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Blow-up Behavior of Ground States of Semilinear Elliptic Equations in R^n Involving Critical Sobolev Exponents

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1. INTRODUCTION

The purpose of this paper is to study the blow-up behavior of the ground states of the following elliptic equation in R^n,

$$\Delta u - u + u^q = 0, \qquad x \in R^n, \tag{1.1}$$

where $n \geqslant 3$, $1 < q < (n+2)/(n-2)$. This equation arises in many areas of applied mathematics including nuclear physics, fluid mechanics, and population genetics (see, e.g., [BL] and references therein) and has been studied extensively in recent years. We also study the generalized equation

$$\Delta u - K(x)u + u^q = 0, \qquad x \in R^n, \tag{1.2}$$

where $K(x)$ is a non-negative C^1 function in R^n.

If a solution $u(x)$ of (1.1) exists in the whole space R^n satisfying

$$u(x) > 0, \qquad u(x) \to 0 \qquad \text{as} \quad |x| \to \infty,$$

it is called a ground state. The existence and uniqueness of ground states of (1.1) has been a very interesting topic of mathematicians for years. The first existence result was proved by Nehari [Ne] for some special cases. The general case $n \geqslant 3$ and $1 < q < (n+2)/(n-2)$ was considered by Berger [B] and the existence of ground states was proved in [B] by an idea of

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Nehari. From [GNN1, GNN2] ground states of (1.1) must be radially symmetric about some point (for simplicity, we always assume that the point is the origin of R^n). The uniqueness of ground states has been studied by many authors, including Coffman [C] and McLeod and Serrin [MS], and was finally proved by Kwong [K] for the full range $1 < q < (n+2)/(n-2)$ (for the same question concerning more general equations, see [Z]). The non-existence of ground states when $q \geqslant (n+2)/(n-2)$ is well known (see, e.g., [NS]).

It is natural to consider the behavior of the ground state of (1.1) as $q \to (n+2)/(n-2)$. As one can easily see, the L^∞ norm of the ground state of (1.1) blows up as $q \to (n+2)/(n-2)$. This is also true for (1.2) under some condition on K which we shall specify later. In this paper we give a description of the blow-up behavior of the ground state(s) of (1.1) and (1.2).

We point out that blow-up problems for elliptic equations involving critical exponents in bounded domains have been studied by, among others, Atkinson and Peletier [AP2], Brezis and Peletier [BP], Han [H], and Rey [R]. Ideas developed by them are used in this paper.

Before we give our results, some notation is introduced. In this paper, we always let $\varepsilon = p - q$, where $p = (n+2)/(n-2)$ is the critical exponent. The following will be used throughout.

$$\|u\|_{L^\infty} = \|u\|_{L^\infty(R^n)}, \qquad |u|_q = \|u\|_{L^q(R^n)}.$$

THEOREM 1. *Let u_ε be the unique ground state of* (1.1). *Then*

(i) $\lim_{\varepsilon \to 0} |\nabla u_\varepsilon|_2^2 / |u_\varepsilon|_{p+1-\varepsilon}^2 = S$, *where S is the best Sobolev constant in R^n, i.e.,*

$$S = \pi n(n-2) \left[\frac{\Gamma(n/2)}{\Gamma(n)} \right]^{2/n};$$

(ii) $|\nabla u_\varepsilon(x)|^2 \to S^{n/2}\, \delta(x)$ *in the sense of distribution as* $\varepsilon \to 0$, *where* $\delta(x)$ *is the Dirac measure*;

(iii) *when* $n > 4$,
$$\lim_{\varepsilon \to 0} \varepsilon\, \|u_\varepsilon\|_{L^\infty}^{4/(n-2)} = \frac{16n(n-1)}{(n-2)^3},$$

when $n = 4$,
$$\lim_{\varepsilon \to 0} \frac{\varepsilon\, \|u_\varepsilon\|_{L^\infty}^2}{\log \|u_\varepsilon\|_{L^\infty}} = 48,$$

when $n = 3$,
$$\lim_{\varepsilon \to 0} \varepsilon\, \|u_\varepsilon\|_{L^\infty}^2 = \frac{384}{\sqrt{3}} \pi^{-5};$$

(iv) *$\|u_\varepsilon\|_{L^\infty} u_\varepsilon(x) \to (1/n)\,\omega_n[n(n-2)]^{n/2}\,\Gamma_1(|x|)$ in $C^2_{\text{loc}}(R^n\backslash\{0\})$ as $\varepsilon \to 0$, where ω_n is the area of the unit sphere in R^n and $\Gamma_1(|x-y|)$ is the fundamental solution of $-\Delta+1$ in R^n.*

Now we turn to Eq. (1.2). The existence of ground states of (1.2) has been studied by many authors; see, e.g., [DN], [L1, L2]. Before we state the blow-up results, we recall a definition in [DN, p. 295], which will play an important role in our discussion.

Let $\Sigma_i \equiv \{x=(x_1, ..., x_n)\in R^n\colon x_i=0\}$, $i=1, ..., n$, be the hyperplanes in R^n. Let e_i denote the unit vector pointing along the positive x_i-axis. For any $\rho \geqslant 0$, define $E(\rho, R^n)$ to be the set of all functions u on R^n satisfying $u(y+te_i) \leqslant u(y+(2\lambda-t)e_i)$ for all $t \geqslant \lambda \geqslant \rho$ or $t \leqslant -\lambda \leqslant -\rho$, $y \in \Sigma_i$, $1 \leqslant i \leqslant n$.

Throughout this paper we assume that $K(x)$ satisfies the following condition:

(K) K is a non-negative, C^1 function in R^n, $K+\frac{1}{2}x\cdot\nabla K \geqslant 0$, $\not\equiv 0$ and is bounded in R^n, $K(x)\geqslant K_0>0$ for large $|x|$, and $-K\in E(\rho, R^n)$ for some $\rho \geqslant 0$.

Under the condition that $-K\in E(\rho, R^n)$ and $K(x)\geqslant K_0$ at $|x|=+\infty$, Ding and Ni [DN] proved the existence of a positive solution u of (1.2) with $u\in E\cap E(\rho, R^n)$, where

$$E=\left\{u\in H^1(R^n)\colon \|u\|_E=\left(\int_{R^n} |\nabla u|^2+K(x)u^2\right)^{1/2}<\infty\right\}.$$

By slightly modifying their proof, we shall prove that (1.2) has a positive solution $u_q\in E\cap E(\rho, R^n)$, which is also a minimizer of the functional

$$I_q(u)=\frac{\|u\|_E^2}{|u|_{q+1}^2}, \qquad u\in E, u\not\equiv 0$$

(see Lemma 2.1 below). We shall call such a solution the ground state of (1.2). In [DN] the existence of a positive solution was proved by using the Mountain Pass Lemma. It is not clear to us whether the solution they obtained is a minimizer of I_q. So far no uniqueness result is available for positive solutions of (1.2). In the following part of this paper, u_q (sometimes denoted by u_ε) is an arbitrary ground state of (1.2), not necessarily the one obtained in Lemma 2.1. Also, whenever ε $(=p-q)$ is used, we always assume that it is positive and small.

Since $u_\varepsilon\in E(\rho, R^n)$ for any fixed $\varepsilon>0$, u_ε is bounded in R^n and assumes its maximum at some $x_\varepsilon\in C(\rho)=\{x=(x_1, ..., x_n)\in R^n\colon |x_i|\leqslant\rho, i=1, ..., n\}$.

THEOREM 2. *Assume* $\varepsilon_j \to 0$ *and* $x_{\varepsilon_j} \to x_0$. *Then*

(i) $\dfrac{|\nabla u_{\varepsilon_j}|_2^2}{|u_{\varepsilon_j}|_{p+1-\varepsilon_j}^2} \to S$ *as* $\varepsilon_j \to 0$;

(ii) $|\nabla u_{\varepsilon_j}|^2 \to S^{n/2}\,\delta(x-x_0)$ *in the sense of distribution as* $\varepsilon_j \to 0$;

(iii) *when* $n > 4$,

$$\varepsilon_j\, \|u_{\varepsilon_j}\|_{L^\infty}^{4/(n-2)} \to \left(K(x_0) + \frac{1}{2}x_0\cdot\nabla K(x_0)\right)\cdot\frac{16n(n-1)}{(n-2)^3};$$

when $n = 3$,

$$\varepsilon_j\, \|u_{\varepsilon_j}\|_{L^\infty}^2 \to \frac{768\pi^3}{\sqrt{3}}\int_{R^3}\left(K + \frac{1}{2}x\cdot\nabla K\right)\Gamma_K^2(x, x_0)\,dx,$$

as $\varepsilon_j \to 0$, *where* Γ_K *is the unique fundamental solution of* $-\Delta + K$ *in* R^n;

(iv) $\|u_{\varepsilon_j}\|_{L^\infty}\, u_{\varepsilon_j}(x) \to (1/n)\,\omega_n[n(n-2)]^{n/2}\,\Gamma_K(x, x_0)$ *in* $C^2_{\mathrm{loc}}(R^n\backslash\{x_0\})$ *as* $\varepsilon_j \to 0$.

Remark 1.1. By saying that Γ_K is a fundamental solution of $-\Delta + K(x)$ in R^n, we mean that for each $y \in R^n$,

$$-\Delta\Gamma_K(\cdot, y) + K(x)\,\Gamma_K(\cdot, y) = \delta(\cdot - y), \qquad x \in R^n,$$

$\Gamma_K(\cdot, y)$ is classical at $x \neq y$, and $\Gamma_K(x, y) \to 0$ as $|x| \to \infty$. The existence of a fundamental solution of $-\Delta + K(x)$ in R^n can be found in Miranda [M, Theorem 20.I, p. 68], where the boundedness of $K(x)$ in R^n is required. However, we can easily obtain a fundamental solution of $-\Delta + K(x)$ with $K(x)$ just non-negative and locally Hölder continuous by using an argument in the Appendix of [KN], which is different from the classical paramatrix method. The proof of this assertion is included in the Appendix of this paper, for the convenience of the readers. The uniqueness of the fundamental solution follows easily from [GS].

Remark 1.2. Theorem 2 does not cover the case $n = 4$. However, we believe in this case that

$$\frac{\varepsilon_j\, \|u_{\varepsilon_j}\|_{L^\infty}^2}{\log \|u_{\varepsilon_j}\|_{L^\infty}} \to 48\left(K(x_0) + \frac{1}{2}x_0\cdot\nabla K(x_0)\right) \tag{1.3}$$

as $\varepsilon_j \to 0$. As we shall see in Section 4, the analysis in this case requires very accurate estimates for u_ε, which seem to be difficult to obtain when we cannot use ODE methods.

Remark 1.3. The assertion of Theorem 2 says that $\{u_{\varepsilon_j}\}$ blows up at some point x_0. In the simple case when $\rho = 0$ (this is true if $x \cdot \nabla K(x) \geqslant 0$ for all $x \in R^n$), $x_\varepsilon = x_0 = 0$ for all ε, and $\{u_\varepsilon\}$ itself blows up at the origin. Therefore Theorem 1 is a special case of Theorem 2, except for (iii) when $n = 4$.

The outline of this paper is as follows. In Section 2 we present some preliminary results for Eq. (1.2). Section 3 is devoted to the proof of Theorem 2. In Section 4 we complete the proof of Theorem 1, that is, prove (iii) when $n = 4$. The existence of a fundamental solution of $-\Delta + K(x)$ in R^n is proved in the Appendix.

2. PRELIMINARY RESULTS

In this section we prove the existence of ground states of Eq. (1.2) and some related results which will be used in the next section. Recall that we always assume that $K(x)$ satisfies condition (K) throughout this paper, and by a ground state of (1.2) we mean a classical positive solution $u \in E \cap E(\rho, R^n)$ which is also a minimizer of the functional I_q defined in Section 1.

LEMMA 2.1. *For each* $1 < q < (n+2)/(n-2)$, (1.2) *has a ground state.*

Remark 2.2. For Lemma 2.1 we do not need the assumption made for $K + \frac{1}{2}x \cdot \nabla K$ in (K).

Proof of Lemma 2.1. For each positive integer i let $B_i = \{x \in R^n \colon |x| \leqslant i\}$ and

$$\alpha_i = \inf\{I_q(u) \colon 0 \not\equiv u \in H_0^1(B_i)\}.$$

Set

$$S_q = \inf\{I_q(u) \colon 0 \not\equiv u \in E\}.$$

Then it is easy to see that $\alpha_i \searrow S_q$ as $i \to +\infty$.

It is standard that α_i is assumed by a positive $u_i \in H_0^1(B_i)$, which is also a classical solution of the Dirichlet problem

$$\Delta u - K(x)u + u^q = 0, \qquad x \in B_i,$$
$$u|_{\partial B_i} = 0.$$

From the above equation we see that

$$\int_{B_i} (|\nabla u_i|^2 + K u_i^2)\, dx = \int_{B_i} u_i^{q+1}\, dx.$$

Hence

$$\alpha_i = I_q(u_i) = \left[\int_{B_i} (|\nabla u_i|^2 + K u_i^2)\,dx\right]^{(1-2/(q+1))}.$$

Since α_i is bounded, then u_i's, if we think of them as elements in E, are bounded in E. Hence, after passing to a subsequence, $\{u_i\}$ converges weakly to some u_q in E, and $u_i \to u_q$ a.e. in R^n as $i \to +\infty$. It is routine to see that $u_q \in E$ is a non-negative solution of (1.2), therefore a classical solution by a boot-strap argument.

Next, we prove that $u_q \not\equiv 0$ (hence $u_q > 0$ by the strong maximum principle) and $u_q \in E(\rho, R^n)$. From the assumption that $K \geqslant 0$ and $K \geqslant K_0$ at infinity, it is not hard to see that there exists $\lambda_0 > 0$ independent of i, such that

$$\int_{B_i} (|\nabla u_i|^2 + K u_i^2)\,dx \geqslant \lambda_0 \int_{B_i} u_i^2\,dx.$$

Hence

$$\lambda_0 \int_{B_i} u_i^2\,dx \leqslant \int_{B_i} u_i^{q+1}\,dx,$$

that is,

$$0 \leqslant \int_{B_i} u_i^2 (u_i^{q-1} - \lambda_0)\,dx.$$

Since $u_i > 0$ in B_i, we have $\max u_i \geqslant \lambda_0^{1/(q-1)}$. On the other hand, by Lemma 3.22 of [DN], $u_i \in E(\rho, B_i)$ for all i such that $C(\rho) \subset B_i$. From this we have $u_q \in E(\rho, R^n)$ and $\max u_i = u_i(y_i)$ for some $y_i \in C(\rho)$. By the standard boot-strap argument we see that $\|u_i\|_{C^{2+\alpha}(C(\rho))}$ is uniformly bounded in i. Hence after passing to a subsequence $u_i \to u_q$ in $C(C(\rho))$ as $i \to \infty$. Now we conclude that $\max u_q \geqslant \lambda_0^{1/(q-1)}$.

It remains to prove that u_q is a minimizer of I_q. We observe that from (1.2),

$$\begin{aligned} S_q \leqslant I_q(u_q) &= \frac{\|u_q\|_E^2}{|u_q|_{q+1}^2} = \|u_q\|_E^{2(1-2/(q+1))} \\ &\leqslant \liminf_{i\to\infty} \|u_i\|_E^{2(1-2/(q+1))} \\ &= \liminf_{i\to\infty} \alpha_i = S_q. \end{aligned}$$

Therefore u_q is a minimizer of I_q. Q.E.D.

The next lemma gives a uniform lower bound for the L^∞ norm of an arbitrary positive solution $u \in E$ of (1.2).

LEMMA 2.3. *There exists a positive constant α_0 depending only on n and $K(x)$ such that for $1 < q < (n+2)/(n-2)$, and for any positive solution $u \in E$ of* (1.2), *we have*

$$\|u\|_{L^\infty} \geqslant \alpha_0.$$

Proof. Multiplying (1.2) by u and integrating on R^n, we have

$$\int_{R^n} (|\nabla u|^2 + K(x) u^2)\, dx = \int_{R^n} |u|^{q+1}\, dx.$$

Take $R > 0$ large enough that $K(x) \geqslant K_0 > 0$ for $|x| \geqslant R$; then

$$\begin{aligned}\int_{R^n} (|\nabla u|^2 + K(x) u^2)\, dx &\geqslant K_0 \int_{|x| \geqslant R} u^2\, dx + \int_{R^n} |\nabla u|^2\, dx \\ &\geqslant K_0 \int_{|x| \geqslant R} u^2\, dx + S\left(\int_{R^n} |u|^{2^*}\, dx\right)^{2/2^*} \\ &\geqslant K_0 \int_{|x| \geqslant R} u^2\, dx + S\left(\int_{B_R} |u|^{2^*}\, dx\right)^{2/2^*} \\ &\geqslant K_0 \int_{|x| \geqslant R} u^2\, dx + S\, |B_R|^{2(1/2^* - 1/2)} \int_{B_R} u^2\, dx \\ &\geqslant \alpha_1 \int_{R^n} u^2\, dx,\end{aligned}$$

where S is the best Sobolev constant, $2^* = 2n/(n-2)$, and α_1 is a positive constant depending only on n and $K(x)$. Hence we have

$$\begin{aligned}\alpha_1 \int_{R^n} u^2\, dx &\leqslant \int_{R^n} (|\nabla u|^2 + K(x) u^2)\, dx \\ &= \int_{R^n} |u|^{q+1}\, dx,\end{aligned}$$

$$0 \geqslant \int_{R^n} u^2 (\alpha_1 - |u|^{q-1})\, dx.$$

Thus $\|u\|_{L^\infty} \geqslant \alpha_1^{1/(q-1)}$. By taking a suitable constant α_0, we complete the proof. Q.E.D.

Remark 2.4. From the proof above we see that E is continuously imbedded into $H^1(R^n)$, that is,

$$\|u\|_{H^1} \leqslant c(n, K)\, \|u\|_E, \qquad \text{for} \quad u \in E. \tag{2.1}$$

LEMMA 2.5. *Let* S_q *be defined as in* (2.1) *for* $1 < q < (n+2)/(n-2)$, *and let* $p = (n+2)/(n-2)$. *Then*

$$\lim_{q \to p} S_q = S.$$

Proof. First, it is easy to see that

$$\sup_{1<q<p} S_q < \infty.$$

In fact, for any $u \in C_0^\infty(B_1)$, $u \geqslant 0$, $\inf_{1<q<p} |u|_{q+1} > 0$, we have

$$S_q \leqslant \frac{\|u\|_E^2}{|u|_{q+1}^2} \leqslant \frac{\|u\|_E^2}{\inf_{1<q<p} |u|_{q+1}^2}.$$

Now we choose a $w_q \in E$ such that $|w_q|_{q+1} = 1$ and $\|w_q\|_E^2 = S_q$. From (2.1) we have

$$|w_q|_2 \leqslant \|w_q\|_{H^1} \leqslant c(n, K)\, \|w_q\|_E \leqslant c(n, K) S_q^{1/2}.$$

By the Hölder inequality we have

$$\begin{aligned} 1 = |w_q|_{q+1}^{q+1} &\leqslant |w_q|_2^{(n-2)(p-q)/2} \cdot |w_q|_{p+1}^{(p+1)(1-(n-2)(p-q)/4)} \\ &\leqslant c(n, K)^{(n-2)(p-q)/2} S_q^{(n-2)(p-q)/4} \cdot |w_q|_{p+1}^{(p+1)(1-(n-2)(p-q)/4)}. \end{aligned}$$

From this and the fact that $\sup_{1<q<p} S_q < \infty$, we infer that

$$\liminf_{q \to p} |w_q|_{p+1} \geqslant 1.$$

Combining this with the fact that

$$S \leqslant \frac{|\nabla w_q|_2^2}{|w_q|_{p+1}^2} \leqslant \frac{\|w_q\|_E^2}{|w_q|_{p+1}^2} = \frac{S_q}{|w_q|_{p+1}^2},$$

we have

$$S \leqslant \liminf_{q \to p} S_q.$$

Now it remains to prove

$$\limsup_{q \to p} S_q \leqslant S. \tag{2.2}$$

This can be proved by using the result in Lemma 1.1 of [BN]. Set $\varepsilon = p - q$. For $n \geqslant 4$, define a radial function w_ε as

$$w_\varepsilon(r) = \varphi(r)(\varepsilon + r^2)^{-(n-2)/2},$$

where $r = |x|$, $\varphi \in C_0^\infty(B_1)$, $\varphi \geqslant 0$, and $\varphi \equiv 1$ in $B_{1/2}$. Then one has as $\varepsilon \to 0$,

$$|\nabla w_\varepsilon|_2^2 = K_1 \varepsilon^{-(n-2)/2} + O(1),$$
$$|w_\varepsilon|_{p+1}^2 = K_2 \varepsilon^{-(n-2)/2} + O(\varepsilon),$$
$$|w_\varepsilon|_2^2 = \begin{cases} K_3 \varepsilon^{-(n-4)/2} + O(1), & n \geqslant 5, \\ K_3 \, |\log \varepsilon| + O(1), & n = 4, \end{cases}$$

where K_i's are positive constants with $K_1/K_2 = S$ (see [BN]). By a simple estimate we see that as $\varepsilon \to 0$,

$$|w_\varepsilon|_{p+1-\varepsilon}^2 = |w_\varepsilon|_{p+1}^2 + o(\varepsilon^{-(n-2)/2}),$$

$$\int_{R^n} K w_\varepsilon^2 \, dx = \begin{cases} O(\varepsilon^{-(n-4)/2}), & n \geqslant 5 \\ O(|\log \varepsilon|), & n = 4. \end{cases}$$

For $n = 3$, let

$$w_\varepsilon(r) = \varphi(r)(\varepsilon + r^2)^{-1/2}$$

where $\varphi \in C_0^\infty(B_1)$, $\varphi(0) = 1$, $\varphi'(0) = 0$, $\varphi(1) = 0$. We have from [BN] that

$$|\nabla w_\varepsilon|_2^2 = K_1 \varepsilon^{-1/2} + O(1),$$
$$|w_\varepsilon|_6^2 = K_2 \varepsilon^{-1/2} + O(\varepsilon^{1/2}),$$
$$|w_\varepsilon|_2^2 = O(1).$$

It is easy to check that

$$|w_\varepsilon|_{6-\varepsilon}^2 = |w_\varepsilon|_6^2 + o(\varepsilon^{-1/2}),$$

$$\int_{R^n} K(x) w_\varepsilon^2 \, dx = O(1).$$

Hence for $n \geqslant 3$ we have

$$S_{p-\varepsilon} \leqslant \frac{|\nabla w_\varepsilon|_2^2 + \int_{R^n} K(x) w_\varepsilon^2 \, dx}{|w_\varepsilon|_{p+1-\varepsilon}^2}$$

$$= \frac{K_1}{K_2} + o(1) = S + o(1), \qquad \text{as} \quad \varepsilon \to 0.$$

Thus (2.2) is verified and the proof is completed. Q.E.D.

From Lemma 2.5 we immediately have the following

COROLLARY 2.6.

$$\int_{R^n} (|\nabla u_q|^2 + K(x) u_q^2)\, dx \to S^{n/2},$$

$$\int_{R^n} u_q^{q+1}\, dx \to S^{n/2} \qquad as \quad q \to p,$$

where u_q is an arbitrary minimizer of I_q and a solution of (1.2).

The following lemma concerns the local properties of subsolutions of the equation

$$\Delta u + a(x) u^q = 0$$

and is useful in our analysis. Set $B(Q, r) = \{x \in R^n\colon |x - Q| \leqslant r\}$.

LEMMA 2.7. *Suppose $u \in H^1_{\text{loc}}(R^n)$ is a non-negative subsolution of $-\Delta u = a(x) u^q$ with $1 < q \leqslant (n+2)/(n-2)$. Let $2^* = 2n/(n-2)$. Then there exists a $\delta_0 > 0$, depending only on n, such that if*

$$\int_{B(Q,2r)} |au^{q-1}|^{n/2}\, dx \leqslant \delta_0,$$

then

$$\|u\|_{L^{(2^*)^2/2}(B(Q,r))} \leqslant c(n) r^{-2/2^*} \|u\|_{L^{2^*}(B(Q,2r))}.$$

Furthermore, if there exists $0 < \delta < 1$ such that

$$au^{q-1} \in L^{n/(2-\delta)}(B(Q, 2r)),$$

then

$$\sup_{B(Q,r)} u \leqslant C \left(\frac{1}{r^n} \int_{B(Q,\, 2r)} u^{2^*}\, dx \right)^{1/2^*},$$

where C depends only on n, δ, and $r^\delta \|au^{q-1}\|_{L^{n/(2-\delta)}}(B(Q, 2r))$.

Remark 2.8. The first part of this lemma is basically covered by Lemma 6 in [H]. The second part can be proved by thinking of u as a subsolution of a linear equation and then using the well-known property of subsolutions (see [T]).

3. Proof of Theorem 2

This section is devoted to the proof of Theorem 2. Keep in mind that we always assume that $K(x)$ satisfies condition (K), u_ε is a ground state of (1.2).

The first lemma in this section concludes that $\{u_\varepsilon\}$ blows up as $\varepsilon \to 0$.

Lemma 3.1. *$\|u_\varepsilon\|_{L^\infty} \to \infty$ as $\varepsilon \to 0$.*

Proof. Suppose the assertion is not true. Then there exists a sequence $\varepsilon_j \to 0$ such that $\|u_{\varepsilon_j}\|_{L^\infty}$ is bounded. Combining this with the fact that $\|u_\varepsilon\|_E$ is bounded (by Corollary 2.6), we have, after passing to a subsequence,

$$\begin{aligned} u_{\varepsilon_j} &\to u_0 \qquad \text{weakly in } E,\\ u_{\varepsilon_j} &\to u_0 \qquad \text{in } C^2_{\text{loc}}(R^n). \end{aligned}$$

Then u_0 is a bounded non-negative classical solution of the equation

$$\Delta u - K(x)u + u^p = 0, \qquad x \in R^n.$$

In fact $u_0 \not\equiv 0$, because $u_\varepsilon \in E(\rho, R^n)$ and $\|u_\varepsilon\|_{L^\infty} \geqslant \alpha_0 > 0$ by Lemma 2.3.

Now we prove that this is impossible by using the Pohozaev identity. First we prove that u_0 and ∇u_0 decay exponentially at ∞. Indeed, by the second part of Lemma 2.7 with $\delta = \frac{1}{2}$, we have

$$\sup_{B(Q,1)} u_0 \leqslant C\left(\int_{B(Q,2)} u_0^{2^*}\,dx\right)^{1/2^*}, \tag{3.1}$$

where C depends only on n and $\|u_0\|_{L^\infty}$. Since $u_0 \in E$, $u_0 \in L^{2^*}(R^n)$. So we have

$$\int_{B(Q,2)} u_0^{2^*}\,dx \to 0 \qquad \text{as} \quad |Q| \to \infty. \tag{3.2}$$

Together, (3.1) and (3.2) imply that $u_0(x) \to 0$ as $|x| \to \infty$. From this fact and the proof of Proposition 4.1 in [GNN2] we have

$$u_0(x), \qquad |\nabla u_0(x)| = o(e^{-a|x|}) \qquad \text{as} \quad |x| \to \infty, \tag{3.3}$$

for some $a > 0$.

On the other hand, applying the Pohozaev identity (see, e.g., [DN]), we have

$$\int_{B_t}\left(K + \frac{1}{2}x \cdot \nabla K\right) u_0^2\,dx = -\int_{\partial B_t}\left[(x, \nabla u_0)\frac{\partial u_0}{\partial \nu} - (x, \nu)\frac{|\nabla u_0|^2}{2}\right.$$
$$\left. + (x, \nu)\left(\frac{Ku_0^2}{2} + \frac{u_0^{p+1}}{p+1}\right) + \frac{n-2}{2}u_0\frac{\partial u_0}{\partial \nu}\right],$$

where $B_i = \{x \in R^n: |x| < i\}$ and ν is the outer normal vector on ∂B_i. Letting $i \to +\infty$, by the decay rate of u_0 and $|\nabla u_0|$ in (3.3) we infer

$$\int_{R^n} \left(K + \frac{1}{2} x \cdot \nabla K \right) u_0^2 \, dx = 0.$$

Since $K(x) + \frac{1}{2} x \cdot \nabla K(x) \geqslant 0$ and $\not\equiv 0$ in R^n, it follows that $u_0 \equiv 0$, which contradicts the fact that $u_0 \not\equiv 0$. Q.E.D.

Now we make a standard rescaling of u_ε as follows. Since $u_\varepsilon \in E(\rho, R^n)$, we can assume that $u_\varepsilon(x_\varepsilon) = \|u_\varepsilon\|_{L^\infty}$ for some $x_\varepsilon \in C(\rho)$. Let $\mu_\varepsilon > 0$ be such that

$$\mu_\varepsilon^{-2/(p-1-\varepsilon)} = \|u_\varepsilon\|_{L^\infty}.$$

Define $v_\varepsilon(x) = \mu_\varepsilon^{2/(p-1-\varepsilon)} u_\varepsilon(x_\varepsilon + \mu_\varepsilon x)$. Then $0 < v_\varepsilon(x) \leqslant 1$, $v_\varepsilon(0) = 1$, and

$$\Delta v_\varepsilon - \mu_\varepsilon^2 K(x_\varepsilon + \mu_\varepsilon x) v_\varepsilon + v_\varepsilon^{p-\varepsilon} = 0, \qquad x \in R^n.$$

From Lemma 3.1, $\mu_\varepsilon \to 0$ as $\varepsilon \to 0$. By the interior elliptic estimates we have

$$\|v_\varepsilon\|_{C^{2,\alpha}(B_i)} \leqslant M_i < \infty$$

for each $i > 0$. Therefore by a standard diagonalization argument, there exists a sequence $\varepsilon_j \to 0$ such that $v_{\varepsilon_j} \to U_1$ in $C^2_{\mathrm{loc}}(R^n)$, where U_1 is the classical positive solution of the equation

$$\Delta u + u^p = 0, \qquad x \in R^n,$$

and $U_1(0) = \|U_1\|_{L^\infty} = 1$. By the uniqueness result in [CGS] we know that

$$U_1(x) = \left(1 + \frac{|x|^2}{n(n-2)} \right)^{-(n-2)/2}, \tag{3.4}$$

and hence

$$v_\varepsilon \to U_1 \quad \text{in} \quad C^2_{\mathrm{loc}}(R^n) \quad \text{as} \quad \varepsilon \to 0.$$

It is well known that U_1 is a minimizer of the functional

$$I(u) = \frac{|\nabla u|_2^2}{|u|_{2^*}^2} \quad \text{in} \quad H^1(R^n), \qquad I(U_1) = S.$$

From this and the equation of U_1, it is easy to see that

$$|\nabla U_1|_2^2 = |U_1|_{2^*}^{2^*} = S^{n/2}.$$

Hence we have

$$\begin{aligned}
S^{n/2} = |\nabla U_1|_2^2 &\leqslant \liminf_{\varepsilon \to 0} |\nabla v_\varepsilon|_2^2 \\
&\leqslant \limsup_{\varepsilon \to 0} |\nabla v_\varepsilon|_2^2 \\
&\leqslant \limsup_{\varepsilon \to 0} \int_{R^n} [|\nabla v_\varepsilon|^2 + \mu_\varepsilon^2 K(x_\varepsilon + \mu_\varepsilon x) v_\varepsilon^2]\, dx \\
&= \limsup_{\varepsilon \to 0} \mu_\varepsilon^{\varepsilon(n-2)^2/(4-\varepsilon(n-2))} \int_{R^n} (|\nabla u_\varepsilon|^2 + K(x) u_\varepsilon^2)\, dx \\
&\leqslant \lim_{\varepsilon \to 0} \int_{R^n} (|\nabla u_\varepsilon|^2 + K(x) u_\varepsilon^2)\, dx \\
&= S^{n/2}.
\end{aligned}$$

The last equality comes from Corollary 2.6. Thus $\lim_{\varepsilon \to 0} |\nabla v_\varepsilon|_2^2 = |\nabla U_1|_2^2 = S^{n/2}$ and $\mu_\varepsilon^\varepsilon \to 1$ as $\varepsilon \to 0$. Therefore we have

LEMMA 3.2. *$\nabla v_\varepsilon \to \nabla U_1$ in $L^2(R^n)$, $v_\varepsilon \to U_1$ in $L^{2^*}(R^n)$, $|\nabla v_\varepsilon|_2^2 \to S^{n/2}$, and $\mu_\varepsilon^\varepsilon \to 1$ as $\varepsilon \to 0$.*

COROLLARY 3.3. *If $x_{\varepsilon_j} \to x_0$ as $\varepsilon_j \to 0$, then*

$$|\nabla u_{\varepsilon_j}|^2 \to S^{n/2}\, \delta(\cdot - x_0)$$

in the sense of distribution.

Proof of Corollary 3.3. For any $\varphi \in C_0^\infty(R^n)$,

$$\begin{aligned}
&\lim_{\varepsilon_j \to 0} \int_{R^n} |\nabla u_{\varepsilon_j}|^2 \varphi\, dx \\
&= \lim_{\varepsilon_j \to 0} \left[\int_{R^n} |\nabla v_{\varepsilon_j}|^2 (y)\, \varphi(x_{\varepsilon_j} + \mu_{\varepsilon_j} y)\, dy \right] \mu_\varepsilon^{-\varepsilon_j(n-2)^2/(4-\varepsilon_j(n-2))} \\
&= \lim_{\varepsilon_j \to 0} \int_{R^n} |\nabla v_{\varepsilon_j}|^2 (y)\, \varphi(x_{\varepsilon_j} + \mu_{\varepsilon_j} y)\, dy \\
&= \lim_{\varepsilon_j \to 0} \int_{R^n} |\nabla U_1|^2 (y)\, \varphi(x_{\varepsilon_j} + \mu_{\varepsilon_j} y)\, dy \\
&= \varphi(x_0) \int_{R^n} |\nabla U_1|^2\, dy \\
&= \varphi(x_0) S^{n/2}. \quad \text{Q.E.D.}
\end{aligned}$$

We shall need the following lemma to obtain a uniform decay rate estimate for u_ε at ∞.

LEMMA 3.4. *$\int_{|x| \geqslant R} u_\varepsilon^{2^*}\, dx \to 0$ uniformly w.r.t. ε as $R \to \infty$.*

Proof. Suppose the assertion is not true. Then there exist two sequences $\varepsilon_j \to \varepsilon_0$ and $R_j \to \infty$ such that

$$\int_{|x| \geqslant R_j} u_{\varepsilon_j}^{2^*}\, dx \geqslant \delta \tag{3.5}$$

for some $\delta > 0$ and $j = 1, 2, ...$ We shall prove that this is impossible by considering the following two cases.

Case 1. $\varepsilon_0 > 0$. Since $\|u_\varepsilon\|_E$ is bounded, passing to a subsequence of $\{u_{\varepsilon_j}\}$ if necessary, we may assume that $u_{\varepsilon_j} \to \bar{u}_{\varepsilon_0}$ weakly in E. By the standard elliptic regularity argument, $u_{\varepsilon_j} \to \bar{u}_{\varepsilon_0}$ in $C_{\mathrm{loc}}^{2+\alpha}(R^n)$, after passing to a subsequence again. Hence $\bar{u}_{\varepsilon_0}$ is a classical solution of (1.2) with $q = p - \varepsilon_0$. Using Lemma 2.3 we can prove as before that $\bar{u}_{\varepsilon_0}$ is a positive solution. Observe that

$$\begin{aligned} S_{p-\varepsilon_0} &\leqslant I_{p-\varepsilon_0}(\bar{u}_{\varepsilon_0}) = \|\bar{u}_{\varepsilon_0}\|_E^{2(1-2/(p+1-\varepsilon_0))} \\ &\leqslant \liminf_{j \to \infty} \|u_{\varepsilon_j}\|_E^{2(1-2/(p+1-\varepsilon_j))} = \liminf_{j \to \infty} S_{p-\varepsilon_j} \\ &\leqslant \limsup_{\varepsilon \to \varepsilon_0} S_{p-\varepsilon} \leqslant S_{p-\varepsilon_0}. \end{aligned}$$

Hence we have

$$S_{p-\varepsilon_0} = I_{p-\varepsilon_0}(\bar{u}_{\varepsilon_0}) = \lim_{j \to \infty} S_{p-\varepsilon_j} = \lim_{j \to \infty} I_{p-\varepsilon_j}(u_{\varepsilon_j}).$$

In particular, $\|u_{\varepsilon_j}\|_E \to \|\bar{u}_{\varepsilon_0}\|_E$ as $j \to \infty$, and therefore $u_{\varepsilon_j} \to \bar{u}_{\varepsilon_0}$ in E and hence in $L^{2^*}(R^n)$ as $j \to \infty$. But this is contrary to (3.5).

Case 2. $\varepsilon_0 = 0$. From (3.5) we have

$$\begin{aligned} \delta &\leqslant \int_{|x| \geqslant R_j} u_{\varepsilon_j}^{2^*}\, dx = \mu_{\varepsilon_j}^n \int_{|x_{\varepsilon_j} + \mu_{\varepsilon_j} y| \geqslant R_j} u_{\varepsilon_j}^{2^*}(x_{\varepsilon_j} + \mu_{\varepsilon_j} y)\, dy \\ &= \mu_{\varepsilon_j}^{n - (2/(p-1-\varepsilon_j)) \cdot 2^*} \int_{|x_{\varepsilon_j} + \mu_{\varepsilon_j} y| \geqslant R_j} v_{\varepsilon_j}^{2^*}(y)\, dy \\ &= \mu_{\varepsilon_j}^{-n\varepsilon_j/(4-\varepsilon_j(n-2))} \int_{|y \geqslant (1/\mu_{\varepsilon_j})(R_j - |x_{\varepsilon_j}|)} v_{\varepsilon_j}^{2^*}(y)\, dy \\ &\to 0 \qquad \text{as} \quad j \to \infty. \end{aligned}$$

because $\mu_\varepsilon^\varepsilon \to 1$, $(1/\mu_{\varepsilon_j})(R_j - |x_{\varepsilon_j}|) \to \infty$, and $v_{\varepsilon_j} \to U_1$ in $L^{2^*}(R^n)$. Again we reach a contradiction. Q.E.D.

Now we are ready to give a decay rate estimate for u_ε and $|\nabla u_\varepsilon|$ uniform in ε.

LEMMA 3.5. *There exist positive constants C, R, and a, independent of ε, such that*

$$u_\varepsilon(x),\ |\nabla u_\varepsilon(x)| \leqslant Ce^{-a|x|} \qquad \textit{for} \quad |x| \geqslant R.$$

Proof. We shall prove that $u_\varepsilon(x) \to 0$ uniformly w.r.t. ε as $|x| \to \infty$. Then the result desired follows from the proof of Proposition 4.1 in [GNN2].

By Lemma 3.4 and the Hölder inequality, there exists $R_0 > 0$ independent of ε such that

$$\int_{B(Q,4)} (u_\varepsilon^{p-1-\varepsilon})^{n/2}\, dx \leqslant \delta_0 \qquad \text{for} \quad |Q| \geqslant R_0,$$

where δ_0 is as in Lemma 2.7. By virtue of the first part of Lemma 2.7, we have that $\|u_\varepsilon\|_{L^{(2^*)^2/2}(B(Q,2))}$ is uniformly bounded w.r.t. ε and $Q \in R^n$ with $|Q| \geqslant R_0$. Since $\frac{1}{2}(2^*)^2 > (n/2)(p-1-\varepsilon)$, by the second part of Lemma 2.7, we have

$$\sup_{B(Q,1)} u_\varepsilon \leqslant C\left(\int_{B(Q,2)} u_\varepsilon^{2^*}\, dx\right)^{1/2^*},$$

where C is independent of ε and $Q \in R^n$ with $|Q| \geqslant R_0$. Now letting $|Q| \to \infty$ and again using Lemma 3.4 we have $u_\varepsilon(x) \to 0$ uniformly w.r.t. ε as $|x| \to \infty$. This completes the proof. Q.E.D.

The following lemma will play a key role in our analysis.

LEMMA 3.6. *There exists a positive constant c, independent of ε, such that*

$$v_\varepsilon(x) \leqslant cU_1(x), \qquad \textit{for} \quad x \in R^n. \tag{3.6}$$

Proof. Let $\bar{w}_\varepsilon(x)$ be the Kelvin transform of v_ε, i.e., $\bar{w}_\varepsilon(x) = |x|^{2-n} v_\varepsilon(x/|x|^2)$. From (3.4) we see that (3.6) is equivalent to the assertion

$$\bar{w}_\varepsilon(x) \leqslant \bar{c} \qquad \text{for} \quad x \in R^n, \tag{3.7}$$

where $\bar{c}$ is independent of ε. Since $v_\varepsilon(y) \leqslant 1$ for $y \in R^n$, (3.7) is true for $|x|$ bounded away from zero. Therefore it suffices to prove that $\{\bar{w}_\varepsilon\}$ is bounded uniformly w.r.t. ε in a neighborhood of $x = 0$.

By direct calculation, we see that $\bar{w}_\varepsilon$ satisfies the equation

$$\Delta\bar{w}_\varepsilon - \mu_\varepsilon^2 |x|^{-4} K\left(\frac{x}{|x|^2}\right)\bar{w}_\varepsilon + |x|^{-(n-2)\varepsilon}\bar{w}_\varepsilon^q = 0,$$

where $q = p - \varepsilon$. Hence

$$-\Delta\bar{w}_\varepsilon \leqslant a(x)\,\bar{w}_\varepsilon^q, \qquad a(x) = |x|^{-(n-2)\varepsilon}.$$

We shall use Lemma 2.7 to obtain the desired upper bound for $\bar{w}_\varepsilon$. First, we estimate the integral

$$\int_{B(r_0)} (a(x)\bar{w}_\varepsilon^{q-1})^{n/2}\,dx = \int_{|x| \leqslant \mu_\varepsilon^2} + \int_{\mu_\varepsilon^2 \leqslant |x| \leqslant r_0} = I_1 + I_2,$$

where $B(r_0) = \{x \in R^n\colon\ |x| < r_0\}$ and $r_0 < 1$ is to be chosen later, and without loss of generality we assume $\mu_\varepsilon^2 \leqslant r_0$. In the following we shall denote by c_i's constants independent of ε and $r_0 < 1$.

For I_2, we observe that

$$\begin{aligned} I_2 &\leqslant \mu_\varepsilon^{-n(n-2)\varepsilon} \int_{\mu_\varepsilon^2 \leqslant |x| \leqslant r_0} \bar{w}_\varepsilon^{(q-1)\cdot(n/2)}\,dx \\ &\leqslant \mu_\varepsilon^{-n(n-2)\varepsilon}\left(\int_{B(r_0)} \bar{w}_\varepsilon^{2^*}\right)^{n(q-1)/(2\cdot 2^*)} \cdot |B(r_0)|^{1-n(q-1)/(2\cdot 2^*)} \\ &\leqslant c_1\left(\int_{B(r_0)} \bar{w}_\varepsilon^{2^*}\,dx\right)^{n(q-1)/(2\cdot 2^*)}. \end{aligned} \tag{3.8}$$

The last inequality is by the fact $\mu_\varepsilon^\varepsilon \to 1$ as $\varepsilon \to 0$. From Lemma 3.2, $v_\varepsilon \to U_1$ in $L^{2^*}(R^n)$, hence $\bar{w}_\varepsilon \to \bar{U}_1$ in $L^{2^*}(R^n)$, where $\bar{U}_1(x) = |x|^{2-n}\,U_1(x/|x|^2)$ is the Kelvin transform of U_1. Thus by (3.8) we can choose a small $r_0 > 0$ such that

$$I_2 \leqslant \tfrac{1}{2}\delta_0, \tag{3.9}$$

where δ_0 is as in Lemma 2.7.

To estimate I_1 we observe that from Lemma 3.5,

$$v_\varepsilon(x) = \mu_\varepsilon^{2/(q-1)} u_\varepsilon(\mu_\varepsilon x) \leqslant c_2\mu_\varepsilon^{2/(q-1)} e^{-a\mu_\varepsilon|x|} \qquad \text{for} \quad |x| \geqslant \frac{R}{\mu_\varepsilon}.$$

Hence

$$\bar{w}_\varepsilon(x) \leqslant c_2\mu_\varepsilon^{2/(q-1)}\,|x|^{2-n}\,e^{-a\mu_\varepsilon/|x|} \qquad \text{for} \quad |x| \leqslant \frac{\mu_\varepsilon}{R}.$$

So

$$\begin{aligned}I_1 &\leqslant \int_{|x|\leqslant\mu_\varepsilon^2} |x|^{-\varepsilon n(n-2)/2}\left(c_2\mu_\varepsilon^{2/(q-1)}\,|x|^{2-n}\,e^{-a\mu_\varepsilon/|x|}\right)^{(n/2)(q-1)}dx\\ &\leqslant c_3\mu_\varepsilon^n\int_{|x|\leqslant\mu_\varepsilon^2}|x|^{-2n}\,e^{-b\mu_\varepsilon/|x|}\,dx\\ &= c_3\mu_\varepsilon^{-n}\int_{|x|\leqslant\mu_\varepsilon^2}\left(\frac{\mu_\varepsilon}{|x|}\right)^{2n} e^{-b(\mu_\varepsilon/|x|)}\,dx\\ &\leqslant c_4\mu_\varepsilon^{-n}\int_{|x|\leqslant\mu_\varepsilon^2}dx\\ &\leqslant c_5\mu_\varepsilon^n\leqslant c_5 r_0^{n/2}\leqslant\frac{\delta_0}{2}\end{aligned}\tag{3.10}$$

if r_0 is small, where b is independent of ε.

Now (3.9) and (3.10) together imply that

$$\int_{B(r_0)}(a(x)\bar{w}_\varepsilon^{q-1})^{n/2}\,dx\leqslant\delta_0.$$

Hence by Lemma 2.7 we have

$$\|\bar{w}_\varepsilon\|_{L^{(2^*)^2/2}(B(r_0/2))}\leqslant c(n)\,r_0^{-2/2^*}\,\|\bar{w}_\varepsilon\|_{L^{2^*}(B(r_0))}.$$

The right-hand side of above inequality is bounded uniformly in ε, and so is the left-hand side.

Since $\frac{1}{2}(2^*)^2>\frac{1}{2}(q-1)n$, we can choose an $0<\delta<1$ such that $\frac{1}{2}(2^*)^2>n(q-1)/(2-\delta)$. For this δ, we estimate the integral

$$\int_{B(r_0/2)}(a\bar{w}_\varepsilon^{q-1})^{n/(2-\delta)}\,dx=\int_{|x|\leqslant\mu_\varepsilon^2}+\int_{\mu_\varepsilon^2\leqslant|x|\leqslant r_0/2}=I_1'+I_2'.$$

By slightly modifying the previous estimates for I_1 and I_2 and using the bound for the $L^{(2^*)^2/2}$ norm of $\bar{w}_\varepsilon$ mentioned in the above paragraph, we can prove that I_1' and I_2', and hence $\int_{B(r_0/2)}(a\bar{w}_\varepsilon^{q-1})^{n/(2-\delta)}\,dx$, are bounded uniformly in small ε. Now the second part of Lemma 2.7 implies the desired upper bound for $\bar{w}_\varepsilon$. Q.E.D.

In the following, we set

$$w_\varepsilon(x)=\|u_\varepsilon\|_{L^\infty}\,u_\varepsilon(x)=\mu_\varepsilon^{-2/(p-1-\varepsilon)}u_\varepsilon(x).$$

Then w_ε satisfies

$$\Delta w_\varepsilon-K(x)w_\varepsilon+\mu_\varepsilon^2 w_\varepsilon^{p-\varepsilon}=0,\qquad x\in R^n.\tag{3.11}$$

LEMMA 3.7. *There exist positive constants* $\bar{c}$, $\bar{R}$, *and* $\bar{a}$ *independent of* ε *such that*

$$w_\varepsilon(x) \leqslant \bar{c} e^{-\bar{a}|x|} \qquad \textit{for} \quad |x| \geqslant \bar{R}, \tag{3.12}$$

$$w_\varepsilon(x) \leqslant \bar{c}\, |x - x_\varepsilon|^{2-n} \qquad \textit{for} \quad x \in R^n, \tag{3.13}$$

where $x_\varepsilon \in C(\rho)$ *such that* $u_\varepsilon(x_\varepsilon) = \|u_\varepsilon\|_{L^\infty}$.

Proof. From Lemma 3.6 we have

$$w_\varepsilon(x) \leqslant c\mu_\varepsilon^{-\varepsilon(n-2)^2/(4-\varepsilon(n-2))}(\mu_\varepsilon^2 + |x - x_\varepsilon|^2)^{-(n-2)/2}.$$

The inequality (3.13) follows. Since $x_\varepsilon \in C(\rho)$, $\{|x_\varepsilon|\}$ is bounded, $w_\varepsilon(x) \to 0$ uniformly in ε as $|x| \to \infty$. Now (3.12) follows from (3.11) and the proof of Proposition 4.1 in [GNN2]. Q.E.D.

LEMMA 3.8. *Suppose* $x_{\varepsilon_j} \to x_0$ *as* $\varepsilon_j \to 0$. *Then*

$$w_{\varepsilon_j} \to \frac{1}{n}\, \omega_n [n(n-2)]^{n/2}\, \Gamma_K(\cdot, x_0) \qquad \textit{in} \quad C^2_{\text{loc}}(R^n - \{x_0\}),$$

where $\Gamma_K(x, y)$ *is the fundamental solution of* $-\Delta + K$.

Proof. From Lemma 3.7, we see that $\{w_\varepsilon\}$ is uniformly bounded in any compact subset of $R^n \backslash \{x_0\}$. By the elliptic regularity argument we can extract a subsequence $\{\bar{\varepsilon}_j\}$ of $\{\varepsilon_j\}$ such that

$$w_{\bar{\varepsilon}_j} \to G \qquad \text{in} \quad C^2_{\text{loc}}(R^n \backslash \{x_0\}).$$

We shall prove that

$$G = \frac{1}{n}\, \omega_n [n(n-2)]^{n/2}\, \Gamma_K(\cdot, x_0).$$

Then by the uniqueness of G, we see that

$$w_{\varepsilon_j} \to G \qquad \text{in} \quad C^2_{\text{loc}}(R^n \backslash \{x_0\}). \tag{3.14}$$

Since $\{w_\varepsilon\}$ is bounded in any compact subset of $R^n \backslash \{x_0\}$ and $\mu_\varepsilon \to 0$, from (3.11) we have

$$-\Delta G + K(x) G = 0 \qquad \text{in} \quad R^n \backslash \{x_0\}.$$

Also, (3.13) implies that

$$G(x) \leqslant \bar{c}\, |x - x_0|^{2-n} \qquad \text{for} \quad x \neq x_0. \tag{3.15}$$

For any $\varphi \in C_0^\infty(R^n)$, by (3.14) we have

$$\int_{R^n} G(-\Delta + K)\varphi\, dx = \lim_{j\to\infty} \int_{R^n} w_{\varepsilon_j}(-\Delta + K)\varphi\, dx.$$

But

$$\begin{aligned}
&\int_{R^n} w_{\varepsilon_j}(-\Delta + K)\varphi\, dx\\
&\quad = \int_{R^n} \varphi(-\Delta + K) w_{\varepsilon_j}\, dx\\
&\quad = \int_{R^n} \varphi(x)\, \mu_\varepsilon^{-2/(p-1-\varepsilon_j)} u_{\varepsilon_j}^{p-\varepsilon_j}(x)\, dx \qquad \text{(by (3.11))}\\
&\quad = \mu_\varepsilon^{-\varepsilon_j(n-2)^2/(4-\varepsilon_j(n-2))} \int_{R^n} \varphi(x_{\varepsilon_j} + \mu_{\varepsilon_j} y)\, v_{\varepsilon_j}^{p-\varepsilon_j}(y)\, dy\\
&\quad \to \varphi(x_0) \int_{R^n} U_1^p(x)\, dx \qquad \text{(by Lemma 3.6)}\\
&\quad = \varphi(x_0)\cdot \frac{1}{n}\, \omega_n [n(n-2)]^{n/2}.
\end{aligned}$$

Thus

$$-\Delta G + K(x) G = \frac{1}{n}\, \omega_n [n(n-2)]^{n/2}\, \delta(\cdot - x_0).$$

By [GS]

$$G = \frac{1}{n}\, \omega_n [n(n-2)]^{n/2}\, \Gamma_K(\cdot, x_0) + g,$$

where g is a regular solution of

$$-\Delta u + K(x) u = 0 \qquad \text{in} \quad R^n.$$

Since both $G(x)$ and $\Gamma_K(x, x_0) \to 0$ as $|x| \to \infty$, $g(x) \to 0$ as $|x| \to \infty$. Now by the maximum principle we have $g \equiv 0$. Q.E.D.

LEMMA 3.9. *Suppose $x_{\varepsilon_j} \to x_0$ as $\varepsilon_j \to 0$. Then for $n > 4$ we have*

$$\lim_{\varepsilon_j \to 0} \varepsilon_j \|u_{\varepsilon_j}\|_{L^\infty}^{4/(n-2)} = \left(K(x_0) + \frac{1}{2} x_0 \cdot \nabla K(x_0)\right) \cdot \frac{16n(n-1)}{(n-2)^3}.$$

Proof. For simplicity we denote ε_j by ε. Applying the Pohozaev identity to (1.5) on a ball B_i, we get

$$\int_{B_i}\left[\left(\frac{n}{p+1-\varepsilon}-\frac{n-2}{2}\right)u_\varepsilon^{p+1-\varepsilon}-\left(K+\frac{1}{2}x\cdot\nabla K\right)u_\varepsilon^2\right]dx$$

$$=\int_{\partial B_i}\left[(x,\nabla u_\varepsilon)\frac{\partial u_\varepsilon}{\partial v}-(x,v)\frac{|\nabla u_\varepsilon|^2}{2}+(x,v)\left(\frac{1}{2}Ku_\varepsilon^2+\frac{u_\varepsilon^{p+1-\varepsilon}}{p+1-\varepsilon}\right)+\frac{n-2}{2}u_\varepsilon\frac{\partial u_\varepsilon}{\partial v}\right].$$

Using Lemma 3.5 and letting $i\to\infty$, we have

$$\int_{R^n}\left[\left(\frac{n}{p+1-\varepsilon}-\frac{n-2}{2}\right)u_\varepsilon^{p+1-\varepsilon}-\left(K+\frac{1}{2}x\cdot\nabla K\right)u_\varepsilon^2\right]dx=0,$$

that is,

$$\begin{aligned}\frac{\varepsilon(n-2)^2}{2(2n-\varepsilon(n-2))}&\int_{R^n}u_\varepsilon^{p+1-\varepsilon}\,dx\\ &=\int_{R^n}\left(K+\frac{1}{2}x\cdot\nabla K\right)u_\varepsilon^2\,dx\\ &=\mu_\varepsilon^{2-\varepsilon(n-2)^2/(4-\varepsilon(n-2))}\int_{R^n}\left[K(x_\varepsilon+\mu_\varepsilon y)+\frac{1}{2}(x_\varepsilon+\mu_\varepsilon y)\right.\\ &\qquad\left.\cdot\nabla K(x_\varepsilon+\mu_\varepsilon y)\right]v_\varepsilon^2(y)\,dy. \qquad (3.16)\end{aligned}$$

Using Lemma 3.6 and the fact $U_1\in L^2(R^n)$ when $n>4$, by the Lebesgue Dominated Convergence Theorem we have

$$\begin{aligned}&\int_{R^n}\left[K(x_\varepsilon+\mu_\varepsilon y)+\frac{1}{2}(x_\varepsilon+\mu_\varepsilon y)\cdot\nabla K(x_\varepsilon+\mu_\varepsilon y)\right]v_\varepsilon^2(y)\,dy\\ &\qquad\to\left(K(x_0)+\frac{1}{2}x_0\cdot\nabla K(x_0)\right)\int_{R^n}U_1^2(y)\,dy \qquad \text{as}\quad \varepsilon\to 0. \qquad (3.17)\end{aligned}$$

Note that

$$\begin{aligned}\int_{R^n}U_1^2(y)\,dy&=\omega_n\int_0^\infty\frac{r^{n-1}}{(1+r^2/n(n-2))^{n-2}}\,dr\\ &=\omega_n[n(n-2)]^{n/2}\int_0^\infty\frac{r^{n-1}}{(1+r^2)^{n-2}}\,dr\\ &=\frac{1}{2}\omega_n[n(n-2)]^{n/2}\int_0^\infty(1+s)^{2-n}s^{-1+n/2}\,ds\end{aligned}$$

$$= \omega_n [n(n-2)]^{n/2} \cdot \frac{2(n-1)}{n-2} \frac{\Gamma(n/2)^2}{\Gamma(n)}$$

$$= 4(\pi n)^{n/2} (n-2)^{(n-2)/2} (n-1) \frac{\Gamma(n/2)}{\Gamma(n)}. \tag{3.18}$$

By Corollary 2.6, we have, as $\varepsilon \to 0$,

$$\int_{R^n} u_\varepsilon^{p+1-\varepsilon}\, dx \to S^{n/2} = [\pi n(n-2)]^{n/2} \frac{\Gamma(n/2)}{\Gamma(n)}. \tag{3.19}$$

Together, (3.16)–(3.19) give

$$\lim_{\varepsilon \to 0} \varepsilon \mu_\varepsilon^{-2} = \left[K(x_0) + \frac{1}{2} x_0 \cdot \nabla K(x_0) \right] \cdot \frac{16n(n-1)}{(n-2)^3},$$

which is what we seek. Q.E.D.

Lemma 3.10. *Suppose* $n = 3$ *and* $x_{\varepsilon_j} \to x_0$ *as* $\varepsilon_j \to 0$. *Then*

$$\lim_{\varepsilon_j \to 0} \varepsilon_j \|u_{\varepsilon_j}\|_{L^\infty}^2 = \frac{768\pi^3}{\sqrt{3}} \int_{R^3} \left(K + \frac{1}{2} x \cdot \nabla K \right) \Gamma_K^2(x, x_0)\, dx. \tag{3.20}$$

Proof. Using the Pohozaev identity as before, we have

$$\frac{\varepsilon}{2(6-\varepsilon)} \int_{R^3} u_\varepsilon^{6-\varepsilon}\, dx = \int_{R^3} \left(K + \frac{1}{2} x \cdot \nabla K \right) u_\varepsilon^2\, dx.$$

Therefore

$$\frac{\varepsilon \|u_\varepsilon\|_{L^\infty}^2}{2(6-\varepsilon)} \int_{R^3} u_\varepsilon^{6-\varepsilon}\, dx = \int_{R^3} \left(K + \frac{1}{2} x \cdot \nabla K \right) w_\varepsilon^2\, dx. \tag{3.21}$$

Recall from Lemma 3.8 with $n = 3$,

$$w_{\varepsilon_j} \to \sqrt{3}\, \omega_3 \Gamma_K(\cdot, x_0) \qquad \text{in} \quad C_{\mathrm{loc}}^2(R^3 \backslash \{x_0\}).$$

From this and Lemma 3.7, it is easily seen that

$$\lim_{\varepsilon_j \to 0} \int_{R^3} \left(K + \frac{1}{2} x \cdot \nabla K \right) w_{\varepsilon_j}^2\, dx$$

$$= 3\omega_3^2 \int_{R^3} \left(K + \frac{1}{2} x \cdot \nabla K \right) \Gamma_K^2(x, x_0)\, dx < +\infty.$$

This, (3.21), and Corollary 2.6 imply

$$\lim_{\varepsilon\to 0} \varepsilon \, \|u_\varepsilon\|^2_{L^\infty} = \frac{768}{\sqrt{3}} \pi^3 \int_{R^3} \left(K + \frac{1}{2} x \cdot \nabla K\right) \Gamma^2_K(x, x_0)\, dx.$$

(Note that for $n=3$, $S^{3/2} = (3\pi)^{3/2}\,(\Gamma(\frac{3}{2})/\Gamma(3)) = 12\sqrt{3}\,\pi$.) Q.E.D.

Remark 3.11. When $n=3$ and $K \equiv 1$ we have

$$\Gamma(x, y) = \left(\frac{\pi}{2}\right)^{1/2} |x-y|^{-1} e^{-|x-y|}.$$

In this case (3.20) becomes

$$\lim_{\varepsilon\to 0} \varepsilon \, \|u\|^2_{L^\infty} = \frac{768\pi^3}{\sqrt{3}} \int_{R^3} \frac{\pi}{4} |x-x_0|^{-2} e^{-2|x-x_0|}\, dx$$
$$= \frac{768\pi^4}{4\sqrt{3}} \omega_3 \int_0^\infty e^{-2r}\, dr = \frac{384}{\sqrt{3}} \pi^6.$$

Remark 3.12. Part (i) of Theorem 2 follows from Corollary 2.6; (ii) comes from Corollary 3.3; (iii) is the same as Lemmas 3.9 and 3.10; and (iv) is nothing but Lemma 3.8.

4. COMPLETION OF THE PROOF FOR THEOREM 1

As we mentioned in Section 1, Theorem 1 follows from Theorem 2 except for (iii) of Theorem 1 when $n=4$. Therefore to complete the proof for Theorem 1, we need only prove the following:

LEMMA 4.1. *Assume* $n=4$ *and* u_ε *is the unique ground state of* (1.1) *with* $u_\varepsilon(0) = \|u_\varepsilon\|_{L^\infty}$. *Then*

$$\lim_{\varepsilon\to 0} \frac{\varepsilon \, \|u_\varepsilon\|^2_{L^\infty}}{\log \|u_\varepsilon\|_{L^\infty}} = 48.$$

Denote $\|u_\varepsilon\|_{L^\infty}$ by α_ε. Then

$$\mu_\varepsilon = \alpha_\varepsilon^{-2/(p-1-\varepsilon)}, \qquad v_\varepsilon(r) = \frac{1}{\alpha_\varepsilon} u_\varepsilon(\mu_\varepsilon r).$$

From Lemmas 3.1 and 3.2 we have

$$\alpha_\varepsilon \to \infty, \qquad \alpha_\varepsilon^\varepsilon \to 1 \qquad \text{as} \quad \varepsilon \to 0.$$

By the mean value theorem we have

$$|\alpha_\varepsilon^\varepsilon - 1| \leqslant c\varepsilon \log \alpha_\varepsilon, \qquad |\mu_\varepsilon^\varepsilon - 1| \leqslant c\varepsilon \log \alpha_\varepsilon, \tag{4.1}$$

where c is independent of ε. Let

$$\varphi_\varepsilon(r) = \alpha_\varepsilon \left[1 + \frac{1}{n(n-2)} \left(\frac{r}{\mu_\varepsilon} \right)^2 (1 - \mu_\varepsilon^2) \right]^{-(n-2)/2}.$$

LEMMA 4.2. *For $n \geqslant 3$ and $r > 0$, we have*

$$u_\varepsilon(r) < \varphi_\varepsilon(r), \tag{4.2}$$

$$\left(1 - \frac{r^2}{2n}\right) u_\varepsilon(r) > \frac{\alpha_\varepsilon^p}{\alpha_\varepsilon^{p-\varepsilon} - \alpha_\varepsilon} \varphi_\varepsilon(r)^{1-\varepsilon} - \alpha_\varepsilon \left[\frac{\alpha_\varepsilon^p}{(\alpha_\varepsilon^{p-\varepsilon} - \alpha_\varepsilon)\, \varphi_\varepsilon^\varepsilon(r)} - 1 \right]. \tag{4.3}$$

Proof. We shall prove this lemma by the method in [AP1]. Define

$$y(t) = u_\varepsilon(r), \qquad t = (n-2)^{n-2} r^{2-n}.$$

Then

$$y'' + t^{-k}(y^{p-\varepsilon} - y) = 0, \qquad \text{for} \quad t > 0,$$

$$\lim_{t \to \infty} y(t) = \alpha_\varepsilon, \qquad y(0) = 0,$$

where $k = 2(n-1)/(n-2)$. Since $u_\varepsilon'(r) < 0$ for $r > 0$, we have $y'(t) > 0$ for $t > 0$. As in [AP1] we have

$$(y' t^{k-1} y^{1-k})' = -2(k-1)\, t^{k-2} y^{-k} H(t), \tag{4.4}$$

where

$$H(t) = \frac{1}{2} t (y')^2 - \frac{1}{2} y y' + \frac{1}{2k-2} t^{1-k} y (y^{p-\varepsilon} - y).$$

We claim that $H(t) > 0$ for $t > 0$. First, since $y'(t) = -u_\varepsilon'(r) t^{-k/2}$ and $k > 2$, we see that $H(t) \to 0$ as $t \to +\infty$. Second, by the fact that $u_\varepsilon(r)$ and $u_\varepsilon'(r)$ decay exponentially at $r = +\infty$ (see Lemma 3.5), we have $H(t) \to 0$ as $t \to 0$. Now we see that to prove the claim, it suffices to show that $H'(t)$ has only one zero on $(0, +\infty)$. This can be seen from the formula

$$H'(t) = \frac{1}{2(k-1)} t^{1-k} y'(t)\, y(t) [2(k-2) - \varepsilon y^{p-1-\varepsilon}(t)]$$

and the fact that $y'(t) > 0$ for $t > 0$. The proof of the claim is completed.

From (4.4) and the claim proved above, we have $(y't^{k-1}y^{1-k})' < 0$, and hence

$$\begin{aligned} y't^{k-1}y^{1-k} &> \lim_{t\to\infty} y'(t)\, t^{k-1}y(t)^{1-k} \\ &= \lim_{t\to\infty} (-u_\varepsilon'(r)t^{-k/2})\, t^{k-1}y(t)^{1-k} \\ &= \lim_{t\to\infty} \frac{-(n-2)}{r} u_\varepsilon'(r)\, y(t)^{1-k} \\ &= -(n-2)\alpha_\varepsilon^{1-k} \lim_{t\to\infty} \frac{u_\varepsilon'(r)}{r} \\ &= \frac{n-2}{n} \alpha_\varepsilon^{1-k}(\alpha_\varepsilon^{p-1-\varepsilon} - \alpha_\varepsilon) \\ &= \frac{1}{k-1} \alpha_\varepsilon^{2-k}(\alpha_\varepsilon^{p-1-\varepsilon} - 1). \end{aligned}$$

So

$$y'y^{1-k} > \frac{1}{k-1} t^{1-k}\alpha_\varepsilon^{2-k}(\alpha_\varepsilon^{p-1-\varepsilon} - 1).$$

Integrating over (t, ∞) we get

$$y(t) < z(t) \equiv \alpha_\varepsilon \left[1 + \frac{1}{k-1} t^{2-k}(\alpha_\varepsilon^{p-1-\varepsilon} - 1) \right]^{-1/(k-2)}, \tag{4.5}$$

which gives (4.2).

It is easy to see that

$$z'' + t^{-k}\alpha_\varepsilon^{-p}(\alpha_\varepsilon^{p-\varepsilon} - \alpha_\varepsilon) z^p = 0$$

$$z(\infty) = \alpha_\varepsilon,$$

$$z(t) = \alpha_\varepsilon - \frac{\alpha_\varepsilon^{p-1-\varepsilon} - 1}{\alpha_\varepsilon^{p-1}} \int_t^\infty (s-t)\, s^{-k} z(s)^p \, ds,$$

and

$$y(t) = \alpha_\varepsilon - \int_t^\infty (s-t)\, s^{-k} y(s)^{p-\varepsilon} \, ds + \int_t^\infty (s-t)\, s^{-k} y(s) \, ds.$$

From the integral equations of z and y above, (4.5), and the fact that $y'(t) < 0$, one easily obtains

$$y(t) > \frac{\alpha_\varepsilon^p}{\alpha_\varepsilon^{p-\varepsilon} - \alpha_\varepsilon} z(t)^{1-\varepsilon} - \alpha_\varepsilon \left[\frac{\alpha_\varepsilon^p}{z(t)^\varepsilon\, (\alpha_\varepsilon^{p-\varepsilon} - \alpha_\varepsilon)} - 1 \right] + \frac{t^{2-k} y(t)}{(k-1)(k-2)},$$

which gives (4.3). Q.E.D.

Now we are ready to give

Proof of Lemma 4.1. First we observe that for fixed $0<a<1$, if $u_\varepsilon(r_0)<(1-a^2)^{1/(p-1-\varepsilon)}$, then $u_\varepsilon(r)\leqslant u_\varepsilon(r_0)e^{-a(r-r_0)}$ for $r>r_0$. This fact follows from the proof of Proposition 4.1 of [GNN2]. Using this fact and Lemma 3.8 we have

$$u_\varepsilon(r)\leqslant u_\varepsilon(1)e^{-(r-1)/2}\leqslant c\alpha_\varepsilon^{-1}e^{-r/2} \qquad \text{for} \quad r>1.$$

From this we have

$$\int_{|x|\geqslant 1} u_\varepsilon^2\,dx\leqslant c_1\alpha_\varepsilon^{-2}. \tag{4.6}$$

The following will be used frequently in the remaining part of this proof,

$$\mu_\varepsilon=\alpha_\varepsilon^{-1+O(\varepsilon)} \quad \textit{and} \quad \varphi_\varepsilon(r)=\alpha_\varepsilon\left[1+\frac{1}{8}\left(\frac{r}{\mu_\varepsilon}\right)^2(1-\mu_\varepsilon^2)\right]^{-1} \qquad \textit{when } n=4.$$

By virtue of (4.2), we have

$$\int_{1/\alpha_\varepsilon\leqslant|x|\leqslant 1} u_\varepsilon^2\,dx\leqslant c_2\alpha_\varepsilon^{-2}\log\alpha_\varepsilon. \tag{4.7}$$

Observe that for any fixed $N>0$,

$$\int_{|x|\leqslant N/\alpha_\varepsilon} u_\varepsilon^2\,dx\leqslant \alpha_\varepsilon^2\left|B\left(0,\frac{N}{\alpha_\varepsilon}\right)\right|\leqslant c_3\alpha_\varepsilon^{-2}N^4. \tag{4.8}$$

Now (4.6)–(4.8) imply that

$$\int_{R^4} u_\varepsilon^2\,dx\leqslant c_5\alpha_\varepsilon^{-2}\log\alpha_\varepsilon. \tag{4.9}$$

By the Pohozaev identity and Corollary 2.6 we have

$$\int_{R^4} u_\varepsilon^2\,dx=\frac{1}{4}S^2\varepsilon+O(\varepsilon). \tag{4.10}$$

From (4.9) and (4.10) we have

$$\varepsilon\leqslant c_6\alpha_\varepsilon^{-2}\log\alpha_\varepsilon. \tag{4.11}$$

By (4.3) we have

$$\frac{\alpha_\varepsilon^3}{\alpha_\varepsilon^{3-\varepsilon}-\alpha_\varepsilon}\,\varphi_\varepsilon(r)<\varphi_\varepsilon^\varepsilon(r)\,u_\varepsilon(r)+\alpha_\varepsilon\left[\frac{\alpha_\varepsilon^3}{\alpha_\varepsilon^{3-\varepsilon}-\alpha_\varepsilon}-\varphi_\varepsilon^\varepsilon(r)\right]. \tag{4.12}$$

For $1/\alpha_\varepsilon\leqslant r\leqslant 1$ we have from the definition of φ_ε that

$$\alpha_\varepsilon^\varepsilon>\varphi_\varepsilon^\varepsilon(r)\geqslant\alpha_\varepsilon^\varepsilon[1+\tfrac{1}{8}r^2\mu_\varepsilon^{-2}]^{-\varepsilon}\geqslant c_7^\varepsilon\alpha_\varepsilon^\varepsilon\mu_\varepsilon^{2\varepsilon}\geqslant c_7^\varepsilon\mu_\varepsilon^\varepsilon.$$

This and (4.1) yield

$$|\varphi_\varepsilon^\varepsilon(r) - 1| \leqslant c_8 \varepsilon \log \alpha_\varepsilon. \tag{4.13}$$

Now observe that for $1/\alpha_\varepsilon \leqslant r \leqslant 1$

$$\begin{aligned}
\frac{\alpha_\varepsilon^3}{\alpha_\varepsilon^{3-\varepsilon} - \alpha_\varepsilon} - \varphi_\varepsilon^\varepsilon &= \frac{1}{1 - \alpha_\varepsilon^{-2+\varepsilon}} \left[\alpha_\varepsilon^\varepsilon - \varphi_\varepsilon^\varepsilon(r) + \alpha_\varepsilon^{-2+\varepsilon} \varphi_\varepsilon^\varepsilon(r)\right] \\
&\leqslant \frac{1}{1 + o(1)} \left[|\alpha_\varepsilon^\varepsilon - 1| + |\varphi_\varepsilon^\varepsilon(r) - 1| + \alpha_\varepsilon^{-2+2\varepsilon}\right] \\
&\leqslant c_9 \varepsilon \log \alpha_\varepsilon \leqslant c_{10} \alpha_\varepsilon^{-2} (\log \alpha_\varepsilon)^2.
\end{aligned}$$

(The last two inequalities follow from (4.1), (4.13), and (4.11).) Combining this with (4.12) and (4.1), we have for $1/\alpha_\varepsilon \leqslant r \leqslant 1$,

$$\frac{\alpha_\varepsilon^3}{\alpha_\varepsilon^{3-\varepsilon} - \alpha_\varepsilon} \varphi_\varepsilon(r) < \alpha_\varepsilon^\varepsilon u_\varepsilon(r) + c_{10} \alpha_\varepsilon^{-1} (\log \alpha_\varepsilon)^2,$$

$$\varphi_\varepsilon(r) < u_\varepsilon(r) + c_{10} \alpha_\varepsilon^{-1} (\log \alpha_\varepsilon)^2.$$

This and (4.2) yield

$$\begin{aligned}
|\varphi_\varepsilon^2(r) - u_\varepsilon^2(r)| &\leqslant c_{10} \alpha_\varepsilon^{-1} (\log \alpha_\varepsilon)^2 \, (\varphi_\varepsilon(r) + u_\varepsilon(r)) \\
&\leqslant c_{11} \alpha_\varepsilon^{-1} (\log \alpha_\varepsilon)^2 \, \varphi_\varepsilon(r).
\end{aligned} \tag{4.14}$$

Now for any fixed $N \geqslant 1$,

$$\begin{aligned}
&\left| \frac{1}{\log \alpha_\varepsilon} \int_{N/\alpha_\varepsilon}^{1/\log \alpha_\varepsilon} r^3 (\alpha_\varepsilon^2 u_\varepsilon^2(r) - \alpha_\varepsilon^2 \varphi_\varepsilon^2(r)) \, dr \right| \\
&\quad \leqslant c_{11} \alpha_\varepsilon \log \alpha_\varepsilon \int_{N/\alpha_\varepsilon}^{1/\log \alpha_\varepsilon} r^3 \varphi_\varepsilon(r) \, dr \\
&\quad \leqslant c_{12} \alpha_\varepsilon \log \alpha_\varepsilon \int_{N/\alpha_\varepsilon}^{1/\log \alpha_\varepsilon} r^3 \alpha_\varepsilon \mu_\varepsilon^2 r^{-2} \, dr \\
&\quad \leqslant c_{13} \log \alpha_\varepsilon \left[\frac{1}{(\log \alpha_\varepsilon)^2} - \left(\frac{N}{\alpha_\varepsilon}\right)^2 \right] \to 0 \qquad \text{as} \quad \varepsilon \to 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{1}{\log \alpha_\varepsilon} \int_{N/\alpha_\varepsilon}^{1/\log \alpha_\varepsilon} r^3 (\alpha_\varepsilon u_\varepsilon(r))^2 \, dr \\
&\quad = \frac{\alpha_\varepsilon^4}{\log \alpha_\varepsilon} \int_{N/\alpha_\varepsilon}^{1/\log \alpha_\varepsilon} \left[1 + \frac{1}{8} \left(\frac{r}{\mu_\varepsilon}\right)^2 (1 - \mu_\varepsilon^2) \right]^{-2} r^3 \, dr + o(1) \\
&\quad = 64 + o\left(\frac{1}{N}\right) + o((\log \alpha_\varepsilon)^{-1/2}) + o(1).
\end{aligned}$$

This and (4.8) give that

$$\lim_{\varepsilon \to 0} \frac{1}{\log \alpha_\varepsilon} \int_0^{1/\log \alpha_\varepsilon} r^3 (\alpha_\varepsilon u_\varepsilon(r))^2 \, dr = 64 + o\left(\frac{1}{N}\right).$$

Letting $N \to \infty$ we get

$$\lim_{\varepsilon \to 0} \frac{1}{\log \alpha_\varepsilon} \int_0^{1/\log \alpha_\varepsilon} r^3 (\alpha_\varepsilon u_\varepsilon(r))^2 \, dr = 64. \tag{4.15}$$

From (4.2) we have

$$\begin{aligned} &\frac{1}{\log \alpha_\varepsilon} \int_{1/\log \alpha_\varepsilon}^{1} r^3 (\alpha_\varepsilon u_\varepsilon(r))^2 \, dr \\ &\leqslant \frac{1}{\log \alpha_\varepsilon} \int_{1/\log \alpha_\varepsilon}^{1} r^3 (\alpha_\varepsilon \varphi_\varepsilon(r))^2 \, dr \\ &\leqslant \frac{c_{14}}{\log \alpha_\varepsilon} \log \log \alpha_\varepsilon \to 0 \qquad \text{as} \quad \varepsilon \to 0. \end{aligned}$$

Combining this with (4.6) and (4.15), we have

$$\frac{\alpha_\varepsilon^2}{\log \alpha_\varepsilon} \int_{R^4} u_\varepsilon^2 \, dx = 64\omega_4 = 128\pi^2.$$

This and (4.10) imply

$$\lim_{\varepsilon \to 0} \frac{\varepsilon \alpha_\varepsilon^2}{\log \alpha_\varepsilon} = 48.$$

(Note that when $n = 4$, $S^2 = \frac{32}{3}\pi^2$.) Q.E.D.

APPENDIX

In this appendix we prove the existence of a fundamental solution (defined in Section 1) of $-\Delta + K(x)$ in R^n, under the assumption that K is a locally Hölder continuous function in R^n and $K(x) \geqslant 0$. We believe this is also true for more general second order elliptic operators, but we do not intend to pursue those. The authors thank Professor Wei-Ming Ni for his suggestion of using the argument in [KN].

Let $B_R = \{x \in R^n \colon |x| < R\}$. For any $f \in C(\partial B_1)$, consider the following Dirichlet problem in the exterior domain:

$$\begin{aligned} &-\Delta w + K(x) w = 0, \qquad |x| > 1, \\ &w|_{\partial B_1} = f, \qquad \lim_{|x| \to \infty} w(x) = 0. \end{aligned} \tag{A1}$$

Problem (A1) is solvable by the method of sub- and super-solutions. In fact, let $\Gamma_0(x)$ be a suitable multiple of the standard fundamental solution of $-\Delta$ such that $\Gamma_0|_{\partial B_1}=1$. Then take $\|f\|_{L^\infty(\partial B_1)}\Gamma_0(x)$ as a super-solution and $-\|f\|_{L^\infty(\partial B_1)}\Gamma_0(x)$ as a sub-solution. The solution w_f obtained above satisfies

$$|w_f(x)|\leqslant\|f\|_{L^\infty(\partial B_1)}\Gamma_0(x). \tag{A2}$$

By the maximum principle this solution is unique.

We consider the following Dirichlet problem in a ball B_R:

$$\begin{aligned}&-\Delta z+K(x)z=0, \qquad x\in B_R,\\&z|_{\partial B_R}=w_f|_{\partial B_R},\end{aligned} \tag{A3}$$

where $R>1$ is to be chosen. The unique solution of (A3) is denoted by z_f. By the maximum principle again we have

$$\|z_f\|_{L^\infty(\partial B_1)}\leqslant\|w_f\|_{L^\infty(\partial B_R)}\leqslant\|f\|_{L^\infty(\partial B_1)}\|\Gamma_0\|_{L^\infty(\partial B_R)}. \tag{A4}$$

Since $\Gamma_0(x)\to 0$ as $|x|\to\infty$, we can choose R large enough that $\|\Gamma_0\|_{L^\infty(\partial B_R)}\leqslant\frac{1}{2}$. Then we can define a linear operator A from $C(\partial B_1)$ to $C(\partial B_1)$ by

$$Af=z_f|_{\partial B_1}, \qquad f\in C(\partial B_1),$$

where z_f is determined by (A1) and (A3) for this R. From (A4) we have $\|A\|\leqslant\frac{1}{2}$.

For R chosen above we consider the problem

$$\begin{aligned}&-\Delta z_1+K(x)z_1=\delta(x), \qquad x\in B_R,\\&z_1|_{\partial B_R}=0;\end{aligned} \tag{A5}$$

i.e., z_1 is the Green function of $-\Delta+K(x)$ in B_R with the Dirichlet condition on ∂B_R, with pole at $x=0$. The existence of z_1 is a well-known fact.

Set $g=z_1|_{\partial B_1}$. Since $\|A\|\leqslant\frac{1}{2}$, there exists an $f\in C(\partial B_1)$ such that $(I-A)f=g$, that is,

$$f-z_f|_{\partial B_1}=z_1|_{\partial B_1}. \tag{A6}$$

Let $w_1=z_f+z_1$. From (A1), (A3), and (A5),

$$\begin{gathered}(-\Delta+K(x))w_1=(-\Delta+K(x))w_f=0 \qquad \text{in} \quad \{1<|x|<R\},\\ w_1|_{\partial B_1}=f=w_f|_{\partial B_1},\\ w_1|_{\partial B_R}=z_f|_{\partial B_R}=w_f|_{\partial B_R}.\end{gathered}$$

Therefore $w_1 \equiv w_f$ in $\{1 < |x| < R\}$. Now we define $\Gamma_K(x, 0)$ as

$$\Gamma_K(x, 0) = \begin{cases} z_f(x) + z_1(x), & \text{if } |x| \leqslant R, \\ w_f(x), & \text{if } |x| > 1. \end{cases}$$

We see that $\Gamma_K(x, 0)$ is well-defined and satisfies

$$-\Delta\Gamma_K(x, 0) + K(x)\,\Gamma_K(x, 0) = \delta(x),$$
$$\Gamma_K(x, 0) \to 0 \qquad \text{as} \quad |x| \to \infty.$$

In the same way, for any $y \in R^n$ we can find $\Gamma_K(x, y)$ such that

$$-\Delta\Gamma_K(\cdot, y) + K(\cdot)\,\Gamma_K(\cdot, y) = \delta(\cdot - y),$$
$$\Gamma_K(x, y) \to 0 \qquad \text{as} \quad |x| \to \infty.$$

Thus we are done. Q.E.D.

Note added in proof. We were informed by Zhenchao Han that he obtained a proof of (1.3).

REFERENCES

[AP1] F. V. Atkinson and L. A. Peletier, Emden–Fowler equations involving critical exponents, *Nonlinear Anal.* **10** (1986), 755–776.

[AP2] F. V. Atkinson and L. A. Peletier, Elliptic equations with nearly critical growth, *J. Differential Equations* **70** (1987), 349–365.

[B] M. S. Berger, On the existence and structure of stationary states for a nonlinear Klein–Gordon equation, *J. Funct. Anal.* **9** (1972), 249–261.

[BL] H. Berestyeki and P. L. Lions, Nonlinear scalar field equations, I, II, *Arch. Rational Mech. Anal.* **82** (1983), 313–345, 347–375.

[BN] H. Brezis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Comm. Pure Appl. Math.* **36** (1983), 437–477.

[BP] H. Brezis and L. A. Peletier, Asymptotics for elliptic equations involving critical growth, *in* "Partial Differential Equations and Calculus of Variations," Vol. 1 (F. Colombini *et al.*, Eds.), pp. 149–192, Birkhäuser, Basel, 1989.

[C] C. V. Coffman, Uniqueness of the ground state solution for $\Delta u - u + u^3 = 0$ and a variational characterization of other solutions, *Arch. Rational Mech. Anal.* **46** (1972), 81–95.

[CGS] L. Caffarelli, B. Gidas, and J. Spruck, Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth, *Comm. Pure Appl. Math.* **42** (1989), 271–297.

[DN] W.-Y. Ding and W.-M. Ni, On the existence of positive entire solutions of a similinear elliptic equation, *Arch. Rational Mech. Anal.* **91** (1986), 283–308.

[GNN1] B. Gidas, W.-M. Ni, and L. Nirenberg, Symmetry and related properties via the maximum principle, *Comm. Math. Phys.* **68** (1979), 209–243.

[GNN2] B. Gidas, W.-M. Ni, and L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in R^n, *Adv. in Math. Stud. A* **7** (1981), 369–402.

[GS] D. GILBARG AND J. SERRIN, On isolated singularities of solutions of second order elliptic differential equations, *J. Analyse Math.* **4** (1955/1956), 309–340.

[H] Z. C. HAN, Asymptotic approach to singular solutions for nonlinear elliptic equations involving critical Sobolev exponent, *Analyse Nonlin.* **8** (1991), 159–174.

[K] M. K. KWONG, Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in R^n. *Arch. Rational Mech. Anal.* **105** (1989), 243–266.

[KN] C. E. KENIG AND W.-M. NI, On the elliptic equation $Lu - k + Ke^{2u} = 0$, *Ann. Scuola Norm. Sup. Pisa Ser. IV* **12** (1985), 191–224.

[L1] P. L. LIONS, The concentration–compactness principle in the calculus of variations. The local compact case, I, II, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **1** (1984), 109–145, 223–283.

[L2] P. L. LIONS, On positive solutions of semilinear elliptic equations in unbounded domains, *in* "Nonlinear Diffusion Equations and Their Equilibrium States, II" (W.-M. Ni *et al.*, Eds.), pp. 85–177, Springer-Verlag, New York, 1988.

[M] C. MIRANDA, "Partial Differential Equations of Elliptic Type," 2nd rev. ed. (Z. C. Motheler, Trans.), Springer-Verlag, New York, 1970.

[MS] K. MCLEOD AND J. SERRIN, Uniqueness of positive radial solutions of $\Delta u + f(u) = 0$ in R^n, *Arch. Rational Mech. Anal.* **99** (1987), 115–145.

[Ne] Z. NEHARI, On a nonlinear differential equation arising in nuclear physics, *Proc. Roy. Irish Acad. Sect. A* **62** (1963), 117–135.

[NS] W.-M. NI AND J. SERRIN, Existence and nonexistence theorems for ground states for quasilinear partial differential equations, *Accad. Naz. Lincei* **77** (1986), 231–257.

[PS] L. A. PELETIER AND J. SERRIN, Uniqueness of solutions of semilinear equations in R^n, *J. Differential Equations* **61** (1986), 380–397.

[R] O. REY, Proof of two conjectures of H. Brezis and L. A. Peletier, *Manuscripta Math.* **65** (1989), 19–37.

[T] N. S. TRUDINGER, On Harnack type inequalities and their applications to quasilinear elliptic equations. *Comm. Pure Appl. Math.* **20** (1967), 721–747.

[Z] L. ZHANG, Uniqueness of ground state solutions, *Acta Mat. Sci.* **8** (1988), 449–468