, and their orbits coincide so that the corresponding orbit space is a manifold.

It is easy to see that, in this case, $\mathcal{K}\backslash B = B/\mathcal{K}'$ is a generalized morphism from $Y_1' \rightrightarrows M'$ to $Y_1 \rightrightarrows M$.

Lemma 2.9. *Strict homomorphisms of groupoid extensions are generalized homomorphisms.*

Proof. Let f be a strict homomorphism of groupoid extensions from $X_1' \xrightarrow{\phi'} Y_1' \rightrightarrows M'$ to $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$. Set $B_f = M' \times_{f,M,\mathbf{s}} X_1$, where $B_f \to M'$ is $(m', x) \mapsto m'$, and $B_f \to M$ is $(m', x) \mapsto \mathbf{t}(x)$. The left $X_\bullet'$-action is given by $x' \cdot (\mathbf{t}(x'), x) = (\mathbf{s}(x'), f(x')x)$ and the right $X_\bullet$-action is given by $(m', x_1) \cdot x_2 = (m', x_1x_2)$. It is simple to see that the induced actions of $\ker\phi'$ and $\ker\phi$ on B are free and their orbits coincide. The corresponding orbit space is the manifold $M' \times_{f,M,\mathbf{s}} Y_1$. This concludes the proof. □

Just like strict homomorphisms, generalized homomorphisms of groupoid extensions can be composed.

Proposition 2.10. *Let $M'' \xleftarrow{f'} B' \xrightarrow{g'} M'$ be a generalized morphism of groupoid extensions from $X_1'' \xrightarrow{\phi''} Y_1'' \rightrightarrows M''$ to $X_1' \xrightarrow{\phi'} Y_1' \rightrightarrows M'$, and $M' \xleftarrow{f} B \xrightarrow{g} M$ a generalized morphism of groupoid extensions from $X_1' \xrightarrow{\phi} Y_1' \rightrightarrows M'$ to $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$, their composition is the generalized morphism of groupoid extensions from $X_1'' \xrightarrow{\phi''} Y_1'' \rightrightarrows M''$ to $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ given by the $X_\bullet''$-$X_\bullet$-bimodule*

$$B'' = (B' \times_{M'} B)/X_1',$$

where $X_1' \rightrightarrows M'$ acts on $B' \times_{M'} B$ diagonally: $(b', b) \cdot x' = (b' \cdot x', x'^{-1} b)$, for all compatible $x' \in X_1'$ and $(b', b) \in B' \times_{M'} B$. In particular, if both $M' \xleftarrow{f} B \xrightarrow{g} M$ and $M'' \xleftarrow{f'} B' \xrightarrow{g'} M'$ are bitorsors, the resulting composition is a $\phi_\bullet$-$\phi_\bullet''$-bitorsor.

Moreover, the composition of generalized morphisms is associative.

As a consequence, we obtain a category, where the objects are groupoid extensions and the morphisms are generalized homomorphisms of groupoid extensions. Invertible morphisms exactly correspond to Morita equivalence of groupoid extensions. As usual, we can decompose a generalized homomorphism of groupoid extensions as the composition of a Morita equivalence with a strict homomorphism.

Proposition 2.11. *Any generalized homomorphism of groupoid extensions $M' \xleftarrow{f} B \xrightarrow{g} M$ from $X_1' \xrightarrow{\phi'} Y_1' \rightrightarrows M'$ to $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ can be decomposed as the composition of the canonical Morita equivalence between $X_\bullet' \to Y_\bullet'$ and $X_\bullet'[B] \to Y_\bullet'[B]$, with a strict homomorphism of groupoid extensions from $X_\bullet'[B] \to Y_\bullet'[B]$ to $X_\bullet \to Y_\bullet$.*

Proof. Denote by $X_1'[B] \to Y_1'[B] \rightrightarrows B$ the pullback extension of $X_1' \to Y_1' \rightrightarrows M'$ via the surjective submersion $B \xrightarrow{f} M'$. Then the projection from $X_\bullet'[B] \to Y_\bullet'[B]$ to $X_\bullet' \to Y_\bullet'$ is a Morita morphism, which induces a Morita equivalence between these two groupoid extensions.

As in the proof of Proposition 2.4, consider the fiber product $X_1' \times_{\mathbf{t},M'} B \times_{M,\mathbf{s}} X_1$. Thus $X_1' \times_{\mathbf{t},M'} B \times_{M,\mathbf{s}} X_1 \rightrightarrows B$ is a Lie groupoid, where the source, target, and multiplication are given by Eqs. (4) and (5). Introduce an equivalence relation in $X_1' \times_{\mathbf{t},M'} B \times_{M,\mathbf{s}} X_1 \rightrightarrows B$ by $(x', b, x) \sim (x'k', b, k^{-1}x)$, iff $k'b = bk$, where $k' \in \mathcal{K}'_{s(b)}$ and $k \in \mathcal{K}_{\mathbf{t}(b)}$. It follows from a direct verification, using Eqs. (4) and (5), that the groupoid structure on $X_1' \times_{\mathbf{t},M'} B \times_{M,\mathbf{s}} X_1 \rightrightarrows B$ descends to the quotient and $X_1' \times_{\mathbf{t},M'} B \times_{M,\mathbf{s}} X_1 \to (X_1' \times_{\mathbf{t},M'} B \times_{M,\mathbf{s}} X_1)/\sim \rightrightarrows B$ is a groupoid extension. Moreover, one has the following commutative diagram

$$
\begin{tikzcd}
X_1'[B] & X_1' \times_{\mathbf{t},M'} B \times_{M,\mathbf{s}} X_1 \arrow[l] \arrow[r] \arrow[d] & X_1 \arrow[d] \\
Y_1'[B] \arrow[u, leftarrow] \arrow[d, shift left] \arrow[d, shift right] & (X_1' \times_{\mathbf{t},M'} B \times_{M,\mathbf{s}} X_1)/\sim \arrow[l] \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & Y_1 \arrow[d, shift left] \arrow[d, shift right] \\
B & B \arrow[l] \arrow[r] & M \\
(b, (x')^{-1}, x'bx) \arrow[d] & (x', b, x) \arrow[l] \arrow[r] \arrow[d] & x \arrow[d] \\
(b, \phi'((x')^{-1}), x'bx) \arrow[d, shift left] \arrow[d, shift right] & {[x', b, x]} \arrow[l] \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \phi(x) \arrow[d, shift left] \arrow[d, shift right] \\
b & b \arrow[l] \arrow[r] & g(b)
\end{tikzcd}
\tag{7}
$$

where the left arrow is an isomorphism of groupoid extensions, while the right arrow is a strict homomorphism of groupoid extensions.

This concludes the proof of the proposition. $\square$

Now we are ready to prove Proposition 2.5.

Proof. $\boxed{\Rightarrow}$ As in the proof of Lemma 2.9, $M' \xleftarrow{f} B_f \xrightarrow{g} M$ is indeed a $\phi'_\bullet$-$\phi_\bullet$-bitorsor. Thus the conclusion follows from Proposition 2.10.

$\boxed{\Leftarrow}$ Note that since $M' \xleftarrow{f} B \xrightarrow{g} M$ is a bitorsor, the right arrow in diagram (7) is indeed a Morita morphism. $\square$

2.4. The outer action

Let G and H be any Lie groups. We define $\mathrm{Iso}(G, H)$ as the set of group isomorphisms $G \xleftarrow{f} H$. Let $\mathbf{Ad}_h \in \mathrm{Aut}(H)$ denote the conjugation by $h \in H$. The map

$$\mathrm{Iso}(G, H) \times H \to \mathrm{Iso}(G, H) : (f, h) \mapsto f \circ \mathbf{Ad}_h$$

defines an action of H on $\mathrm{Iso}(G, H)$. The quotient of this action is the set $\mathrm{Out}(G, H)$ of all outer isomorphisms from H to G. Since, for all $f \in \mathrm{Aut}(G)$ and $g \in G$, $f \circ \mathbf{Ad}_g = \mathbf{Ad}_{f(g)} \circ f$, the set $\mathrm{Out}(G, G)$ is a Lie group which will be denoted by $\mathrm{Out}(G)$.

For any group bundle $\mathcal{G} \to M$, let $\mathrm{Iso}(\mathcal{G}, \mathcal{G}) = \coprod_{m,n \in M} \mathrm{Iso}(\mathcal{G}_m, \mathcal{G}_n)$, where $\mathcal{G}_m$ stands for the group fiber of $\mathcal{G}$ at the point m. Similarly, $\mathrm{Out}(\mathcal{G}, \mathcal{G}) = \coprod_{m,n \in M} \mathrm{Out}(\mathcal{G}_m, \mathcal{G}_n)$.

The following proposition is obvious.

Proposition 2.12.

1. *The maps* $s : \mathrm{Iso}(\mathcal{G},\mathcal{G}) \to M : (\mathcal{G}_m \xleftarrow{f} \mathcal{G}_n) \mapsto m$, $t : \mathrm{Iso}(\mathcal{G},\mathcal{G}) \to M : (\mathcal{G}_m \xleftarrow{f} \mathcal{G}_n) \mapsto n$ *and* $m : \mathrm{Iso}(\mathcal{G},\mathcal{G}) \times_{\mathbf{t},M,\mathbf{s}} \mathrm{Iso}(\mathcal{G},\mathcal{G}) \to \mathrm{Iso}(\mathcal{G},\mathcal{G}) : (\mathcal{G}_m \xleftarrow{f} \mathcal{G}_n, \mathcal{G}_n \xleftarrow{g} \mathcal{G}_p) \mapsto (\mathcal{G}|_m \xleftarrow{f\circ g} \mathcal{G}|_p)$ *endow* $\mathrm{Iso}(\mathcal{G},\mathcal{G}) \overset{s}{\underset{t}{\rightrightarrows}} M$ *with a groupoid structure.*
2. *These maps descend to the quotient* $\mathrm{Out}(\mathcal{G},\mathcal{G})$*, yielding a groupoid* $\mathrm{Out}(\mathcal{G},\mathcal{G}) \rightrightarrows M$.
3. *The quotient map* $\mathrm{Iso}(\mathcal{G},\mathcal{G}) \to \mathrm{Out}(\mathcal{G},\mathcal{G})$ *is a groupoid morphism.*

Definition 2.13. Let $\Gamma_1 \rightrightarrows M$ be a Lie groupoid and $\mathcal{G} \to M$ a bundle of Lie groups over the same base.

1. An action by isomorphisms of $\Gamma_1 \rightrightarrows M$ on $\mathcal{G} \to M$ is a Lie groupoid morphism

$$\begin{tikzcd} \Gamma_1 \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \mathrm{Iso}(\mathcal{G},\mathcal{G}) \arrow[d, shift left] \arrow[d, shift right] \\ M \arrow[r, equal] & M. \end{tikzcd}$$

2. An action by outer isomorphisms of $\Gamma_1 \rightrightarrows M$ on $\mathcal{G} \to M$ is a Lie groupoid morphism

$$\begin{tikzcd} \Gamma_1 \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \mathrm{Out}(\mathcal{G},\mathcal{G}) \arrow[d, shift left] \arrow[d, shift right] \\ M \arrow[r, equal] & M. \end{tikzcd}$$

Proposition 2.14. *Let* $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ *be a Lie groupoid extension with kernel* $\mathcal{K}$*. Then,*

1. *the groupoid* $X_1 \rightrightarrows M$ *acts on* $\mathcal{K} \to M$ *by conjugation. I.e.*

$$\begin{tikzcd} X_1 \arrow[r, "\mathbf{Ad}"] \arrow[d, shift left] \arrow[d, shift right] & \mathrm{Iso}(\mathcal{K},\mathcal{K}) \arrow[d, shift left] \arrow[d, shift right] \\ M \arrow[r, equal] & M \end{tikzcd} \tag{8}$$

given by $\mathbf{Ad} : x \to \mathbf{Ad}_x, \forall x \in X_1$ *is a groupoid morphism;*

2. *the composition of the groupoid morphism* (8) *with the quotient morphism*

$$\begin{tikzcd} \mathrm{Iso}(\mathcal{K},\mathcal{K}) \arrow[r, "q"] \arrow[d, shift left] \arrow[d, shift right] & \mathrm{Out}(\mathcal{K},\mathcal{K}) \arrow[d, shift left] \arrow[d, shift right] \\ M \arrow[r, equal] & M \end{tikzcd}$$

factorizes through ϕ*:*

$$\begin{tikzcd} X_1 \arrow[d, "\phi"'] \arrow[dr, "q\circ\mathbf{Ad}"] & \\ Y_1 \arrow[r, "\overline{\mathbf{Ad}}"'] & \mathrm{Out}(\mathcal{K},\mathcal{K}), \end{tikzcd}$$

yielding an action by outer isomorphisms $\overline{\mathbf{Ad}}$ of $Y_1 \rightrightarrows M$ on $\mathcal{K} \to M$, which is called the ***outer action****.*

Remark 2.15. Let $Y_1 \rightrightarrows M$ be a Lie groupoid and $X_1 \xrightarrow{\phi} Y_1$ a surjective submersion. Assume m_1 and m_2 are two groupoid multiplications on X_1 making $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ into an extension of the groupoid $Y_1 \rightrightarrows M$ in two different ways. Then the outer actions $\overline{\mathbf{Ad}}_1$ and $\overline{\mathbf{Ad}}_2$ induced by the two groupoid structures on X_1 are equal.

3. Differentiable G-gerbes as groupoid G-extensions

3.1. Groupoid G-extensions

Let G be a fixed Lie group.

Definition 3.1. A Lie groupoid extension $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ is called a G**-extension** if its kernel $\mathcal{K} \to M$ is a locally trivial bundle of groups with fibers isomorphic to G.

It is simple to see that this notion is invariant under Morita equivalence. Namely, any groupoid extension Morita equivalent to a groupoid G-extension must be a groupoid G-extension itself.

The following proposition shows that any G-extension admits a Morita equivalent G-extension whose kernel is a trivial bundle of groups.

Proposition 3.2. *Let $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ be a groupoid extension. Then ϕ is a G-extension if and only if there is an open covering $\{U_i\}_{i\in I}$ of M such that the kernel of the pullback extension $X_1[\coprod U_i] \xrightarrow{\phi'} Y_1[\coprod U_i] \rightrightarrows \coprod U_i$ via the covering map $\coprod_{i\in I} U_i \to M$ is isomorphic to the trivial bundle of groups $\coprod_{i\in I} U_i \times G$.*

Proof. Let $\mathcal{K}$ denote the kernel of ϕ. Since ϕ is a G-extension, there exists an open cover $\{U_i\}_{i\in I}$ of M such that the pullback $\mathcal{K}'$ of $\mathcal{K} \to M$ to $\coprod_{i\in I} U_i$ is isomorphic to the trivial bundle $\coprod_{i\in I} U_i \times G \to \coprod_{i\in I} U_i$. To conclude, it suffices to notice that $\mathcal{K}'$ is the kernel of the pullback extension $X_1[\coprod U_i] \xrightarrow{\phi'} Y_1[\coprod U_i] \rightrightarrows \coprod U_i$. □

Finally we say that an extension is a **trivial** G**-extension** if it is isomorphic to the extension $Y_1 \times G \to Y_1 \rightrightarrows M$, where $Y_1 \times G$ is equipped with the product groupoid structure.

3.2. Band of a groupoid G-extension

Now we introduce an important notion for a groupoid G-extension, namely, the band. It corresponds to the band of a G-gerbe in terms of stack language. First of all, let us recall the definition of G-torsors (also known as G-principal bundles) over a groupoid [35].

Definition 3.3. Let $\Gamma_1 \rightrightarrows \Gamma_0$ be a Lie groupoid and G a Lie group. A G**-torsor**, or G**-principal bundle over** $\Gamma_\bullet$ consists of a right principal G-bundle $P \xrightarrow{J} \Gamma_0$ endowed with a left action of $\Gamma_1 \rightrightarrows \Gamma_0$ on P with momentum map J such that the actions of G and $\Gamma_\bullet$ on P commute.

Let $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ be a Lie groupoid G-extension with kernel $\mathcal{K}$. Let $\mathrm{Iso}(\mathcal{K}, G)$ (resp. $\mathrm{Out}(\mathcal{K}, G)$) be the group bundle $\coprod_{m\in M} \mathrm{Iso}(\mathcal{K}_m, G)$ (resp. $\coprod_{m\in M} \mathrm{Out}(\mathcal{K}_m, G)$) and J (resp. $\bar{J}$) the canonical projection of $\mathrm{Iso}(\mathcal{K}, G)$ (resp. $\mathrm{Out}(\mathcal{K}, G)$) onto M. First note that $\mathrm{Iso}(\mathcal{K}, G) \xrightarrow{J} M$ (resp. $\mathrm{Out}(\mathcal{K}, G) \xrightarrow{\bar{J}} M$) is a right principal $\mathrm{Aut}(G)$-bundle (resp. $\mathrm{Out}(G)$-bundle).

On the other hand, $\mathrm{Iso}(\mathcal{K}, G) \xrightarrow{J} M$ (resp. $\mathrm{Out}(\mathcal{K}, G) \xrightarrow{\bar{J}} M$) admits a natural left action of the groupoid $\mathrm{Iso}(\mathcal{K}, \mathcal{K}) \rightrightarrows M$ (resp. $\mathrm{Out}(\mathcal{K}, \mathcal{K}) \rightrightarrows M$). Moreover, these two actions commute. Hence $\mathrm{Iso}(\mathcal{K}, G) \xrightarrow{J} M$ (resp. $\mathrm{Out}(\mathcal{K}, G) \xrightarrow{\bar{J}} M$) is an $\mathrm{Aut}(G)$-torsor (resp. $\mathrm{Out}(G)$-torsor) over $\mathrm{Iso}(\mathcal{K}, \mathcal{K}) \rightrightarrows M$ (resp. $\mathrm{Out}(\mathcal{K}, \mathcal{K}) \rightrightarrows M$). According to Proposition 2.14, there is a groupoid morphism $\mathbf{Ad} : X_\bullet \to \mathrm{Iso}(\mathcal{K}, \mathcal{K})_\bullet$ (resp. $\overline{\mathbf{Ad}} : Y_\bullet \to \mathrm{Out}(\mathcal{K}, \mathcal{K})_\bullet$). Therefore, one may pull back the $\mathrm{Out}(G)$-torsor $\mathrm{Out}(\mathcal{K}, G) \xrightarrow{\bar{J}} M$ over $\mathrm{Out}(\mathcal{K}, \mathcal{K}) \rightrightarrows M$ to an $\mathrm{Out}(G)$-torsor over $Y_\bullet$ via $\overline{\mathbf{Ad}} : Y_\bullet \to \mathrm{Out}(\mathcal{K}, \mathcal{K})_\bullet$. More precisely, one defines the left $Y_\bullet$-action on $\mathrm{Out}(\mathcal{K}, G) \xrightarrow{J} M$ by

$$y \cdot f = \overline{\mathbf{Ad}}(y) \circ f, \quad \forall f \in \mathrm{Out}(\mathcal{K}, G)|_{\mathbf{t}(y)}. \tag{9}$$

Definition 3.4. Let $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ be a Lie groupoid G-extension with kernel $\mathcal{K}$. Then $\mathrm{Out}(\mathcal{K}, G) \xrightarrow{\bar{J}} M$, considered as an $\mathrm{Out}(G)$-torsor over $Y_\bullet$, is called the **band** of the G-extension ϕ.

It is well known that for a given Lie group G there is a bijection between G-torsors over a pair of Morita equivalent groupoids. They are called Morita equivalent G-torsors. The following proposition shows that bands are preserved under Morita morphisms, thus also by Morita equivalences in the above sense.

Proposition 3.5. *Let f be a Morita morphism of Lie groupoid extensions from $X_1' \xrightarrow{\phi'} Y_1' \rightrightarrows M'$ to $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$. Let $\mathcal{K}'$ and $\mathcal{K}$ denote the kernels of ϕ' and ϕ respectively. If ϕ and ϕ' are groupoid G-extensions, the band of ϕ':*

$$\begin{tikzcd} Y_1' \arrow[dr, shift left] \arrow[dr, shift right] & \mathrm{Out}(\mathcal{K}', G) \arrow[d] \\ & M' \end{tikzcd}$$

is isomorphic to the pullback of the band of ϕ:

$$\begin{tikzcd} Y_1 \arrow[dr, shift left] \arrow[dr, shift right] & \mathrm{Out}(\mathcal{K}, G) \arrow[d] \\ & M \end{tikzcd}$$

via $M' \xrightarrow{f} M$.

Proof. First, we observe that $\mathcal{K}'$ is isomorphic to the pullback bundle $M' \times_M \mathcal{K}$ and f restricts to a bundle map $\mathcal{K}' \xrightarrow{f} \mathcal{K}$, which is a fiberwise group isomorphism. Therefore, the torsor

$$\begin{tikzcd} \mathrm{Out}(\mathcal{K}', \mathcal{K}') \arrow[dr, shift left] \arrow[dr, shift right] & \mathrm{Out}(\mathcal{K}', G) \arrow[d] \\ & M' \end{tikzcd}$$

is isomorphic to the pullback of the torsor

$$\begin{tikzcd} \mathrm{Out}(\mathcal{K}, \mathcal{K}) \arrow[dr, shift left] \arrow[dr, shift right] & \mathrm{Out}(\mathcal{K}, G) \arrow[d] \\ & M \end{tikzcd}$$

via $M' \xrightarrow{f} M$. Let

$$\begin{tikzcd} Y_1' \arrow[r, "\overline{\mathbf{Ad}}'"] \arrow[d, shift left] \arrow[d, shift right] & \mathrm{Out}(\mathcal{K}', \mathcal{K}') \arrow[d, shift left] \arrow[d, shift right] \\ M' \arrow[r, equal] & M' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Y_1 \arrow[r, "\overline{\mathbf{Ad}}"] \arrow[d, shift left] \arrow[d, shift right] & \mathrm{Out}(\mathcal{K}, \mathcal{K}) \arrow[d, shift left] \arrow[d, shift right] \\ M \arrow[r, equal] & M \end{tikzcd}$$

be the outer actions of ϕ' and ϕ respectively. The morphism f induces a Morita morphism

$$\begin{tikzcd} \mathrm{Out}(\mathcal{K}', \mathcal{K}') \arrow[r, "\mathbf{f}"] \arrow[d, shift left] \arrow[d, shift right] & \mathrm{Out}(\mathcal{K}, \mathcal{K}) \arrow[d, shift left] \arrow[d, shift right] \\ M' \arrow[r, "f"'] & M \end{tikzcd}$$

such that the diagram

$$\begin{tikzcd} Y_1' \arrow[r, "\overline{\mathbf{Ad}}'"] \arrow[d, "f"'] & \mathrm{Out}(\mathcal{K}', \mathcal{K}') \arrow[d, "\mathbf{f}"] \\ Y_1 \arrow[r, "\overline{\mathbf{Ad}}"'] & \mathrm{Out}(\mathcal{K}, \mathcal{K}) \end{tikzcd}$$

commutes. This completes the proof. □

Remark 3.6. In terms of stack and gerbe language, the band of a G-gerbe over a stack is an $\mathrm{Out}(G)$-torsor over this stack [26].

3.3. G-bound gerbes and groupoid central extensions

This subsection is devoted to the study of an important class of G-gerbes, namely, G-bound gerbes. These correspond to groupoid central extensions.

Recall that when G is abelian, a groupoid G-central extension is a groupoid extension $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ such that $M \times G \simeq \ker\phi$, $(m, g) \mapsto g_m \in \ker\phi|_m$ and

$$x \cdot g_{\mathbf{t}(x)} = g_{\mathbf{s}(x)} \cdot x.$$

Example 3.7. Consider the particular case of S^1-extensions. In this case, $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ is an S^1-central extension if and only if there exists a trivialization of the kernel

$$M \times S^1 \to \ker\phi : (m, g) \mapsto g_m$$

such that

$$x \cdot g_{\mathbf{t}(x)} = g_{\mathbf{s}(x)} \cdot x, \quad \forall x \in X_1,\ \forall g \in G. \tag{10}$$

In other words, $X_1 \xrightarrow{\phi} Y_1$ is an S^1-principal bundle such that

$$(g_1 x_1) \cdot (g_2 x_2) = (g_1 g_2) \cdot (x_1 x_2), \quad \forall g_1, g_2 \in S^1,\ (x_1, x_2) \in X_2.$$

See [56] for details. Gerbes represented by S^1-central extensions are called S^1-bound gerbes.[3]

However, when G is non-abelian, the situation is more subtle. First of all, we need to introduce the following

Definition 3.8. Let $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ be a groupoid G-extension with kernel $\mathcal{K} \to M$. Its band $\mathrm{Out}(\mathcal{K}, G) \xrightarrow{J} M$ is said to be trivial (as a $Y_\bullet$-torsor) if there exists a section $\bar{\eta}$ of $\mathrm{Out}(\mathcal{K}, G) \to M$ invariant under the $Y_\bullet$-action:

$$y \cdot \bar{\eta}\big(\mathbf{t}(y)\big) = \bar{\eta}\big(\mathbf{s}(y)\big), \quad \forall y \in Y_1. \tag{11}$$

Such a section is called a **trivialization of the band**.

Proposition 3.9. *If $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ and $X_1' \xrightarrow{\phi'} Y_1' \rightrightarrows M'$ are Morita equivalent groupoid extensions, then the band of the first one is trivial if, and only if, the band of the second one is trivial.*

Proof. As in Proposition 3.5, we can restrict ourselves to the case that there is a Morita morphism of groupoid extensions from $X_1' \xrightarrow{\phi'} Y_1' \rightrightarrows M'$ to $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$. Since $\mathcal{K}' \simeq M' \times_M \mathcal{K}$, there is a natural 1–1 correspondence between the $Y_\bullet$-invariant sections of $\mathrm{Out}(\mathcal{K}, G)$ and the $Y_\bullet'$-invariant

[3] In [7,8] they are called S^1-gerbes for simplicity. When $Y_\bullet$ is Morita equivalent to a manifold, they are called bundle gerbes by Hitchin [29] and Murray [42].

sections of $\mathrm{Out}(\mathcal{K}', G)$. Hence $X_1' \xrightarrow{\phi'} Y_1' \rightrightarrows M'$ has trivial band if, and only if, $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ has trivial band. $\square$

Any section η of $\mathrm{Iso}(\mathcal{K}, G) \to M$ is called a **trivialization of the kernel**. For it defines a map $M \times G \to \mathcal{K} : (m, g) \mapsto g_m := \eta_m(g)$, which is an isomorphism of bundles of groups over M. i.e. $\eta_m(gh) = \eta_m(g)\eta_m(h)$.

The following proposition indicates that when the band is trivial, a trivialization of the kernel always exists when passing to a Morita equivalent extension.

Proposition 3.10. *Let $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ be a groupoid G-extension with kernel $\mathcal{K} \to M$ whose band is trivial. Then there exists a Morita equivalent groupoid G-extension $X_1' \xrightarrow{\phi'} Y_1' \rightrightarrows M'$ with kernel $\mathcal{K}' \to M'$ together with a section η' of* $\mathrm{Iso}(\mathcal{K}', G) \to M'$ *such that its induced section $\bar{\eta}'$ of* $\mathrm{Out}(\mathcal{K}', G) \to M'$ *is invariant under the $Y_\bullet'$-action.*

Proof. Choose a trivialization $\bar{\eta}$ of the band $\mathrm{Out}(\mathcal{K}, G) \to M$ of ϕ. Take a good open covering $\{U_i\}_{i\in I}$ of M and consider the pullback extension $X_1[\coprod U_i] \xrightarrow{\phi'} Y_1[\coprod U_i] \rightrightarrows \coprod U_i$ of ϕ by the projection $\coprod_{i\in I} U_i \to M$. Let us call $\bar{\eta}'$, the $Y_\bullet[\coprod U_i]$-invariant section of $\mathrm{Out}(\mathcal{K}', G) \to \coprod U_i$ associated to $\bar{\eta}$ as in Proposition 3.9 above. Since the U_i's are contractible, $\bar{\eta}'$ can be lifted to a section η' of $\mathrm{Iso}(\mathcal{K}', G) \to \coprod U_i$. $\square$

The following proposition can be verified directly.

Proposition 3.11. *Let H be a subgroup of $Z(G)$. Assume that $\tilde{X}_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ is an H-central extension. Let*

$$X_1 = \frac{\tilde{X}_1 \times G}{H},$$

where H acts on $\tilde{X}_1 \times G$ diagonally: $(\tilde{x}, g) \cdot h = (\tilde{x} h_{\mathbf{t}(x)}, h^{-1} g)$, $\forall h \in H$. Then $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ is a G-extension endowed with a trivialization of its kernel, called the induced G-extension.

We are now ready to state the main result of this subsection.

Theorem 3.12. *Let $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ be a groupoid G-extension with kernel $\mathcal{K} \to M$, and let η be a trivialization of the kernel. The following assertions are equivalent.*

1. *The section $\bar{\eta}$ of* $\mathrm{Out}(\mathcal{K}, G) \to M$ *induced by the trivialization of the kernel $\eta : M \to$* $\mathrm{Iso}(\mathcal{K}, G)$ *is a trivialization of the band.*
2. *There exists a groupoid morphism*

$$\begin{array}{ccc} X_1 & \xrightarrow{r} & G/Z(G) \\ \downdownarrows & & \downdownarrows \\ M & \longrightarrow & * \end{array}$$

with kernel $Z(\mathcal{K})$ *such that*

$$x \cdot g_{\mathbf{t}(x)} = (\underline{\mathbf{Ad}}_{r(x)} g)_{\mathbf{s}(x)} \cdot x, \quad \forall x \in X_1,\ \forall g \in G,$$

where $\underline{\mathbf{Ad}}$ *denotes the canonical isomorphism from* $G/Z(G)$ *to* $\mathrm{Inn}(G)$*:*

$$\underline{\mathbf{Ad}}_{[g']}(g) = g' g (g')^{-1}, \quad \forall g, g' \in G.$$

3. *There exists a global section* σ *of the induced morphism* $X_1/Z(\mathcal{K}) \to Y_1$ *such that, for any local lift* $\tilde{\sigma} : Y_1 \to X_1$ *of* σ*,*

$$\tilde{\sigma}(y) \cdot g_{\mathbf{t}(y)} = g_{\mathbf{s}(y)} \cdot \tilde{\sigma}(y), \quad \forall y \in Y_1,\ \forall g \in G.$$

   Moreover, σ *is a groupoid morphism.*
4. *The extension* $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ *is isomorphic to the induced* G*-extension* $\frac{\tilde{X}_1 \times G}{Z(G)} \to Y_1 \rightrightarrows M$ *of a* $Z(G)$*-central extension* $\tilde{X}_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$*.*

When any of the conditions above is satisfied, the groupoid G-extension is called **central**. Differentiable gerbes represented by G-central extensions are called G**-bound gerbes**.

Proof. $\boxed{1 \Rightarrow 2}$ It is well known that the group morphism $\mathbf{Ad} : G \to \mathrm{Inn}(G) : g \mapsto \mathbf{Ad}_g$ factorizes through $G/Z(G)$:

$$\begin{tikzcd} G \arrow[d] \arrow[dr, "\mathbf{Ad}"] & \\ G/Z(G) \arrow[r, "\underline{\mathbf{Ad}}"'] & \mathrm{Inn}(G), \end{tikzcd}$$

and $\underline{\mathbf{Ad}} : G/Z(G) \to \mathrm{Inn}(G)$ is an isomorphism.

For any $x \in X_1$, it is clear that $\eta^{-1}_{\mathbf{s}(x)} \circ \mathbf{Ad}_x \circ \eta_{\mathbf{t}(x)}$ is an automorphism of G. According to Eq. (11), it must be an inner automorphism, and therefore corresponds to an element in $G/Z(G)$, which is defined to be $r(x)$. Here we used the fact that $\underline{\mathbf{Ad}} : \mathrm{Inn}(G) \to G/Z(G)$ is an isomorphism. Thus we have

$$\mathbf{Ad}_x \circ \eta_{\mathbf{t}(x)} = \eta_{\mathbf{s}(x)} \circ \underline{\mathbf{Ad}}_{r(x)} \tag{12}$$

or

$$x \cdot g_{\mathbf{t}(x)} = (\underline{\mathbf{Ad}}_{r(x)} g)_{\mathbf{s}(x)} \cdot x, \quad \forall x \in X_1,\ \forall g \in G. \tag{13}$$

From its definition, it is simple to see that

$$\begin{tikzcd} X_1 \arrow[r, "r"] \arrow[d, shift left] \arrow[d, shift right] & G/Z(G) \arrow[d, shift left] \arrow[d, shift right] \\ M \arrow[r] & * \end{tikzcd}$$

is indeed a Lie groupoid morphism.

$\boxed{2 \Rightarrow 3}$ Since the groupoid morphism $\mathbf{Ad}$ has kernel $Z(\mathcal{K})$, it factorizes through $X_1/Z(\mathcal{K})$:

$$\begin{tikzcd} X_1 \arrow[d] \arrow[dr, "\mathbf{Ad}"] & \\ X_1/Z(\mathcal{K}) \arrow[r, "\underline{\mathbf{Ad}}"'] & \mathrm{Iso}(\mathcal{K}, \mathcal{K}). \end{tikzcd}$$

Also the trivialization of the kernel $\eta : M \to \mathrm{Iso}(\mathcal{K}, G)$ induces a unique section $\underline{\eta}$ of $\mathrm{Iso}(\mathcal{K}/Z(\mathcal{K}), G/Z(G)) \to M$ making the diagram

$$\begin{tikzcd} \mathcal{K}_m \arrow[d] & G \arrow[l, "\eta_m"'] \arrow[d] \\ (\mathcal{K}/Z(\mathcal{K}))_m & G/Z(G) \arrow[l, "\underline{\eta}_m"'] \end{tikzcd}$$

commute. It thus follows that $\eta_m \circ \mathbf{Ad}_g = \mathbf{Ad}_{\eta_m(g)} \circ \eta_m = \underline{\mathbf{Ad}}_{[\eta_m(g)]} \circ \eta_m$. Hence

$$\eta_m \circ \underline{\mathbf{Ad}}_{[g]} = \underline{\mathbf{Ad}}_{\underline{\eta}_m[g]} \circ \eta_m, \tag{14}$$

where $[q]$ denotes the class of an element $q \in X_1$ (resp. G) in $X_1/Z(\mathcal{K})$ (resp. $G/Z(G)$). Using Eq. (14), the formula (12) becomes $\mathbf{Ad}_x \circ \eta_{\mathbf{t}(x)} = \eta_{\mathbf{s}(x)} \circ \underline{\mathbf{Ad}}_{r(x)}$. Hence $\underline{\mathbf{Ad}}_{[x]} \circ \eta_{\mathbf{t}(x)} = \underline{\mathbf{Ad}}_{\underline{\eta}_{\mathbf{s}(x)}(r(x))} \circ \eta_{\mathbf{s}(x)}$. Therefore

$$\underline{\mathbf{Ad}}_{\underline{\eta}_{\mathbf{s}(x)}((r(x))^{-1} \cdot [x])} \circ \eta_{\mathbf{t}(x)} = \eta_{\mathbf{s}(x)}. \tag{15}$$

For any $y \in Y_1$, take $x \in X_1$ any element such that $\phi(x) = y$. Set

$$\sigma(y) = \underline{\eta}_{\mathbf{s}(x)}\big(r(x)\big)^{-1} \cdot [x].$$

It is easy to see that σ defines a section of $X_1/Z(\mathcal{K}) \to Y_1$. Then Eq. (15) becomes

$$\underline{\mathbf{Ad}}_{\sigma(\phi(x))} \circ \eta_{\mathbf{t}(x)} = \eta_{\mathbf{s}(x)}, \quad \forall x \in X_1. \tag{16}$$

Therefore, for any local lift $\tilde{\sigma} : Y_1 \to X_1$ of σ, we have

$$\mathbf{Ad}_{\tilde{\sigma}(\phi(x))} \circ \eta_{\mathbf{t}(x)} = \eta_{\mathbf{s}(x)}, \quad \forall x \in X_1, \tag{17}$$

or, equivalently,

$$\tilde{\sigma}\big(\phi(x)\big)\cdot g_{\mathbf{t}(x)} = g_{\mathbf{s}(x)}\cdot\tilde{\sigma}\big(\phi(x)\big),\quad \forall x\in X_1,\ g\in G. \tag{18}$$

I.e.

$$\tilde{\sigma}(y)\cdot g_{\mathbf{t}(y)} = g_{\mathbf{s}(y)}\cdot\tilde{\sigma}(y),\quad \forall y\in Y_1,\ g\in G. \tag{19}$$

Moreover, Eq. (16) implies that $\forall(x,y)\in X_2$, $\eta^{-1}_{\mathbf{t}(y)}\circ\underline{\mathbf{Ad}}_{(\sigma((\phi(xy))^{-1})\sigma(\phi(x))\sigma(\phi(y)))}\circ\eta_{\mathbf{t}(y)} = \mathrm{id}$. Hence,

$$\sigma\big(\big(\phi(xy)\big)^{-1}\big)\sigma\big(\phi(x)\big)\sigma\big(\phi(y)\big) = 1 \quad \text{in } X_1/Z(\mathcal{K}),$$

since $\underline{\mathbf{Ad}}$ is injective. It thus follows that σ is a groupoid morphism.

$\boxed{3\Rightarrow 4}$ Take $\tilde{X}_1 = \pi^{-1}(\sigma(Y_1))$, where π denotes the projection $X_1\to X_1/Z(\mathcal{K})$. Since σ is a Lie groupoid morphism, it is simple to see that $\tilde{X}_1\rightrightarrows M$ is a Lie subgroupoid of $X_1\rightrightarrows M$ and $\tilde{X}_1\to Y_1\rightrightarrows M$ is a groupoid $Z(G)$-extension. The trivialization of the kernel $\eta_m: G\to\mathcal{K}_m$, when being restricted to the center $Z(G)$, induces a trivialization of the kernel $\tilde{\eta}_m: Z(G)\to Z(\mathcal{K})_m$ of the $Z(G)$-extension $\tilde{X}_1\to Y_1\rightrightarrows M$. Moreover, Eq. (17) implies that

$$\mathbf{Ad}_{\tilde{x}}\circ\eta_{\mathbf{t}(\tilde{x})} = \eta_{\mathbf{s}(\tilde{x})},\quad \forall\tilde{x}\in\tilde{X}_1. \tag{20}$$

It thus follows that $\tilde{X}_1\to Y_1\rightrightarrows M$ is a $Z(G)$-central extension. Consider the map

$$\tau:\tilde{X}_1\times G\to X_1:(\tilde{x},g)\mapsto\tilde{x}\cdot g_{\mathbf{t}(\tilde{x})}.$$

From Eq. (19), it follows that τ is a groupoid morphism. According to Eq. (16), we have

$$[x] = \underline{\eta}_{\mathbf{s}(x)}\big(r(x)\big)\cdot\sigma\big(\phi(x)\big) = \sigma\big(\phi(x)\big)\cdot\underline{\eta}_{\mathbf{t}(x)}\big(r(x)\big),\quad \forall x\in X_1.$$

It thus follows that $x = \sigma(\phi(x))\cdot\underline{\eta}_{\mathbf{t}(x)}(r(x))\cdot k$ for some $k\in Z(\mathcal{K})$. Hence τ is surjective. And its kernel $\{(z_m, z^{-1}) \mid z\in Z(G)\}$ is isomorphic to $Z(G)$. Therefore $X_\bullet\xrightarrow{\phi}Y_\bullet$ is isomorphic to the induced G-extension $\frac{\tilde{X}_1\times G}{Z(G)}\to Y_1\rightrightarrows M$.

$\boxed{4\Rightarrow 1}$ For any $x = [(\tilde{x},g)]\in\frac{\tilde{X}_1\times G}{Z(G)}$, it follows from a simple computation that

$$\eta^{-1}_{\mathbf{s}(x)}\circ\mathbf{Ad}_x\circ\eta_{\mathbf{t}(x)} = \mathbf{Ad}_g.$$

The conclusion thus follows. □

As an immediate consequence, Morita equivalent classes of G-extensions with trivial band are in one–one correspondence with Morita equivalent classes of $Z(G)$-central extensions. The latter is classified by $H^2(Y_\bullet, Z(G))$. Thus we have recovered the following result of Giraud [26] in the context of differential geometry.

Theorem 3.13. *Morita equivalent classes of G-extensions with trivial band (i.e. G-bound gerbes) over the groupoid $Y_1\rightrightarrows M$ are in one–one correspondence with $H^2(Y_\bullet, Z(G))$.*

3.4. G-gerbes over manifolds

In this subsection, we study G-gerbes over a manifold. This corresponds to a G-extension over a groupoid which is Morita equivalent to a manifold, i.e. a groupoid of the form $M \times_N M \rightrightarrows M$ for a surjective submersion $M \to N$ (see also [46]). When $\{U_i\}_{i\in I}$ is an open covering of N and M is the disjoint union $\coprod_i U_i$ with the covering map $\coprod_i U_i \to N$, the resulting groupoid $M \times_N M \rightrightarrows M$, which is easily seen to be isomorphic to $\coprod_{ij} U_{ij} \rightrightarrows \coprod_i U_i$, is called the Čech groupoid associated to the open covering $\{U_i\}_{i\in I}$ of the manifold N.

Let $\{U_i\}_{i\in I}$ be a **good open covering** of N, namely all U_i's and their finite intersections are contractible. We form its Čech groupoid $Y_\bullet : \coprod_{ij} U_{ij} \rightrightarrows \coprod_i U_i$, and consider a G-extension $\mathcal{K}_\bullet \to X_\bullet \xrightarrow{\phi} Y_\bullet$. A point x in N will be denoted by x_i when considered as a point in U_i and by x_{ij} when considered as a point in U_{ij}, etc. The source, target and multiplication maps of $Y_\bullet$ are given, respectively, by

$$\mathbf{s}(x_{ij}) = x_i, \qquad \mathbf{t}(x_{ij}) = x_j, \qquad x_{ij} \cdot x_{jk} = x_{ik}.$$

Since U_i is contractible for all $i \in I$, we can identify $\mathcal{K}$ with the trivial bundle of groups $\mathcal{K} \simeq \coprod_i U_i \times G$. Since U_{ij} is contractible for all $i, j \in I$, there exists a global section ρ (which needs not be a groupoid morphism) of $X_1 \xrightarrow{\phi} Y_1$ such that

$$\rho(x_{ii}) = \varepsilon(x_i), \quad \forall i \in I, \quad \text{and} \quad \rho(x_{ji}) = \big(\rho(x_{ij})\big)^{-1}, \quad \forall i, j \in I,$$

where $\varepsilon : M \to X_1$ is the unit map.

We identify any element $\tilde{x} \in X_1$ with a pair $x_{ij} = \phi(\tilde{x}) \in Y_1$ and $g \in \mathcal{K}_{x_j}$ such that $\rho(x_{ij}) \cdot g = \tilde{x}$. In other words, we identify X_1 with $\coprod_{ij} U_{ij} \times G$ in such a way that the multiplication of elements in X_1 by elements of $\mathcal{K} \simeq \coprod_i U_i \times G$ from the right coincides with the multiplication from the right on the group G, i.e.

$$(x_{ij}, g) \cdot (x_j, h) = (x_{ij}, gh), \quad \forall g, h \in G. \tag{21}$$

First, we define $C^\infty(G, G)$-valued functions λ_{ij} on the 2-intersections U_{ij} by comparing the right and the left actions of the group bundle $\mathcal{K} \to \coprod_i U_i$. Indeed for any $i, j \in I$, $x_{ij} \in U_{ij}$ and $g \in G$, there is a unique element $\lambda_{ij}(x_{ij})(g) \in G$ such that

$$(x_{ij}, 1) \cdot (x_j, g) = \big(x_i, \lambda_{ij}(x_{ij})(g)\big) \cdot (x_{ij}, 1). \tag{22}$$

Note that $g \to \lambda_{ij}(x_{ij})(g)$ is a smooth diffeomorphism by construction.

Second, we define G-valued functions g_{ijk} on the 3-intersections U_{ijk} by

$$(x_{ij}, 1) \cdot (x_{jk}, 1) = \big(x_{ik}, g_{ijk}(x_{ijk})\big).$$

These functions measure the default of ρ from being a groupoid homomorphism. In the sequel, the reference to the point x where λ_{ij} and g_{ijk} are evaluated will be omitted.

The data (λ_{ij}, g_{ijk}) determine completely the multiplication on the groupoid $X_1 \rightrightarrows \coprod_i U_i$. More precisely, one has

$$(x_{ij}, g)(x_{jk}, h) = \big(x_{ik}, g_{ijk}\lambda_{jk}^{-1}(g)h\big), \quad \forall g, h \in G. \tag{23}$$

The associativity of the groupoid multiplication defined on X_1 imposes the following relations on the functions λ_{ij} and g_{ijk}:

$$\lambda_{ij}(gh) = \lambda_{ij}(g)\lambda_{ij}(h), \quad \forall g, h \in G; \tag{24}$$

$$\lambda_{ij} \circ \lambda_{jk} = \mathbf{Ad}_{g_{ijk}} \circ \lambda_{ik}; \tag{25}$$

$$g_{ijl}g_{jkl} = g_{ikl}\lambda_{kl}^{-1}(g_{ijk}); \tag{26}$$

which are immediate consequences of the following identities, reflecting the associativity of the groupoid product:

$$\begin{gathered}
\big((x_{ij}, 1)(x_j, g)\big)(x_j, h) = (x_{ij}, 1)\big((x_j, g)(x_j, h)\big), \\
\big((x_i, g)(x_{ij}, 1)\big)(x_{jk}, 1) = (x_i, g)\big((x_{ij}, 1)(x_{jk}, 1)\big), \\
\big((x_{ij}, 1)(x_{jk}, 1)\big)(x_{kl}, 1) = (x_{ij}, 1)\big((x_{jk}, 1)(x_{kl}, 1)\big).
\end{gathered}$$

Eq. (24) means that λ_{ij} is an $\mathrm{Aut}(G)$-valued function on U_{ij}. Alternatively, one may write Eq. (22) as

$$\mathbf{Ad}_{(x_{ij},1)}(x_j, g) = \big(x_i, \lambda_{ij}(x_{ij})(g)\big).$$

From the fact that $\mathbf{Ad}_{(x_{ij},1)}$ is a group isomorphism from $\mathcal{K}_{x_j}$ to $\mathcal{K}_{x_i}$, it follows immediately that λ_{ij} is an $\mathrm{Aut}(G)$-valued function.

Conversely, if we are given some $\lambda_{ij} : U_{ij} \to \mathrm{Aut}(G)$, $g_{ijk} : U_{ijk} \to G$ satisfying Eqs. (25) and (26), then the product defined in Eq. (23) defines a groupoid structure on $X_1 \rightrightarrows \coprod_{ij} U_{ij}$, which makes $X_\bullet \to Y_\bullet$ into a groupoid G-extension.

The above discussion can be summarized by the following

Proposition 3.14. *Assume that $\{U_i\}_{i\in I}$ is a good open covering of a manifold N. Then there is a one–one correspondence between G-extensions of the Čech groupoid $\coprod_{ij} U_{ij} \rightrightarrows \coprod_i U_i$ and the data (λ_{ij}, g_{ijk}), where $\lambda_{ij} : U_{ij} \to \mathrm{Aut}(G)$ and $g_{ijk} : U_{ijk} \to G$ satisfy Eqs.* (25)–(26).

The data (λ_{ij}, g_{ijk}) is called a non-abelian 2-cocycle, as in [10,22,45].

Remark 3.15. Non-abelian 2-cocycles have recently appeared in many places in deformation quantization theory. See [15,16,21,30–32,34,48,59,60].

Now let us consider the band of this G-extension. According to Definition 3.4, the band is an $\mathrm{Out}(G)$-torsor over the Čech groupoid $Y_\bullet : \coprod_{ij} U_{ij} \rightrightarrows \coprod_i U_i$, which is Morita equivalent to the manifold N. Thus the band is a principal $\mathrm{Out}(G)$-bundle over the manifold N. To describe it more explicitly, by $\underline{\lambda}_{ij}$ we denote the composition of $\lambda_{ij} : U_{ij} \to \mathrm{Aut}(G)$ with the projection $\mathrm{Aut}(G) \to \mathrm{Out}(G)$. Then Eq. (25) implies that

$$\underline{\lambda}_{ij} \circ \underline{\lambda}_{jk} \circ \underline{\lambda}_{ki} = 1.$$

In other words, $\underline{\lambda}_{ij} : U_{ij} \to \mathrm{Out}(G)$ is a Čech 2-cocycle, which defines an $\mathrm{Out}(G)$-principal bundle over N. This is exactly the band of the G-extension.

Proposition 3.16. *Let $\{U_i\}_{i\in I}$ be a good open covering of a manifold N.*

1. *A G-extension $X_1 \xrightarrow{\phi} \coprod_{ij} U_{ij} \rightrightarrows \coprod_i U_i$ of the Čech groupoid $\coprod_{ij} U_{ij} \rightrightarrows \coprod_i U_i$ has a trivial band if and only if there exists a trivialization of the kernel $\mathcal{K} \simeq \coprod_i U_i \times G$ and a section ρ of $X_1 \xrightarrow{\phi} \coprod_{ij} U_{ij}$ such that the associated data (λ_{ij}, g_{ijk}) satisfies $\lambda_{ij} = \mathrm{id}$ and $(g_{ijk}) \in \check{Z}^2(N, Z(G))$.*
2. *Moreover, if (g_{ijk}) is a coboundary, then the section ρ can be modified so that $\lambda_{ij} = \mathrm{id}$ and $g_{ijk} = 1$. That is, the G-extension is isomorphic to the trivial G-extension.*

Proof. 1. Let $\bar{\eta} \in \Gamma(\mathrm{Out}(\mathcal{K}, G) \to \coprod U_i)$ be a trivialization of the band. Since the U_i's are contractible, $\bar{\eta}$ can be lifted to a section η of $\mathrm{Iso}(\mathcal{K}, G) \to \coprod U_i$. This gives the desired trivialization of the kernel. According to Theorem 3.12, since the band of the G-extension is trivial, and all U_{ij}'s are contractible, there exists a section ρ of $X_1 \xrightarrow{\phi} \coprod U_{ij}$ such that

$$\begin{aligned} \rho(x_{ij}) \cdot g_{x_j} &= g_{x_i} \cdot \rho(x_{ij}), \\ \rho(x_{ii}) &= 1, \\ \rho(x_{ji}) &= \rho(x_{ij})^{-1} \end{aligned} \tag{27}$$

for all $x_{ij} \in U_{ij}$ and $g \in G$. Now, we identify X_1 with $\coprod U_{ij} \times G$ through

$$\coprod U_{ij} \times G \to X_1 : (x_{ij}, g) \mapsto \rho(x_{ij}) \cdot g_{x_j}.$$

Then Eq. (27) becomes $(x_{ij}, 1)(x_j, g) = (x_i, g)(x_{ij}, 1)$. Hence $\lambda_{ij} = \mathrm{id}$. From $(x_{ij}, 1)(x_{jk}, 1) = (x_{ik}, g_{ijk})$, it follows that $\rho(x_{ki})\rho(x_{ij})\rho(x_{jk}) = g_{ijk}$. By Eq. (27), we have $g_{ijk} \in Z(G)$.

2. Assume that (g_{ijk}) is a coboundary: $\rho(x_{ki})\rho(x_{ij})\rho(x_{jk}) = g_{ijk} = h_{jk}h_{ik}^{-1}h_{ij}$, where $h_{ij} : U_{ij} \to Z(G)$. Using Eq. (27), this can be rewritten as $\rho(x_{ki})h_{ki}^{-1} \cdot \rho(x_{ij})h_{ij}^{-1} \cdot \rho(x_{jk})h_{jk}^{-1} = 1$. Define a new section $\rho' : \coprod U_{ij} \to X_1 : x_{ij} \mapsto \rho(x_{ij})h_{ij}^{-1}$. It is easy to check that relatively to the associated identification of X_1 with $\coprod U_{ij} \times G$, one has $\lambda'_{ij} = \lambda_{ij} \circ \mathbf{Ad}_{h_{ij}^{-1}} = 1$ and $g'_{ijk} = \rho'(x_{ki})\rho'(x_{ij})\rho'(x_{jk}) = 1$. $\square$

As an immediate consequence, we see that a G-extension with trivial band over the Čech groupoid of a good cover is completely determined by a Čech 2-cocycle in $\check{Z}^2(N, Z(G))$. Therefore we have derived the following result of Giraud [26] by a direct argument.

Corollary 3.17. *Isomorphism classes of G-bound gerbes over a manifold N are in one–one correspondence with $H^2(N, Z(G))$.*

Proposition 3.18. *Any groupoid G-extension of the Čech groupoid associated to a good open covering of a contractible manifold is isomorphic to the trivial G-extension.*

Proof. Let N be a contractible manifold, (U_i) a good covering of N, and $X_1 \to \coprod U_{ij} \rightrightarrows \coprod U_i$ a groupoid G-extension of the associated Čech groupoid. Its band must be trivial, since any principal bundle over a contractible manifold is trivial. According to Proposition 3.16, the multiplication on X_1 is entirely determined by the 2-cocycle (g_{ijk}) in $\check{Z}^2(N, Z(G))$. Since the manifold N is contractible and (U_i) is a good covering, (g_{ijk}) must be a 2-coboundary, which implies that $X_1 \to \coprod U_{ij} \rightrightarrows \coprod U_i$ is isomorphic to the trivial G-extension. □

By a **refinement** of a G-extension $X_1 \to \coprod_{i,j} U_{ij} \rightrightarrows \coprod_j U_j$ of a Čech groupoid, we mean the pullback of this G-extension through a refinement of the covering $(U_j)_{j\in J}$ of N.

As an immediate consequence of Proposition 3.18, we have the following

Corollary 3.19. *Any groupoid G-extension of a Čech groupoid over a contractible manifold N has a refinement which is isomorphic to a trivial G-extension.*

4. Connections on groupoid extensions

4.1. Connections as horizontal distributions

Recall that a **horizontal distribution** on a fiber bundle $X \xrightarrow{\phi} Y$ is an assignment to each point $x \in X$ of a subspace H_x of $T_x X$ transversal to the fiber of ϕ containing x.

Recall also that, to any Lie groupoid $\Gamma_1 \rightrightarrows \Gamma_0$, one can associate its **tangent Lie groupoid** $T\Gamma_1 \rightrightarrows T\Gamma_0$ whose structure maps are the differentials of those of $\Gamma_\bullet$ [39,40]. More precisely, if $\mathbf{s} : \Gamma_1 \to \Gamma_0$, $\mathbf{t} : \Gamma_1 \to \Gamma_0$ and $m : \Gamma_2 \to \Gamma_1$ denote the source, target and multiplication maps of $\Gamma_\bullet$, then $\mathbf{s}_* : T\Gamma_1 \to T\Gamma_0$ and $\mathbf{t}_* : T\Gamma_1 \to T\Gamma_0$ are respectively the source and target maps of $(T\Gamma)_\bullet$, and the multiplication map $T\Gamma_1 \times_{\mathbf{t}_*,\Gamma_0,\mathbf{s}_*} T\Gamma_1 \to T\Gamma_1$ is the composition of the canonical isomorphism $T\Gamma_1 \times_{\mathbf{t}_*,\Gamma_0,\mathbf{s}_*} T\Gamma_1 \simeq T(\Gamma_1 \times_{\mathbf{t},\Gamma_0,\mathbf{s}} \Gamma_1)$ with the differential of $m : \Gamma_1 \times_{\mathbf{t},\Gamma_0,\mathbf{s}} \Gamma_1 \to \Gamma_1$.

Definition 4.1. A **connection** on a Lie groupoid extension $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ is a horizontal distribution H on $X_1 \xrightarrow{\phi} Y_1$ which is also a Lie subgroupoid of the tangent groupoid $TX_1 \rightrightarrows TM$. I.e. we have

$$H_{x_1} \cdot H_{x_2} \subset H_{x_1 \cdot x_2} \quad \text{for all } (x_1, x_2) \in X_2, \tag{28}$$

$$H_x^{-1} \subset H_{x^{-1}} \quad \text{for all } x \in X_1. \tag{29}$$

Lemma 4.2.

1. *The distribution H contains the entire unit space TM of X_1.*
2. $H_{x_1} \cdot H_{x_2} = H_{x_1 \cdot x_2}$.
3. $(H_x)^{-1} = H_{x^{-1}}$.

Proof. 1. Since H is a Lie subgroupoid of $TX_1 \rightrightarrows TM$, it contains its unit space, which is $\mathbf{s}_* H$. On the other hand, we have $\mathbf{s}_* H = \mathbf{s}_* \phi_* H = \mathbf{s}_* TY_1 = TM$. The conclusion follows.

2. Since $T_{\mathbf{t}(x)}M \subset H_{x^{-1}} \cdot H_x$ and $H_{x_1}^{-1} \subset H_{x_1^{-1}}$, we have

$$H_{x_1 \cdot x_2} = H_{x_1 \cdot x_2} \cdot T_{\mathbf{t}(x_2)}M \subset H_{x_1 \cdot x_2} \cdot H_{x_2^{-1}} \cdot H_{x_2} \subset H_{x_1} \cdot H_{x_2}.$$

3. Substituting x^{-1} for x in Eq. (29), one gets $H_{x^{-1}}^{-1} \subset H_x$. Thus, $H_{x^{-1}} \subset H_x^{-1}$. □

As before, let $\mathcal{K}$ be the kernel of $X_\bullet \xrightarrow{\phi} Y_\bullet$. Then $\mathcal{K} \to M$ is a group bundle, which is also a Lie subgroupoid of $X_\bullet$. Set

$$H^{\mathcal{K}} = T\mathcal{K} \cap H. \tag{30}$$

Since $\ker(\phi_*) \subset T_k\mathcal{K}$ for any $k \in \mathcal{K}$, $H^{\mathcal{K}}$ defines a connection for the bundle $\mathcal{K} \xrightarrow{\phi} M$, which is also a groupoid extension connection when $\mathcal{K} \to M$ is considered as a groupoid extension $\mathcal{K} \to M \rightrightarrows M$. Since this groupoid extension is simply a bundle of groups, $H^{\mathcal{K}}$ is a group bundle connection that we call the **induced connection on the bundle of groups** $\mathcal{K} \to M$.

Let $\mathfrak{K} \to M$ denote the Lie algebra bundle associated to the group bundle $\mathcal{K} \to M$. Let $\exp : \mathfrak{K} \to \mathcal{K}$ denote the pointwise exponential map. The connection $H^{\mathcal{K}}$ on the group bundle $\mathcal{K} \xrightarrow{\phi} M$ induces a connection on the Lie algebra bundle $\mathfrak{K} \to M$ that we call the **induced horizontal distribution $H^{\mathfrak{K}}$ on the bundle of Lie algebras** $\mathfrak{K} \to M$ defined as follows: $v \in T\mathfrak{K}$ is horizontal if $\exp_* v \in H^{\mathcal{K}}$. This connection is compatible with the Lie algebra bundle structure, i.e. it is given by a covariant derivative on the vector bundle $\mathfrak{K} \to M$ and satisfies Proposition 4.21(4).

4.2. Horizontal paths for groupoid extension connections

It is natural to ask what is the geometrical meaning of a connection on a groupoid extension. This subsection aims at answering this question.

Recall that, given a horizontal distribution on a fiber bundle, a path is said to be **horizontal** if it is tangent to the horizontal distribution. Also recall that, given a groupoid, two paths $\tau \mapsto \gamma_1(\tau)$, $\tau \mapsto \gamma_2(\tau)$ on the space of arrows are said to be **compatible** if, and only if, $\gamma_1(\tau)$, $\gamma_2(\tau)$ are composable for all values of τ. The **product** of these paths is the path $\tau \to \gamma_1(\tau) \cdot \gamma_2(\tau)$.

The following proposition gives an alternative definition of Lie groupoid extension connection.

Proposition 4.3. *Let $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ be a Lie groupoid extension. A horizontal distribution on $X_1 \xrightarrow{\phi} Y_1$ is a groupoid extension connection if and only if the product of any pair of horizontal composable paths in X_1 is still a horizontal path, and the inverse of any horizontal path is still a horizontal path.*

Here is yet another alternative description of groupoid extension connections.

For any manifold N, denote N^I the set of smooth paths from $[0, 1]$ to N. If $\Gamma_1 \rightrightarrows \Gamma_0$ is a groupoid, then $\Gamma_1^I \rightrightarrows \Gamma_0^I$ inherits a groupoid structure with the source map $\gamma \mapsto \mathbf{s} \circ \gamma$, the target map $\gamma \mapsto \mathbf{t} \circ \gamma$, and the product $\tau \mapsto \gamma_1(\tau) \cdot \gamma_2(\tau)$, for any composable $\gamma_1, \gamma_2 \in \Gamma_1^I$. We call this groupoid the **path groupoid**. Proposition 4.3 can be reinterpreted as follows: a horizontal distribution is a groupoid extension connection if, and only if, the horizontal paths form a subgroupoid of the path groupoid $X_1^I \rightrightarrows M^I$.

Recall that an **Ehresmann connection** on a fiber bundle $X \xrightarrow{\phi} Y$ is a horizontal distribution satisfying the following technical assumption: any path on Y starting from $y \in Y$ has a unique horizontal lift on X starting from a given point x in the fiber of X over y. This additional assumption is required to avoid horizontal lifts going to infinity in a finite time.

Let γ be a path in Y_1. The parallel transport τ_t^γ along γ is a transformation satisfying the following properties:

- $\tau_0^\gamma(x) = x$;
- for any $x \in X_1$ with $\phi(x) = \gamma(s)$, the relation $\phi(\tau_t^\gamma(x)) = \gamma(s+t)$ holds;
- the path $t \mapsto \tau_t^\gamma(x)$ is horizontal.

Proposition 4.4. *Groupoid extension connections are Ehresmann connections.*

Proof. Let $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ be a groupoid extension endowed with a groupoid extension connection. Let us fix a path $\gamma : t \mapsto \gamma_t$ in Y_1 defined on an open interval I of $\mathbb{R}$ containing $[0, 1]$. For each $x \in X_1$ in the fiber over γ_0, we will denote by $\bar{\gamma}^x : t \to \bar{\gamma}_t^x$ the path (if any) in X_1, which passes through x and lifts γ horizontally.

$\boxed{\text{Step 1}}$ Given a point b in $\phi^{-1}(\gamma_0)$, we will show that there exists a positive real number ε such that, for all points x in the same connected component as b in $\phi^{-1}(\gamma_0)$, the path $\bar{\gamma}^x$ is defined for $t \in (-\varepsilon, \varepsilon)$.

Indeed, since the solutions of ODEs depend smoothly on the parameters, there exists a connected neighborhood U of b in its fiber $\phi^{-1}(\gamma_0)$ and a positive real number ε such that, for all $u \in U$, the horizontal lift $\bar{\gamma}^u$ is defined for all $t \in (-\varepsilon, \varepsilon)$.

For each $u \in U$, define the path $\delta^{b^{-1}\cdot u} \in C^\infty((-\varepsilon, \varepsilon), \mathcal{K})$ by $\delta_t^{b^{-1}\cdot u} = (\bar{\gamma}_t^b)^{-1} \cdot (\bar{\gamma}_t^u)$. Since the horizontal distribution on X_1 is compatible with the groupoid multiplication, the paths $\delta^{b^{-1}\cdot u}$ with $u \in U$ as well as all products of such paths in the groupoid $C^\infty((-\varepsilon, \varepsilon), X_1)$ are horizontal.

Observe that $\{\delta_0^{b^{-1}\cdot u} \mid u \in U\}$ is the neighborhood $L_{b^{-1}}U$ of the identity in the Lie group $\mathcal{K}_{\mathbf{t}(x)}$. Now, taking any element x in the same connected component of $\phi^{-1}(\gamma_0)$ as b, the product $b^{-1} \cdot x$ lies in the connected component of the Lie group $\mathcal{K}_{\mathbf{t}(x)}$ containing its unit element. Hence $b^{-1} \cdot x$ can be written as a finite product $k_1 \cdot k_2 \cdot \cdots \cdot k_n$ of elements in the neighborhood $L_{b^{-1}}U$.

Therefore, the horizontal lift $\bar{\gamma}^x$ is defined for all $t \in (-\varepsilon, \varepsilon)$ because

$$\bar{\gamma}_t^x = \bar{\gamma}_t^b \cdot \delta_t^{k_1} \cdot \delta_t^{k_2} \cdot \cdots \cdot \delta_t^{k_n}.$$

$\boxed{\text{Step 2}}$ We show that for any point x in the connected component of b in the fiber $\phi^{-1}(\gamma_0)$, the path $\bar{\gamma}^x$ is defined for all $t \in [0, 1]$.

By the first step, it is clear that there exists an open covering

$$\bigcup_{s \in I,\, U \in \mathcal{C}_s} \bigl\{ \bigl(s+t, \tau_t^\gamma(u)\bigr) \bigm| t \in (-\varepsilon_U, \varepsilon_U),\ u \in U \bigr\}$$

of $I \times_{\gamma, Y_1, \phi} X_1$, where $\mathcal{C}_s$ denotes the set of connected components of the fiber $\phi^{-1}(\gamma_s)$, ε_U depends on U and τ denotes the parallel transport. Any connected component of $I \times_{\gamma, Y_1, \phi} X_1$ is thus a union of such open sets. Any horizontal lift of γ does entirely lie in such a connected component. Consider the connected component B of $I \times_{\gamma, Y_1, \phi} X_1$ containing b. Since the parallel transport maps fibers homeomorphically, it does preserve the connected components of the fibers. Hence the intersection of B with any fiber must be a connected component of this fiber. It is thus clear that

$$B = \bigcup_{s \in I} \bigl\{ \bigl(s+t, \tau_t^{\gamma}(u)\bigr) \bigm| t \in (-\varepsilon_s, \varepsilon_s),\ u \in B \cap \phi^{-1}\bigl(\gamma(s)\bigr) \bigr\},$$

where ε_s does now only depend on s.

Since $\bigcup_{s \in I}(s - \varepsilon_s, s + \varepsilon_s)$ is an open covering of the compact interval $[0,1]$, there exists a finite subcovering $\bigcup_{i=0,\dots,n}(s_i - \varepsilon_{s_i}, s_i + \varepsilon_{s_i})$ of $[0,1]$ with $0 = s_0 < s_1 < s_2 < \cdots < s_n = 1$ and $(s_{i-1} - \varepsilon_{s_{i-1}}, s_{i-1} + \varepsilon_{s_{i-1}}) \cap (s_i - \varepsilon_{s_i}, s_i + \varepsilon_{s_i}) \neq \varnothing$. Let $t_0 = 0$, $t_{n+1} = 1$ and choose n real numbers $t_1, \dots, t_n$ such that $t_i \in (s_{i-1} - \varepsilon_{s_{i-1}}, s_{i-1} + \varepsilon_{s_{i-1}}) \cap (s_i - \varepsilon_{s_i}, s_i + \varepsilon_{s_i})$. Then, for any x in the same connected component of $\phi^{-1}(\gamma_0)$ as b, the path recursively defined by $\bar{\gamma}_t^x = \tau_{t-t_i}^{\bar{\gamma}_{t_i}^x}$ for $t_i \leqslant t \leqslant t_{i+1}$ is the horizontal lift of γ through x. Modifying the choice of the point b and the path γ in the two steps above, the conclusion follows. □

The following result is an immediate consequence of Proposition 4.3 applied to $H^{\mathcal{K}}$.

Corollary 4.5. *Let $H^{\mathcal{K}}$ be the induced connection on the group bundle $\mathcal{K} \to M$. Then the parallel transport in $\mathcal{K} \to M$ preserves the group structure on the fibers.*

Since the horizontal distribution on $\mathcal{K}$ is the image of the horizontal distribution on $\mathfrak{K}$ under the differential of the exponential map, we have the following lemma.

Lemma 4.6. *The following diagram*

$$\begin{tikzcd} \mathfrak{K}_{\gamma(0)} \arrow[r, "\tau_t^{\gamma}"] \arrow[d, "\exp"'] & \mathfrak{K}_{\gamma(t)} \arrow[d, "\exp"'] \\ \mathcal{K}_{\gamma(0)} \arrow[r, "\tau_t^{\gamma}"] & \mathcal{K}_{\gamma(t)} \end{tikzcd} \tag{31}$$

commutes, where τ_t^{γ} stands, with an abuse of notation, for the parallel transportation in both $\mathcal{K}$ and $\mathfrak{K}$ over some path γ in M.

The following proposition shows that any groupoid extension of a connected Lie groupoid that admits a connection must be a groupoid G-extension for some fixed Lie group G.

Proposition 4.7. *Let $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ be a Lie groupoid extension with kernel $\mathcal{K}$. Assume that the orbit space M/Y_1 is path connected. If there exists a groupoid extension connection, then*

1. *the groups $\mathcal{K}_m$, $m \in M$, are all isomorphic;*
2. *ϕ is a Lie groupoid G-extension for a fixed Lie group G.*

Proof. 1. Since M/X_1 is path connected, any two points $m, n \in M$ can be connected by a family of points $\{m_i\}_{i=1,\dots,l}$ such that any two consecutive elements in the family are either in the same connected component of M, or are the source and target of some element in X_1.

For any pair of points m and n in the same connected component of M, an isomorphism $\mathcal{K}_m \simeq \mathcal{K}_n$ can be constructed by taking the parallel transport over a path joining m and n. And if $x \in X_1$, the conjugation by x induces an automorphism $\mathcal{K}_{\mathbf{s}(x)} \simeq \mathcal{K}_{\mathbf{t}(x)}$.

2. Choose a Riemannian metric on M and take an open covering of M by contractible open normal neighborhoods $\{U_i\}_{i\in I}$. Then $h_i : U_i \times [0,1] \to U_i : (\exp_{u_i} \xi, t) \mapsto \exp_{u_i} t\xi$, where ξ is in a small neighborhood of zero in $T_{u_i}U_i$, is a smooth deformation retraction of U_i onto a fixed point u_i. Consider the pullback $X_1[\coprod U_i] \to Y_1[\coprod U_i] \rightrightarrows \coprod U_i$ of $X_1 \to Y_1 \rightrightarrows M$. Its kernel is $\coprod_{i\in I} U_i \times_M \mathcal{K}$. As each $U_i \times_M \mathcal{K}$ can be locally identified to $U_i \times \mathcal{K}_{u_i}$ using parallel transport along the paths $t \mapsto h_i(m,t)$, $m \in U_i$, as above, it follows that $\coprod_{i\in I} U_i \times_M \mathcal{K}$ can be identified with $\coprod_{i\in I} U_i \times G$ for a fixed Lie group G. Hence $X_1[\coprod U_i] \to Y_1[\coprod U_i] \rightrightarrows \coprod U_i$ is a Lie groupoid G-extension. □

From now on, if a Lie groupoid extension admits a groupoid extension connection, it is always assumed to be a G-extension. Take an arbitrary path γ in X_1^I. The horizontal distribution being an Ehresmann connection, there exists an unique horizontal path $\bar{\gamma}$ starting at $\gamma(0)$ and satisfying $\phi \circ \gamma = \phi \circ \bar{\gamma}$. There is therefore a unique map $g : [0,1] \to \mathcal{K}$ such that $\bar{\gamma}(\tau) = \gamma(\tau) \cdot g_\gamma(\tau)$; note that, by construction, $g_\gamma(\tau) \in \mathcal{K}_{\mathbf{t}\circ\gamma(\tau)}$ for all $\tau \in [0,1]$.

We call **right holonomy** of a path γ, denoted by $\mathrm{hol}(\gamma)$, the element $g_\gamma(1) \in \mathcal{K}_{\mathbf{t}\circ\gamma(1)}$. The left holonomy can be defined similarly using the left action of $\mathcal{K}$.

Proposition 4.8. *For any pair of paths γ_1, γ_2 composable in X_1, the following relation holds:*

$$\mathrm{hol}(\gamma_1 \cdot \gamma_2) = \left(\mathbf{Ad}_{(\gamma_2(1))^{-1}} \mathrm{hol}(\gamma_1)\right) \cdot \mathrm{hol}(\gamma_2).$$

Proof. The paths γ_1, γ_2 being composable, so are the associated horizontal paths $\bar{\gamma}_1, \bar{\gamma}_2$. The product of horizontal paths being horizontal, their product $\bar{\gamma}_1 \cdot \bar{\gamma}_2$ is a horizontal path. Moreover, this path starts at the point $\bar{\gamma}_1(0)\bar{\gamma}_2(0) = (\gamma_1 \cdot \gamma_2)(0)$. Therefore $\overline{\gamma_1\gamma_2} = \bar{\gamma}_1 \cdot \bar{\gamma}_2$. It is now a simple matter to check that:

$$\begin{aligned}
(\gamma_1\gamma_2)(1) \cdot \mathrm{hol}_{\gamma_1\gamma_2} &= \overline{\gamma_1\gamma_2}(1) \\
&= \gamma_1(1) \cdot \mathrm{hol}(\gamma_1) \cdot \gamma_2(1) \cdot \mathrm{hol}(\gamma_2) \\
&= \gamma_1(1) \cdot \gamma_2(1) \cdot \left(\mathbf{Ad}_{(\gamma_2(1))^{-1}} \mathrm{hol}(\gamma_1)\right) \cdot \mathrm{hol}(\gamma_2). \qquad \square
\end{aligned}$$

4.3. Connections as 1-forms

Recall that any Lie groupoid $X_1 \rightrightarrows M$ gives rise to a simplicial manifold

$$\cdots \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; X_2 \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; X_1 \;\rightrightarrows\; X_0, \tag{32}$$

where

$$X_n = \left\{(x_1, \dots, x_n) \;\middle|\; \mathbf{t}(x_i) = \mathbf{s}(x_{i+1}),\ i = 1, \dots, n-1\right\}$$

is the set of composable n-tuples of elements of X_1, and $X_0 = M$ and the face maps are defined as follows [52]. The maps $\varepsilon_i^n : X_n \to X_{n-1}$ are given by, for $n > 1$,

$$\varepsilon_0^n(x_1, x_2, \ldots, x_n) = (x_2, \ldots, x_n),$$
$$\varepsilon_n^n(x_1, x_2, \ldots, x_n) = (x_1, \ldots, x_{n-1}),$$
$$\varepsilon_i^n(x_1, x_2, \ldots, x_n) = (x_1, \ldots, x_i x_{i+1}, \ldots, x_n), \quad 1 \leqslant i \leqslant n-1,$$

and, for $n = 1$, $\varepsilon_0^1(x) = \mathbf{s}(x)$, $\varepsilon_1^1(x) = \mathbf{t}(x)$. They satisfy the simplicial relations

$$\varepsilon_i^{n-1} \circ \varepsilon_j^n = \varepsilon_{j-1}^{n-1} \circ \varepsilon_i^n \quad \forall i < j.$$

We also define the maps $\mathbf{s} : X_n \to M : (x_1, \ldots, x_n) \mapsto \mathbf{s}(x_1)$ and $\mathbf{t} : X_n \to M : (x_1, \ldots, x_n) \mapsto \mathbf{t}(x_n)$. When $n = 1$, we recover the source and target of the groupoid, justifying the notation.

Consider a Lie groupoid extension (1). Since $X_1 \rightrightarrows M$ acts on $\mathcal{K} \to M$ by conjugation, it acts on $\mathfrak{K} \to M$ by adjoint action. Therefore one obtains a left representation of the groupoid $X_1 \rightrightarrows M$ on $\mathfrak{K} \to M$. For any $x \in X_1$, the adjoint representation is denoted by $\mathrm{Ad}_x : \mathfrak{K}_{\mathbf{t}(x)} \to \mathfrak{K}_{\mathbf{s}(x)}$, which, by definition, is the derivative at the identity of the conjugation $\mathbf{Ad}_x : \mathcal{K}_{\mathbf{t}(x)} \to \mathcal{K}_{\mathbf{s}(x)}$. Therefore one can talk about Lie groupoid cohomology with values in $\mathfrak{K}$ [37,38]. Here a $\mathfrak{K}$-valued cochain is a smooth map which associates an element in $\mathfrak{K}_{\mathbf{s}(x_1)}$ to a composable n-tuple $(x_1, \ldots, x_n)$. Thus the space of n-cochains can be identified with $C^\infty(X_n, \mathbf{s}^*\mathfrak{K}) = \Gamma(\mathbf{s}^*\mathfrak{K} \to X_n)$, the space of smooth sections of the vector bundle $\mathbf{s}^*\mathfrak{K} \to X_n$, i.e. the pullback bundle of $\mathfrak{K} \to M$ via $\mathbf{s} : X_n \to M$. The differential $\partial^{\lhd} : C^\infty(X_{n-1}, \mathbf{s}^*\mathfrak{K}) \to C^\infty(X_n, \mathbf{s}^*\mathfrak{K})$ is given by

$$\partial^{\lhd}|_{(x_1,\ldots,x_n)} = \mathrm{Ad}_{x_1} \circ \left(\varepsilon_0^n\right)^* + \sum_{i=1}^{n} (-1)^i \left(\varepsilon_i^n\right)^*. \tag{33}$$

In general, as a cochain complex, one can consider $\Omega^l(X_n, \mathbf{s}^*\mathfrak{K})$, the space of differential forms on X_n with values in the vector bundle $\mathbf{s}^*\mathfrak{K} \to X_n$, i.e. the space of smooth sections of the vector bundle $\wedge^l T^*X_n \otimes \mathbf{s}^*\mathfrak{K} \to X_n$ and the operator $\partial^{\lhd}$ given by exactly the same formula (33) as the differential. Thus $(\partial^{\lhd})^2 = 0$. Similarly, by taking the inverse of the adjoint action of $X_1 \rightrightarrows M$ on $\mathfrak{K} \to M$, one obtains a right representation of $X_1 \rightrightarrows M$ on $\mathfrak{K} \to M$, and thus can consider the cochain complex formed by $\Omega^l(X_n, \mathbf{t}^*\mathfrak{K})$, the space of differential forms on X_n with values in the vector bundle $\mathbf{t}^*\mathfrak{K} \to X_n$, and the differential $\partial^{\rhd} : \Omega^l(X_{n-1}, \mathbf{t}^*\mathfrak{K}) \to \Omega^l(X_n, \mathbf{t}^*\mathfrak{K})$ given by

$$\partial^{\rhd}|_{(x_1,\ldots,x_n)} = \sum_{i=0}^{n-1} (-1)^i \left(\varepsilon_i^n\right)^* + (-1)^n \mathrm{Ad}_{(x_n)^{-1}} \circ \left(\varepsilon_n^n\right)^*. \tag{34}$$

We thus have $(\partial^{\rhd})^2 = 0$.

In the sequel, if ξ is an element of $\mathfrak{K}$, the right and left fundamental vector fields generated by ξ will be denoted by $\xi^{\rhd}$ and $\xi^{\lhd}$ respectively. Thus,

$$\xi_x^{\rhd} = \frac{d}{d\tau} x \exp(\tau\xi)\bigg|_{\tau=0}, \qquad \xi_y^{\lhd} = \frac{d}{d\tau} \exp(\tau\xi) y\bigg|_{\tau=0}, \tag{35}$$

where $x, y \in X_1$ with $\mathbf{t}(x) = m = \mathbf{s}(y)$ and $\xi \in \mathfrak{K}_m$.

Definition 4.9. Let $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ be a Lie groupoid extension. A 1-form $\alpha \in \Omega^1(X_1, \mathbf{t}^*\mathfrak{K})$ is said to be a **right connection 1-form** if $\alpha(\xi^{\rhd}) = \xi$, $\forall \xi \in \mathfrak{K}$ and $\partial^{\rhd}\alpha = 0$. Similarly, a 1-form $\beta \in \Omega^1(X_1, \mathbf{s}^*\mathfrak{K})$ is said to be a **left connection 1-form** if $\beta(\xi^{\lhd}) = \xi$, $\forall \xi \in \mathfrak{K}$ and $\partial^{\lhd}\beta = 0$.

Remark 4.10.

1. An l-form $\alpha \in \Omega^l(X_1, \mathbf{t}^*\mathfrak{K})$ satisfies $\partial^{\rhd}\alpha = 0$ if and only if, for any composable pair $(x_1, x_2) \in X_2$ and for any l-tuples of composable pairs $(u_1, v_1), \ldots, (u_l, v_l)$ in the tangent groupoid $TX_1 \rightrightarrows TM$, with $u_i \in T_{x_1}X_1$, $v_i \in T_{x_2}X_1$ for all $i \in \{1, \ldots, l\}$, one has
$$\alpha(u_1 \cdot v_1, \ldots, u_l \cdot v_l) = \alpha(v_1, \ldots, v_l) + \mathrm{Ad}_{x_2^{-1}} \alpha(u_1, \ldots, u_l). \tag{36}$$
In particular, for $l = 1$, one has
$$\alpha(u \cdot v) = \alpha(v) + \mathrm{Ad}_{x_2^{-1}} \alpha(u) \tag{37}$$
for any composable $u \in T_{x_1}X_1$ and $v \in T_{x_2}X_1$.
Eq. (37) can be interpreted as follows. The tangent groupoid $TX_1 \rightrightarrows TM$ acts on $\mathfrak{K} \to M$ from the right by $u \cdot k = Ad_x^{-1}k$ for any $u \in T_xX_1$ and any $k \in \mathfrak{K}_{\mathbf{s}(x)}$. Then $\partial^{\rhd}\alpha = 0$ if, and only if, $\alpha : TX_1 \to \mathfrak{K}$ is a 1-cocycle with respect to this action.
2. In the case of G-extensions over a Čech groupoid, the condition $\partial^{\rhd}\alpha = 0$ should be equivalent to the condition given by Breen and Messing in [14, Eq. (6.1.9)]. See [13].

Lemma 4.11. *Let $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ be a Lie groupoid extension endowed with a Lie groupoid extension connection H. Let (p, q) be any point in X_2, and $v_p \in T_pX_1$ and $v_q \in T_qX_1$ any pair of composable horizontal vectors.*

1. *If ξ and η are elements of the fibers of $\mathfrak{K} \to M$ at the points $\mathbf{t}(p)$ and $\mathbf{t}(q)$, respectively, then $(\xi_p^{\rhd} + v_p)$ and $(\eta_q^{\rhd} + v_q)$ are composable and*
$$\big(\xi_p^{\rhd} + v_p\big) \cdot \big(\eta_q^{\rhd} + v_q\big) = \big(\mathrm{Ad}_{q^{-1}} \xi + \eta\big)_{pq}^{\rhd} + (v_p \cdot v_q).$$
2. *If ξ and η are elements of the fibers of $\mathfrak{K} \to M$ at the points $s(p)$ and $s(q)$ respectively, then $(\xi_p^{\lhd} + v_p)$ and $(\eta_q^{\lhd} + v_q)$ are composable and*
$$\big(\xi_p^{\lhd} + v_p\big) \cdot \big(\eta_q^{\lhd} + v_q\big) = \big(\xi + \mathrm{Ad}_p \eta\big)_{pq}^{\lhd} + (v_p \cdot v_q).$$

Proof. We prove (1) only since the argument for (2) is similar. Choose a path $t \mapsto (\gamma(t), \delta(t))$ in X_2 such that $\frac{d}{dt}|_0 \gamma(t) = v_p$ and $\frac{d}{dt}|_0 \delta(t) = v_q$. Then,

$$\xi_p^{\rhd} + v_p = \frac{d}{dt}\bigg|_0 \big(\gamma(t)\exp(t\xi)\big), \qquad \eta_q^{\rhd} + v_q = \frac{d}{dt}\bigg|_0 \big(\delta(t)\exp(t\eta)\big).$$

Hence the result follows from the identity

$$\frac{d}{dt}\bigg|_0 \big(\gamma(t)\exp(t\xi)\delta(t)\exp(t\eta)\big) = \frac{d}{dt}\bigg|_0 \big(p\exp(t\xi)q\exp(t\eta)\big) + \frac{d}{dt}\bigg|_0 \big(\gamma(t)\delta(t)\big). \qquad \square$$

Theorem 4.12. *Let $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ be a Lie groupoid extension. Then the following are all equivalent.*

1. *Lie groupoid extension connections $H \subset TX_1$;*
2. *right connection 1-forms $\alpha \in \Omega^1(X_1, \mathbf{t}^*\mathfrak{K})$; and*
3. *left connection 1-forms $\beta \in \Omega^1(X_1, \mathbf{s}^*\mathfrak{K})$.*

Proof. It suffices to prove the one–one correspondence between (1) and (2). The equivalence between (1) and (3) can be proved similarly.

$\boxed{1 \Rightarrow 2}$ Define $\alpha \in \Omega^1(X_1, \mathbf{t}^*\mathfrak{K})$ by setting

$$\alpha\big(\xi^{\triangleright}\big) = \xi, \quad \forall \xi \in \mathfrak{K} \quad \text{and} \quad \alpha(v) = 0, \quad \forall v \in H.$$

From Lemma 4.11 and Eq. (28), it follows that for all $(p,q) \in X_2$, $v_p \in H_p$ and $v_q \in H_q$ with v_p, v_q composable,

$$\big(\big(\xi_p^{\triangleright} + v_p\big)\cdot\big(\eta_q^{\triangleright} + v_q\big)\big) \lrcorner\, \alpha = \mathrm{Ad}_{q^{-1}}\,\xi + \eta = \mathrm{Ad}_{q^{-1}}\big(\big(\xi_p^{\triangleright} + v_p\big) \lrcorner\, \alpha\big) + \big(\eta_q^{\triangleright} + v_q\big) \lrcorner\, \alpha.$$

Thus we have

$$\partial^{\triangleright}\alpha|_{(p,q)} = \big(\varepsilon_0^2\big)^*\alpha - \big(\varepsilon_1^2\big)^*\alpha + \mathrm{Ad}_{q^{-1}} \circ \big(\varepsilon_2^2\big)^*\alpha = 0.$$

Hence α is a right connection 1-form.

$\boxed{2 \Rightarrow 1}$ Set $H = \ker\alpha$. It is clear that H is a horizontal distribution for the fiber bundle $X_1 \xrightarrow{\phi} Y_1$. Since $\partial^{\triangleright}\alpha = 0$, H is a subgroupoid of the tangent groupoid $TX_1 \rightrightarrows TM$ by Eq. (37). $\square$

The right and left connection 1-forms of a Lie groupoid extension connection are related in a simple manner as described in the following

Proposition 4.13. *Let $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ be a Lie groupoid extension. And let $H \subset TX_1$ be a Lie groupoid extension connection, $\alpha \in \Omega^1(X_1, \mathbf{t}^*\mathfrak{K})$, and $\beta \in \Omega^1(X_1, \mathbf{s}^*\mathfrak{K})$ the right and left connection 1-forms associated to H, respectively. Then α and β are related by the following equation: $\beta|_x = \mathrm{Ad}_x \circ \alpha|_x$, $\forall x \in X_1$.*

Proof. This equation is obviously true on horizontal vectors. Moreover, the equation $\xi_x^{\triangleright} = (\mathrm{Ad}_x\,\xi)_x^{\triangleleft}$, $\forall \xi \in \mathfrak{K}$, implies the expected result on vertical vectors. $\square$

4.4. Connections on groupoid G-extensions

In this subsection, we consider a Lie groupoid G-extension $X_\bullet \xrightarrow{\phi} Y_\bullet$ whose kernel $\mathcal{K}$ is endowed with a trivialization $\chi : M \times G \to \mathcal{K}$. Thus $X_1 \xrightarrow{\phi} Y_1$ is a principal G-G bibundle. Assume that there exists a groupoid extension connection H on ϕ. A natural question is when this connection is a connection for the right (or left) principal G-bundle $X_1 \xrightarrow{\phi} Y_1$.

According to Eq. (30), there is an induced connection $H^{\mathcal{K}}$ on the kernel $\mathcal{K} \to M$. Via the trivialization χ, it in turn induces a connection on the group bundle $M \times G \to M$, which is denoted by the symbol H'. Then H' is a connection for the trivial extension $M \times G \to M \rightrightarrows M$.

For any $\xi \in \mathfrak{g}$, ξ^R denotes the right invariant vector field on G corresponding to ξ. Associated to H', there exists a family $\{F_g\}_{g\in G}$ of $\mathfrak{g}$-valued 1-forms on M such that $(v_m, (F_g(v_m))^R_g) \in T_mM \times T_gG$ is the unique element of $H'_{(m,g)}$ whose projection in T_mM is v_m. In other words, $\{F_g\}$ measures the defect between $H'_{(m,g)}$ and $T_{(m,g)}(M \times \{g\})$. Note that, by definition

$$F_g(v_m) = -\beta|_{(m,g)}(g_* v_m), \tag{38}$$

where g_* stands for the differential of the constant section $m \mapsto (m, g)$ of $M \times G \to M$.

Lemma 4.14. *The condition*

$$H'_{(m,g)} \cdot H'_{(m,h)} = H'_{(m,gh)}$$

is equivalent to

$$F_{gh} = F_g + \mathrm{Ad}_g F_h, \quad \forall g, h \in G. \tag{39}$$

I.e. $F : G \to \Omega^1(M) \otimes \mathfrak{g}$ *is a Lie group* 1*-cocycle, where* G *acts on* $\Omega^1(M) \otimes \mathfrak{g}$ *by adjoint action on the second factor. In particular,* $F_e = 0$*, where* e *denotes the unit element of* G*.*

Proof. It is simple to see that

$$\begin{aligned}\big(v_m, \big(F_g(v_m)\big)^R_g\big) \cdot \big(v_m, \big(F_h(v_m)\big)^R_h\big) &= \big(v_m, R_{h*}\big(F_g(v_m)\big)^R_g + L_{g*}\big(F_h(v_m)^R_h\big)\big) \\ &= \big(v_m, \big(F_g(v_m) + \mathrm{Ad}_g F_h(v_m)\big)^R_{gh}\big).\end{aligned}$$

Thus the right-hand side belongs to $H'_{(m,gh)}$ if, and only if, Eq. (39) is satisfied. □

Corollary 4.15. *Under the same hypothesis as in Lemma* 4.14*, if* $G = S^1$*, then* $F_g = 0$*,* $\forall g \in G$*.*

Proof. Since S^1 is abelian, Eq. (39) becomes $F_{gh} = F_g + F_h$. Since there is no non-trivial group homomorphism between S^1 and $\mathbb{R}$, F must vanish. In other words, for S^1-extensions, each subspace $M \times \{e^{i\theta}\}$ must be a horizontal section for the group bundle $M \times S^1 \to M$. □

Proposition 4.16. *Let* $X_\bullet \xrightarrow{\phi} Y_\bullet$ *be a Lie groupoid* G*-extension whose kernel* $\mathcal{K}$ *is endowed with a trivialization* $\chi : M \times G \to \mathcal{K}$*. Assume that* H *is a Lie groupoid extension connection with its associated right and left connection* 1*-forms* α *and* β *respectively.*

1. *If $\xi_x \in H_x$, then*

$$R_{g*}\xi_x + \left(\mathrm{Ad}_{g^{-1}} F_g(\mathbf{t}_*\xi_x)\right)^{\rhd}_{xg} \in H_{xg}$$

and

$$L_{g*}\xi_x + \left(F_g(\mathbf{s}_*\xi_x)\right)^{\lhd}_{gx} \in H_{gx}.$$

2. *The connection 1-forms satisfy*

$$R_g^*\alpha - \mathrm{Ad}_{g^{-1}}\alpha + \mathbf{t}^*(\mathrm{Ad}_{g^{-1}} F_g) = 0$$

and

$$L_g^*\beta - \mathrm{Ad}_g\beta + \mathbf{s}^* F_g = 0.$$

Proof.

1. It is simple to check that $\forall \xi \in H_x$,

$$\xi_x \cdot \left(\mathbf{t}_*\xi_x, R_{g*}F_g(\mathbf{t}_*\xi_x)\right) = R_{g*}\xi_x + \left(\mathrm{Ad}_{g^{-1}} F_g(\mathbf{t}_*\xi_x)\right)^{\rhd}_{xg},$$

where $\cdot$ on the left-hand side stands for the groupoid multiplication on $TX_1 \rightrightarrows TM$. It thus follows that $R_{g*}\xi_x + (\mathrm{Ad}_{g^{-1}} F_g(\mathbf{t}_*\xi_x))^{\rhd}_{xg} \in H_{xg}$.
Similarly, $\forall \xi \in H_x$,

$$\left(\mathbf{s}_*\xi_x, R_{g*}F_g(\mathbf{s}_*\xi_x)\right) \cdot \xi_x = L_{g*}\xi_x + F_g(\mathbf{s}_*\xi_x)^{\lhd}_{gx}.$$

Thus $L_{g*}\xi_x + (F_g(\mathbf{s}_*\xi_x))^{\lhd}_{gx} \in H_{gx}$.
2. Easy consequence of (1). □

As an immediate consequence, we have the following

Corollary 4.17. *The following assertions are equivalent:*

1. $F_g = 0$, $\forall g \in G$*;*
2. $M \times \{g\}$, $\forall g \in G$, *is horizontal;*
3. *H is right G-invariant ($R_{g*}H = H$, $\forall g \in G$), i.e. $\alpha \in \Omega^1(X_1) \otimes \mathfrak{g}$ is a connection one-form for the right G-principal bundle $X_1 \to Y_1$;*
4. *H is left G-invariant ($L_{g*}H = H$, $\forall g \in G$), i.e. $\beta \in \Omega^1(X_1) \otimes \mathfrak{g}$ is a connection one-form for the left G-principal bundle $X_1 \to Y_1$.*

By Corollary 4.15, we are led to the following result, as expected.

Corollary 4.18. *A connection on a Lie groupoid S^1-extension $X_\bullet \to Y_\bullet$ must be a principal left (and right) S^1-bundle connection.*

When the kernel of the groupoid extension $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ is not identified to $M \times G$, Lemma 4.14 and Proposition 4.16 can be generalized by replacing the section $M \times \{g\}$ by any section σ of $\mathcal{K} \to M$.

Define the transformations $R_\sigma : x_1 \to x_1 \cdot \sigma_{\mathbf{t}(x_1)}$ and $L_\sigma : x_1 \to \sigma_{\mathbf{s}(x_1)} \cdot x_1$ of X_1, and let $F_\sigma := -\sigma^*\beta$. By construction, we have $F_\sigma \in \Omega^1(M, \mathfrak{K})$. According to Eq. (38), we recover the previous definition F_g when $\mathcal{K} = M \times G$ and $\sigma(m) = (m, g)$, which justifies the notation.

Lemma 4.19. *The condition*

$$H^{\mathcal{K}}{}_{\sigma_1} \cdot H^{\mathcal{K}}{}_{\sigma_2} = H^{\mathcal{K}}{}_{\sigma_1 \cdot \sigma_2}, \quad \forall \sigma_1, \sigma_2 \in \mathcal{K} \text{ such that } \pi(\sigma_1) = \pi(\sigma_2) \tag{40}$$

is equivalent to

$$F_{\sigma_1\sigma_2} = F_{\sigma_1} + \mathrm{Ad}_{\sigma_1} F_{\sigma_2}, \quad \forall \sigma_1, \sigma_2 \in \Gamma(\mathcal{K}). \tag{41}$$

I.e. $F : \Gamma(\mathcal{K}) \to \Omega^1(M) \otimes \Gamma(\mathfrak{K})$ *is a group* 1*-cocycle, where* $\Gamma(\mathcal{K})$ *acts on* $\Omega^1(M) \otimes \Gamma(\mathfrak{K})$ *by the adjoint action on the second factor. In particular,* $F_e = 0$*, where* e *denotes the unit section of* $\mathcal{K}$*.*

Proof. The proof is similar to that of Lemma 4.14 and is omitted. □

Given $\sigma \in \Gamma(\mathcal{K})$ and $x \in X_1$, we use the shorthand σx (resp. $x\sigma$) for $\sigma_{\mathbf{s}(x)} \cdot x$ (resp. $x \cdot \sigma_{\mathbf{t}(x)}$).

Proposition 4.20. *Let* $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ *be a groupoid* G*-extension endowed with a groupoid extension connection* H *with associated right and left connection* 1*-forms* α *and* β *respectively. Let* $\sigma \in \Gamma(\mathcal{K})$.

1. *If* $v_x \in H_x$*, then*

$$R_{\sigma*} v_x + \left(\mathrm{Ad}_{\sigma(\mathbf{t}(x))^{-1}} F_\sigma(\mathbf{t}_* v_x)\right)^{\rhd}_{x\sigma} \in H_{x\sigma}$$

and

$$L_{\sigma*} v_x + \left(F_\sigma(\mathbf{s}_* v_x)\right)^{\lhd}_{\sigma x} \in H_{\sigma x}.$$

2. *The connection* 1*-forms satisfy*

$$R_\sigma^* \alpha - \mathrm{Ad}_{\sigma^{-1}} \alpha + \mathbf{t}^*(\mathrm{Ad}_{\sigma^{-1}} F_\sigma) = 0 \tag{42}$$

and

$$L_\sigma^* \beta - \mathrm{Ad}_\sigma \beta + \mathbf{s}^* F_\sigma = 0. \tag{43}$$

Proof. The proof is similar to that of Proposition 4.16 and is omitted. □

4.5. Covariant derivative on the bundle of Lie algebras

Using parallel transport on the Lie algebra bundle $\mathfrak{K} \to M$, one can introduce a covariant derivative on its space of sections:

$$(\nabla_{\overset{\bullet}{\gamma}}\xi)_{\gamma(0)} = \frac{d}{dt}\tau^{\gamma}_{-t}(\xi_{\gamma(t)})\bigg|_0, \tag{44}$$

where $\gamma \in M^I$, $\xi \in \Gamma(\mathfrak{K} \to M)$ and τ^{γ}_{-t} denotes the parallel transport from $\mathfrak{K}_{\gamma(t)}$ to $\mathfrak{K}_{\gamma(0)}$ along the path γ. The following proposition is obvious.

Proposition 4.21.

1. $(\nabla_{\overset{\bullet}{\gamma}}\xi)_{\gamma(0)} = \frac{d}{dt}|_0 \frac{d}{ds}|_0 \tau^{\gamma}_{-t}(\exp s\xi_{\gamma(t)})$;
2. $\nabla_X\xi$ *is* $C^\infty(M)$*-linear in* X *and* $\mathbb{R}$*-linear in* ξ;
3. $\nabla_X(f\xi) = X(f)\cdot\xi + f\cdot\nabla_X\xi$; *and*
4. $\nabla_X[\xi,\eta] = [\nabla_X\xi,\eta] + [\xi,\nabla_X\eta]$,

where $\xi,\eta \in \Gamma(\mathfrak{K})$ *and* $X \in \mathfrak{X}(M)$.

Proposition 4.22. $(\nabla_X\xi)_x = \frac{d}{ds}|_0 F_{\exp s\xi}(X_x)$.

Proof. First, one easily checks that

$$\frac{d}{dt}\tau^{\gamma}_{-t}(\exp s\xi_{\gamma(t)})\bigg|_0 = \mathrm{Ver}\left(\frac{d}{dt}\exp s\xi_{\gamma(t)}\bigg|_0\right) = \left(\beta\left(\frac{d}{dt}\exp s\xi_{\gamma(t)}\bigg|_0\right)\right)^{\lhd}_{\exp s\xi_{\gamma(0)}}.$$

Second, one has

$$\begin{aligned}
(\nabla_{\overset{\bullet}{\gamma}}\xi)_{\gamma(0)} &= \frac{d}{dt}\bigg|_0\frac{d}{ds}\bigg|_0\tau^{\gamma}_{-t}(\exp s\xi_{\gamma(t)})\\
&= \frac{d}{dt}\bigg|_0\frac{d}{ds}\bigg|_0\big(\tau^{\gamma}_{-t}(\exp s\xi_{\gamma(t)})\cdot\exp(-s\xi_{\gamma(0)})\big)\cdot\exp(s\xi_{\gamma(0)})\\
&= \frac{d}{dt}\bigg|_0\left(\frac{d}{ds}\tau^{\gamma}_{-t}(\exp s\xi_{\gamma(t)})\cdot\exp(-s\xi_{\gamma(0)})\bigg|_0 + \xi_{\gamma(0)}\right)\\
&= \frac{d}{ds}\bigg|_0\frac{d}{dt}\bigg|_0\tau^{\gamma}_{-t}(\exp s\xi_{\gamma(t)})\cdot\exp(-s\xi_{\gamma(0)})\\
&= \frac{d}{ds}\bigg|_0\left(R_{\exp(-s\xi_{\gamma(0)})*}\left(\beta\left(\frac{d}{dt}\exp s\xi_{\gamma(t)}\bigg|_0\right)\right)^{\lhd}_{\exp s\xi_{\gamma(0)}}\right)\\
&= \frac{d}{ds}\bigg|_0\beta\left(\frac{d}{dt}\exp s\xi_{\gamma(t)}\bigg|_0\right)\\
&= \frac{d}{ds}\bigg|_0 F_{\exp s\xi}\big(\overset{\bullet}{\gamma}(0)\big). \qquad \square
\end{aligned}$$

The connection $H^{\mathfrak{K}}$ defined on the Lie algebra bundle $\mathfrak{K} \to M$ naturally induces a connection on the pullback bundle $\mathbf{t}^*\mathfrak{K} \to X_1$, given by $H^{\mathbf{t}^*\mathfrak{K}} = (\hat{\mathbf{t}}_*)^{-1} H^{\mathfrak{K}}$, where $\hat{\mathbf{t}} : \mathbf{t}^*\mathfrak{K} \to \mathfrak{K}$ is the projection. The associated covariant derivatives are related by

$$\nabla^{\mathbf{t}}_v(\mathbf{t}^*\xi) = \mathbf{t}^*(\nabla_{\mathbf{t}_* v}\xi), \quad \forall v \in TX_1,\ \forall \xi \in \Gamma(\mathfrak{K}),$$

where $\mathbf{t}^*\xi \in \Gamma(\mathbf{t}^*\mathfrak{K} \to M)$ denotes the pullback through $\mathbf{t}$ of a section $\xi \in \Gamma(\mathfrak{K} \to M)$. Similarly, we have the pullback connection $H^{\mathbf{s}^*\mathfrak{K}}$ on $\mathbf{s}^*\mathfrak{K} \to X_1$, whose covariant derivative is denoted by $\nabla^{\mathbf{s}}$.

Proposition 4.23.

1. *For every $\eta \in \Gamma(\mathbf{t}^*\mathfrak{K} \to X_1)$, we define an associated vertical vector field on X_1:*

$$\eta_x^{\blacktriangleright} := \frac{d}{d\tau}\bigg|_0 x \cdot \exp\bigl(\tau\eta(x)\bigr).$$

Then

$$\nabla^{\mathbf{t}}_X \eta = (\mathcal{L}_{\eta^{\blacktriangleright}}\alpha)(X) + \bigl[\eta, \alpha(X)\bigr]. \tag{45}$$

2. *For every $\eta \in \Gamma(\mathbf{s}^*\mathfrak{K} \to X_1)$, we define an associated vertical vector field on X_1:*

$$\eta_x^{\blacktriangleleft} := \frac{d}{d\tau}\bigg|_0 \exp\bigl(\tau\eta(x)\bigr) \cdot x.$$

Then

$$\nabla^{\mathbf{s}}_X \eta = (\mathcal{L}_{\eta^{\blacktriangleleft}}\beta)(X) - \bigl[\eta, \beta(X)\bigr]. \tag{46}$$

Here the notations $\eta_x^{\blacktriangleright}$ and $\eta_x^{\blacktriangleleft}$ generalize those in Eq. (35)*. More precisely, for any $\eta \in \Gamma(\mathfrak{K} \to M)$, we have $(t^*\eta)^{\blacktriangleright} = \eta^{\rhd}$ and $(s^*\eta)^{\blacktriangleleft} = \eta^{\lhd}$.*

Proof. We will prove (1). The argument for (2) is similar.

Let ξ be a section of $\mathfrak{K} \to M$. Setting $\sigma = \exp(u\xi)$ in Eq. (42) and evaluating on a tangent vector X_x, we get

$$\alpha(R_{\exp(\tau\xi)*}X_x) - \mathrm{Ad}^{-1}_{\exp(\tau\xi)}\alpha(X_x) + \mathrm{Ad}^{-1}_{\exp(\tau\xi)} F_{\exp(\tau\xi)}(\mathbf{t}_* X_x) = 0.$$

Differentiating with respect to τ at $u = 0$ and using Proposition 4.22, we obtain

$$-(\mathcal{L}_{\xi^{\rhd}}\alpha)(X_x) + \bigl[\mathbf{t}^*\xi, \alpha(X_x)\bigr] + (\nabla_{\mathbf{t}_* X}\xi)_{\mathbf{t}(x)} = 0,$$

where $\xi_x^{\rhd} := \frac{d}{d\tau}|_0 x \cdot \exp(\tau\xi(\mathbf{t}(x)))$. Hence $\nabla^{\mathbf{t}}_X \mathbf{t}^*\xi = (\mathcal{L}_{\xi^{\lhd}}\alpha)(X) + [\mathbf{t}^*\xi, \alpha(X)]$.

Now, for any function $f \in C^\infty(X_1)$, we have

$$\begin{aligned}\nabla^{\mathbf{t}}_X(f\cdot\mathbf{t}^*\xi) &= X(f)\cdot\mathbf{t}^*\xi + f\nabla^{\mathbf{t}}_X(\mathbf{t}^*\xi)\\ &= X(f)\cdot\mathbf{t}^*\xi + f\big(\mathcal{L}_{\xi^{\triangleright}}\big(\alpha(X)\big)-\alpha\big(\big[\xi^{\triangleright},X\big]\big)+\big[\mathbf{t}^*\xi,\alpha(X)\big]\big)\\ &= \mathcal{L}_{f\xi^{\triangleright}}\big(\alpha(X)\big)-\alpha\big(\big[f\xi^{\triangleright},X\big]\big)+\big[f\cdot\mathbf{t}^*\xi,\alpha(X)\big]\\ &= (\mathcal{L}_{f\xi^{\triangleright}}\alpha)(X)+\big[f\cdot\mathbf{t}^*\xi,\alpha(X)\big].\end{aligned}$$

The result follows from the fact that any section $\eta\in\Gamma(\mathbf{t}^*\mathfrak{K}\to X_1)$ is a linear combination of sections of the type $f\cdot\mathbf{t}^*\xi$ for $\xi\in\Gamma(\mathfrak{K}\to M)$ and $f\in C^\infty(X_1)$. □

Remark 4.24. If X is horizontal, $\alpha(X)=0$ and $\beta(X)=0$ and therefore we have

$$\nabla^{\mathbf{t}}_X\eta=\alpha\big(\big[X,\eta^{\blacktriangleright}\big]\big),\quad \forall\eta\in\Gamma(t^*\mathfrak{K}\to X_1) \tag{47}$$

and

$$\nabla^{\mathbf{s}}_X\eta=\beta\big(\big[X,\eta^{\blacktriangleleft}\big]\big),\quad \forall\eta\in\Gamma(s^*\mathfrak{K}\to X_1). \tag{48}$$

4.6. Ehresmann curvature

In this subsection, we study the curvature of a connection H on a Lie groupoid extension. We denote the horizontal and vertical parts of a vector $v\in TX_1$ by $\mathrm{Hor}(v)$ and $\mathrm{Ver}(v)$ respectively. Recall that the Ehresmann curvature of a horizontal distribution H, on the bundle $X_1\xrightarrow{\phi}Y_1$, is the 2-form on X_1, valued in the vertical space $\ker\phi_*$, which is defined by

$$\omega(u,v)=-\mathrm{Ver}\big(\big[\mathrm{Hor}(\tilde u),\mathrm{Hor}(\tilde v)\big]\big), \tag{49}$$

where $u,v\in T_xX_1$ and $\tilde u,\tilde v$ are vector fields on X_1 such that $\tilde u_x=u$ and $\tilde v_x=v$. It is easy to check that the right-hand side of Eq. (49) is well defined, i.e. independent of the choice of the vector fields $\tilde u,\tilde v$.

Using the right (resp. left) action of $\mathcal{K}$ on X_1, one can identify the vertical space $\ker\phi_*\subset T_xX_1$ of the groupoid extension $X_1\xrightarrow{\phi}Y_1$ with $\mathbf{t}^*\mathfrak{K}$ (resp. $\mathbf{s}^*\mathfrak{K}$) via the right (resp. left) action of $\mathcal{K}$ on X_1. Therefore, the curvature of the connection H can be seen as a 2-form either in $\Omega^2(X_1,\ker\phi_*)$, $\Omega^2(X_1,\mathbf{t}^*\mathfrak{K})$ or $\Omega^2(X_1,\mathbf{s}^*\mathfrak{K})$. Note that for any Lie algebra bundle $\mathfrak{K}\to M$ over a smooth manifold M, there is a graded Lie bracket $\Omega^k(M,\mathfrak{K})\otimes\Omega^l(M,\mathfrak{K})\xrightarrow{[\cdot,\cdot]}\Omega^{k+l}(M,\mathfrak{K})$ given by $[\omega_1\otimes a_1,\omega_2\otimes a_2]=(\omega_1\wedge\omega_2)\otimes[a_1,a_2]$, $\forall a_1,a_2\in\Gamma(\mathfrak{K})$, $\omega_1\in\Omega^k(M)$, $\omega_2\in\Omega^l(M)$. In particular, for any $\alpha\in\Omega^1(M,\mathfrak{K})$, and any vector fields $X,Y\in\mathfrak{X}(M)$, we have $\frac{1}{2}[\alpha,\alpha](X,Y)=[\alpha(X),\alpha(Y)]$.

The following lemma indicates that, similar to the case of principal bundles, the Ehresmann curvature can be computed using the standard formula.

Lemma 4.25.

1. *If $\alpha\in\Omega^1(X_1,\mathbf{t}^*\mathfrak{K})$ is a left connection 1-form of a groupoid extension connection, then its curvature 2-form $\omega\in\Omega^2(X_1,\mathbf{t}^*\mathfrak{K})$ is given by*

$$\omega=d^{\nabla^{\mathbf{t}}}\alpha+\frac{1}{2}[\alpha,\alpha],$$

where $d^{\nabla^{\mathbf{t}}} : \Omega^\bullet(X_1, \mathbf{t}^*\mathfrak{K}) \to \Omega^{\bullet+1}(X_1, \mathbf{t}^*\mathfrak{K})$ *denotes the exterior covariant derivative induced by* $\nabla^{\mathbf{t}}$.

2. *Similarly, if* $\beta \in \Omega^1(X_1, \mathbf{s}^*\mathfrak{K})$ *is a left connection* 1*-form of a groupoid extension connection, then its curvature* 2*-form* $\omega \in \Omega^2(X_1, \mathbf{s}^*\mathfrak{K})$ *is given by*

$$\omega = d^{\nabla^{\mathbf{s}}}\beta - \frac{1}{2}[\beta, \beta],$$

where $d^{\nabla^{\mathbf{s}}} : \Omega^\bullet(X_1, \mathbf{s}^*\mathfrak{K}) \to \Omega^{\bullet+1}(X_1, \mathbf{s}^*\mathfrak{K})$ *denotes the exterior covariant derivative induced by* $\nabla^{\mathbf{s}}$.

Proof. The proof is straightforward and is omitted. □

Now let us recall a general fact regarding horizontal distributions. I.e. the Ehresmann curvature is the holonomy of infinitesimal loops (see [55, p. 118]). For any point m in a manifold N and any tangent vectors $u, v \in T_m N$, we say that a smooth map C from a neighborhood of 0 in $\mathbb{R}^2$ to a neighborhood of $m \in N$ is *adapted to* (u, v) if

$$C_*\left(\frac{\partial}{\partial x}\bigg|_0\right) = u \quad \text{and} \quad C_*\left(\frac{\partial}{\partial y}\bigg|_0\right) = v,$$

where x, y are the standard coordinates on $\mathbb{R}^2$.

Consider now, for any small enough $\varepsilon_1, \varepsilon_2$, the loop $R_{\varepsilon_1,\varepsilon_2} : [0, 1] \to \mathbb{R}^2$ obtained by turning anti-clockwise along the rectangle with edges $(0, 0)$, $(\varepsilon_1, 0)$, $(\varepsilon_1, \varepsilon_2)$, $(0, \varepsilon_2)$. Define, for any (small enough) $\varepsilon_1, \varepsilon_2$, a family of loops $L^C_{\varepsilon_1,\varepsilon_2}$ on N by

$$L^C_{\varepsilon_1,\varepsilon_2} = C \circ R_{\varepsilon_1,\varepsilon_2}. \tag{50}$$

The curvature can be computed from the holonomy according to the following formula:

Proposition 4.26. *For any C adapted to (u, v), we have*

$$\frac{\partial^2}{\partial\varepsilon_1\partial\varepsilon_2}\bigg|_{\varepsilon_1=\varepsilon_2=0} \mathrm{Hol}\big(L^C_{\varepsilon_1,\varepsilon_2}\big) = \omega(u, v). \tag{51}$$

From now on, we shall work only with the right-connection 1-form α. Below we list two important identities that ω satisfies, which we call **Bianchi identities**. They will turn out to be of fundamental importance in the sequel.

Theorem 4.27. *Let* $\omega \in \Omega^2(X_1, \mathbf{t}^*\mathfrak{K})$ *be the Ehresmann curvature for a Lie groupoid extension connection* $\alpha \in \Omega^1(X_1, \mathbf{t}^*\mathfrak{K})$*. Then we have the Bianchi identities:*

$$d^{\nabla^t}\omega + [\alpha, \omega] = \mathbf{t}^*\omega^{\mathfrak{K}}(\alpha), \tag{52}$$

$$\partial^{\triangleright}\omega = 0, \tag{53}$$

where $\omega^{\mathfrak{K}} \in \Omega^2(M, \mathrm{End}(\mathfrak{K}))$ *is the curvature of the induced connection* ∇ *on the Lie algebra bundle* $\mathfrak{K} \to M$*, and* $\mathbf{t}^*\omega^{\mathfrak{K}}(\alpha) \in \Omega^3(X_1, \mathbf{t}^*\mathfrak{K})$ *is the* $\mathbf{t}^*\mathfrak{K}$*-valued* 3*-form on* X_1 *obtained* *by composing* $\mathbf{t}^*\omega^{\mathfrak{K}} \in \Omega^2(X_1, \mathrm{End}(\mathbf{t}^*\mathfrak{K}))$ *with* $\alpha \in \Omega^1(X_1, \mathbf{t}^*\mathfrak{K})$ *under the natural pairing* $\Omega^2(X_1, \mathrm{End}(\mathbf{t}^*\mathfrak{K})) \otimes \Omega^1(X_1, \mathbf{t}^*\mathfrak{K}) \to \Omega^3(X_1, \mathbf{t}^*\mathfrak{K})$.

Proof. Let us prove Eq. (52) first. By Lemma 4.25(1), we have

$$\begin{aligned}(d^{\nabla^{\mathbf{t}}} + \mathrm{ad}_\alpha)\omega &= (d^{\nabla^{\mathbf{t}}} + \mathrm{ad}_\alpha) \circ \left(d^{\nabla^{\mathbf{t}}} + \frac{1}{2}\,\mathrm{ad}_\alpha\right)(\alpha),\\ &= (d^{\nabla^{\mathbf{t}}})^2\alpha + \frac{1}{2} d^{\nabla^{\mathbf{t}}}[\alpha, \alpha] + [\alpha, d^{\nabla^{\mathbf{t}}}\alpha] + \frac{1}{2}[\alpha, [\alpha, \alpha]]. \end{aligned} \tag{54}$$

The graded Jacobi identity implies that $[\alpha, [\alpha, \alpha]] = 0$. Since $\nabla^{\mathbf{t}}$ is a connection on the Lie algebra bundle $\mathbf{t}^*\mathfrak{K} \to X_1$ we have $\frac{1}{2} d^{\nabla^{\mathbf{t}}}[\alpha, \alpha] = [d^{\nabla^{\mathbf{t}}}\alpha, \alpha] = -[\alpha, d^{\nabla^{\mathbf{t}}}\alpha]$. Eq. (54) then becomes $(d^{\nabla^{\mathbf{t}}} - \mathrm{ad}_\alpha)\omega = (d^{\nabla^{\mathbf{t}}})^2\alpha$. On the other hand, the curvature of $\nabla^{\mathbf{t}}$ is $\mathbf{t}^*\omega^{\mathfrak{K}}$. Hence we have $(d^{\nabla^{\mathbf{t}}})^2\alpha = \mathbf{t}^*\omega^{\mathfrak{K}}(\alpha)$. This concludes the proof of Eq. (52).

Let us prove Eq. (53). Denote by p_1, m, p_2 the three face maps from X_2 to X_1 (previously denoted by $\varepsilon_0^1, \varepsilon_1^1, \varepsilon_2^1$): $p_1(x_1, x_2) = x_1$, $m(x_1, x_2) = x_1 x_2$, $p_2(x_1, x_2) = x_2$. We denote by (u, v), with $u \in T_{x_1}X_1$, $v \in T_{x_2}X_1$ and $\mathbf{t}_* u = \mathbf{s}_* v$, an element in $T_{(x_1,x_2)}X_2$. Note that $m_*(u, v) = u \cdot v$, where the dot on the right-hand side stands for the multiplication in the tangent groupoid $TX_1 \rightrightarrows TM$.

For any composable pair $(x_1, x_2) \in X_2$, and any $(u_1, u_2), (v_1, v_2) \in T_{(x_1,x_2)}X_2$, let us choose a smooth map c from a neighborhood $\mathcal{U}$ of 0 in $\mathbb{R}^2$ to a neighborhood of $(x_1, x_2) \in X_2$ adapted to $((u_1, u_2), (v_1, v_2))$. Then $p_1 \circ c$, $m \circ c$ and $p_2 \circ c$ are smooth maps from $\mathcal{U}$ to X_1 adapted to (u_1, v_1) in $T_{x_1}X_1$, $(u_1 u_2, v_1 v_2)$ in $T_{x_1 \cdot x_2}X_1$ and (u_2, v_2) in $T_{x_2}X_1$, respectively.

For any small enough $\varepsilon_1, \varepsilon_2$, the loops $L^{p_1\circ c}_{\varepsilon_1,\varepsilon_2}$ and $L^{p_2\circ c}_{\varepsilon_1,\varepsilon_2}$, as defined in Eq. (50), are compatible and their product is precisely $L^{m\circ c}_{\varepsilon_1,\varepsilon_2}$. According to Proposition 4.8, we have

$$\mathrm{Hol}(L^{m\circ c}_{\varepsilon_1,\varepsilon_2}) = \mathbf{Ad}_{x_2^{-1}} \mathrm{Hol}(L^{p_2\circ c}_{\varepsilon_1,\varepsilon_2}) \cdot \mathrm{Hol}(L^{p_1\circ c}_{\varepsilon_1,\varepsilon_2}).$$

Applying $\frac{\partial^2}{\partial\varepsilon_1\partial\varepsilon_2}|_{\varepsilon_1=\varepsilon_2=0}$ to this equation and using Eq. (51), one obtains

$$\omega(u_1 \cdot u_2, v_1 \cdot v_2) = \omega(u_2, v_2) + \mathrm{Ad}_{x_2^{-1}}\, \omega(u_1, v_1).$$

The result now follows. □

Remark 4.28. The relation between Theorem 4.27 and Eqs. (6.1.12)–(6.1.15) in [14] is investigated in [13].

5. Induced connections on the band

5.1. Induced horizontal distributions on the band

Let $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ be a Lie groupoid G-extension endowed with a connection $H \subset TX_1$. The purpose of this subsection is to construct an induced connection on the band out of the connection H. First of all, let us recall some basic notions.

Definition 5.1. (See [35].) Let $\Gamma_1 \rightrightarrows \Gamma_0$ be a Lie groupoid with Lie algebroid A. A connection on a principal G-bundle $P \xrightarrow{J} \Gamma_0$ over $\Gamma_1 \rightrightarrows \Gamma_0$ is a G-invariant horizontal distribution $H \subset TP$ satisfying the following two conditions:

1. for each $p \in P$, we have the inclusion $\hat{A}_p \subset H_p$, where $\hat{A}_p$ denotes the subspace of T_pP generated by the infinitesimal action of the Lie algebroid $A \to \Gamma_0$;
2. the distribution $\{H_p\}_{p\in P}$ on P is preserved under the action of $U_{\mathrm{loc}}(\Gamma_\bullet)$, the pseudo-group of local bisections [38] of $\Gamma_1 \rightrightarrows \Gamma_0$, which naturally acts on P locally.

First we consider the right $\mathrm{Aut}(G)$-principal bundle $\mathrm{Iso}(\mathcal{K}, G) \to M$. For any $g \in G$, let

$$\mathrm{ev}_g : \mathrm{Iso}(\mathcal{K}, G) \to \mathcal{K} : f \mapsto f(g)$$

be the evaluation map. Differentiating it with respect to f yields, for all fixed $g \in G$, a map

$$T_f \,\mathrm{Iso}(\mathcal{K}, G) \xrightarrow{\mathrm{ev}_{g*}} T_{f(g)}\mathcal{K}.$$

We define a horizontal distribution H^{iso} on $\mathrm{Iso}(\mathcal{K}, G) \to M$ by

$$H^{\mathrm{iso}}_f = \bigl\{ v \in T_f \,\mathrm{Iso}(\mathcal{K}, G) \bigm| \mathrm{ev}_{g*}\, v \in {H^{\mathcal{K}}}_{f(g)},\ \forall g \in G \bigr\} \subset T_f \,\mathrm{Iso}(\mathcal{K}, G), \tag{55}$$

where, as defined in Eq. (30), $H^{\mathcal{K}}$ denotes the induced connection on the kernel $\mathcal{K} \to M$. It is obvious that H^{iso} is invariant under the $\mathrm{Aut}(G)$-action on $\mathrm{Iso}(\mathcal{K}, G)$.

Set

$$H^{\mathrm{out}} = \rho_* H^{\mathrm{iso}},$$

where $\rho : \mathrm{Iso}(\mathcal{K}, G) \to \mathrm{Out}(\mathcal{K}, G)$ denotes the projection. It is clear that H^{out} is a horizontal distribution on the bundle $\mathrm{Out}(\mathcal{K}, G) \to M$.

The following lemma is immediate.

Lemma 5.2.

1. *The horizontal paths in* $\mathrm{Iso}(\mathcal{K}, G)$ *are the paths* f_t *such that, for any* $g \in G$, $\mathrm{ev}_g\, f_t = f_t(g)$ *is a horizontal path in* $\mathcal{K}$.
2. *The horizontal paths in* $\mathrm{Out}(\mathcal{K}, G)$ *are the images of horizontal paths in* $\mathrm{Iso}(\mathcal{K}, G)$ *under the projection* $\rho : \mathrm{Iso}(\mathcal{K}, G) \to \mathrm{Out}(\mathcal{K}, G)$.

Proposition 5.3. *The horizontal distribution* H^{out} *defines a connection on the band.*

Proof. First, as H^{iso} is invariant under the right $\mathrm{Aut}(G)$-action, H^{out} is invariant under the right $\mathrm{Out}(G)$-action.

Second, given $m \in M$, let f be any element in $\mathrm{Iso}(\mathcal{K}, G)|_m$ and $\tau \mapsto \gamma(\tau)$ a path in Y_1 lying in the target fiber $t^{-1}(m)$ over m. Consider the horizontal lift $\tau \mapsto \bar{\gamma}(\tau)$ of γ in X_1. Fix $f \in \mathrm{Iso}(\mathcal{K}, G)|_m$. For all $g \in G$, the paths $\tau \mapsto \bar{\gamma}(\tau) f(g) (\bar{\gamma}(\tau))^{-1}$ in $\mathcal{K}$ are horizontal paths. Therefore, the path $\tau \mapsto \rho(\mathbf{Ad}_{\bar{\gamma}(\tau)} \circ f)$ in $\mathrm{Out}(\mathcal{K}, G)$ is horizontal by Lemma 5.2. Since any element of $\hat{A}_f$ is tangent to such a path at its origin, we have $\hat{A}_f \subset H^{\mathrm{out}}_f$.

For any $f \in \mathrm{Iso}(\mathcal{K}, G)|_m$, let m_τ be any path in M starting at the point m. For any $g \in G$, by $\bar{m}_\tau^{f(g)}$, we denote the horizontal lift of m_τ in $\mathcal{K}$ starting at the point $f(g) \in \mathcal{K}_m$. Let f_τ be a path in $\mathrm{Iso}(\mathcal{K}, G)$ defined by $f_\tau(g) = \bar{m}_\tau^{f(g)}$, $\forall g \in G$. By definition, f_τ is the horizontal lift of m_τ in $\mathrm{Iso}(\mathcal{K}, G)$ through the point f. Hence, $\rho(f_\tau)$ is the horizontal lift of m_τ in $\mathrm{Out}(\mathcal{K}, G)$ starting at the point $\rho(f)$.

To show that H^{out} is preserved under the action of $U_{\mathrm{loc}}(Y_\bullet)$, it suffices to show that, for any $L \in U_{\mathrm{loc}}(Y_\bullet)$, $L \cdot \rho(f_\tau)$ is still a horizontal path in $\mathrm{Out}(\mathcal{K}, G)$. For this purpose, let σ_τ be the unique path in L such that $\mathbf{t} \circ \sigma_\tau = m_\tau$, and let $\bar{\sigma}_\tau$ be any of its horizontal lifts on X_1. By definition,

$$L \cdot \rho(f_\tau) = \sigma_\tau \cdot \rho(f_\tau) = \rho(\mathbf{Ad}_{\bar{\sigma}_\tau} \circ f_\tau),$$

which is clearly still a horizontal path since the paths

$$(\mathbf{Ad}_{\bar{\sigma}_\tau} \circ f_\tau)(g) = \bar{\sigma}_\tau \cdot \bar{m}_\tau^{f(g)} \cdot (\bar{\sigma}_\tau)^{-1}$$

are horizontal in $\mathcal{K}$, for all $g \in G$. This concludes the proof. □

5.2. *Connection* 1*-forms on the band*

A connection on a principal bundle over a Lie groupoid can be equivalently described by a 1-form, called the **connection 1-form**.

Definition 5.4. (See [35].) Let $P \xrightarrow{J} \Gamma_0$ be a principal G-bundle over a Lie groupoid $\Gamma_1 \rightrightarrows \Gamma_0$. A connection 1-form is a usual connection 1-form $\theta \in \Omega^1(P) \otimes \mathfrak{g}$ of the principal G-bundle $P \to \Gamma_0$ (ignoring the groupoid action), satisfying the additional equation $\mathbf{t}^*\theta - \mathbf{s}^*\theta = 0$. Here $\mathbf{s}$ and $\mathbf{t}$ are the source and target maps of the transformation groupoid $\Gamma_1 \times_{\mathbf{t}, \Gamma_0, J} P \rightrightarrows P$ associated to the $\Gamma_\bullet$-action on P.

Just like the usual principal G-bundles, we have the following (see Proposition 3.6 in [35]):

Proposition 5.5. *For a principal G-bundle over a Lie groupoid, a connection is equivalent to a connection* 1*-form.*

The purpose of this subsection is to construct the connection 1-form for the induced connection on the band, and to prove directly that it satisfies the conditions in Definition 5.4. Hence, this subsection can be considered as an alternative approach to obtain, from a connection on a groupoid G-extension, an induced connection on its band.

Let $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ be a Lie groupoid G-extension endowed with a groupoid G-extension connection. Let $\alpha \in \Omega^1(X_1, \mathbf{t}^*\mathfrak{K})$ be its corresponding right connection 1-form and $\alpha^{\mathcal{K}} \in \Omega^1(\mathcal{K}, \mathbf{t}^*\mathfrak{K})$ the induced connection 1-form on the group bundle $\mathcal{K} \xrightarrow{\pi} M$ obtained by restricting α to $\mathcal{K}$.

Denote by ∂ the Lie group cohomology differential of G with values in its Lie algebra $\mathfrak{g}$, where G acts on $\mathfrak{g}$ by the adjoint action. In particular, for any $\xi \in \mathfrak{g}$, $\partial\xi$ is the $\mathfrak{g}$-valued function on G given by $(\partial\xi)(g) = \xi - \mathrm{Ad}_{g^{-1}}\xi$, $\forall g \in G$.

The following lemma is standard.

Lemma 5.6. *Let G be a Lie group with Lie algebra $\mathfrak{g}$.*

1. *The Lie algebra* $\operatorname{Lie}\operatorname{Aut}(G)$ *of* $\operatorname{Aut}(G)$ *is naturally identified with the space of* 1*-cocycles* $Z^1(G,\mathfrak{g})$ *with the bracket:*

$$[z_1,z_2](g)=z_{2*}\big(z_1(g)\big)-z_{1*}\big(z_2(g)\big)-\big[z_1(g),z_2(g)\big],\quad \forall z_1,z_2\in Z^1(G,\mathfrak{g}),\ g\in G,$$

where for any $z\in Z^1(G,\mathfrak{g})$, $z_*:\mathfrak{g}\to\mathfrak{g}$ is the differential of z at the identity.
The isomorphism from $\operatorname{Lie}\operatorname{Aut}(G)$ *to* $Z^1(G,\mathfrak{g})$ *is given as follows. Let f_t be any C^1-path in* $\operatorname{Aut}(G)$ *with $f_0=\mathrm{id}$. Then the element $z(g)$ in $Z^1(G,\mathfrak{g})$ corresponding to* $\frac{df_t}{dt}|_{t=0}\in \operatorname{Lie}\operatorname{Aut}(G)$ *is*

$$z(g)=(L_{g^{-1}})_*\frac{df_t(g)}{dt}\bigg|_{t=0}. \tag{56}$$

2. *The Lie algebra* $\operatorname{Lie}\operatorname{Inn}(G)$ *of* $\operatorname{Inn}(G)$ *is naturally identified with the space of* 1*-coboundaries* $B^1(G,\mathfrak{g})$.
3. *Let* $\operatorname{Ad}:\mathfrak{g}\to B^1(G,\mathfrak{g})$ *be the Lie algebra morphism corresponding to the Lie group morphism $G\to\operatorname{Inn}(G)$ given by $x\to\mathbf{Ad}_x$. Then* $\operatorname{Ad}\xi=\partial\xi$, $\forall\xi\in\mathfrak{g}$.
4. *The Lie algebra* $\operatorname{Lie}\operatorname{Out}(G)$ *of* $\operatorname{Out}(G)$ *is naturally identified with the first cohomology group* $H^1(G,\mathfrak{g})$.

Define a $C^\infty(G,\mathfrak{g})$-valued 1-form α^{iso} on $\operatorname{Iso}(\mathcal{K},G)$ by

$$\big(v\lrcorner\alpha^{\mathrm{iso}}\big)(g)=f_*^{-1}\big(\mathrm{ev}_{g*}(v)\lrcorner\alpha^{\mathcal{K}}\big),\quad f\in\operatorname{Iso}(\mathcal{K},G),\ \text{and } v\in T_f\operatorname{Iso}(\mathcal{K},G),$$

where $f_*:\mathfrak{g}\to\mathfrak{K}_m$ is the Lie algebra isomorphism corresponding to the Lie group isomorphism $f:G\to\mathcal{K}_m$, and $\alpha^{\mathcal{K}}\in\Omega^1(\mathcal{K},\mathfrak{K})$ is the pullback of the right connection 1-form α on $\mathcal{K}$.

Lemma 5.7. *The* 1*-form α^{iso} is $Z^1(G,\mathfrak{g})$-valued. I.e.* $\alpha^{\mathrm{iso}}\in\Omega^1(\operatorname{Iso}(\mathcal{K},G))\otimes Z^1(G,\mathfrak{g})$.

Proof. Denote by $m:\mathcal{K}\times_M\mathcal{K}\to\mathcal{K}$ the multiplication in the Lie group bundle $\mathcal{K}$ and by p_1 and p_2 the projections of $\mathcal{K}\times_M\mathcal{K}$ onto its first and second factors respectively. The relation $\partial^{\triangleright}\alpha=0$ implies that $\partial^{\triangleright}_{\mathcal{K}}\alpha^{\mathcal{K}}=0$, where $\partial^{\triangleright}_{\mathcal{K}}$ stands for the restriction of $\partial^{\triangleright}$ to the simplicial manifold associated to the groupoid $\mathcal{K}\rightrightarrows M$. Therefore, for any $(k_1,k_2)\in\mathcal{K}\times_M\mathcal{K}$,

$$m^*\alpha^{\mathcal{K}}=p_2^*\alpha^{\mathcal{K}}+\operatorname{Ad}_{k_2^{-1}}p_1^*\alpha^{\mathcal{K}}. \tag{57}$$

For any $g_1,g_2\in G$ and $f\in\operatorname{Iso}(\mathcal{K},G)$, differentiating with respect to f the relation $\mathrm{ev}_{g_1g_2}f=m(\mathrm{ev}_{g_1}f,\mathrm{ev}_{g_2}f)$, one obtains that for any $v\in T_f\operatorname{Iso}(\mathcal{K},G)$,

$$\mathrm{ev}_{g_1g_2*}(v)=m_*(\mathrm{ev}_{g_1*}v,\mathrm{ev}_{g_2*}v).$$

Eq. (57) implies that

$$\mathrm{ev}_{g_1g_2*}(v)\lrcorner\alpha^{\mathcal{K}}=\mathrm{ev}_{g_2*}(v)\lrcorner\alpha^{\mathcal{K}}+\operatorname{Ad}_{(f(g_2))^{-1}}\big(\mathrm{ev}_{g_1*}(v)\lrcorner\alpha^{\mathcal{K}}\big).$$

Applying f_*^{-1} yields that

$$\big(v \lrcorner \alpha^{\mathrm{iso}}\big)(g_1 g_2) = \big(v \lrcorner \alpha^{\mathrm{iso}}\big)(g_2) + \mathrm{Ad}_{g_2^{-1}}\big(v \lrcorner \alpha^{\mathrm{iso}}\big)(g_1).$$

Therefore α^{iso} takes its values in the Lie algebra $Z^1(G, \mathfrak{g})$. □

Proposition 5.8. *The* 1*-form* $\alpha^{\mathrm{iso}} \in \Omega^1(\mathrm{Iso}(\mathcal{K}, G)) \otimes Z^1(G, \mathfrak{g})$ *defines a connection on the* $\mathrm{Aut}(G)$*-principal bundle* $\mathrm{Iso}(\mathcal{K}, G) \to M$.

Proof. For any $\eta \in Z^1(G, \mathfrak{g})$, denote by $\eta_f^{\triangleright}$ the tangent vector in $T_f \mathrm{Iso}(\mathcal{K}, G)$ induced by the infinitesimal action of the Lie algebra $Z^1(G, \mathfrak{g})$.

Differentiating the relation $\mathrm{ev}_g(f \cdot \psi) = (f \circ \psi)(g)$ with respect to $\psi \in \mathrm{Aut}(G)$ at the identity, one obtains the relation $\mathrm{ev}_{g*}(\eta_f^{\triangleright}) = (f_*\eta(g))^{\triangleright}$ for any $\eta \in Z^1(G, \mathfrak{g})$ and $g \in G$. Hence $\alpha^{\mathcal{K}}(\mathrm{ev}_{g*}\,\eta^{\triangleright}) = f_*\eta(g)$. Applying f_*^{-1}, we obtain, for any $\eta \in Z^1(G, \mathfrak{g})$,

$$\eta^{\triangleright} \lrcorner \alpha^{\mathrm{iso}} = \eta. \tag{58}$$

Let us now check that α^{iso} is $\mathrm{Aut}(G)$-equivariant. Fix $\psi \in \mathrm{Aut}(G)$. Differentiating the relation $\mathrm{ev}_g(f \cdot \psi) = (f \circ \psi)(g)$ with respect to f, one obtains the relation $\mathrm{ev}_{g*}(R_\psi)_* v = \mathrm{ev}_{\psi(g)*}\, v$ for all $v \in T_f \mathrm{Iso}(\mathcal{K}, G)$. Applying $(f \circ \psi)_*^{-1} \circ \alpha^{\mathcal{K}}$ yields

$$\big(\psi_* v \lrcorner \alpha^{\mathrm{iso}}\big)(g) = \psi_*^{-1}\big(v \lrcorner \alpha^{\mathrm{iso}}\big(\psi(g)\big)\big) = \big(\mathrm{Ad}_{\psi^{-1}}\big(v \lrcorner \alpha^{\mathrm{iso}}\big)\big)(g). \tag{59}$$

Eqs. (58)–(59) imply that α^{iso} is a connection 1-form. □

Proposition 5.9. *The following are equivalent characterizations of a connection on the* $\mathrm{Aut}(G)$*-principal bundle* $\mathrm{Iso}(\mathcal{K}, G) \to M$*:*

1. *a connection* 1*-form* $\alpha^{\mathrm{iso}} \in \Omega^1(\mathrm{Iso}(\mathcal{K}, G)) \otimes Z^1(G, \mathfrak{g})$ *as in Proposition* 5.8*;*
2. *a distribution* $H^{\mathrm{iso}} \subset T\,\mathrm{Iso}(\mathcal{K}, G)$ *as in Eq.* (55)*;*
3. *horizontal paths in* $\mathrm{Iso}(\mathcal{K}, G)$ *are those paths* f_t *such that, for any* $g \in G$, $\mathrm{ev}_g\, f_t = f_t(g)$ *is a horizontal path in* $\mathcal{K}$.

Proof. $\boxed{1 \Leftrightarrow 2}$ It is straightforward to check that $\ker(\alpha^{\mathrm{iso}}) = H^{\mathrm{iso}}$. Hence H^{iso} defines the same connection as α^{iso}. In particular, it is an $\mathrm{Aut}(G)$-invariant horizontal distribution.

$\boxed{2 \Leftrightarrow 3}$ A path f_t is horizontal if, and only if, its tangent vectors are in H^{iso}. In other words, the tangent vectors of the paths $f_t(g)$ are in $H^{\mathcal{K}}$ for all $g \in G$. Therefore the horizontal paths of the connection defined by H^{iso} or α^{iso} are the paths in $\mathrm{Iso}(\mathcal{K}, G)$ such that, for any $g \in G$, $\mathrm{ev}_g(f_t) = f_t(g)$ is a horizontal path in $\mathcal{K}$. This completes the proof. □

Assume that we are given a short exact sequence of Lie groups $1 \to R \to G \to H \to 1$. Then, for any principal G-bundle $P \to M$, $P/R \to M$ is a principal H-bundle. Given a connection 1-form $\theta \in \Omega^1(P) \otimes \mathfrak{g}$, then $\bar{\theta} := \mathrm{pr}(\theta)$ is an $\mathfrak{h}$-valued 1-form on P which is R-basic. Here $\mathrm{pr} : \mathfrak{g} \to \mathfrak{h}$ is the natural projection. Therefore, it descends to an $\mathfrak{h}$-valued 1-form on P/R, which is a connection 1-form on the principal H-bundle $P/R \to M$.

Applying this construction to the particular case of the exact sequence $1 \to \mathrm{Inn}(G)/Z(G) \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1$ and the connection 1-form α^{iso} on $\mathrm{Iso}(\mathcal{K}, G)$, we obtain a connection 1-form $\alpha^{\mathrm{out}} \in \Omega^1(\mathrm{Out}(\mathcal{K}, G)) \otimes H^1(G, \mathfrak{g})$.

Recall that $X_1 \rightrightarrows M$ acts on $\mathrm{Iso}(\mathcal{K}, G) \xrightarrow{\pi} M$ by conjugation, so $\mathrm{Iso}(\mathcal{K}, G) \xrightarrow{\pi} M$ is indeed an $\mathrm{Aut}(G)$-torsor over $X_1 \rightrightarrows M$. One can consider the transformation groupoid:

$$X_1 \times_M \mathrm{Iso}(\mathcal{K}, G) \rightrightarrows \mathrm{Iso}(\mathcal{K}, G).$$

Similarly, $Y_1 \rightrightarrows M$ acts on $\mathrm{Out}(\mathcal{K}, G) \xrightarrow{\pi} M$, so $\mathrm{Out}(\mathcal{K}, G) \to M$ is an $\mathrm{Out}(G)$-torsor over $Y_1 \rightrightarrows M$. And one can consider the transformation groupoid

$$Y_1 \times_M \mathrm{Out}(\mathcal{K}, G) \rightrightarrows \mathrm{Out}(\mathcal{K}, G).$$

Proposition 5.10. *The connection* 1*-form* $\alpha^{\mathrm{out}} \in \Omega^1(\mathrm{Out}(\mathcal{K}, G)) \otimes H^1(G, \mathfrak{g})$ *is compatible with the* $Y_\bullet$*-action. More precisely, the following relation holds:*

$$\mathbf{s}^*\alpha^{\mathrm{out}} - \mathbf{t}^*\alpha^{\mathrm{out}} = 0.$$

Here $\mathbf{s}$ *and* $\mathbf{t}$ *denote the source and target maps of the transformation groupoid* $Y_1 \times_M \mathrm{Out}(\mathcal{K}, G) \rightrightarrows \mathrm{Out}(\mathcal{K}, G)$.

The above proposition is an immediate consequence of the following lemma.

Lemma 5.11. *For all* $(x, f) \in X_1 \times_M \mathrm{Iso}(\mathcal{K}, G)$*, we have*

$$\mathbf{s}^*\alpha^{\mathrm{iso}} - \mathbf{t}^*\alpha^{\mathrm{iso}} = \partial\big(p^*\big(f_*^{-1} \circ \alpha_x\big)\big), \tag{60}$$

where both sides are 1*-forms on* $X_1 \times_M \mathrm{Iso}(\mathcal{K}, G)$ *with values in* $Z^1(G, \mathfrak{g})$*. Here,* $\mathbf{s}$ *and* $\mathbf{t}$ *are the source and target maps of the transformation groupoid* $X_1 \times_M \mathrm{Iso}(\mathcal{K}, G) \rightrightarrows \mathrm{Iso}(\mathcal{K}, G)$ *and* $p : X_1 \times_M \mathrm{Iso}(\mathcal{K}, G) \to X_1$ *is the projection on the first component. Note that* $\mathbf{s}^*\alpha^{\mathrm{iso}}$ *and* $\mathbf{t}^*\alpha^{\mathrm{iso}}$ *are* 1*-forms on* $X_1 \times_M \mathrm{Iso}(\mathcal{K}, G)$ *with values in* $Z^1(G, \mathfrak{g})$*. On the right-hand side, the pull-back via* p *of the composition of the* $\mathfrak{K}_{\mathbf{t}(x)}$*-valued covector* α_x *on* X_1 *and the isomorphism* $f_*^{-1} : \mathfrak{K}_{\mathbf{t}(x)} \to \mathfrak{g}$ *is a covector on* $X_1 \times_M \mathrm{Iso}(\mathcal{K}, G)$ *at* (x, f) *with values in* $\mathfrak{g}$*. Therefore,* $\partial(p^*(f_*^{-1} \circ \alpha_x))$ *is a covector of* $X_1 \times_M \mathrm{Iso}(\mathcal{K}, G)$ *at* (x, f) *with values in* $B^1(G, \mathfrak{g})$*.*

Proof. The tangent space $T_{(x,f)}(X_1 \times_M \mathrm{Iso}(\mathcal{K}, G))$ consists of the pairs $(u, v) \in T_x X_1 \times T_f \mathrm{Iso}(\mathcal{K}, G)$ such that $\mathbf{t}_* u = \pi_* v$, and therefore is the direct sum of the following three vector spaces:

$$E_1 := \big\{\big(0, \eta_f^{\triangleright}\big) \bigm| \eta \in Z^1(G, \mathfrak{g})\big\},$$
$$E_2 := \big\{\big(\xi_x^{\triangleright}, 0\big) \bigm| \xi \in \mathfrak{K}_{\mathbf{t}(x)}\big\},$$
$$E_3 := \big\{(u, v) \bigm| u \in H_x,\ v \in H_f^{\mathrm{iso}},\ \mathbf{t}_* u = \pi_* v\big\}.$$

Below we check that Eq. (60) holds on each of the direct summands.

- By construction,

$$\big(0,\eta^{\rhd}\big)\lrcorner\big(\mathbf{s}^*\alpha^{\mathrm{iso}}-\mathbf{t}^*\alpha^{\mathrm{iso}}\big)=\eta-\eta=0,$$

  and

$$\big(0,\eta^{\rhd}\big)\lrcorner p^*\alpha=0.$$

  Therefore, Eq. (60) holds on E_1.
- Fixing $(x,f)\in X_1\times_M \mathrm{Iso}(\mathcal{K},G)$ and differentiating the relation

$$\mathbf{s}(x\cdot k,f)=\mathbf{Ad}_{xk}\circ f=(\mathbf{Ad}_x\circ f)\cdot\mathbf{Ad}_{f^{-1}(k)}\in \mathrm{Iso}(\mathcal{K},G)$$

  with respect to $k\in\mathcal{K}_{\mathbf{t}(x)}$ at the identity, we obtain

$$\mathbf{s}_*\big(\xi^{\rhd},0\big)\big|_{(x,f)}=\big(\partial\big(f_*^{-1}\xi\big)\big)^{\rhd}_{\mathbf{Ad}_x\circ f}.$$

  Here $\mathbf{Ad}_{f^{-1}(k)}$ is considered as an element in $\mathrm{Inn}(G)\subset\mathrm{Aut}(G)$ and the dot refers to the right $\mathrm{Aut}(G)$-action on $\mathrm{Iso}(\mathcal{K},G)$. Hence, we have

$$\big(\xi^{\rhd},0\big)\lrcorner\big(\mathbf{s}^*\alpha^{\mathrm{iso}}-\mathbf{t}^*\alpha^{\mathrm{iso}}\big)=\partial\big(f_*^{-1}\xi\big),$$

  while, on the other hand,

$$\big(\xi^{\rhd},0\big)\lrcorner\partial\big(p^*f_*^{-1}(\alpha_x)\big)=\partial\big(f_*^{-1}\xi\big).$$

  Therefore, Eq. (60) holds on E_2.
- Let $(u,v)\in E_3$. We have

$$(u,v)\lrcorner\partial\big(p^*\big(f_*^{-1}\circ\alpha\big)\big)=\partial\big(f_*^{-1}(v\lrcorner\alpha_x)\big)=0 \tag{61}$$

  and

$$(u,v)\lrcorner\mathbf{t}^*\alpha^{\mathrm{iso}}=v\lrcorner\alpha^{\mathrm{iso}}=0. \tag{62}$$

  Let x_t and f_t be horizontal paths in X_1 and $\mathrm{Iso}(\mathcal{K},G)$, respectively, such that $\pi\circ f_t=\mathbf{t}\circ x_t$, for all t, and $\frac{d}{dt}(x_t,f_t)|_{t=0}=(u,v)$. For any $g\in G$, $f_t(g)$ is a horizontal path in $\mathcal{K}$ by Proposition 5.9(3). Since a path in $\mathcal{K}$ is horizontal in $\mathcal{K}$ if, and only if, it is horizontal when considered as a path in X_1, and the product of horizontal paths in X_1 is horizontal by Proposition 4.3, $x_t\cdot f_t(g)\cdot x_t^{-1}$ is a horizontal path in $\mathcal{K}$ for all $g\in G$. Since $\mathbf{s}(x,f)=\mathbf{Ad}_x\circ f$, it thus follows that $\mathbf{s}_*(u,v)=\frac{d}{dt}(x_t\cdot f_t(g)\cdot x_t^{-1})|_{t=0}$ is a horizontal vector in $\mathrm{Iso}(\mathcal{K},G)$. Together with Eqs. (61)–(62), this implies that Eq. (60) holds on E_3. □

Proposition 5.10 means that α^{out} is indeed a connection 1-form for the principal $\mathrm{Out}(G)$-bundle $\mathrm{Out}(\mathcal{K},G)\to M$ over $Y_\bullet$, i.e. a connection 1-form on the band.

In summary, we have the following

Theorem 5.12. *The* 1*-form* $\alpha^{\mathrm{out}} \in \Omega^1(\mathrm{Out}(\mathcal{K}, G)) \otimes H^1(G, \mathfrak{g})$ *is a connection* 1*-form on the band of the* G*-extension, whose corresponding horizontal distribution is* H^{out}.

Remark 5.13. As shown in [35], connections behave well with respect to Morita equivalence. More precisely, there is a 1–1 correspondence between connections on principal bundles over Morita equivalent groupoids. Therefore, a connection in our sense indeed yields a connection on the corresponding torsor over the stack. This implies that a connection on a gerbe induces a connection on its band.

5.3. Curvature on the band

This section is devoted to describing the relation between the Ehresmann curvature $\omega^{\mathcal{K}}$ on the group bundle $\mathcal{K} \to M$ and the curvature ω^{iso} on the $\mathrm{Aut}(G)$-principal bundle $\mathrm{Iso}(\mathcal{K}, G) \xrightarrow{\pi} M$, as well as the curvature ω^{out} on the band $\mathrm{Out}(\mathcal{K}, G) \xrightarrow{\pi} M$.

Since $\omega^{\mathcal{K}}$ is a horizontal form, it can be considered as a form on the manifold M. More precisely, one defines $\omega^{\mathcal{K}} \in \Omega^2(M, Z^1(\mathcal{K}, \mathfrak{K}))$ as follows. For any $k \in \mathcal{K}_m$ and $u, v \in T_m M$,

$$\big(i_{u\wedge v}\omega^{\mathcal{K}}\big)(k) = i_{\tilde{u}\wedge\tilde{v}}(\omega|_k), \tag{63}$$

where $\tilde{u}, \tilde{v} \in T_k\mathcal{K}$ are any tangent vectors such that $\phi_*(\tilde{u}) = u$ and $\phi_*(\tilde{v}) = v$. This definition requires some justification.

- First, it is clear that $(i_{u\wedge v}\omega^{\mathcal{K}})(k)$ is independent of the choice of $\tilde{u}, \tilde{v}$ because $\omega|_{\mathcal{K}}$ is a horizontal 2-form, and is therefore well defined.
- Second, we need to check that, for any fixed $u,\ v$, the map $k \mapsto (i_{u\wedge v}\omega^{\mathcal{K}})(k)$ is an element in $Z^1(\mathcal{K}, \mathfrak{K})$. For any $k_1, k_2 \in \mathcal{K}_m$, let $\tilde{u}_1, \tilde{v}_1 \in T_{k_1}\mathcal{K}$ and $\tilde{u}_2, \tilde{v}_2 \in T_{k_2}\mathcal{K}$ be any tangent vectors such that $\phi_*(\tilde{u}_1) = \phi_*(\tilde{u}_2) = u$ and $\phi_*(\tilde{v}_1) = \phi_*(\tilde{v}_2) = v$. Then $\tilde{u}_1 \cdot \tilde{u}_2$ and $\tilde{v}_1 \cdot \tilde{v}_2$ (where the dot stands for the product in the tangent groupoid $TX_1 \rightrightarrows TM$) are elements in $T_{k_1k_2}\mathcal{K}$ such that $\phi_*(\tilde{u}_1 \cdot \tilde{u}_2) = u$ and $\phi_*(\tilde{v}_1 \cdot \tilde{v}_2) = v$. The Bianchi identity Eq. (53) implies that $\omega(\tilde{u}_1 \cdot \tilde{u}_2, \tilde{v}_1 \cdot \tilde{v}_2) = \omega(\tilde{u}_2, \tilde{v}_2) + \mathrm{Ad}_{k_2^{-1}}\,\omega(\tilde{u}_1, \tilde{v}_1)$. Hence

$$\big(i_{u\wedge v}\omega^{\mathcal{K}}\big)(k_1k_2) = \big(i_{u\wedge v}\omega^{\mathcal{K}}\big)(k_2) + \mathrm{Ad}_{k_2^{-1}}\big(i_{u\wedge v}\omega^{\mathcal{K}}\big)(k_1).$$

Note that any $f \in \mathrm{Iso}(\mathcal{K}, G)|_m$ induces an isomorphism $T_f : Z^1(\mathcal{K}_m, \mathfrak{K}_m) \to Z^1(G, \mathfrak{g})$ given by, $\forall z \in Z^1(\mathcal{K}_m, \mathfrak{K}_m)$,

$$T_f(z)(g) = f_*^{-1} z\big(f(g)\big), \quad \forall g \in G. \tag{64}$$

Tensoring with $\wedge^2 T_m^* M$, this extends naturally to a map (denoted again by the same symbol by abuse of notation)

$$T_f : \wedge^2 T_m^* M \otimes Z^1(\mathcal{K}_m, \mathfrak{K}_m) \to \wedge^2 T_m^* M \otimes Z^1(G, \mathfrak{g}).$$

Proposition 5.14. *For any* $f \in \mathrm{Iso}(\mathcal{K}, G)|_m$*, we have the relation*

$$\omega^{\mathrm{iso}}\big|_f = \pi^* T_f \omega^{\mathcal{K}}\big|_m,$$

where $\omega^{\mathrm{iso}}|_f \in \wedge^2 T_f^* \operatorname{Iso}(\mathcal{K}, G) \otimes Z^1(G, \mathfrak{g})$ *is the curvature of the connection* α^{iso}, $\omega^{\mathcal{K}}|_m \in \wedge^2 T_m^* M \otimes Z^1(\mathcal{K}, \mathfrak{K})$ *is as in Eq.* (63) *and* $\pi : \operatorname{Iso}(\mathcal{K}, G) \to M$ *is the projection.*

Proof. For any path $f_t : [0,1] \to \operatorname{Iso}(\mathcal{K}, G)$, let $\overline{f_t}$ be the unique horizontal path in $\operatorname{Iso}(\mathcal{K}, G)$ starting at the same point f_0 as f_t and satisfying $\pi \circ \gamma = \pi \circ \bar{\gamma}$. Then the holonomy $\operatorname{Hol}(f_t) \in \operatorname{Aut}(G)$ of the path f_t is given by

$$f_1 \cdot \operatorname{Hol}(f_t) = \overline{f}_1. \tag{65}$$

The curvature ω^{iso} can be expressed by

$$\omega^{\mathrm{iso}}(u, v) = \frac{\partial^2}{\partial \varepsilon_1 \partial \varepsilon_2}\bigg|_{\varepsilon_1 = \varepsilon_2 = 0} \operatorname{Hol}\big(L^C_{\varepsilon_1, \varepsilon_2}\big) \tag{66}$$

for any smooth map C from $\mathbb{R}^2$ to $\operatorname{Iso}(\mathcal{K}, G)$ adapted to the tangent vectors $u, v \in T_f \operatorname{Iso}(\mathcal{K}, G)$ (see Eq. (50) for the notation $L^C_{\varepsilon_1, \varepsilon_2}$). According to Lemma 5.6(1), Eq. (66) can be written as

$$\omega^{\mathrm{iso}}(u, v)(g) = \frac{\partial^2}{\partial \varepsilon_1 \partial \varepsilon_2}\bigg|_{\varepsilon_1 = \varepsilon_2 = 0} g^{-1} \operatorname{Hol}\big(L^C_{\varepsilon_1, \varepsilon_2}\big)(g), \quad \forall g \in G. \tag{67}$$

By Lemma 5.2, for any path f_t in $\operatorname{Iso}(\mathcal{K}, G)$ and any $g \in G$, $\mathrm{ev}_g(\overline{f_t})$ is a horizontal path starting at $\mathrm{ev}_g(f_0)$. Hence $\mathrm{ev}_g\, \overline{f_t} = \overline{\mathrm{ev}_g\, f_t}$ holds for all $t \in [0,1]$. In particular, for $t = 1$, one obtains

$$\mathrm{ev}_g\big(f_1 \cdot \operatorname{Hol}(f_t)\big) = (\mathrm{ev}_g\, f_1) \cdot \operatorname{Hol}(\mathrm{ev}_g\, f_t), \quad \forall g \in G,$$

and therefore

$$f_1\big(g^{-1}\big) \cdot \big[f_1 \cdot \operatorname{Hol}(f_t)\big](g) = \operatorname{Hol}(\mathrm{ev}_g\, f_t). \tag{68}$$

The latter can be re-written as

$$f_1\big(g^{-1} \operatorname{Hol}(f_t)(g)\big) = \operatorname{Hol}(\mathrm{ev}_g\, f_t). \tag{69}$$

For any smooth map C from a neighborhood $\mathcal{U} \subset \mathbb{R}^2$ of 0 to a neighborhood of $f \in \operatorname{Iso}(\mathcal{K}, G)$ adapted to the tangent vectors $u, v \in T_f \operatorname{Iso}(\mathcal{K}, G)$, the smooth map $\mathrm{ev}_g \circ C \in C^\infty(\mathcal{U}, \mathcal{K})$ is adapted to $\mathrm{ev}_{g*}\, u, \mathrm{ev}_{g*}\, v \in T_{f(g)}\mathcal{K}$. Take f_t to be the loop $L^C_{\varepsilon_1, \varepsilon_2}$. Then $\mathrm{ev}_g\, f_t$ is the loop $L^{\mathrm{ev}_g \circ C}_{\varepsilon_1, \varepsilon_2}$. Now, according to Eq. (69), one obtains that, for any $\varepsilon_1, \varepsilon_2$,

$$f\big(g^{-1} \operatorname{Hol}\big(L^C_{\varepsilon_1, \varepsilon_2}\big)(g)\big) = \operatorname{Hol}\big(L^{\mathrm{ev}_g \circ C}_{\varepsilon_1, \varepsilon_2}\big), \quad \forall g \in G.$$

Applying $\frac{\partial^2}{\partial \varepsilon_1 \partial \varepsilon_2}|_{\varepsilon_1 = \varepsilon_2 = 0}$ to both sides and using Proposition 4.26 and Eq. (67), we have

$$f_*\big(\omega^{\mathrm{iso}}(u, v)(g)\big) = \omega(\mathrm{ev}_{g*}\, u, \mathrm{ev}_{g*}\, v)|_{f(g)}, \quad \forall g \in G.$$

By Eq. (63), we obtain

$$f_*\big(\omega^{\mathrm{iso}}(u,v)(g)\big) = \omega^{\mathcal{K}}(\pi_* u, \pi_* v)\big(f(g)\big), \quad \forall g \in G.$$

This concludes the proof. □

Denote by $[\omega^{\mathcal{K}}] \in \Omega^2(M, H^1(\mathcal{K}, \mathfrak{K}))$ the class of $\omega^{\mathcal{K}}$. Since for any $f \in \mathrm{Iso}(\mathcal{K}, G)|_m$, the map T_f defined in Eq. (64) maps $B^1(\mathcal{K}_m, \mathfrak{K}_m)$ to $B^1(G, \mathfrak{g})$, it induces a map $T_f : H^1(\mathcal{K}_m, \mathfrak{K}_m) \to H^1(G, \mathfrak{g})$. It is simple to check that T_f only depends on the class of $\overline{f} \in \mathrm{Out}(\mathcal{K}, G)|_m$. Therefore, for any $\overline{f} \in \mathrm{Out}(\mathcal{K}, G)|_m$, we have a well-defined map $T_{\overline{f}} : H^1(\mathcal{K}_m, \mathfrak{K}_m) \to H^1(G, \mathfrak{g})$. Alternatively, one may also obtain $T_{\overline{f}}$ as follows. For any $\overline{f} \in \mathrm{Out}(\mathcal{K}, G)|_m$, conjugation by $\overline{f}$ is a group homomorphism $\mathrm{Out}(\mathcal{K})|_m \to \mathrm{Out}(G)$. Then $T_{\overline{f}} : H^1(\mathcal{K}_m, \mathfrak{K}_m) \to H^1(G, \mathfrak{g})$ is the corresponding Lie algebra homomorphism.

Tensoring with $\wedge^2 T_m^* M$ as before, $T_{\overline{f}}$ extends naturally to a map

$$T_{\overline{f}} : \wedge^2 T_m^* M \otimes H^1(\mathcal{K}_m, \mathfrak{K}_m) \to \wedge^2 T_m^* M \otimes H^1(G, \mathfrak{g}).$$

Proposition 5.14 immediately implies the following

Theorem 5.15. *For any $\overline{f} \in \mathrm{Out}(\mathcal{K}, G)|_m$, the curvature on the band $\omega^{\mathrm{out}}|_{\overline{f}} \in \wedge^2 T_m^* \mathrm{Out}(\mathcal{K}, G) \otimes H^1(G, \mathfrak{g})$ and the class $[\omega^{\mathcal{K}}] \in \wedge^2 T_m^* M \otimes H^1(\mathcal{K}, \mathfrak{K})$ are related by the following equation*

$$\omega^{\mathrm{out}}|_{\overline{f}} = \pi^* T_{\overline{f}}\big[\omega^{\mathcal{K}}\big],$$

where $\pi : \mathrm{Out}(\mathcal{K}, G) \to M$ is the bundle projection.

As an immediate consequence, we have the following

Corollary 5.16. *The band of a G-extension $X_\bullet \xrightarrow{\phi} Y_\bullet$ is flat if and only if $[\omega^{\mathcal{K}}] = 0$.*

To end this section, we describe an important relation between the curvatures $\omega \in \Omega^2(X_1, \mathbf{t}^*\mathfrak{K})$ and $\omega^{\mathcal{K}}$.

Let $\omega^{\mathfrak{K}} \in \Omega^2(M, \mathrm{Lie}\,\mathrm{Aut}(\mathfrak{K}))$ be the curvature of the induced connection ∇ on the Lie algebra bundle $\mathfrak{K} \to M$. Note that $\mathrm{Lie}\,\mathrm{Aut}(\mathfrak{K})$ can be naturally identified with the space of derivations of $\mathfrak{K}$, which can also be viewed as Lie algebra 1-cocycles relative to the adjoin action and with values in the Lie algebra itself. Therefore we have an identification $\mathrm{Lie}\,\mathrm{Aut}(\mathfrak{K}) \simeq Z^1(\mathfrak{K}, \mathfrak{K})$.

First we need to give a relation between $\omega^{\mathfrak{K}}$ and the curvature $\omega^{\mathcal{K}}$ on the group bundle $\mathcal{K} \to M$. For any Lie group G with Lie algebra $\mathfrak{g}$, differentiating a Lie group cocycle in $Z^1(G, \mathfrak{g})$ at the identity, one obtains an element in $Z^1(\mathfrak{g}, \mathfrak{g})$. More generally, for any group bundle $\mathcal{K} \to M$, this yields a map $Z^1(\mathcal{K}, \mathfrak{K}) \to Z^1(\mathfrak{K}, \mathfrak{K})$, which extends to a map $D : \Omega^2(M, Z^1(\mathcal{K}, \mathfrak{K})) \to \Omega^2(M, Z^1(\mathfrak{K}, \mathfrak{K}))$. The following can be easily verified:

Lemma 5.17.

$$D\omega^{\mathcal{K}} = \omega^{\mathfrak{K}}. \tag{70}$$

Recall that for any Lie group G with Lie algebra $\mathfrak{g}$, we have $\partial : \mathfrak{g} \to B^1(G, \mathfrak{g}) : \xi \mapsto (g \mapsto \xi - \mathrm{Ad}_{g^{-1}} \xi)$. Applying ∂ fiberwise, one obtains a map $\Gamma(\mathbf{t}^*\mathfrak{K}) \to \Gamma(\mathbf{t}^* B^1(\mathcal{K}, \mathfrak{K})) \subset \Gamma(\mathbf{t}^* Z^1(\mathcal{K}, \mathfrak{K}))$ that we denote by $\partial_{\mathcal{K}}$. By abuse of notation, we denote by $\partial_{\mathcal{K}}$ again the induced map

$$\partial_{\mathcal{K}} : \Omega^2(X_1, \mathbf{t}^*\mathfrak{K}) \to \Omega^2\big(X_1, \mathbf{t}^* B^1(\mathcal{K}, \mathfrak{K})\big) \subset \Omega^2\big(X_1, \mathbf{t}^* Z^1(\mathcal{K}, \mathfrak{K})\big).$$

We can now prove the following:

Proposition 5.18. *The following relations hold for any $x \in X_1$:*

$$\begin{aligned} \mathrm{Ad}_{x^{-1}} \mathbf{s}^*\omega^{\mathcal{K}}\big|_{\mathbf{s}(x)} - \mathbf{t}^*\omega^{\mathcal{K}}\big|_{\mathbf{t}(x)} &= -\partial_{\mathcal{K}}\omega|_x, \\ \mathrm{Ad}_{x^{-1}} \mathbf{s}^*\omega^{\mathfrak{K}}\big|_{\mathbf{s}(x)} - \mathbf{t}^*\omega^{\mathfrak{K}}\big|_{\mathbf{t}(x)} &= \mathrm{ad}_{\omega_x}, \end{aligned} \tag{71}$$

where $\mathrm{Ad}_{x^{-1}}$ stands for the natural isomorphism from $Z^1(\mathcal{K}_{\mathbf{s}(x)}, \mathfrak{K}_{\mathbf{s}(x)})$ to $Z^1(\mathcal{K}_{\mathbf{t}(x)}, \mathfrak{K}_{\mathbf{t}(x)})$, and from $Z^1(\mathfrak{K}_{\mathbf{s}(x)}, \mathfrak{K}_{\mathbf{s}(x)})$ to $Z^1(\mathfrak{K}_{\mathbf{t}(x)}, \mathfrak{K}_{\mathbf{t}(x)})$ induced by $\mathbf{Ad}_x : \mathcal{K}_{\mathbf{t}(x)} \to \mathcal{K}_{\mathbf{s}(x)}$. Note that $\omega|_x \in \wedge^2 T_x^* X_1 \otimes \mathfrak{K}_{\mathbf{t}(x)}$, $\partial\omega|_x \in \wedge^2 T_x^* X_1 \otimes Z^1(\mathcal{K}_{\mathbf{t}(x)}, \mathfrak{K}_{\mathbf{t}(x)})$, $\omega^{\mathcal{K}}|_{\mathbf{t}(x)} \in \wedge^2 T_{\mathbf{t}(x)}^* M \otimes Z^1(\mathcal{K}_{\mathbf{t}(x)}, \mathfrak{K}_{\mathbf{t}(x)})$, and $\omega^{\mathcal{K}}|_{\mathbf{s}(x)} \in \wedge^2 T_{\mathbf{s}(x)}^* M \otimes Z^1(\mathcal{K}_{\mathbf{s}(x)}, \mathfrak{K}_{\mathbf{s}(x)})$. Similarly, $\mathrm{ad}_{\omega_x} \in \wedge^2 T_x^* X_1 \otimes Z^1(\mathfrak{K}_{\mathbf{t}(x)}, \mathfrak{K}_{\mathbf{t}(x)})$.

Proof. The second identity in Eq. (71) follows from the first one by applying D and using Eq. (70). Let us now prove the first one. This equation is equivalent to

$$\mathrm{Ad}_{x^{-1}} \mathbf{s}^*\omega^{\mathcal{K}}(\mathbf{Ad}_x k) - \mathbf{t}^*\omega^{\mathcal{K}}(k) = -\omega_x + \mathrm{Ad}_{k^{-1}} \omega_x \in \wedge^2 T_x^* X_1 \otimes \mathfrak{K}_{\mathbf{t}(x)}. \tag{72}$$

Let $u_1, u_2 \in T_x X_1$ be any two tangent vectors at the point x, and $\varepsilon_1, \varepsilon_2 \in T_k\mathcal{K}$ any two tangent vectors of $\mathcal{K}$ at the point k such that $\mathbf{t}_* u_i = \pi_* \varepsilon_i$, $i \in \{1, 2\}$ (i.e. (u_i, ε_i) is a composable pair in the tangent groupoid $TX_1 \rightrightarrows TM$ for $i \in \{1, 2\}$).

By using Eq. (53) repeatedly, we obtain

$$\omega_{\mathbf{Ad}_x k}\big(u_1\varepsilon_1 u_1^{-1}, u_2\varepsilon_2 u_2^{-1}\big) = \mathrm{Ad}_{xk^{-1}} \omega_x(u_1, u_2) + \mathrm{Ad}_x \omega_k(\varepsilon_1, \varepsilon_2) - \mathrm{Ad}_x \omega_x(u_1, u_2).$$

Hence

$$\mathrm{Ad}_{x^{-1}} \omega_{\mathbf{Ad}_x k}\big(u_1\varepsilon_1 u_1^{-1}, u_2\varepsilon_2 u_2^{-1}\big) = \mathrm{Ad}_{k^{-1}} \omega_x(u_1, u_2) - \omega_x(u_1, u_2) + \omega_k(\varepsilon_1, \varepsilon_2). \tag{73}$$

Since $\varepsilon_1, \varepsilon_2 \in T_k\mathcal{K}$, $u_i\varepsilon_i u_i^{-1} \in T_{\mathbf{Ad}_x k}\mathcal{K}$ for $i \in \{1, 2\}$. According to Eq. (63),

$$\omega_k(\varepsilon_1, \varepsilon_2) = \omega_k^{\mathcal{K}}(\phi_*\varepsilon_1, \phi_*\varepsilon_2) = \omega_k^{\mathcal{K}}(\mathbf{t}_* u_1, \mathbf{t}_* u_2) \tag{74}$$

and

$$\omega_{\mathbf{Ad}_x k}\big(u_1\varepsilon_1u_1^{-1}, u_2\varepsilon_2u_2^{-1}\big) = \omega^{\mathcal{K}}_{\mathbf{Ad}_x k}\big(\phi_*\big(u_1\varepsilon_1u_1^{-1}\big), \phi_*\big(u_2\varepsilon_2u_2^{-1}\big)\big) = \omega^{\mathcal{K}}_{\mathbf{Ad}_x k}(\mathbf{s}_*u_1, \mathbf{s}_*u_2). \tag{75}$$

The result now follows from Eqs. (73)–(75). □

Remark 5.19. For the Lie algebra $\mathfrak{g}$ of any connected Lie group G, denote by $Z(\mathfrak{g})$ its center. Since G is connected, one has $Z(\mathfrak{g}) = \{\xi \in \mathfrak{g} \mid \mathrm{Ad}_g\,\xi = \xi, \forall g \in G\} = \ker(\partial)$.

Similarly, let $Z(\mathfrak{K}) = \coprod_{m\in M} Z(\mathfrak{K})_m$ be the vector bundle over M obtained by taking the center of each fiber, and $\mathbf{t}^*Z(\mathfrak{K}) \to X_1$, its pullback by the target map $\mathbf{t}$. The space of sections of the vector bundle $\mathbf{t}^*Z(\mathfrak{K}) \to X_1$ is precisely the kernel of $\partial_{\mathcal{K}}$.

Proposition 5.18 implies that the image of ω under $\partial_{\mathcal{K}}$ is entirely determined by $\omega^{\mathcal{K}}$. Equivalently, this means that the class of ω in $\Omega^2(X_1, \mathbf{t}^*\frac{\mathfrak{K}}{Z(\mathfrak{K})})$ is entirely determined by $\omega^{\mathcal{K}}$.

6. Cohomology theory of connections

The purpose of this section is to develop a cohomology theory for groupoid extensions, which appears naturally while studying connections and curvings.

First of all, in Section 6.1, we introduce the cohomological object in question, which we call horizontal cohomology of a Lie groupoid extension. We also discuss in Section 6.1 how and when the horizontal cohomology can be pulled back by a morphism of Lie groupoid extensions. In Section 6.3, we show that the horizontal cohomology can actually be defined for a gerbe, i.e. it is invariant under Morita equivalence. In Section 6.4, we introduce an obstruction class [obs] in the horizontal cohomology, which characterizes the existence of connections. As a consequence, we show that, if a groupoid extension admits a connection, then any Morita equivalent groupoid extension admits a connection as well. In the language of stacks and gerbes, it means that the existence of a connection "goes down" to a "gerbe" notion. In Section 6.6, we compute the horizontal cohomology for G-gerbes over a manifold, i.e. for a G-extension of a Čech groupoid. We show that, in this case, the class [obs] vanishes, and therefore, any G-extension of a Čech groupoid (i.e. any G-gerbe over a manifold) admits a connection. Section 6.7 is devoted to the study of flat gerbes. In Section 6.8, we study connections and curvings on central extensions.

6.1. Horizontal cohomology

Recall that, for any fiber bundle $T \xrightarrow{\phi} B$, with T and B being finite dimensional manifolds, an l-form ξ on T (possibly valued in some vector bundle over T) is said to be **horizontal** if $v \lrcorner \xi = 0$ for any $v \in \ker(\phi_*)$. In other words, a form is horizontal if it vanishes when contracted with a vector tangent to the fiber of ϕ.

Remark 6.1.

1. The reader should not confuse horizontal forms with basic forms. Basic forms are simply obtained by pulling back forms on the base space, while horizontal forms form a much bigger space in general.
2. Given a fiber bundle $T \xrightarrow{\phi} B$, horizontal l-forms on T can also be considered as l-forms on the base manifold B with values in, for any $b \in B$, $C^\infty(\phi^{-1}(b))$. We will use both viewpoints all along this section.

We now introduce the horizontal forms relevant to our situation. Given a Lie groupoid extension $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ with kernel $\mathcal{K} \to M$, for any $n \in \mathbb{N}$, the manifold X_n of n-tuples of composable arrows of $X_1 \rightrightarrows M$ is a fiber bundle over Y_n, with respect to the projection

$$\phi_n : (x_1, \dots, x_n) \mapsto \big(\phi(x_1), \dots, \phi(x_n)\big).$$

A typical fiber over a point $(y_1, \dots, y_n) \in Y_n$ is isomorphic to $\mathcal{K}_{\mathbf{t}(y_1)} \times \mathcal{K}_{\mathbf{t}(y_2)} \times \cdots \times \mathcal{K}_{\mathbf{t}(y_n)}$. By a **horizontal $\mathfrak{K}$-valued l-form on X_n**, we mean a $\mathfrak{K}$-valued l-form on X_n (i.e. a section of the vector bundle $\wedge^l T^* X_n \otimes \mathbf{t}^*\mathfrak{K} \to X_n$, which is horizontal with respect to the fiber bundle $\phi_n : X_n \to Y_n$). From now on, for any $n, l \in \mathbb{N}$, we denote by $\Omega^l_{\mathrm{hor}}(X_n, \mathbf{t}^*\mathfrak{K})$ the space of horizontal $\mathfrak{K}$-valued l-form on X_n.

We list below two important examples of horizontal forms.

Proposition 6.2. *The Ehresmann curvature ω of a connection on a groupoid extension $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ is a horizontal 2-form.*

Proof. This follows immediately from Eq. (49). □

Proposition 6.3. *Let $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ be a Lie groupoid extension with kernel $\mathcal{K} \to M$, whose corresponding Lie algebra bundle is $\mathfrak{K} \to M$. Let $\theta \in \Omega^1(X_1, \mathbf{t}^*\mathfrak{K})$ be a $\mathfrak{K}$-valued 1-form on X_1 satisfying*

$$\theta\big(\xi^{\triangleright}\big) = \xi, \quad \forall \xi \in \mathfrak{K}, \tag{76}$$

where, as in Section 4.3, for any $\xi \in \mathfrak{K}$ we denote by $\xi^{\triangleright}$ the fundamental vector field on X_1 corresponding to the infinitesimal right action of $\mathcal{K} \to M$ on X_1. Then $\partial^{\triangleright}\theta \in \Omega^1(X_2, \mathbf{t}^*\mathfrak{K})$ is a horizontal 1-form on X_2.

Proof. Let us introduce some notations. The group bundle $\mathcal{K} \to M$ acts on X_2 from the right in two different ways. The first one is given, for all $(x_1, x_2) \in X_2$ and $k \in \mathcal{K}_{\mathbf{t}(x_1)}$, by $(x_1, x_2) \cdot k = (x_1 \cdot k, x_2)$; the second one is given, for all $(x_1, x_2) \in X_2$ and $k \in \mathcal{K}_{\mathbf{t}(x_2)}$, by $(x_1, x_2) \cdot k = (x_1, x_2 \cdot k)$. For any $(x_1, x_2) \in X_2$, and any $\xi \in \mathfrak{K}_{\mathbf{t}(x_1)}$ (resp. $\xi \in \mathfrak{K}_{\mathbf{t}(x_2)}$), we denote by $\xi_1^{\triangleright}$ (resp. $\xi_2^{\triangleright}$) the tangent vectors in $T_{(x_1,x_2)}X_2$ corresponding infinitesimally to these two actions. Denote by p_1, m, p_2 the three face maps from X_2 to X_1. For any $\xi \in \mathfrak{K}$, we have

$$p_{1*}\xi_1^{\triangleright} = \xi^{\triangleright}, \qquad m_*\xi_1^{\triangleright} = (\mathrm{Ad}_{x_2^{-1}}\xi)^{\triangleright}, \qquad p_{2*}\xi_1^{\triangleright} = 0,$$
$$p_{1*}\xi_2^{\triangleright} = 0, \qquad m_*\xi_2^{\triangleright} = \xi^{\triangleright}, \qquad p_{2*}\xi_2^{\triangleright} = \xi^{\triangleright}.$$

Thus it is routine to check that the two relations $\xi_1^{\triangleright} \lrcorner\, \partial^{\triangleright}\theta = 0$ and $\xi_2^{\triangleright} \lrcorner\, \partial^{\triangleright}\theta = 0$ hold, which implies that $\partial^{\triangleright}\theta$ is horizontal. □

Definition 6.4. The **$\partial^{\triangleright}$-cohomology of a Lie groupoid extension** $X_\bullet \xrightarrow{\phi} Y_\bullet$ is the cohomology of the cochain complex $((\Omega^l(X_n, \mathbf{t}^*\mathfrak{K}))_{n\in\mathbb{N}}, \partial^{\triangleright})$. It is denoted by $H^{n,l}_{\partial^{\triangleright}}(X_\bullet \to Y_\bullet)$.

The following proposition can be verified directly.

Proposition 6.5. *Let $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ be a Lie groupoid extension. Then for any fixed $l \in \mathbb{N}$, $\Omega^l_{\mathrm{hor}}(X_\bullet, \mathbf{t}^*\mathfrak{K})$ is stable under $\partial^{\triangleright}$, i.e.*

$$\partial^{\triangleright}\big(\Omega^l_{\mathrm{hor}}(X_n, \mathbf{t}^*\mathfrak{K})\big) \subset \Omega^l_{\mathrm{hor}}(X_{n+1}, \mathbf{t}^*\mathfrak{K}).$$

Therefore, for any fixed $l \in \mathbb{N}$, the horizontal l-forms $((\Omega^l_{\mathrm{hor}}(X_n, \mathbf{t}^*\mathfrak{K}))_{n\in\mathbb{N}}, \partial^{\triangleright})$ form a cochain subcomplex.

Definition 6.6. The **horizontal cohomology of a Lie groupoid extension** $X_\bullet \xrightarrow{\phi} Y_\bullet$ is the cohomology of the cochain complex $((\Omega^l_{\mathrm{hor}}(X_n, \mathbf{t}^*\mathfrak{K}))_{n\in\mathbb{N}}, \partial^{\triangleright})$. It is denoted by $H^{n,l}_{\mathrm{hor}}(X_\bullet \to Y_\bullet)$.

6.2. Horizontal cohomology and strict homomorphisms

Let f be a homomorphism of Lie groupoid extensions from $X'_\bullet \to Y'_\bullet$ to $X_\bullet \to Y_\bullet$. As usual, denote by $\mathfrak{K} \to M$ (resp. $\mathfrak{K}' \to M'$) the Lie algebra bundle associated to the groupoid extension $X_1 \to Y_1 \rightrightarrows M$ (resp. $X'_1 \to Y'_1 \rightrightarrows M'$). In general, there is no straightforward way to make sense of the pullback through f of a $\mathfrak{K}$-valued form on X_1 (or, more generally on X_n for some $n \in \mathbb{N}$) such that the pullback form is $\mathfrak{K}'$-valued. Indeed, if ζ is an l-form on X_n with values in $\mathbf{t}^*\mathfrak{K}$, the usual pullback $f^*\zeta$ takes its values in $(f^* \circ \mathbf{t}^*)\mathfrak{K}$ rather than in $\mathbf{t}'^*\mathfrak{K}'$. To build up a form on X'_n with values in $\mathbf{t}'^*\mathfrak{K}'$, one needs to identify $(f^* \circ \mathbf{t}^*)\mathfrak{K}$ with $\mathbf{t}'^*\mathfrak{K}'$ at any point $(x'_1, \ldots, x'_n) \in X'_n$. When f is a strict homomorphism of Lie groupoid extensions, in particular a Morita morphism, f establishes an isomorphism between $\mathcal{K}_{\mathbf{t}(x'_n)}$ and $\mathcal{K}_{\mathbf{t}(f(x_n))}$. Hence its differential at the identity $f_{\mathbf{t}(x'_n)*}$ gives the required identification. This leads to the following definition of the pullback $\Omega^l(X_n, \mathbf{t}^*\mathfrak{K}) \xrightarrow{f^*} \Omega^l(X'_n, \mathbf{t}'^*\mathfrak{K}')$:

$$(f^*\zeta)(e_1, \ldots, e_l) = f^{-1}_{\mathbf{t}(x'_n)*}\big(\zeta(f_*e_1, \ldots, f_*e_l)\big), \quad \forall e_1, \ldots, e_l \in T_{(x_1,\ldots,x_n)}X_n. \tag{77}$$

The following lemma can be verified directly.

Lemma 6.7. *Let f be a strict homomorphism of Lie groupoid extensions from $X'_\bullet \to Y'_\bullet$ to $X_\bullet \to Y_\bullet$. Then*

1. *the pullback map $\Omega^l(X_n, \mathbf{t}^*\mathfrak{K}) \xrightarrow{f^*} \Omega^l(X'_n, \mathbf{t}'^*\mathfrak{K}')$ is a chain map with respect to $\partial^{\triangleright}$;*
2. *the pullback of a horizontal form is again horizontal;*
3. *$\theta \in \Omega^1(X_1, \mathbf{t}^*\mathfrak{K})$ satisfies $\xi^{\triangleright} \lrcorner\, \theta = \xi$, for all $\xi \in \mathfrak{K}$, if, and only if, $f^*\theta \in \Omega^1(X'_1, \mathbf{t}^*\mathfrak{K}')$ satisfies $\xi'^{\triangleright} \lrcorner\, f^*\theta = \xi'$, for all $\xi' \in \mathfrak{K}'$.*

As an immediate consequence, we have the following

Corollary 6.8. *Let f be a strict homomorphism of Lie groupoid extensions from $X'_\bullet \to Y'_\bullet$ to $X_\bullet \to Y_\bullet$. Then the pullback map f^* gives rise to morphisms*

$$f^* : H^{n,l}_{\partial^{\triangleright}}(X_\bullet \to Y_\bullet) \to H^{n,l}_{\partial^{\triangleright}}\big(X'_\bullet \to Y'_\bullet\big), \tag{78}$$

$$f^* : H^{n,l}_{\mathrm{hor}}(X_\bullet \to Y_\bullet) \to H^{n,l}_{\mathrm{hor}}\big(X'_\bullet \to Y'_\bullet\big), \tag{79}$$

for all $n, l \in \mathbb{N}$.

6.3. Morita invariance

In this subsection, we show that both the horizontal cohomology and the $\partial^{\rhd}$-cohomology are invariant under Morita equivalence. In other words, they turn out to be cohomological objects associated to a gerbe. Our proof relies on the notion of generalized morphisms introduced in Section 2.3, and is similar to the proof of Proposition 2.5 in [51].

Theorem 6.9.

1. *Morita equivalent groupoid extensions have isomorphic $\partial^{\rhd}$-cohomologies.*
2. *Morita equivalent groupoid extensions have isomorphic horizontal cohomologies.*
3. *In particular, if f is a Morita morphism of groupoid extensions, the maps f^* defined in Eqs.* (78)–(79) *are isomorphisms.*

We need two lemmas.

Lemma 6.10. *If f is a Morita morphism of Lie groupoid extensions from $X_1' \to Y_1' \rightrightarrows M'$ to $X_1 \to Y_1 \rightrightarrows M$ such that $f : M' \to M$ is an étale map, then the pullback map f^* as in Eqs.* (78)–(79) *induces isomorphisms on both horizontal and $\partial^{\rhd}$-cohomologies.*

Proof. Note that $U \mapsto \Omega^l_{\mathrm{hor}}(U, \mathbf{t}|_U^* \mathfrak{K})$ (resp. $U \mapsto \Omega^l(U, \mathbf{t}|_U^* \mathfrak{K})$) is a sheaf over the simplicial manifold $(X_n)_{n\in\mathbb{N}}$ (see [50]). And the horizontal cohomology $H^{n,l}_{\mathrm{hor}}(X_\bullet \to Y_\bullet)$ (resp. the $\partial^{\rhd}$-cohomology $H^{n,l}_{\partial^{\rhd}}(X_\bullet \to Y_\bullet)$) is the corresponding sheaf cohomology. When f is an étale map, it is clear that the pullback sheaf $f^*(\Omega^l_{\mathrm{hor}}(-, \mathbf{t}^*\mathfrak{K}))$ (resp. $f^*(\Omega^l(-, \mathbf{t}^*\mathfrak{K}))$) is isomorphic to $\Omega^l_{\mathrm{hor}}(-, \mathbf{t}'^*\mathfrak{K}')$ (resp. $\Omega^l(-, \mathbf{t}'^*\mathfrak{K}')$). According to Theorem 8.1 in [50], f^* induces an isomorphism in cohomology:

$$f^* : H^{n,l}_{\mathrm{hor}}(X_\bullet \to Y_\bullet) \xrightarrow{\simeq} H^{n,l}_{\mathrm{hor}}\big(X'_\bullet \to Y'_\bullet\big), \tag{80}$$

$$f^* : H^{n,l}_{\partial^{\rhd}}(X_\bullet \to Y_\bullet) \xrightarrow{\simeq} H^{n,l}_{\partial^{\rhd}}\big(X'_\bullet \to Y'_\bullet\big). \qquad \square \tag{81}$$

Proposition 6.11.

1. *Any generalized homomorphism of Lie groupoid extensions F from $X'_\bullet \xrightarrow{\phi'} Y'_\bullet$ to $X_\bullet \xrightarrow{\phi} Y_\bullet$ induces canonical homomorphisms*

$$F^* : H^{n,l}_{\partial^{\rhd}}(X_\bullet \to Y_\bullet) \to H^{n,l}_{\partial^{\rhd}}\big(X'_\bullet \to Y'_\bullet\big), \tag{82}$$

$$F^* : H^{n,l}_{\mathrm{hor}}(X_\bullet \to Y_\bullet) \to H^{n,l}_{\mathrm{hor}}\big(X'_\bullet \to Y'_\bullet\big), \tag{83}$$

   for all $n, l \in \mathbb{N}$.
2. *In particular, if F is a generalized homomorphism of Lie groupoid extensions induced by a strict morphism of groupoid extensions f, then F^* coincides with the pullback map f^* in Corollary* 6.8.
3. *Moreover, the relation*

$$(F_1 \circ F_2)^* = F_2^* \circ F_1^*$$

holds for any composable pair F_1, F_2 of generalized homomorphisms of Lie groupoid extensions.

Proof. 1. Assume that F is given by the bimodule $M' \xleftarrow{g} B \xrightarrow{f} M$. According to Proposition 2.11, F is the composition of the canonical Morita equivalence between $X'_\bullet \to Y'_\bullet$ and $X'_\bullet[B] \to Y'_\bullet[B]$, with a strict homomorphism of groupoid extensions from $X'_\bullet[B] \to Y'_\bullet[B]$ to $X_\bullet \to Y_\bullet$. Since $B \xrightarrow{g} M'$ is a surjective submersion, it admits local sections. Hence, there exists an open cover $(U_i)_{i\in I}$ of M' and local sections σ_i of $B \xrightarrow{g} M'$ relative to this open cover. The sections (σ_i) induce a strict homomorphism $\psi_{\mathfrak{U}}$ from $X'_\bullet[\mathfrak{U}] \to Y'_\bullet[\mathfrak{U}]$ to $X_\bullet \to Y_\bullet$ by composing the natural strict homomorphism from $X'_\bullet[\mathfrak{U}] \to Y'_\bullet[\mathfrak{U}]$ to $X'_\bullet[B] \to Y'_\bullet[B]$ with the one from $X'_\bullet[B] \to Y'_\bullet[B]$ to $X_\bullet \to Y_\bullet$. Denote by $\chi_{\mathfrak{U}}$ the Morita morphism from $X'_\bullet[\mathfrak{U}] \to Y'_\bullet[\mathfrak{U}]$ to $X'_\bullet \to Y'_\bullet$.

We now define F^* as the composition

$$H^{n,l}_{\mathrm{hor}}(X_\bullet \to Y_\bullet) \xrightarrow{\psi^*_{\mathfrak{U}}} H^{n,l}_{\mathrm{hor}}\big(X'_\bullet[\mathfrak{U}] \to Y'_\bullet[\mathfrak{U}]\big) \xrightarrow[\simeq]{(\chi^*_{\mathfrak{U}})^{-1}} H^{n,l}_{\mathrm{hor}}\big(X'_\bullet \to Y'_\bullet\big)$$

and similarly

$$H^{n,l}_{\partial\triangleright}(X_\bullet \to Y_\bullet) \xrightarrow{\psi^*_{\mathfrak{U}}} H^{n,l}_{\partial\triangleright}\big(X'_\bullet[\mathfrak{U}] \to Y'_\bullet[\mathfrak{U}]\big) \xrightarrow[\simeq]{(\chi^*_{\mathfrak{U}})^{-1}} H^{n,l}_{\partial\triangleright}\big(X'_\bullet \to Y'_\bullet\big).$$

Here we have used the fact that $\chi^*_{\mathfrak{U}}$ is an isomorphism by Lemma 6.10.

We need to check that F^* does not depend on the choice of the open cover and the local sections. Let $\mathfrak{V} := (V_j, \tilde{\tau}_j)_{j\in J}$ be another choice of such local sections. The union of $\mathfrak{U}$ and $\mathfrak{V}$ is another such open cover of M' together with a set of local sections of $B \to M'$. It is simple to see that we have the following commutative diagram, where all the arrows are strict homomorphisms of groupoid extensions:

$$\begin{tikzcd}
X'_\bullet \arrow[r, equal] & X'_\bullet \arrow[r, equal] & X'_\bullet \\
X'_\bullet[\mathfrak{U}] \arrow[u, "\chi_{\mathfrak{U}}"] \arrow[r] \arrow[d, "\psi_{\mathfrak{U}}"'] & X'_\bullet[\mathfrak{U}\cup\mathfrak{V}] \arrow[u, "\chi_{\mathfrak{U}\cup\mathfrak{V}}"] \arrow[d, "\psi_{\mathfrak{U}\cup\mathfrak{V}}"'] & X'_\bullet[\mathfrak{V}] \arrow[l] \arrow[u, "\chi_{\mathfrak{V}}"] \arrow[d, "\psi_{\mathfrak{V}}"'] \\
X_\bullet \arrow[r, equal] & X_\bullet \arrow[r, equal] & X_\bullet\,.
\end{tikzcd}$$

It follows from a diagram chasing argument that

$$(\chi^*_{\mathfrak{U}})^{-1} \circ \psi^*_{\mathfrak{U}} = (\chi^*_{\mathfrak{U}\cup\mathfrak{V}})^{-1} \circ \psi^*_{\mathfrak{U}\cup\mathfrak{V}} = (\chi^*_{\mathfrak{V}})^{-1} \circ \psi^*_{\mathfrak{V}}.$$

Hence F^* is indeed well defined.

2. If F is induced from a strict homomorphism of Lie groupoid extensions f, in other words, if F is given by the bimodule $M' \xleftarrow{\mathrm{id}} M' \xrightarrow{f} M$, then $\chi = \mathrm{id}$ and ψ coincides with the map f^* defined in Corollary 6.8.

3. Let $X_\bullet \to Y_\bullet$, $X'_\bullet \to Y'_\bullet$ and $X''_\bullet \to Y''_\bullet$ be Lie groupoid extensions with base manifolds M, M' and M'', respectively. Let F_1 and F_2 be generalized homomorphisms from $X'_\bullet \to Y'_\bullet$ to $X_\bullet \to Y_\bullet$ and from $X''_\bullet \to Y''_\bullet$ to $X'_\bullet \to Y'_\bullet$, respectively. According to the first part of the proof, there exists an open covering $\mathfrak{U} = (U_k)$ of M' and a strict homomorphism $\psi_{\mathfrak{U}}$ of groupoid extensions from $X'_\bullet[\mathfrak{U}] \to Y'_\bullet[\mathfrak{U}]$ to $X_\bullet \to Y_\bullet$ such that $F_1^* = (\chi_{\mathfrak{U}}^*)^{-1} \circ \psi_{\mathfrak{U}}$. The same holds for F_2. However, we can choose an open covering $\mathfrak{V} = (V_l)_{l\in L}$ of M'' and a strict homomorphism of groupoid extensions $\psi_{\mathfrak{V}}$ such that $\psi_{\mathfrak{V}}(V)$ is contained in an open set $U \in \mathfrak{U}$, for all $V \in \mathfrak{V}$. Composing $\psi_{\mathfrak{V}} : \coprod_l V_l \to M'$ with these inclusions yields a map $j : \coprod_l V_l \to \coprod_k U_k$. We can then construct a strict homomorphism μ_j of groupoid extensions from $X''_\bullet[\mathfrak{V}] \to Y''_\bullet[\mathfrak{V}]$ to $X'_\bullet[\mathfrak{U}] \to Y'_\bullet[\mathfrak{U}]$:

$$\begin{cases} \mu_j(v_1, x, v_2) = \big(j(v_1), \psi_{\mathfrak{V}}(v_1, x, v_2), j(v_2)\big), \\ \mu_j(v_1, y, v_2) = \big(j(v_1), \psi_{\mathfrak{V}}(v_1, y, v_2), j(v_2)\big) \end{cases}$$

for all $x \in X''_1$, $y \in Y''_1$, $v_1, v_2 \in \coprod_l V_l$ with $\chi_{\mathfrak{V}}(v_1) = \mathbf{s}(x) = \mathbf{s}(y)$ and $\chi_{\mathfrak{V}}(v_2) = \mathbf{t}(x) = \mathbf{t}(y)$. We then have the following commutative diagram of strict homomorphisms of Lie groupoid extensions:

$$\begin{tikzcd} X''_\bullet & & X'_\bullet & & X_\bullet \\ & X''_\bullet[\mathfrak{V}] \arrow[ul, "\chi_{\mathfrak{V}}"] \arrow[ur, "\psi_{\mathfrak{V}}"] \arrow[rr, "\mu_j"'] & & X'_\bullet[\mathfrak{U}]. \arrow[ul, "\chi_{\mathfrak{U}}"'] \arrow[ur, "\psi_{\mathfrak{U}}"'] & \end{tikzcd} \tag{84}$$

The composition $\psi_{\mathfrak{U}} \circ \mu_j$ is a strict homomorphism of Lie groupoid extensions from $X''_\bullet[\mathfrak{V}] \to Y''_\bullet[\mathfrak{V}]$ to $X_\bullet \to Y_\bullet$. It is obvious from diagram (84) that

$$X''_\bullet \xleftarrow{\chi_{\mathfrak{V}}} X''_\bullet[\mathfrak{V}] \xrightarrow{\psi_{\mathfrak{U}} \circ \mu_j} X_\bullet$$

is a bimodule representing the generalized homomorphism $F_1 \circ F_2$. Therefore,

$$\begin{aligned} (F_1 \circ F_2)^* &= (\chi_{\mathfrak{V}}^*)^{-1} \circ (\psi_{\mathfrak{U}} \circ \mu_j)^* = (\chi_{\mathfrak{V}}^*)^{-1} \circ \mu_j^* \circ \psi_{\mathfrak{U}}^* \\ &= (\chi_{\mathfrak{V}}^*)^{-1} \circ (\psi_{\mathfrak{V}})^* \circ (\chi_{\mathfrak{U}}^*)^{-1} \circ \psi_{\mathfrak{U}}^* = F_2^* \circ F_1^*. \qquad \square \end{aligned}$$

Proof of Theorem 6.9. If F is a Morita equivalence from $X'_\bullet \to Y'_\bullet$ to $X_\bullet \to Y_\bullet$ given by the bitorsor $M' \xleftarrow{f} B \xrightarrow{g} M$, then, according to Proposition 2.10, $F^{-1} \circ F$ is a Morita equivalence from $X_\bullet \to Y_\bullet$ to itself, whose corresponding bitorsor is given by $M \leftarrow (B \times_{M'} B)/X'_1 \to M$. The later is canonically isomorphic to $M \leftarrow X_1 \to M$, which is indeed the generalized morphism corresponding to the identity strict homomorphism. Thus it follows from Proposition 6.11 that $(F^{-1} \circ F)^* = \mathrm{id}$. Therefore the conclusion follows from Proposition 6.11 immediately. $\square$

Remark 6.12. Note that the isomorphism between the horizontal cohomologies of Morita equivalent groupoid extensions is not canonical. It depends on the choice of the Morita equivalence bitorsors, as seen in the above proof.

6.4. Obstruction class to the existence of connections

The main purpose of this subsection is to introduce a characteristic class [obs] in $H^{2,1}_{\mathrm{hor}}(X_\bullet \to Y_\bullet)$, which measures the existence of connections on a Lie groupoid extension $X_\bullet \xrightarrow{\phi} Y_\bullet$.

Forgetting about the groupoid structure, consider any horizontal distribution H on the fiber bundle $X_1 \xrightarrow{\phi} Y_1$. Let $\theta \in \Omega^1(X_1, \mathbf{t}^*\mathfrak{K})$ be its associated $\mathfrak{K}$-valued 1-form, i.e. the unique 1-form that vanishes on H and satisfies Eq. (76). According to Proposition 6.3, $\partial^\rhd\theta$ is a $\mathfrak{K}$-valued horizontal 1-form on X_2. Moreover, since $(\partial^\rhd)^2 = 0$, $\partial^\rhd\theta$ is a 2-cocycle of the horizontal cohomology, and hence it defines a class in $H^{2,1}_{\mathrm{hor}}(X_\bullet \to Y_\bullet)$ that we call the **obstruction class** and denote simply by [obs]. This terminology is justified by the following

Proposition 6.13.

1. *The class* [obs] $\in H^{2,1}_{\mathrm{hor}}(X_\bullet \to Y_\bullet)$ *does not depend on the particular choice of a horizontal distribution H on the fiber bundle $X_1 \xrightarrow{\phi} Y_1$.*
2. *The class* [obs] *vanishes if, and only if, the groupoid extension admits a groupoid extension connection.*

Proof. 1. Let H and H' be any two horizontal distributions on the fiber bundle $X_1 \xrightarrow{\phi} Y_1$, and $\theta, \theta' \in \Omega^1(X_1, \mathbf{t}^*\mathfrak{K})$ their associated $\mathfrak{K}$-valued 1-forms. By construction, $\theta - \theta'$ is a horizontal 1-form on X_1. Therefore $\partial^\rhd\theta$ and $\partial^\rhd\theta'$ differ by a horizontal coboundary. Hence $[\partial^\rhd\theta] = [\partial^\rhd\theta']$. This proves (1).

2. Assume that [obs] $= 0$. The $\mathfrak{K}$-valued 1-form $\theta \in \Omega^1(X_1, \mathbf{t}^*\mathfrak{K})$ associated to any horizontal distribution H on the fiber bundle $X_1 \xrightarrow{\phi} Y_1$ gives a representative $\partial^\rhd\theta$ for the obstruction class [obs]. Hence, by assumption, there exists a horizontal $\mathfrak{K}$-valued 1-form ζ on X_1 such that $\partial^\rhd\theta = \partial^\rhd\zeta$. It is simple to check that $\alpha = \theta - \zeta \in \Omega^1(X_1, \mathbf{t}^*\mathfrak{K})$ is indeed a right connection 1-form for the groupoid extension $X_\bullet \xrightarrow{\phi} Y_\bullet$. □

Combining Proposition 6.13 with the Morita invariance of the horizontal cohomology obtained in Proposition 6.9, we are led to the following main result of this subsection.

Theorem 6.14.

1. *Given a Morita morphism f of Lie groupoid extensions from $X'_\bullet \to Y'_\bullet$ to $X_\bullet \to Y_\bullet$ as in Eq.* (2)*, let* [obs′] *and* [obs] *be their obstruction classes. Then* [obs′] $= f^*$[obs]*, where $f^* : H^{2,1}_{\mathrm{hor}}(X_\bullet \to Y_\bullet) \to H^{2,1}_{\mathrm{hor}}(X'_\bullet \to Y'_\bullet)$ is the homomorphism given by Eq.* (79)*.*
2. *If a Lie groupoid extension admits a connection, so does any Morita equivalent Lie groupoid extension.*

Proof. Observe that (2) is a trivial consequence of (1). Now let θ be a $\mathfrak{K}$-valued 1-form on X_1 satisfying Eq. (76). Then, according to Lemma 6.7(3), the pullback $f^*\theta \in \Omega^1(X'_1, \mathbf{t}'^*\mathfrak{K}')$ satisfies Eq. (76) as well. By Lemma 6.7(1), the relation $\partial^\rhd(f^*\theta) = f^*(\partial^\rhd\theta)$ holds, which implies that [obs′] $= f^*$[obs]. □

6.5. Moduli space of connections on a Lie groupoid extension

We now study the space of right connection 1-forms of a Lie groupoid extension $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ assuming that the obstruction class vanishes. Denote by $Z^1_{\mathrm{hor}}(X_1, \mathbf{t}^*\mathfrak{K})$ (resp. $Z^1_{\partial^{\rhd}}(X_1, \mathbf{t}^*\mathfrak{K})$) the space of horizontal 1-forms in $\Omega^1_{\mathrm{hor}}(X_1, \mathbf{t}^*\mathfrak{K})$ (resp. 1-forms in $\Omega^1(X_1, \mathbf{t}^*\mathfrak{K})$) which are $\partial^{\rhd}$-closed.

Proposition 6.15. *For any Lie groupoid extension $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ with vanishing obstruction class, the space of right connection 1-forms is an affine subspace of $Z^1(X_1, \mathbf{t}^*\mathfrak{K})$ with underlying vector space $Z^1_{\mathrm{hor}}(X_1, \mathbf{t}^*\mathfrak{K})$.*

Proof. Since the obstruction class vanishes, there exists a right connection form $\alpha \in \Omega^1(X_1, \mathbf{t}^*\mathfrak{K})$. It is easy to see that a 1-form $\alpha' \in \Omega^1(X_1, \mathbf{t}^*\mathfrak{K})$ is a right connection 1-form (i.e. is $\partial^{\rhd}$-closed and satisfies Eq. (76)) if, and only if, $\alpha - \alpha' \in Z^1_{\mathrm{hor}}(X_1, \mathbf{t}^*\mathfrak{K})$. □

Two right connection 1-forms α and α' on a Lie groupoid extension $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ are said to be equivalent if, and only if, $\alpha - \alpha' = \partial^{\rhd}\beta$ for some $\beta \in \Omega^1(M, \mathfrak{K})$. By $\mathcal{M}$ we denote the **moduli space of right connection 1-forms on a Lie groupoid extension**[4] $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$. The following proposition is thus immediate.

Proposition 6.16. *For a Lie groupoid extension $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ with vanishing obstruction class, the moduli space of right connection 1-forms $\mathcal{M}$ is an affine subspace of $H^{1,1}_{\partial^{\rhd}}(X_\bullet \to Y_\bullet)$ with underlying vector space $H^{1,1}_{\mathrm{hor}}(X_\bullet \to Y_\bullet)$.*

We now describe the relation between the moduli spaces of right connection 1-forms of Morita equivalent extensions. We start with a lemma.

Lemma 6.17. *Assume that f is a Morita morphism of Lie groupoid extensions from $X'_\bullet \xrightarrow{\phi'} Y'_\bullet$ to $X_\bullet \xrightarrow{\phi} Y_\bullet$. A 1-form $\alpha \in \Omega^1(X_1, \mathbf{t}^*\mathfrak{K})$ is a right connection 1-form of the Lie groupoid extension $X \xrightarrow{\phi} Y \rightrightarrows M$ if, and only if, $f^*\alpha \in \Omega^1(X'_1, \mathbf{t}'^*\mathfrak{K}')$ is a right connection 1-form of the Lie groupoid extension $X'_\bullet \xrightarrow{\phi} Y'_\bullet$.*

Proof. For any $\alpha \in \Omega^1(X_1, \mathbf{t}^*\mathfrak{K})$, according to Lemma 6.7(1), we have $f^*\partial^{\rhd}\alpha = \partial^{\rhd} f^*\alpha = 0$. Hence α is $\partial^{\rhd}$-closed if, and only if, so is $f^*\alpha$. According to Lemma 6.7(3), Eq. (76) holds for α if, and only if, it holds for $f^*\alpha$. Hence it follows that α is a right connection 1-form on $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ if, and only if, $f^*\alpha$ is a right connection 1-form on $X' \xrightarrow{\phi} Y' \rightrightarrows M'$. □

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[4] Readers should not confuse this with moduli space of flat connections in gauge theory. Here there are no gauge groups involved.

Theorem 6.18.

1. *For any Morita morphism of Lie groupoid extensions f from $X'_1 \xrightarrow{\phi'} Y'_1 \rightrightarrows M'$ to $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$, the pullback map $f^* : H^{1,1}_{\partial^{\rhd}}(X_\bullet \to Y_\bullet) \to H^{1,1}_{\partial^{\rhd}}(X'_\bullet \to Y'_\bullet)$ induces an isomorphism of their corresponding moduli spaces of right connection 1-forms $\mathcal{M}$ and $\mathcal{M}'$.*
2. *The moduli spaces of right connection 1-forms of Morita equivalent extensions are isomorphic.*

Proof. By Theorem 6.9, the map $f^* : H^{1,1}_{\partial^{\rhd}}(X_\bullet \to Y_\bullet) \to H^{1,1}_{\partial^{\rhd}}(X'_\bullet \to Y'_\bullet)$ is an isomorphism, which, according to Lemma 6.17, sends $\mathcal{M}$ into $\mathcal{M}'$. It is thus enough to show that the restriction of f^* to $\mathcal{M}$ is surjective onto $\mathcal{M}'$. Assume that $[\alpha'] \in \mathcal{M}'$. Since $f^* : H^{1,1}_{\partial^{\rhd}}(X_\bullet \to Y_\bullet) \to H^{1,1}_{\partial^{\rhd}}(X'_\bullet \to Y'_\bullet)$ is surjective by Theorem 6.9(1), there exists $\alpha \in Z^1_{\partial^{\rhd}}(X_1, \mathbf{t}^*\mathfrak{K})$ such that $f^*[\alpha] = [\alpha']$. That is, $f^*\alpha = \alpha' + \partial^{\rhd}\beta'$ for some $\beta' \in \Omega^1(M', \mathfrak{K}')$. However, since $\alpha' + \partial^{\rhd}\beta'$ is also a right connection 1-form on the groupoid extension $X'_1 \xrightarrow{\phi} Y'_1 \rightrightarrows M'$, according to Lemma 6.17, α must be a right connection 1-form on the groupoid extension $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$. This concludes the proof. □

6.6. Connections for G-gerbes over manifolds

The main goal of this subsection is to compute the horizontal cohomology of a groupoid G-extension of a Čech groupoid, i.e. of a G-gerbe over a manifold, where G is a connected Lie group. As a consequence, we show the existence of connections.

First we need to introduce some preliminary constructions. We do not have to restrict ourselves to the case of G-gerbes over manifolds at this point, although this is the only case that we are concerned about in applications.

Let $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ be a Lie groupoid G-extension. Its kernel is the Lie group bundle $\mathcal{K} \to M$ with Lie algebra bundle $\mathfrak{K} \to M$. For any $n \in \mathbb{N}$, consider the fiberwise group cohomology $\coprod_m H^n(\mathcal{K}_m, \mathfrak{K}_m)$ of $\mathcal{K}$ with values in $\mathfrak{K}$ relative to the adjoint action, which is clearly a vector bundle over M, denoted by $H^n(\mathcal{K}, \mathfrak{K}) \to M$.

As we have seen in Section 2.1, any element $x \in X_1$ defines a Lie group isomorphism $\mathbf{Ad}_x : \mathcal{K}_{\mathbf{t}(x)} \to \mathcal{K}_{\mathbf{s}(x)}$, and thus a Lie algebra isomorphism $\mathrm{Ad}_x : \mathfrak{K}_{\mathbf{t}(x)} \to \mathfrak{K}_{\mathbf{s}(x)}$. These isomorphisms induce an isomorphism of cochain complexes mapping $f \in C^\infty(\mathcal{K}_{\mathbf{s}(x)} \times \cdots \times \mathcal{K}_{\mathbf{s}(x)}, \mathfrak{K}_{\mathbf{s}(x)})$ to $(k_1, \ldots, k_n) \mapsto \mathrm{Ad}_{x^{-1}} f(\mathrm{Ad}_x k_1, \ldots, \mathrm{Ad}_x k_n)$ in $C^\infty(\mathcal{K}_{\mathbf{t}(x)} \times \cdots \times \mathcal{K}_{\mathbf{t}(x)}, \mathfrak{K}_{\mathbf{t}(x)})$. Therefore it induces an isomorphism $H^n(\mathcal{K}_{\mathbf{s}(x)}, \mathfrak{K}_{\mathbf{s}(x)}) \xrightarrow{\sim} H^n(\mathcal{K}_{\mathbf{t}(x)}, \mathfrak{K}_{\mathbf{t}(x)})$, which is still denoted by $\mathbf{Ad}_x$ by abuse of notation. In other words, $H^n(\mathcal{K}, \mathfrak{K}) \to M$ is a vector bundle over M on which the groupoid $X_1 \rightrightarrows M$ acts from the left. That is, $H^n(\mathcal{K}, \mathfrak{K}) \to M$ is a left vector bundle over the groupoid $X_1 \rightrightarrows M$. In fact, this action descends and induces an action of the groupoid $Y_1 \rightrightarrows M$ since any element in the kernel $\mathcal{K}$ of $X_\bullet \xrightarrow{\phi} Y_\bullet$ acts trivially on $H^n(\mathcal{K}, \mathfrak{K}) \to M$. Indeed for a connected Lie group G, it is well known that the inner automorphisms of G preserve the cohomology classes of $H^n(G, \mathfrak{g})$. Thus we have proved the following

Proposition 6.19. *For all $n \in \mathbb{N}$, $H^n(\mathcal{K}, \mathfrak{K}) \to M$ is a vector bundle over the groupoid $Y_1 \rightrightarrows M$.*

Considering $\mathcal{K}$ as a Lie subgroupoid of $X_1 \rightrightarrows M$, one obtains a monomorphism of groupoid extensions i from $\mathcal{K} \to M \rightrightarrows M$ to $X_1 \to Y_1 \rightrightarrows M$. The horizontal cohomology of $\mathcal{K} \to M \rightrightarrows M$ can be easily described by the following

Lemma 6.20. *We have*

$$H^{n,l}_{\mathrm{hor}}(\mathcal{K}_\bullet \to M_\bullet) \simeq \Omega^l\big(M, H^n(\mathcal{K}, \mathfrak{K})\big), \tag{85}$$

where $H^n(\mathcal{K}, \mathfrak{K}) \to M$ is the vector bundle constructed above (ignoring the $Y_\bullet$-action).

Proof. The natural identification $\mathfrak{X}^l_{\mathrm{hor}}(\mathcal{K}_n) \simeq \Gamma(\pi^* \wedge^l TM)$ given by the differential of $\pi : \mathcal{K}_n \to M$ induces an isomorphism $\Omega^l_{\mathrm{hor}}(\mathcal{K}_n, \mathbf{t}^*\mathfrak{K}) \simeq \Omega^l(M, E_n)$, where E_n is the vector bundle $\coprod_{m\in M} C^\infty((\mathcal{K}|_m)^n, \mathfrak{K}_m) \to M$. It is simple to check that, under this isomorphism, the differential $\partial^\triangleright : \Omega^l_{\mathrm{hor}}(\mathcal{K}_n, \mathbf{t}^*\mathfrak{K}) \to \Omega^l_{\mathrm{hor}}(\mathcal{K}_{n+1}, \mathbf{t}^*\mathfrak{K})$ becomes the operator $\partial : \Omega^l(M, E_n) \to \Omega^l(M, E_{n+1})$, which is the fiberwise group cohomology differential $C^\infty((\mathcal{K}|_m)^n, \mathfrak{K}_m) \to C^\infty((\mathcal{K}|_m)^{n+1}, \mathfrak{K}_m)$. Taking its cohomology, one obtains an isomorphism $H^{n,l}_{\mathrm{hor}}(\mathcal{K}_\bullet \to M_\bullet) \simeq \Omega^l(M, H^n(\mathcal{K}, \mathfrak{K}))$. $\square$

For all $n \in \mathbb{N}$, from the induced map $i : \mathcal{K}_n \to X_n$, we obtain a chain map $i^* : \Omega^l_{\mathrm{hor}}(X_n, \mathbf{t}^*\mathfrak{K}) \to \Omega^l_{\mathrm{hor}}(\mathcal{K}_n, \mathbf{t}^*\mathfrak{K})$ and hence a map in cohomology

$$i^* : H^{n,l}_{\mathrm{hor}}(X_\bullet \to Y_\bullet) \to H^{n,l}_{\mathrm{hor}}(\mathcal{K}_\bullet \to M_\bullet). \tag{86}$$

Composing the maps given by Eq. (86) with Eq. (85), we are led to

Proposition 6.21. *Let $X_1 \to Y_1 \rightrightarrows M$ be a Lie groupoid G-extension, with kernel being the Lie group bundle $\mathcal{K} \to M$ with corresponding Lie algebra bundle $\mathfrak{K} \to M$. For any $l \in \mathbb{N}$, the inclusion map $i : \mathcal{K}_\bullet \to X_\bullet$ induces a natural restriction map*

$$i^* : H^{n,l}_{\mathrm{hor}}(X_\bullet \to Y_\bullet) \to \Omega^l\big(M, H^n(\mathcal{K}, \mathfrak{K})\big), \tag{87}$$

where $H^n(\mathcal{K}, \mathfrak{K}) \to M$ is the vector bundle constructed above (ignoring the $Y_\bullet$-action).

It is simple to see that this restriction map i^* is stable with respect to Morita morphisms of Lie groupoid extensions.

Lemma 6.22. *Let f be a Morita morphism of Lie groupoid extensions from $X'_\bullet \xrightarrow{\phi'} Y'_\bullet$ to $X_\bullet \xrightarrow{\phi} Y_\bullet$ as in diagram* (2). *Let $\mathcal{K}$ and $\mathcal{K}'$ denote the kernels of ϕ and ϕ' respectively. Then the following diagram commutes:*

$$\begin{array}{ccc} H^{n,l}_{\mathrm{hor}}(X_\bullet \to Y_\bullet) & \xrightarrow{\;f^*\;} & H^{n,l}_{\mathrm{hor}}(X'_\bullet \to Y'_\bullet) \\ \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i'^*} \\ \Omega^l(M, H^n(\mathcal{K}, \mathfrak{K})) & \xrightarrow[\;f^*\;]{} & \Omega^l(M', H^n(\mathcal{K}', \mathfrak{K}')). \end{array}$$

Let us turn our attention to the case of a G-extension of a Čech groupoid $\coprod_{ij} U_{ij} \rightrightarrows \coprod_i U_i$ associated to an open covering $(U_j)_{j\in J}$ of a manifold N. In this case, vector bundles over the

Čech groupoid $\coprod_{ij} U_{ij} \rightrightarrows \coprod_i U_i$ are in 1–1 correspondence with ordinary vector bundles over the manifold N. For all $n \in \mathbb{N}$, we denote by $H^n(\mathcal{K}, \mathfrak{K})_N$ the vector bundle over N corresponding to the vector bundle $H^n(\mathcal{K}, \mathfrak{K})$ over $\coprod_{ij} U_{ij} \rightrightarrows \coprod_i U_i$ obtained as in Proposition 6.19.

Theorem 6.23. *Let $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ denote a Lie groupoid G-extension $X_1 \to \coprod_{ij} U_{ij} \rightrightarrows \coprod_j U_j$ of a Čech groupoid $\coprod_{ij} U_{ij} \rightrightarrows \coprod_j U_j$ associated to an open covering $\{U_j\}_{j\in J}$ of a manifold N. Then, for any pair $n, l \in \mathbb{N}$, the map in Eq.* (87) *factorizes through $\Omega^l(N, H^n(\mathcal{K}, \mathfrak{K})_N) \to \Omega^l(M, H^n(\mathcal{K}, \mathfrak{K}))$:*

$$\begin{tikzcd} & H^{n,l}_{\mathrm{hor}}(X_\bullet \to Y_\bullet) \arrow[dl, "\tilde{\imath}^*"'] \arrow[d, "i^*"] \\ \Omega^l(N, H^n(\mathcal{K}, \mathfrak{K})_N) \arrow[r, "p^*"'] & \Omega^l(M, H^n(\mathcal{K}, \mathfrak{K})). \end{tikzcd} \tag{88}$$

Moreover, $\tilde{\imath}^* : H^{n,l}_{\mathrm{hor}}(X_\bullet \to Y_\bullet) \to \Omega^l(N, H^n(\mathcal{K}, \mathfrak{K})_N)$ is an isomorphism and the restriction map $i^* : H^{n,l}_{\mathrm{hor}}(X_\bullet \to Y_\bullet) \to \Omega^l(M, H^n(\mathcal{K}, \mathfrak{K}))$ is injective.

We need some preliminaries. Any smooth function f on the manifold N can be pulled back to a function $\tilde{f}_n$ on X_n using any appropriate composition of consecutive face maps together with the projection $\coprod_i U_i \xrightarrow{p} N$. In other words, $\tilde{f}_n = p_n^* f$, where p_n is the composition $X_n \xrightarrow{\phi} \coprod_{j_1,\ldots,j_n} U_{j_1\cdots j_n} \to N$. The following result is straightforward.

Lemma 6.24. *For any $\zeta \in \Omega^l(X_n, \mathbf{t}^*\mathfrak{K})$ and any function $f \in C^\infty(N)$, we have*

$$\partial^{\rhd}(\tilde{f}_n \zeta) = \tilde{f}_{n+1} \partial^{\rhd} \zeta.$$

For any open set $U \subset N$, we denote by $X_1[p^{-1}U] \rightrightarrows p^{-1}(U)$ and $Y_1[p^{-1}U] \rightrightarrows p^{-1}(U)$ the restriction of the groupoids $X_1 \rightrightarrows M$ and $Y_1 \rightrightarrows M$ to the open submanifold $p^{-1}(U)$ of the unit space M, respectively. It is clear that $X_1[p^{-1}U] \to Y_1[p^{-1}U] \rightrightarrows p^{-1}(U)$ is a G-extension.

For any fixed integers n and l, we define a pre-sheaf $\mathcal{E}^{n,l}$ over N by $U \mapsto H^{n,l}_{\mathrm{hor}}(X_\bullet[p^{-1}U] \to Y_\bullet[p^{-1}U])$, the restriction maps $r^U_V : H^{n,l}_{\mathrm{hor}}(X_\bullet[p^{-1}U] \to Y_\bullet[p^{-1}U]) \to H^{n,l}_{\mathrm{hor}}(X_\bullet[p^{-1}V] \to Y_\bullet[p^{-1}V])$ being the pullbacks of the natural inclusion of $X_\bullet[p^{-1}V] \to Y_\bullet[p^{-1}V]$ into $X_\bullet[p^{-1}U] \to Y_\bullet[p^{-1}U]$ for any open subsets $V \subset U$ of N.

Lemma 6.25. *For any $n, l \in \mathbb{N}$, the pre-sheaf $\mathcal{E}^{n,l}$ is a sheaf.*

Proof. First, let $(U_i)_{i\in I}$ be open subsets of N, $U = \bigcup_{i\in I} U_i$ and $(f_i)_{i\in I}$ a partition of unity of U with $\mathrm{supp}(f_i) \subset U_i$ for all $i \in I$ (which exists since we consider paracompact manifolds only). First, let $[\omega_i] \in H^{n,l}_{\mathrm{hor}}(X_\bullet[U_i] \to Y_\bullet[U_i])$ be the cohomology class of some horizontal cocycle $\omega_i \in \Omega^l(X_n[U_i], \mathbf{t}^*\mathfrak{K})$, for all $i \in I$. If $r^{U_i}_{U_i\cap U_j}[\omega_i] = r^{U_j}_{U_i\cap U_j}[\omega_j]$ for all $i, j \in I$, then, thanks to Lemma 6.24, $\omega = \sum_{i\in I} \widetilde{(f_i)}_n \omega_i$ is a horizontal cocycle in $\Omega^l(X_n[U], \mathbf{t}^*\mathfrak{K})$. By construction, its class $[\omega] \in H^{n,l}_{\mathrm{hor}}(X_\bullet[U] \to Y_\bullet[U])$ satisfies $r^U_{U_i}[\omega] = [\omega_i]$, for all $i \in I$.

Secondly, let $[\omega] \in H^{n,l}_{\mathrm{hor}}(X_\bullet[U] \to Y_\bullet[U])$ be the class of some horizontal cocycle $\omega \in \Omega^l(X_n, \mathbf{t}^*\mathfrak{K})$. If $r^U_{U_i}[\omega] = 0$ for all $i \in I$, then there exists a horizontal form $\eta_i \in \Omega^l(X_{n-1}[U_i], \mathbf{t}^*\mathfrak{K})$ such that $\partial^{\rhd}\eta_i = \omega|_{U_i}$ for all $i \in I$. Set $\eta = \sum_{i\in I} \widetilde{(f_i)}_{n-1}\eta_i$. By Lemma 6.24, we have

$$\partial^{\rhd}\eta = \sum_{i\in I} \widetilde{(f_i)}_n \partial^{\rhd}\eta_i = \sum_{i\in I} \widetilde{(f_i)}_n \omega|_{U_i} = \omega.$$

As a consequence, we have $[\omega] = 0$. Therefore, $\mathcal{E}^{n,l}$ is a sheaf. $\square$

We define two other sheaves over N:

$$\mathcal{F}^{n,l} : U \mapsto \Omega^l\big(U, H^n(\mathcal{K}, \mathfrak{K})_N\big),$$
$$\mathcal{G}^{n,l} : U \mapsto \Omega^l\big(p^{-1}U, H^n(\mathcal{K}, \mathfrak{K})\big).$$

The covering $M \xrightarrow{p} N$ induces a morphism of sheaves $\mathcal{F}^{n,l} \xrightarrow{p^*} \mathcal{G}^{n,l}$ and, by Proposition 6.21, the inclusion $\mathcal{K}_\bullet \xrightarrow{i} X_\bullet$ induces a morphism of sheaves $\mathcal{E}^{n,l} \xrightarrow{i^*} \mathcal{G}^{n,l}$.

Now we need a general fact about sheaves over a manifold.

Lemma 6.26. *Let $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$ be sheaves over a manifold N, and let $h : \mathcal{E} \to \mathcal{G}$ and $p : \mathcal{F} \to \mathcal{G}$ be morphisms of sheaves with p injective. Assume that for any contractible open set U, the map $h_U : \mathcal{E}(U) \to \mathcal{G}(U)$ factorizes through*

$$\begin{tikzcd} & \mathcal{E}(U) \arrow[dl, "\tilde{h}_U"'] \arrow[d, "h_U"] \\ \mathcal{F}(U) \arrow[r, "p_U"'] & \mathcal{G}(U) \end{tikzcd} \tag{89}$$

where the map $\tilde{h}_U : \mathcal{E}(U) \to \mathcal{F}(U)$ is an isomorphism. Then the morphism of sheaves $h : \mathcal{E} \to \mathcal{G}$ factorizes through

$$\begin{tikzcd} & \mathcal{E} \arrow[dl, "\tilde{h}"'] \arrow[d, "h"] \\ \mathcal{F} \arrow[r, "p"'] & \mathcal{G} \end{tikzcd}$$

where $\tilde{h} : \mathcal{E} \to \mathcal{F}$ is an isomorphism of sheaves.

We can now turn to the proof of the our main results in this section.

Proof of Theorem 6.23. It remains to show that the morphism of sheaves i^* factorizes through the monomorphism p^* as an isomorphism of sheaves $\tilde{\imath}^*$:

$$\begin{tikzcd} & \mathcal{E}^{n,l} \arrow[dl, "\tilde{\imath}^*"'] \arrow[d, "i^*"] \\ \mathcal{F}^{n,l} \arrow[r, "p^*"'] & \mathcal{G}^{n,l}. \end{tikzcd} \tag{90}$$

Indeed, evaluation of the pre-sheaves $\mathcal{E}^{n,l}$, $\mathcal{F}^{n,l}$ and $\mathcal{G}^{n,l}$ in diagram (90) on N, seen as an open subset of itself, yields diagram (88).

As a consequence of Lemma 6.26, since $\mathcal{E}^{n,l}$, $\mathcal{F}^{n,l}$ and $\mathcal{G}^{n,l}$ are sheaves, it suffices to prove that, for any contractible open subset U of N, there exists a group isomorphism $\tilde{\imath}_U^*$ such that

$$\begin{tikzcd} & H^{n,l}_{\mathrm{hor}}(X_\bullet[p^{-1}U] \to Y_\bullet[p^{-1}U]) \arrow[dl, "\tilde{\imath}_U^*"'] \arrow[d, "i_U^*"] \\ \Omega^l(U, H^n(\mathcal{K}, \mathfrak{K})_N) \arrow[r, "p_U^*"'] & \Omega^l(p^{-1}U, H^n(\mathcal{K}, \mathfrak{K})) \end{tikzcd} \tag{91}$$

commutes.

However, according to Corollary 3.19, the G-extension $X_\bullet[p^{-1}U] \to Y_\bullet[p^{-1}U]$ of the Čech groupoid $\coprod_{i,j} U_{ij} \cap U \rightrightarrows \coprod_i U_i \cap U$ of the contractible open manifold U has a refinement which is isomorphic to a trivial G-extension. In other words, there exists a refinement $\mathfrak{V} = \coprod_{\alpha \in A} V_\alpha$ of the open covering $\coprod_{i \in I} U_i \cap U$ of $p^{-1}U$ together with a Morita equivalence

$$\begin{tikzcd} U \times G \arrow[d] & \coprod V_{\alpha\beta} \times G \arrow[l] \arrow[r] \arrow[d] & X_1[p^{-1}U] \arrow[d] \\ U \arrow[d, shift left] \arrow[d, shift right] & \coprod V_{\alpha\beta} \arrow[l] \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & Y_1[p^{-1}U] \arrow[d, shift left] \arrow[d, shift right] \\ U & \coprod V_\alpha \arrow[l] \arrow[r] & p^{-1}(U). \end{tikzcd}$$

Therefore, applying Lemma 6.22 to each Morita morphism, one obtains the commutative diagram

$$\begin{tikzcd} H^{n,l}_{\mathrm{hor}}((U \times G)_\bullet \to U_\bullet) \arrow[r, "(2)"] \arrow[d, "(3)"'] & H^{n,l}_{\mathrm{hor}}((U \times G)_\bullet[\mathfrak{V}] \to U_\bullet[\mathfrak{V}]) \arrow[d] & H^{n,l}_{\mathrm{hor}}(X_\bullet[p^{-1}U] \to Y_\bullet[p^{-1}U]) \arrow[l, "(1)"'] \arrow[d, "i_U^*"] \\ \Omega^l(U, H^n(G, \mathfrak{g})) \arrow[r, "(4)"'] & \Omega^l(\mathfrak{V}, H^n(G, \mathfrak{g})) & \Omega^l(p^{-1}U, H^n(G, \mathfrak{g})) \arrow[l, "(5)"] \end{tikzcd}$$

where, the restrictions of $H^n(\mathcal{K}, \mathfrak{K})$ to contractible open sets have been identified with the trivial vector bundles with fiber $H^n(G, \mathfrak{g})$. Here (3) is an isomorphism by Lemma 6.9, (1) and (2) are isomorphisms by Theorem 6.9 and the injection (4) factorizes through the injection (5) to yield p_U^*. Thus one obtains the commutative diagram (91). □

We can now state the main corollary of this result.

Corollary 6.27. *Any groupoid G-extension of a Čech groupoid admits a connection.*

Proof. It suffices to prove that the obstruction class [obs] of any groupoid G-extension of a Čech groupoid vanishes. Since the vanishing of [obs] is preserved under Morita equivalence, we can assume that $(U_j)_{j\in J}$ is a good open covering and $X_1 \simeq \coprod_{ij} U_{ij} \times G$, $Y_1 \simeq \coprod_{ij} U_{ij}$. In particular $\mathcal{K} \simeq \coprod_i U_i \times G$ is the trivial group bundle. Consider the $\mathfrak{g}$-valued 1-form on X_1 given by $\theta = p_2^*\theta_G$, where $p_2 : \coprod_{ij} U_{ij} \times G \to G$ is the projection onto the second component and θ_G is the right Maurer–Cartan form on the Lie group G. Since θ satisfies Eq. (76), we have $[\mathrm{obs}] = [\partial^{\rhd}\theta]$. Moreover, by construction, one has $i^*[\mathrm{obs}] = i^*[\partial^{\rhd}\theta] = [\partial\theta_G] = 0$, where $i^* : H^{2,1}_{\mathrm{hor}}(X_\bullet \to Y_\bullet) \to H^{2,1}_{\mathrm{hor}}(\mathcal{K}_\bullet \to M_\bullet)$ is the restriction map. By Theorem 6.23, i^* is injective. Hence the obstruction class [obs] vanishes. □

Note that for bundle gerbes, the existence of connections was proved by Murray [42]. Another consequence of Theorem 6.23 is the following

Corollary 6.28. *Let $X_\bullet \xrightarrow{\phi} Y_\bullet$ be a Lie groupoid G-extension $X_1 \to \coprod_{ij} U_{ij} \rightrightarrows \coprod_j U_j$ of a Čech groupoid $\coprod_{ij} U_{ij} \rightrightarrows \coprod_j U_j$ associated to an open covering $(U_j)_{j\in J}$ of a manifold N. Assume that α is a right connection 1-form with Ehresmann curvature $\omega \in \Omega^2(X_1, \mathbf{t}^*\mathfrak{K})$.*

1. *The class of ω in $H^{1,2}_{\mathrm{hor}}(X_\bullet \to Y_\bullet)$ is zero if, and only if, the band is flat.*
2. *If N is simply connected and $[\omega]$ vanishes in $H^{1,2}_{\mathrm{hor}}(X_\bullet \to Y_\bullet)$, then the band is trivial. And therefore the extension is central.*

Proof. 1. According to Corollary 5.16, the band is flat if, and only if, $[\omega^{\mathcal{K}}] = 0$ in $H^{1,2}_{\mathrm{hor}}(\mathcal{K}_\bullet \to M_\bullet)$ or, equivalently, if, and only if, $[\omega] = 0$ in $H^{1,2}_{\mathrm{hor}}(X_\bullet \to Y_\bullet)$ since $i^* : H^{1,2}_{\mathrm{hor}}(\mathcal{K}_\bullet \to M_\bullet) \to H^{1,2}_{\mathrm{hor}}(X_\bullet \to Y_\bullet)$ is injective.

2. The band can be identified with an $\mathrm{Out}(G)$-principal bundle over N. Hence, since N is simply connected, it must be trivial if it is flat. □

6.7. Flat gerbes

First of all, we introduce the following definition which generalizes the same notion in the case of abelian gerbes [8,17,29,42].

Definition 6.29. Given a Lie groupoid extension $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$, and a right connection 1-form $\alpha \in \Omega^1(X_1, \mathbf{t}^*\mathfrak{K})$ with Ehresmann curvature $\omega \in \Omega^2(X_1, \mathbf{t}^*\mathfrak{K})$,

1. a **curving** is a two-form $B \in \Omega^2(M, \mathfrak{K})$ such that $\partial^{\rhd} B = \omega$;
2. and given (α, B), $\Omega = d^{\nabla} B \in \Omega^3(M, \mathfrak{K})$ is called the **3-curvature** of (α, B), where $d^{\nabla} : \Omega^\bullet(M, \mathfrak{K}) \to \Omega^{\bullet+1}(M, \mathfrak{K})$ is the exterior covariant derivative with respect to the induced connection ∇ on the Lie algebra bundle $\mathfrak{K} \to M$ as in Section 4.5.

Proposition 6.30.

1. *Given a Lie groupoid extension* $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$*, and a right connection* 1*-form* $\alpha \in \Omega^1(X_1, \mathbf{t}^*\mathfrak{K})$ *with Ehresmann curvature* $\omega \in \Omega^2(X_1, \mathbf{t}^*\mathfrak{K})$*, a curving exists if, and only if,* $[\omega] \in H^{1,2}_{\mathrm{hor}}(X_\bullet \to Y_\bullet)$ *vanishes.*
2. *If* $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ *is the groupoid extension of a Čech groupoid, then a curving exists, if and only if, the band is flat.*

Proof. The first assertion is straightforward. And the second follows from the first and Corollary 6.28. □

Remark 6.31. The above proposition indicates that the existence of both connections and curvings on a G-gerbe over a manifold would force it to be close to being a G-bound gerbe (or an abelian gerbe).

The following lemma will be useful. We denote by $\mathrm{Ad}_{x^{-1}}$ the isomorphism $\Gamma(\mathbf{s}^*\mathfrak{K} \to M) \xrightarrow{\simeq} \Gamma(\mathbf{t}^*\mathfrak{K} \to M)$ obtained by mapping a section $\sigma \in \Gamma(\mathbf{s}^*\mathfrak{K} \to M)$ to the section $x \to \mathrm{Ad}_{x^{-1}}\sigma(x)$ in $\Gamma(\mathbf{t}^*\mathfrak{K} \to M)$.

Lemma 6.32. *For any* $\eta \in \Omega^k(M, \mathfrak{K})$*, we have*

$$\big(d^{\nabla^{\mathbf{t}}}\partial^{\triangleright} - \partial^{\triangleright} d^{\nabla}\big)\eta = [\alpha, \mathrm{Ad}_{x^{-1}}\mathbf{s}^*\eta].$$

Proof. Since $\nabla^{\mathbf{t}}$ is the pullback of ∇ via $\mathbf{t}$, one has $(d^{\nabla^{\mathbf{t}}}\mathbf{t}^* - \mathbf{t}^* d^{\nabla})\eta = 0$. Therefore, it suffices to prove the following relation

$$\big(d^{\nabla^{\mathbf{t}}}\mathrm{Ad}_{x^{-1}}\mathbf{s}^* - \mathrm{Ad}_{x^{-1}}\mathbf{s}^* d^{\nabla}\big)\eta = [\alpha, \mathrm{Ad}_{x^{-1}}\mathbf{s}^*\eta]. \tag{92}$$

The latter is equivalent to:

$$\nabla^{\mathbf{t}}_u \mathrm{Ad}_{x^{-1}}\mathbf{s}^*\sigma - \mathrm{Ad}_{x^{-1}}\mathbf{s}^*\nabla_{\mathbf{s}_* u}\sigma = \big[\alpha(u), \mathrm{Ad}_{x^{-1}}\mathbf{s}^*\sigma\big], \quad \forall u \in T_x X_1, \tag{93}$$

where σ is any local section of $\mathfrak{K} \to M$ in a neighborhood of $\mathbf{s}(x)$.

If $u = \xi^{\triangleright}$ is vertical, then

$$\nabla^{\mathbf{t}}_u \mathrm{Ad}_{x^{-1}}\mathbf{s}^*\sigma = \frac{d}{d\tau}\bigg|_{\tau=0} \mathrm{Ad}_{(x\cdot\exp(\tau\xi))^{-1}}\mathbf{s}^*\sigma = [\xi, \mathrm{Ad}_{x^{-1}}\mathbf{s}^*\sigma] = \big[\alpha(u), \mathrm{Ad}_{x^{-1}}\mathbf{s}^*\sigma\big].$$

Hence Eq. (93) holds.

On the other hand, if $u \in H_x$ is horizontal, and $X \in \mathfrak{X}(X_1)$ is a horizontal vector field through u, then according to Eq. (47), we have $\nabla^{\mathbf{t}}_u(\mathrm{Ad}_{x^{-1}}\mathbf{s}^*\sigma) = \alpha([X, (\mathrm{Ad}_{x^{-1}}\mathbf{s}^*\sigma)^{\triangleright}])|_x$. Since $(\mathrm{Ad}_{x^{-1}}\mathbf{s}^*\sigma)^{\triangleright} = \mathbf{s}^*\sigma^{\triangleleft}$ and $\alpha = \mathrm{Ad}_{x^{-1}} \circ \beta$, we have $\nabla^{\mathbf{t}}_u(\mathrm{Ad}_{x^{-1}}\mathbf{s}^*\sigma) = \mathrm{Ad}_{x^{-1}} \circ \beta([X, \mathbf{s}^*\sigma^{\triangleleft}])|_x$. According to Eq. (48), we have $\nabla^{\mathbf{t}}_u(\mathrm{Ad}_{x^{-1}}\mathbf{s}^*\sigma) = \mathrm{Ad}_{x^{-1}}(\nabla^{\mathbf{s}}_u(\mathbf{s}^*\sigma)) = \mathrm{Ad}_{x^{-1}}\mathbf{s}^*\nabla_{\mathbf{s}_* u}\sigma$. Eq. (93) thus holds in this case. This concludes the proof. □

Let $Z(\mathfrak{K}) \to M$ be the subbundle of $\mathfrak{K} \to M$ defined in Remark 5.19. We have the following:

Theorem 6.33. *Assume that $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ is a Lie groupoid extension with connection and curving. Let $\Omega \in \Omega^3(M, \mathfrak{K})$ be the corresponding 3-curvature. Then $\Omega \in \Omega^3(M, Z(\mathfrak{K}))$. Moreover, we have*

$$d^\nabla \Omega = 0, \qquad \partial^\triangleright \Omega = 0. \tag{94}$$

Proof. Denote the curving by $B \in \Omega^2(M, \mathfrak{K})$. When being restricted to the group bundle $\mathcal{K} \xrightarrow{\pi} M$, the relation $\partial^\triangleright B = \omega$ yields that $\pi^* B - \mathrm{Ad}_{k^{-1}} \pi^* B = \omega^{\mathcal{K}}(k)$, $\forall k \in \mathcal{K}$. Differentiating this relation with respect to k at the identity, and using Eq. (70), one obtains that

$$-\mathrm{ad}_B = \omega^{\mathfrak{K}}. \tag{95}$$

Recall that $\omega^{\mathfrak{K}} \in \Omega^2(M, \mathrm{End}(\mathfrak{K}))$ is the curvature of the covariant derivative on the Lie algebra bundle $\mathfrak{K} \to M$. By the Bianchi identity, the relation $d^\nabla \omega^{\mathfrak{K}} = 0$ holds. We therefore have

$$0 = d^\nabla \omega^{\mathfrak{K}} = -d^\nabla \mathrm{ad}_B = -\mathrm{ad}_{d^\nabla B} = -\mathrm{ad}_\Omega .$$

As a consequence, the 3-curvature takes its values in the subbundle $Z(\mathfrak{K}) \to M$.

We have $d^\nabla \Omega = (d^\nabla)^2 B = -\omega^{\mathfrak{K}}(B) = -\mathrm{ad}_B(B) = -[B, B] = 0$ since B is a 2-form. According to Eq. (95), the Bianchi identity (52) reads $d^{\nabla^{\mathbf{t}}} \partial^\triangleright B + [\alpha, \partial^\triangleright B] = -\mathrm{ad}_{\mathbf{t}^* B}(\alpha) = [\alpha, \mathbf{t}^* B]$. The latter is equivalent to $d^{\nabla^{\mathbf{t}}} \partial^\triangleright B - [\alpha, \mathrm{Ad}_{x^{-1}} \mathbf{s}^* B] = 0$. According to Lemma 6.32, we have $\partial^\triangleright \Omega = \partial^\triangleright d^\nabla B = 0$. This concludes the proof. □

Definition 6.34. A Lie groupoid extension $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ is called **flat** if there exists a right connection 1-form $\alpha \in \Omega^1(X_1, \mathbf{t}^* \mathfrak{K})$, a curving $B \in \Omega^2(M, \mathfrak{K})$ such that the induced connection on the group bundle $\mathcal{K} \xrightarrow{\pi} M$ is flat, and the 3-curvature Ω vanishes.

Recall that, from Corollary 5.16, the band is flat if, and only if, $[\omega^{\mathcal{K}}] = 0$ in $H^{1,2}_{\mathrm{hor}}(\mathcal{K}_\bullet \to M_\bullet)$. Hence when a Lie groupoid extension is flat, its band must be flat. In fact, requiring the band to be flat is slightly weaker than requiring the group bundle $\mathcal{K} \xrightarrow{\pi} M$ to be flat.

The following proposition is obvious.

Proposition 6.35. *Given a connection on a Lie groupoid G-extension $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$, where M is a disjoint union of contractible manifolds, the following are equivalent:*

- *the group bundle is flat, i.e. $\omega^{\mathcal{K}} = 0$;*
- *there exists a trivialization $\chi : \mathcal{K} \simeq M \times G$ such that $F_g = 0$ for all $g \in G$.*

Let $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ be a G-extension with kernel $\mathcal{K} \xrightarrow{\pi} M$. Denote by $Z(\mathcal{K}) \xrightarrow{\pi} M$ its bundle of centers. Assume that G is a connected Lie group with compact center $Z(G)$, i.e. $Z(G)$ is isomorphic to a torus. Hence the automorphism group $\mathrm{Aut}(Z(G))$ is discrete. It thus follows that, for any connection on the extension $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$, the induced connection on the kernel $\mathcal{K} \xrightarrow{\phi} M$ defines a flat connection on the subgroup bundle $Z(\mathcal{K}) \xrightarrow{\phi} M$. Therefore, the induced connection on the Lie algebra bundle $Z(\mathfrak{K}) \to M$ is also flat, which implies that its pullback connection on $\mathbf{t}^* Z(\mathfrak{K}) \to (n \in \mathbb{N})$ is flat as well. By

$$D : \Omega^k\big(Y_n, \mathbf{t}^* Z(\mathfrak{K})\big) \to \Omega^{k+1}\big(Y_n, \mathbf{t}^* Z(\mathfrak{K})\big),$$

we denote its exterior covariant differential.

It is simple to see that the adjoint action of $X_1 \rightrightarrows M$ on $\mathfrak{K} \to M$ induces an action of $Y_1 \rightrightarrows M$ on $Z(\mathfrak{K}) \to M$. Thus we have a differential

$$\partial : \Omega^k\big(Y_n, \mathbf{t}^* Z(\mathfrak{K})\big) \to \Omega^k\big(Y_{n+1}, \mathbf{t}^* Z(\mathfrak{K})\big). \tag{96}$$

Lemma 6.36. *The following relations hold:*

$$\partial^2 = 0, \qquad D^2 = 0, \qquad [\partial, D] = 0.$$

Proof. The first relation is a general fact for any Lie groupoid representation. Let us prove the second one.

Since the fibers of $Z(\mathcal{K}) \to M$ are tori, for any $m \in M$, the inverse image of the exponential map $\exp : Z(\mathfrak{K})_m \to Z(\mathcal{K})_m$ of the identity element in $Z(\mathcal{K})_m$ is a lattice $L_m \subset Z(\mathfrak{K})_m$ of maximal rank. This lattice is smooth, i.e. it is generated by smooth sections. The identity section in $Z(\mathcal{K}) \to M$ is horizontal and by Lemma 4.6, a section of $Z(\mathfrak{K}) \to M$ is horizontal if, and only if, its image under the exponential map is horizontal. Hence it follows that any smooth section of the lattice $L \to M$ is horizontal. As a consequence, the pullback $\mathbf{t}^* l$ of a section l of the lattice L is a parallel section of $\mathbf{t}^* Z(\mathfrak{K}) \to Y_n$.

This fact allows us to give an explicit local expression of D with the help of sections of the lattice $L \to M$ as follows. Choose $l_1, \ldots, l_k$ local smooth generators of the lattice $L \to M$ on some open set $U \subset M$. Any $\omega \in \Omega^m(\mathbf{t}^{-1}(U), \mathbf{t}^* Z(\mathfrak{K}))$ can then be written uniquely as $\omega = \sum_{i=1}^k \omega_i \otimes \mathbf{t}^* l_i$ (where $\omega_1, \ldots, \omega_k$ are m-forms on $\mathbf{t}^{-1}(U)$ and $\mathbf{t}^* l_i$ denotes the pullback of l_i through $\mathbf{t}$, for all $i \in \{1, \ldots, k\}$). Hence, the covariant differential of ω is given by

$$D\omega = \sum_{i=1}^k (d\omega_i) \otimes \mathbf{t}^* l_i, \tag{97}$$

where d stands for the usual de Rham differential. Hence it follows that $D^2 = 0$.

Next we give an explicit expression of ∂ with the help of sections of the lattice $L \to M$. Consider any point $(y_1, \ldots, y_n) \in Y_n$. Let $(l_1, \ldots, l_k)$ be smooth generators of the lattice L in a neighborhood U of $\mathbf{t}(y_n)$ and $l'_1, \ldots, l'_k$ smooth generators in a neighborhood V of $\mathbf{s}(y_n)$. The adjoint action of any $y \in \mathbf{s}^{-1}(V) \cap \mathbf{t}^{-1}(U) \subset Y_1$ must preserve the lattice L. That is, for any $l \in L_{\mathbf{s}(y)}$, we have $(\mathrm{Ad}_y)^{-1} l \in L_{\mathbf{t}(y)}$. As a consequence, there exists a matrix $(M_a^j)_{a,j=1}^k$ with constant integral coefficients such that, on $\mathbf{s}^{-1}(V) \cap \mathbf{t}^{-1}(U)$, $\mathrm{Ad}_{y^{-1}} l'_j = \sum_{a=1}^k M_a^j l_j$. Taking the pullback under $\mathbf{t}$, one obtains that

$$\mathrm{Ad}_{y_n^{-1}} \mathbf{t}^* l'_j = \sum_{a=1}^k M_a^j \mathbf{t}^* l_j. \tag{98}$$

The restrictions of $\omega \in \Omega^m(Y_n, \mathbf{t}^* Z(\mathfrak{K}))$ to $\mathbf{t}^{-1}(U)$ and $\mathbf{t}^{-1}(V)$, can be written in an unique way as $\omega = \sum_{j=1}^k \omega_j \otimes \mathbf{t}^* l_j$ and $\omega = \sum_{j=1}^k \omega'_j \otimes \mathbf{t}^* l'_j$, respectively, where $\omega_1, \ldots, \omega_k \in \Omega^m(\mathbf{t}^{-1}(U))$

and $\omega'_1, \ldots, \omega'_i \in \Omega^m(\mathbf{t}^{-1}(V))$. By the definition of ∂, for any $(y_1, \ldots, y_n) \in Y_n$ with $y_n \in \mathbf{s}^{-1}(V) \cap \mathbf{t}^{-1}(U)$, we have:

$$\begin{aligned}
\partial\omega &= \sum_{i=0}^{n-1}(-1)^i \left(\varepsilon_i^n\right)^* \omega + (-1)^n \operatorname{Ad}_{y_n}^{-1} \left(\varepsilon_n^n\right)^* \omega \\
&= \sum_{i=0}^{n-1}\sum_{j=1}^{k}(-1)^i \left(\varepsilon_i^n\right)^* \omega_j \otimes \mathbf{t}^* l_j + (-1)^n \sum_{j=1}^{k}\left(\varepsilon_n^n\right)^* \omega'_j \otimes \operatorname{Ad}_{y_n}^{-1} \mathbf{t}^* l'_i \\
&= \sum_{i=0}^{n-1}\sum_{j=1}^{k}(-1)^i \left(\varepsilon_i^n\right)^* \omega_j \otimes \mathbf{t}^* l_j + (-1)^n \sum_{j=1}^{k}\sum_{a=1}^{k} M_a^j \left(\varepsilon_n^n\right)^* \omega'_j \otimes \mathbf{t}^* l_j.
\end{aligned} \tag{99}$$

The relation $[\partial, D] = 0$ thus follows immediately from Eqs. (97)–(99) together with the fact that the de Rham differential commutes with the pullback maps $(\varepsilon_i^n)^*$ for all $i \in \{0, \ldots, n\}$. □

If $Y_1 \rightrightarrows M$ is a Čech groupoid associated to an open covering of a manifold N, the Lie algebra bundle $Z(\mathfrak{K}) \to M$ over $Y_1 \rightrightarrows M$ corresponds to a Lie algebra bundle over N, denoted by $Z(\mathfrak{K})_N \to N$.

Lemma 6.37. *Assume that $Y_1 \rightrightarrows M$ is a Čech groupoid associated to an open covering of a manifold N. Then the cohomology of the cochain complex* (96) *is given by*

$$H^n\left(\Omega^k\left(Y_\bullet, \mathbf{t}^* Z(\mathfrak{K})\right), \partial\right) = \begin{cases} \Omega^k(N, Z(\mathfrak{K})_N), & n = 0, \\ 0, & n > 0. \end{cases} \tag{100}$$

Proof. $Y_1 \rightrightarrows M$ is Morita equivalent to $N \rightrightarrows N$, and under this Morita equivalence, the module $Z(\mathfrak{K}) \to M$ over $Y_1 \rightrightarrows M$ becomes the trivial one $Z(\mathfrak{K})_N \to N$. Thus the conclusion follows from a general fact regarding groupoid cohomology of Morita equivalent groupoids [20]. □

Now we are ready to state the main result of this section.

Theorem 6.38. *Let $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ be a G-extension with kernel $\mathcal{K} \xrightarrow{\phi} M$, and $\alpha \in \Omega^1(X_1, \mathbf{t}^*\mathfrak{K})$ a right connection 1-form with Ehresmann curvature $\omega \in \Omega^2(X_1, \mathbf{t}^*\mathfrak{K})$. Assume that $\omega^{\mathcal{K}} = 0$, i.e. the group bundle $\mathcal{K} \xrightarrow{\phi} M$ is flat. Then*

1. *ω is valued in the center $\mathbf{t}^* Z(\mathfrak{K})$, i.e. $\omega \in \Omega^2(X_1, \mathbf{t}^* Z(\mathfrak{K}))$;*
2. *$d^{\nabla^{\mathbf{t}}}\omega = 0$;*
3. *there exists an $\eta \in \Omega^2(Y_1, \mathrm{t}^* Z(\mathfrak{K}))$ such that $\phi^*\eta = \omega$, and satisfies*

$$\partial\eta = 0, \qquad D\eta = 0;$$

4. *if $Y_1 \rightrightarrows M$ is the Čech groupoid associated to an open cover of a manifold N, then there exists a curving $B \in \Omega^2(M, Z(\mathfrak{K}))$, and the 3-curvature Ω descends to a 3-form Ω_N in $\Omega^3(N, Z(\mathfrak{K})_N)$ such that $\pi^*\Omega_N = \Omega$, where $\pi : M \to N$ is the projection.*

Proof. 1. According to Eq. (71), if $\omega^{\mathcal{K}} = 0$, then $\delta_{\mathcal{K}}\omega = 0$. Hence ω must take its values in the center of the Lie algebra $\mathfrak{K}$.

2. This follows from 1. and the Bianchi identity (52).

3. One needs to prove that ω is basic with respect to the right action of $\mathcal{K}$. Note that ω is a horizontal 2-form according to Proposition 6.2. For any $x \in X_1$ and $k \in \mathcal{K}_{\mathbf{t}(x)}$, let σ be any local section of $\mathcal{K} \to M$ through k. For any $v_i \in T_x X_1$, $i = 1, 2$, the elements $u_i = \sigma_* \circ \mathbf{s}_*(v_i)$ and v_i, $i = 1, 2$ are composable in the tangent groupoid $TX_1 \rightrightarrows TM$, and $u_i \cdot v_i = (R_\sigma)_* u_i$, $i = 1, 2$. Then the relation $\partial^{\triangleright}\omega = 0$ implies that

$$\begin{aligned}\omega|_{k\cdot x}\big((R_\sigma)_*v_1, (R_\sigma)_*v_2\big) &= \omega|_{k\cdot x}(u_1\cdot v_1, u_2\cdot v_2) = \omega|_k(u_1, u_2) + \mathrm{Ad}_k^{-1}\cdot\omega|_x(v_1, v_2)\\ &= \omega|_x(v_1, v_2),\end{aligned}$$

where, in the second equality, we used the assumption that $\omega^{\mathcal{K}} = 0$. Thus it follows that ω is basic. Hence there exists an $\eta \in \Omega^2(Y_1, \mathbf{t}^* Z(\mathfrak{K}))$ such that $\omega = \phi^*\eta$.

Since the groupoid morphism $\phi : X_\bullet \to Y_\bullet$ commutes with their actions on $Z(\mathfrak{K}) \xrightarrow{\phi} M$, it follows that $\phi^*\partial = \partial^{\triangleright}\phi^*$. Thus we have $\partial\eta = 0$ by Eq. (53). On the other hand, we have $0 = d^{\nabla^{\mathbf{t}}}\omega = d^{\nabla^{\mathbf{t}}}\phi^*\eta = \phi^* D\eta$. It thus follows that $D\eta = 0$.

4. This follows from 3. together with Lemma 6.37. □

Remark 6.39. It would be interesting to compare our definition of curving and 3-curvature with the one defined by Breen and Messing [14]. It is clear that our definition reduces to the standard one in [29,42] for bundle gerbes.

6.8. *Connections on G-central extensions*

We now study connections and curvings on Lie groupoid G-central extensions. By Theorem 3.12, to any central G-extension $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$ corresponds a $Z(G)$-central extension $\tilde{X}_1 \to Y_1 \rightrightarrows M$ so that

$$X_1 \simeq \frac{\tilde{X}_1 \times G}{Z(G)}, \tag{101}$$

where $Z(G)$ acts on $\tilde{X}_1 \times G$ diagonally: $(\tilde{x}, g)\cdot h = (\tilde{x}h_{\mathbf{t}(x)}, h^{-1}g)$, $\forall h \in Z(G)$. Recall that X_1 is a G-G principal bibundle, where the actions are given, respectively, by

$$g\cdot[x, g'] = [x, gg'], \qquad [x, g']g = [x, g'g].$$

Denote by $\pi : \tilde{X}_1 \times G \to X_1$ the quotient map, by $\mathrm{pr}_1 : \tilde{X}_1 \times G \to \tilde{X}_1$ and $\mathrm{pr}_2 : \tilde{X}_1 \times G \to G$ the projections. Also, denote by $\tau : \tilde{X}_1 \to X_1$, the embedding defined by $x \to \pi(x, 1)$.

The following result describes the precise relation between connections on these extensions. Note that in this case $\Omega^m(X_n, \mathbf{t}^*\mathfrak{K})$ (resp. $\Omega^m(X_n, \mathbf{t}^* Z(\mathfrak{K}))$) can be naturally identified with the space of $\mathfrak{g}$ (resp. $Z(\mathfrak{g})$)-valued m-forms on X_n. Denote by θ the left Maurer–Cartan form on G.

Theorem 6.40. *For a groupoid central G-extension $X_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$, there is a one–one correspondence between right connection 1-forms $\alpha \in \Omega^1(X_1, \mathfrak{g})$ which are also right principal G-*

bundle connections, and connections $\tilde{\alpha} \in \Omega^1(\tilde{X}_1, Z(\mathfrak{g}))$ *of the groupoid central* $Z(G)$*-extension* $\tilde{X}_1 \xrightarrow{\phi} Y_1 \rightrightarrows M$.

Proof. Assume that $\alpha \in \Omega^1(X_1, \mathfrak{g})$ is a connection 1-form for the right principal G-bundle $X_1 \to Y_1$. It is simple to see that $\tau^*\alpha$ must be $Z(\mathfrak{g})$-valued and $\tilde{\alpha} = \tau^*\alpha \in \Omega^1(\tilde{X}_1, Z(\mathfrak{g}))$ is a connection 1-form for the $Z(G)$-principal bundle $\tilde{X}_1 \to Y_1$. Conversely, given a connection 1-form $\tilde{\alpha} \in \Omega^1(\tilde{X}_1, Z(\mathfrak{g}))$ for the $Z(G)$-principal bundle $\tilde{X}_1 \to Y_1$, $\mathrm{pr}_1^* \tilde{\alpha} + \mathrm{pr}_2^* \theta \in \Omega^1(\tilde{X}_1 \times G, \mathfrak{g})$ is basic with respect to the diagonal $Z(G)$-action. Hence it defines a 1-form $\alpha \in \Omega^1(X_1, \mathfrak{g})$ such that $\pi^*\alpha = \mathrm{pr}_1^* \tilde{\alpha} + \mathrm{pr}_2^* \theta$. It is simple to check directly that α is a connection 1-form for the right principal G-bundle $X_1 \to Y_1$.

Now we have

$$\pi^* \partial^{\lhd} \alpha = \partial^{\lhd}\big(\mathrm{pr}_1^* \tilde{\alpha} + \mathrm{pr}_2^* \theta\big) = \mathrm{pr}_1^* \partial^{\lhd} \tilde{\alpha} + \mathrm{pr}_2^* \partial^{\lhd} \theta = \mathrm{pr}_1^* \partial^{\lhd} \tilde{\alpha},$$

since θ is a left Maurer–Cartan form. Hence it follows that $\partial^{\lhd}\alpha = 0$ if, and only if, $\partial^{\lhd}\tilde{\alpha} = 0$. Therefore the conclusion follows. $\square$

We end this section with the following important

Corollary 6.41.

1. *There is a one–one correspondence between connections* α *on the central* G*-extension* $X_1 \to Y_1 \rightrightarrows M$ *such that for all* $g \in G$, $M \times \{g\}$ *is horizontal, and connections* $\tilde{\alpha}$ *on the central* $Z(G)$*-extension* $\tilde{X}_1 \to Y_1 \rightrightarrows M$.
2. *The form* $B \in \Omega^2(M, Z(\mathfrak{g}))$ *is a curving for* α *if, and only if, it is a curving for* $\tilde{\alpha}$*. In this case, their* 3*-curvatures coincide.*
3. *There is a one–one correspondence between flat central* G*-extensions* $X_1 \to Y_1 \rightrightarrows M$ *such that for all* $g \in G$, $M \times \{g\}$ *is horizontal, and flat central* $Z(G)$*-extensions* $\tilde{X}_1 \to Y_1 \rightrightarrows M$.

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