SHORT COMMUNICATION

A NOTE ON BILINEAR TIME SERIES MODELS

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Conditions for the existence of a stationary solution for certain forms of bilinear difference equations are derived.

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In [1] certain non-linear 'superdiagonal' time series models are discussed, of the form

$$\sum_0^p \alpha_j x(t-j) + \sum_{j=0}^q \beta_j \varepsilon(t-j) + \sum_{1=k<l=2}^{QP}\sum \gamma_{kl}\varepsilon(t-k)x(t-l) = 0. \tag{1}$$

Here we assume

$$\alpha_0 = \beta_0 = 1, \qquad \sum \alpha_j z^j \neq 0, \qquad |z| \leq 1 \tag{2}$$

and that the $\varepsilon(t)$ are normally and independently distributed with zero mean and variance σ^2. In [1] the question of the existence of a stationary solution to (1), in terms of the $\varepsilon(s)$, $s \leq t$, is discussed but not completely. In [2] the somewhat more difficult question is discussed as to whether, given a stationary solution, the $\varepsilon(t)$ are the (non-linear) innovations, i.e., whether $\varepsilon(t) = x(t) - \mathscr{E}\{x(t)|\mathscr{F}_{t-1}\}$ where $\mathscr{F}_t$ is the σ-algebra determined by $x(s)$, $s \leq t$. Thus it has to be determined whether the $\varepsilon(t)$ are measurable, $\mathscr{F}_t$. In [1, p. 41] it is said that a necessary and sufficient condition for the existence of a second order stationary solution to (1) is, when $p = q = 0$, that

$$\sigma^2 \sum_{k<l}\sum \gamma_{kl}^2 z^l = 1$$

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should have all solutions outside of the closed unit disc. This condition is derived from the equation

$$\mathscr{E}\{x(t)^2\} = \sigma^2 \sum\sum \gamma_{kl}^2 \mathscr{E}\{x(t-l)^2\} + \sigma^2$$

which is, however, incorrect. In any case the argument would be incomplete. However it is fairly easy to obtain a necessary and sufficient condition in this superdiagonal case. If $x(t)$ is to be a stationary solution of the required form then successive substitutions using (1) show that this solution must be

$$x(t) = \sum_{r=1}^{\infty} \Sigma_r \, \mathrm{d}_r(j_1, \ldots, j_r)\varepsilon(t-j_1) \cdots \varepsilon(t-j_r), \tag{3}$$

where Σ_r is over $0 < j_1 < \cdots < j_r < \infty$ except for $r = 1$ when $0 \leqslant j_1 < \infty$. Indeed at any stage in this process of substitution there will be a single term, $x(t-m)$ let us say, in any product, any other factors being a constant or an $\varepsilon(t-j)$. Moreover m will be larger than any such lag, j, for an $\varepsilon(t-j)$. The form of (3) now follows.

Because of the condition $j_1 < j_2 < \cdots < j_r$ the individual terms in (3) are easily seen to be orthogonal. Moreover if $r \neq s$ it also follows that

$$\mathscr{E}[\Sigma_r\{\mathrm{d}_r(j_1, \ldots, j_r)\varepsilon(t-j_1) \cdots \varepsilon(t-j_r)\} \times \Sigma_s\{\mathrm{d}_s(k_1, \ldots, k_s)\varepsilon(\tau-k_1) \cdots \varepsilon(\tau-k_s)\}] = 0 \tag{4}$$

for any t, τ.

We consider the necessary and sufficient condition for mean square convergence, i.e.,

$$\sum_1^{\infty} \sigma^{2r} \Sigma_r \, \mathrm{d}_r(j_1, \ldots, j_r)^2 < \infty. \tag{5}$$

Put

$$\alpha(z) = \sum \alpha_j z^j, \qquad \beta(z) = \sum \beta_j z^j, \qquad \gamma(z_1, z_2) = \sum\sum \gamma_{k,l} z_1^k z_2^l,$$

$$\delta_r(z_1, \ldots, z_r) = \Sigma_r \, \mathrm{d}_r(j_1, \ldots, j_r) z_1^{j_1} \cdots z_r^{j_r}.$$

Now substituting in (1) we see that

$$\delta_1(z_1) = -\beta(z_1)/\alpha(z_1)$$

and that

$$\alpha(z_1 z_2 \cdots z_r)\delta_r(z_1, \ldots, z_r) + \gamma(z_1, z_2 \cdots z_r)\delta_{r-1}(z_2, \ldots, z_r) = 0$$

so that we obtain, in general,

$$\delta_r(z_1, \ldots, z_r) = (-)^r \frac{\gamma(z_1, z_2 z_3 \cdots z_r)\gamma(z_2, z_3 \cdots z_r) \cdots \gamma(z_{r-1}, z_r)\beta(z_r)}{\alpha(z_1 z_2 \cdots z_r)\alpha(z_2 z_3 \cdots z_r) \cdots \alpha(z_{r-1} z_r)\alpha(z_r)}. \tag{6}$$

It is evident that this gives the unique solution of (1) in the form (3). Now using (6) the left side of (5) is

$$\sum_{r=1}^{\infty}\left(\frac{\sigma^2}{2\pi}\right)^r \int_{-\pi}^{\pi}\cdots\int |\delta_r(e^{i\omega_1}, e^{i\omega_2}, \ldots, e^{i\omega_r})|^2 \prod_1^r d\omega_j. \tag{7}$$

It follows easily that a simple sufficient condition for (5) is

$$\frac{\sigma^2}{2\pi}\int_{-\pi}^{\pi}\left|\frac{\gamma'(e^{i\omega}, e^{i\phi})}{\alpha(e^{i(\omega+\phi)})}\right|^2 d\omega \leq c < 1, \quad \phi \in [-\pi, \pi] \tag{8}$$

since then (7) is dominated by a geometrically convergent series. The expression (6) may also be used to evaluate the spectral density of a stationary, mean square convergent, solution to (1). Because of (4) it is

$$\sum_{r=0}^{\infty}\left(\frac{\sigma^2}{2\pi}\right)^r \int_{\Sigma\omega_j=\omega} |\delta_r(e^{i\omega_1}, \ldots, e^{i\omega_r})|^2 \prod d\omega_j.$$

Put $\xi(r) = \Sigma_r\, d_r(j_1, \ldots, j_r)\varepsilon(t-j_1)\cdots\varepsilon(t-j_r)$. Then under (8) the series $\sum \xi(r)$ converges almost surely, since $\mathscr{E}\{|\xi(r)|\} \leq [\mathscr{E}\{\xi(r)^2\}]^{1/2}$ and thus $\sum \mathscr{E}\{|\xi(r)|\}$ is dominated by a geometrically convergent series. Also

$$\sum_{1=j_1<\cdots<j_r=r}^{N} d_r(j_1, \ldots, j_r)\varepsilon(t-j_1)\cdots\varepsilon(t-j_r) \tag{9}$$

converges almost surely as can be seen by rearranging (9), for example for $r=2$, as

$$\sum_{j_2=2}^{N}\varepsilon(t-j_2)\sum_{j_1=1}^{j_2-1} d_2(j_1, j_2)\varepsilon(t-j_1).$$

This shows that (9) is, for each fixed t, a square integrable martingale, with bounded mean square, with respect to a sequence of σ-algebras G_N where G_N is generated by $\varepsilon(t-1), \ldots, \varepsilon(t-N)$.

If the assumption that the model is superdiagonal is eliminated, when the last term in (1) becomes

$$\sum_{k=0}^{Q}\sum_{l=1}^{P}\gamma_{kl}\varepsilon(t-k)x(t-l),$$

then (6) still holds but, because $0<j_1<\cdots<j_r$ in (3) no longer holds, then (7) is invalid. Thus it seems difficult in general to use (6) and to study convergence in general. However in certain special cases, not superdiagonal, a treatment is given in [3].

The discussion of the question as to whether $\varepsilon(t)$ is measurable $\mathscr{F}_t$ seems intrinsically difficult. One might consider cases for which (7) is finite and express $\varepsilon(t)$ in terms of the $x(t-j)$ by a formula of the form of (3) but with the places of $x(t)$ and $\varepsilon(t)$ reversed. Of course $j_1<j_2<\cdots<j_r$ will not hold. Then the coefficients

are generated by formulae of the form of

$$(-1)^r \frac{\gamma(z_2, z_3, \ldots, z_r, z_1) \cdots \gamma(z_r, z_{r-1})\alpha(z_r)}{\beta(z_1 z_2 \cdots z_r) \cdots \beta(z_{r-1} z_r)\beta(z_r)}.$$

However the lack of orthogonality makes this difficult to use. Again some special cases have been dealt with in [3].

References

[1] C.W.J. Granger and A.P. Andersen, An Introduction to Bilinear Time Series Models (Vandenhoeck and Ruprecht, Göttingen, 1978).

[2] C.W.J. Granger and A.P. Andersen, On the invertibility of time series models, Stochastic Process. Appl. 8 (1978) 87–92.

[3] T.D. Phan and L.T. Tran, Quelques résultats sur les modèles bilinéaires de séries chronologique, C.R. Acad. Sci. Paris 290, Série A (1980) 335–338.