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# The $k$-edge intersection graphs of paths in a tree

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**Abstract**

We consider a generalization of edge intersection graphs of paths in a tree. Let $\mathscr{P}$ be a collection of nontrivial simple paths in a tree $T$. We define the $k$-edge ($k \geqslant 1$) intersection graph $\Gamma_k(\mathscr{P})$, whose vertices correspond to the members of $\mathscr{P}$, and two vertices are joined by an edge if the corresponding members of $\mathscr{P}$ share $k$ edges in $T$. An undirected graph $G$ is called a $k$-edge intersection graph of paths in a tree, and denoted by $k$-EPT, if $G = \Gamma_k(\mathscr{P})$ for some $\mathscr{P}$ and $T$. It is known that the recognition and the coloring of the 1-EPT graphs are NP-complete. We extend this result and prove that the recognition and the coloring of the $k$-EPT graphs are NP-complete for any fixed $k \geqslant 1$. We show that the problem of finding the largest clique on $k$-EPT graphs is polynomial, as was the case for 1-EPT graphs, and determine that there are at most $O(n^3)$ maximal cliques in a $k$-EPT graph on $n$ vertices. We prove that the family of 1-EPT graphs is contained in, but is not equal to, the family of $k$-EPT graphs for any fixed $k \geqslant 2$. We also investigate the hierarchical relationships between related classes of graphs, and present an infinite family of graphs that are not $k$-EPT graphs for every $k \geqslant 2$. The edge intersection graphs are used in network applications. Scheduling undirected calls in a tree is equivalent to coloring an edge intersection graph of paths in a tree. Also assigning wavelengths to virtual connections in an optical network is equivalent to coloring an edge intersection graph of paths in a tree.


*Keywords:* Paths of a tree; Intersection graphs; EPT-graphs; Chordal graphs; Coloring

## 1. Introduction

**Definition 1.** Let $\mathscr{P}$ be a collection of nontrivial simple paths in a tree $T$. We define the vertex intersection graph $\Omega(\mathscr{P})$ of $\mathscr{P}$ to have vertices which correspond to the members of $\mathscr{P}$, such that two vertices are adjacent in $\Omega(\mathscr{P})$ if the corresponding paths in $\mathscr{P}$ share a vertex in $T$. An undirected graph $G$ is called a vertex intersection graph of paths in a tree (VPT) if $G = \Omega(\mathscr{P})$ for some $\mathscr{P}$ and $T$. We define $\langle \mathscr{P}, T \rangle$ to be a VPT representation of $G$.

**Definition 2.** Let $\mathscr{P}$ be a collection of nontrivial simple paths in a tree $T$. We define its $k$-edge ($k \geqslant 1$) intersection graph $\Gamma_k(\mathscr{P})$ of $\mathscr{P}$ to have vertices which correspond to the members of $\mathscr{P}$, such that two vertices are adjacent in $\Gamma_k(\mathscr{P})$ if the corresponding paths in $\mathscr{P}$ share $k$ edges in $T$. An undirected graph $G$ is called a $k$-edge intersection graph of paths in a tree ($k$-EPT) if $G = \Gamma_k(\mathscr{P})$ for some $\mathscr{P}$ and $T$. We define $\langle \mathscr{P}, T \rangle$ to be a $k$-EPT representation of $G$. The case of $k = 1$ (known as the EPT graphs) was introduced by Golumbic and Jamison [10,11]. Finally, we denote by EPT* the class of graphs consisting of the $k$-EPT graphs for all possible $k \geqslant 1$.

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Clearly, for a given collection $\mathscr{P}$, the vertices of the $k$-EPT graph $\Gamma_k(\mathscr{P})$ and those of VPT graph $\Omega(\mathscr{P})$ are the same, and every edge of $\Gamma_k(\mathscr{P})$ is also an edge of $\Omega(\mathscr{P})$, but not conversely.

The $k$-EPT graphs are widely used in network applications. The problem of scheduling undirected calls in a tree network is equivalent to the problem of coloring a 1-EPT graph (see e.g. [10]). The communication network is represented as an undirected interconnection graph, where each edge is associated with a physical link between two nodes. An undirected call is a path in the network. In the restricted setting, two calls compete on network resources if they share $k$ physical links, and therefore must be scheduled not at the same time. When the network is a tree, this model is clearly a $k$-EPT representation. Coloring the $k$-EPT graph, such that two adjacent vertices have different colors, implies that paths sharing at least $k$ edges in the $k$-EPT representation have different colors. Therefore, the coloring may be associated with a scheduling of the calls, such that calls of the same color can be scheduled at the same time. There are different extensions of this model for different network problems. For example in [5], the call scheduling problem is examined in two different variants of the model. First, considering bidirectional calls, where both connections associated with a call use the same undirected path in the network, and the second, where a call is directed from source to target, without need for a connection in the opposite direction. Both variations of call scheduling problem are proved to be NP-hard for trees and rings in [5].

Another application is assigning wavelengths to connections in an optical network, where virtual connections share physical links by wavelength-division multiplexing. This problem is equivalent to the problem of coloring an EPT graph as follows. In the optical interconnection graph, every edge is associated with optical link between two vertices. Virtual connections that share the same optical link must have different wavelengths. The coloring may be associated with assignment of wavelengths, such that connections of the same color can be assigned the same wavelength. For a survey on related work on this optical model see e.g. [2]. In [6], optimization problems on optical networks with multiple parallel fibers between adjacent nodes are investigated. In such networks, two paths on the link can use the same wavelength if they are carried on different fibers.

A polynomial algorithm for recognizing VPT graphs was introduced by Gavril in [8]. Golumbic and Jamison showed that the recognition problem and the coloring problem are both NP-complete for EPT graphs [10,11]. Tarjan gave a $\frac{3}{2}$-approximation algorithm for coloring EPT graphs [18].

We generalize EPT graphs into $k$-EPT graphs, by quantizing the number of intersecting edges of paths. The vertices of the $k$-EPT graph $\Gamma_k(\mathscr{P})$ and those of the EPT graph $\Gamma_1(\mathscr{P})$ are the same, but the adjacency of the vertices is different. Namely, the edge sets are nested: $E(\Gamma_k(\mathscr{P})) \subseteq E(\Gamma_{k-1}(\mathscr{P})) \subseteq \cdots \subseteq E(\Gamma_1(\mathscr{P})) \subseteq E(\Omega(\mathscr{P}))$.

A totally different approach is adopted in tolerance models of paths in a tree. Intersection graphs of intervals on the line (interval graphs) are generalized to tolerance graphs, by adding an edge between two vertices in the tolerance graph when the size of the intersection of their intervals exceeds at least one of the tolerances. In the same fashion, one can generalize intersection graphs of paths in a tree (EPT) to tolerance intersection graphs of paths in a tree, by adding an edge between two vertices in the model, when the size of the intersection of their paths exceeds at least one of the tolerances. Thus, $k$-EPT graphs are tolerance intersection graphs of paths in a tree with constant tolerance corresponding to $k$ edges in the tree. See [12] for more details.

In [14–16], Jamison and Mulder have placed these tolerance models into a more general setting. An $(h, s, p)$-subtree representation consists of a collection of subtrees of a tree, such that (i) the maximum degree of $T$ is at most $h$, (ii) every subtree has maximum degree at most $s$, (iii) there is an edge between two vertices in the graph if the corresponding subtrees have at least $p$ vertices in common in $T$. The class [3, 3, 3] is studied in [15]. In this paper, we investigate the class $[\infty, 2, k+1]$, which is exactly the class of $k$-EPT graphs.

We continue with preliminaries in Section 2. In Section 3, we characterize the clique structure of $k$-EPT graphs and show that a clique with maximum cardinality can be found in polynomial time. We prove the NP-completeness of the recognition problem and the coloring problem for $k$-EPT graphs in Sections 4 and 5. We investigate the hierarchy of $k$-EPT graphs in Section 6 and conclude with open problems in Section 7.

All standard definitions of terms we use can be found in [9,17].

## 2. Preliminaries

The claw graph $K_{1,3}$, which consists of one central vertex and three edges that intersect on the central vertex, as shown in Fig. 1, will be called the $T_1$ graph. More generally, the $k$-claw graph $T_k$ consists of one central vertex and

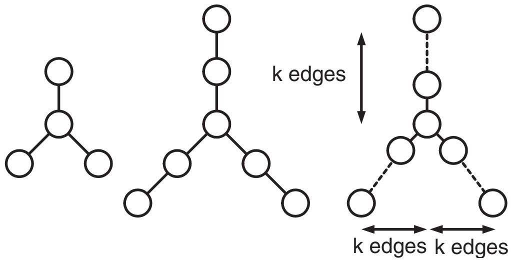


Fig. 1. The graphs $T_1$, $T_2$, $T_k$.

three edge disjoint paths of $k$ edges intersecting only on the central vertex, as shown in Fig. 1. We call these paths the *legs* of the $k$-claw.

Let $\langle \mathscr{P}, T\rangle$ be a $k$-EPT representation of $G$. For any simple path of $k$ edges $[e_1, e_2, \ldots, e_k]$ in the tree $T$, let $\mathscr{P}[e_1, e_2, \ldots, e_k] = \{P \in \mathscr{P} | e_1, e_2, \ldots, e_k \in P\}$. For any copy of the $k$-claw $T_k$ in $T$, let $\mathscr{P}[T_k] = \{P \in \mathscr{P} | P$ *contains two legs of* $T_k\}$. The collection $\mathscr{P}[e_1, e_2, \ldots, e_k]$ corresponds to a clique in $G$ and is called a *k-edge clique*. Similarly, $\mathscr{P}[T_k]$ also corresponds to a clique in $G$ and is called a *k-claw clique*.

**Lemma 3.** *Let G be a* 1*-EPT graph, then G is a k-EPT graph, for every* $k \geqslant 2$.

**Proof.** Let $\langle \mathscr{P}, T\rangle$ be a 1-EPT representation of $G$, i.e., $G = \Gamma_1(\mathscr{P})$. It is simple to construct a $k$-EPT representation $\langle \mathscr{P}', T'\rangle$ of $G$, such that $G = \Gamma_k(\mathscr{P}')$. For every edge $e$ in $T$, we add $k-1$ dummy vertices, dividing the edge $e$ into $k$ edges in $T'$.

Every two paths, which intersect on at least one edge in $\Gamma_1(\mathscr{P})$, intersect on at least $k$ edges in $\Gamma_k(\mathscr{P}')$. Every two paths, that intersect on less than one edge in $\Gamma_1(\mathscr{P})$, intersect on less than $k$ edges in $\Gamma_k(\mathscr{P}')$. Therefore, $G = \Gamma_k(\mathscr{P}')$ is a $k$-EPT graph. $\square$

**Definition 4.** Let $\Psi = \{S_i\}_{i \in I}$ be a collection of subsets of a set $S$. We say that $\Psi$ has *Helly number h* if for all $J \subseteq I$, $\cap\{S_i | i \in J\} = \emptyset$ implies that there exist $h$ indices $i_1, \ldots, i_h \in J$ such that $S_{i_1} \cap \cdots \cap S_{i_h} = \emptyset$.

**Theorem 5** (*Berge [3]*). *Any collection of subtrees of a tree* (*and therefore any collection of paths of a tree*) *has Helly number* 2.

**Definition 6.** Let $\Psi = \{S_i\}_{i \in I}$ be a collection of subsets of a set $S$. We say that $\Psi$ has *strong Helly number s* if for all $J \subseteq I$, there exist $s$ indices $i_1, \ldots, i_s \in J$ such that $S_{i_1} \cap \cdots \cap S_{i_s} = \cap\{S_i | i \in J\}$.

**Theorem 7** (*Golumbic [10]*). *A finite collection of paths in a tree has strong Helly number* 3.

Clearly, the Helly number is less than or equal to the strong Helly number.

## 3. Cliques in $k$-EPT graphs

In this section, we characterize the maximal cliques of the $k$-EPT graphs. This characterization will allow us to state a bound on the number of maximal cliques in a $k$-EPT graph and to find a maximum clique in polynomial time.

The following generalizes a result of Golumbic and Jamison, who proved the case of $k = 1$ [10].

**Proposition 8.** *Let* $G = \Gamma_k(\mathscr{P})$ *be a k-EPT graph, and let* $\langle \mathscr{P}, T\rangle$ *be a k-EPT representation of G. Any maximal clique of G corresponds to a subcollection of paths of the form* $\mathscr{P}[e_1, e_2, \ldots, e_k]$ *for some edges* $e_1, e_2, \ldots, e_k$ *in T* (*a k*-edge clique) *or* $\mathscr{P}[T_k]$ *for some copy of the k-claw* $T_k$ *in T* (*a k*-claw clique).

**Proof.** Let $C$ be a maximal clique of $G = \Gamma_k(\mathscr{P})$, and let $\mathscr{J} \subseteq \mathscr{P}$ be the subcollection corresponding to $C$, whose members intersect pairwise on at least $k$ edges in $T$. Let $Q = \cap\{P | P \in \mathscr{J}\}$. By Helly number 2, $Q$ contains at least one vertex and, by strong Helly number 3, there are paths $P_1, P_2, P_3 \in \mathscr{J}$ such that $Q = P_1 \cap P_2 \cap P_3$. Obviously, the intersection of any two paths is also a connected path, so $Q$ is a path in $T$.

If $Q$ contains at least $k$ edges, then $C$ is a $k$-edge clique.

Suppose $Q$ contains less than $k$ edges, but at least one edge. Due to the connectivity of the intersection $P_1 \cap P_2$ and since $|P_1 \cap P_2| \geqslant k$, there exists an edge $e_{12} \in P_1 \cap P_2$, $e_{12} \notin Q$, which has common endpoint $u$ with $Q$. There also exists an edge $e_{23} \in P_2 \cap P_3$, $e_{23} \notin Q$ ($e_{23} \neq e_{12}$ since otherwise $e_{23}$ would be also in $Q$), such that the edge $e_{23}$ has a common endpoint $v$ with $Q$. Moreover, $u \neq v$ because $P_2$ is a path. Finally, there must be an edge $e_{13} \in P_1 \cap P_3$, $e_{13} \notin Q$, ($e_{13} \neq e_{12}$ and $e_{13} \neq e_{23}$ since otherwise $e_{13}$ would be also in $Q$), such that $e_{13}$ has a common endpoint with $Q$. If the common endpoint of $e_{13}$ and $Q$ is $u$, then $P_1$ is not a path. If the common endpoint of $e_{13}$ and $Q$ is $v$, then $P_3$ is not a path. Contradiction!

Otherwise, $Q$ consists of one single vertex $q$. Suppose $q$ is not an endpoint of the path $P_1 \cap P_2$, meaning that there are two edges in $P_1 \cap P_2$, whose endpoint is $q$. Since $P_3$ contains $q$ and $|P_1 \cap P_3| \geqslant k$, the path $P_3$ intersects with $P_1 \cap P_2$ on an edge, whose endpoint is $q$, and, therefore, $Q$ has more than a single vertex. Hence, $q$ is an endpoint of the path $P_1 \cap P_2$. Symmetrically, $q$ is also an endpoint of the paths $P_1 \cap P_3$ and $P_2 \cap P_3$. Thus, there exist legs $E_{12}$, $E_{13}$, $E_{23}$ which intersect only on $q$, such that $E_{12} \subseteq P_1 \cap P_2$, $E_{13} \subseteq P_1 \cap P_3$, $E_{23} \subseteq P_2 \cap P_3$. Since any other $P \in \mathscr{J}$ must share $k$ edges with each of the $P_1, P_2, P_3$, then $P$ intersects with exactly two of the three legs $E_{12}, E_{13}, E_{23}$. Thus, $P$ contains the central vertex $q$ and contains two of the three legs. Therefore, $C$ is a $k$-claw clique. □

Let $C$ be a maximal clique of a $k$-EPT graph $G = \Gamma_k(\mathscr{P})$, and let $\mathscr{J} \subseteq \mathscr{P}$ be the subcollection corresponding to $C$, whose members intersect pairwise on at least $k$ edges in $T$. By Theorem 7, there exist $P_1, P_2, P_3 \in \mathscr{J}$, such that $\cap\{P | P \in \mathscr{J}\} = P_1 \cap P_2 \cap P_3$. We say that the triple $P_1, P_2, P_3$ *strongly defines* $C$.

**Proposition 9.** *For any triple of vertices $u, v, w$, the paths $P_u, P_v, P_w$ strongly define at most one maximal clique.*

**Proof.** Suppose there exist paths $P_u, P_v, P_w$ that strongly define two maximal cliques $C$ and $C'$ corresponding to subcollections $\mathscr{J}$ and $\mathscr{J}'$ of $\mathscr{P}$.

Let $y \in C'$. If $C$ is a $k$-edge clique, then $\cap\{P | P \in \mathscr{J}\} = P_u \cap P_v \cap P_w \subseteq P_y$. Hence, $P_y$ intersects with every path $P \in \mathscr{J}$ on at least $k$ edges, and therefore $y \in C$. If $C$ is a $k$-claw clique with three legs $L_1, L_2, L_3$, then every path $P \in \mathscr{J}$ contains two legs of the $k$-claw with the central vertex $\{q\} = P_1 \cap P_2 \cap P_3$. Without loss of generality, let $L_1 \subseteq P_u \cap P_v$, $L_2 \subseteq P_v \cap P_w$, $L_3 \subseteq P_w \cap P_u$. Since $|P_y \cap P_u| \geqslant k$, $|P_y \cap P_v| \geqslant k$ and $|P_y \cap P_w| \geqslant k$, the path $P_y$ contains two of the legs $L_1, L_2, L_3$, and therefore $y \in C$. Thus, we have shown $C' \subseteq C$. However, since $C'$ is a maximal clique, $C' = C$. Contradiction! □

**Corollary 10.** *A $k$-EPT graph $G$ has at most $\mathrm{O}(n^3)$ maximal cliques, where $n$ is the number of vertices in $G$.*

**Proof.** This follows immediately from Proposition 9. □

**Corollary 11.** *The problem of finding a clique of maximum cardinality can be solved in polynomial time for $k$-EPT graphs.*

**Proof.** One can enumerate all the maximal cliques in the graph by a standard clique enumeration algorithm [1] and choose the largest. This can be done in polynomial time, since according to Corollary 10 there are at most $\mathrm{O}(n^3)$ maximal cliques. □

## 4. Recognition of $k$-EPT graphs

**Definition 12.** Let $C$ be any subset of the vertices of a graph $G$. The *branch graph* $B(G/C)$ of $G$ over $C$ has a vertex set, $V(B)$, consisting of all the vertices of $G$ not in $C$ but adjacent to some member of $C$, i.e., $V(B) = \bigcup_{v \in C}\{\mathrm{Adj}(v) - C\}$. Adjacency in $B(G/C)$ is defined as follows: we join $x$ and $y$ by an edge in $E(B)$ if and only if in $G$,

(1) $x$ and $y$ are not adjacent,
(2) $x$ and $y$ have a common neighbor $u \in C$,

(3) the sets $\mathrm{Adj}(x) \cap C$ and $\mathrm{Adj}(y) \cap C$ are not comparable, i.e., there exist $w, z \in C$ such that $w$ is adjacent to $x$ but not to $y$, and $z$ is adjacent to $y$ but not to $x$.

The following generalizes a result of Golumbic and Jamison, who proved the case of $k = 1$ [10].

**Theorem 13.** *Let $\langle \mathscr{P}, T\rangle$ be a k-EPT representation of G, and let C be a maximal clique of G. If C corresponds to a k-edge clique in $\langle \mathscr{P}, T\rangle$, then the branch graph $B(G/C)$ can be 2-colored. If C corresponds to a k-claw clique in $\langle \mathscr{P}, T\rangle$, then the branch graph $B(G/C)$ can be 3-colored.*

**Proof.** Using Proposition 8, we consider two cases.

*Case* 1: Suppose the clique $C$ corresponds to a subcollection of paths that form a $k$-edge clique $[e_1, \dots, e_k]$ in $\langle \mathscr{P}, T\rangle$. Let $S_1$ be the subtree containing $e_1$, obtained by removing $e_k$ from $T$. Let $S_2$ be the subtree containing $e_k$, obtained by removing $e_1$ from $T$.

We color the graph $B(G/C)$ as follows. Let $x \in V(B)$ and let $P_x$ be the corresponding path in $\langle \mathscr{P}, T\rangle$. If $P_x$ is contained in $S_i$, then we color $x$ with the color $i$, for $i = 1, 2$, as follows. Since $x$ has a neighbor in the clique, $P_x$ is contained in at least one of $S_1$, $S_2$. Hence, $x$ is colored. Since $P_x$ is not in the clique $C$, $P_x$ contains at most one of $e_1, e_k$ and therefore is contained in at most one of $S_1$ or $S_2$. Hence, $x$ is colored with exactly one color. Therefore, the coloring is well-defined.

We now prove that the coloring of $B(G/C)$ is proper. Suppose $(x, y) \in E(B)$ and $x, y$ have the same color, say 1. Let $u, w, z$ be as given in the definition of the branch graph. Obviously, $P_u \cap P_x$ and $P_u \cap P_y$ do not contain each other. Without loss of generality, suppose that $P_x$ is closer to $e_k$ than $P_y$. The path $P_z$ intersects with $P_y$ and contains both $e_1$ and $e_k$. Therefore, $P_z$ contains $P_u \cap P_x$, meaning that $P_z$ and $P_x$ intersect on at least $k$ edges. Contradiction!

*Case* 2: Suppose the clique $C$ corresponds to a collection $C = \mathscr{P}[T_k]$ where $T_k$ is a $k$-claw with central vertex $q$ and three legs $L_1, L_2, L_3$ of $k$ edges each. Recall that any path in $\mathscr{P}[T_k]$ contains two of the legs $L_1, L_2, L_3$. Let $S_1, S_2, S_3$ be the subtrees of $T$ rooted at $q$, where $L_i$ is contained in $S_i$. We color the graph $B(G/C)$ as follows.

Let $x \in V(B)$ and let $P_x$ be the corresponding path in $\langle \mathscr{P}, T\rangle$. We color $x$ with color $i$, for $i = 1, 2, 3$, if any one of the following holds:

- $P_x \subset S_i$,
- $L_i \subseteq P_x$,
- $P_x \subset (S_j \cup S_l)$ and $0 < |L_j \cap P_x| < k$ and $0 < |L_l \cap P_x| < k$ $(j, l \neq i)$.

We first prove that the coloring is well defined. If $P_x \subset S_i$ or $L_i \subseteq P_x$, for some $i$, then $x$ is colored with the color $i$. Otherwise, consider a neighbor $u \in C$ of $x$ and let $j$ and $l$ be the indices such that $(L_j \cup L_l) \subseteq P_u$. Since $|P_x \cap P_u| \geqslant k$, it follows that $P_x \subset (S_j \cup S_l)$. Since $P_x$ contains neither $L_j$ nor $L_l$, it is colored with color $i$ $(i \neq j, l)$. Therefore, $x$ is colored with at least one color.

Suppose $x$ is colored with two colors $i$ and $j$. If $P_x \subset S_i$ then $P_x \not\subseteq S_j$ and $L_j \not\subseteq P_x$ and $|P_x \cap L_l| = 0$, and therefore $x$ is not colored with $j$. If $L_i \subseteq P_x$ then $P_x \not\subseteq S_j$ and $L_j \not\subseteq P_x$ and $|P_x \cap L_i| \geqslant k$, and therefore $x$ is not colored with color $j$. If $P_x \subset (S_j \cup S_l)$ and $0 < |P_x \cap L_j| < k$ and $0 < |P_x \cap L_l| < k$, then $P_x \not\subseteq S_j$ and $L_j \not\subseteq P_x$ and $|P_x \cap L_i| = 0$, and therefore $x$ is not colored with color $j$. Contradiction! Therefore, $P_x$ is colored with exactly one color and the coloring is well-defined.

We now prove that the coloring of $B(G/C)$ is proper. Suppose $x, y \in V(B)$ have the same color, say 1, and $(x, y) \in E(B)$. Then $(x, y) \notin E(G)$, so $|P_x \cap P_y| < k$. Let $u, w, z$ as given in the definition of the branch graph. We will handle each of the following cases separately.

(1) Suppose $P_x \subset S_1$ and $P_y \subset S_1$. Clearly, $P_x \cap P_u$ and $P_y \cap P_u$ do not contain each other, since each has at least $k$ edges. Without loss of generality, suppose that $P_x$ is closer to $q$ than $P_y$. The path $P_z$ intersects with $P_y$ and contains $q$, therefore it contains $P_x \cap P_u$. Hence, $|P_z \cap P_x| \geqslant k$. Contradiction!
(2) Suppose $L_1 \subseteq P_x$ and $L_1 \subseteq P_y$. Then $L_1 \subseteq (P_x \cap P_y)$ and therefore $|P_x \cap P_y| \geqslant k$. Contradiction!
(3) Suppose $L_1 \subseteq P_x$ and $P_y \subset S_1$. Then $L_1 \subseteq P_z$, since $z$ is in the clique $C$ and $|P_z \cap P_y| \geqslant k$.It means that $L_1 \subset (P_z \cap P_x)$, and therefore $|P_z \cap P_x| \geqslant k$. Contradiction!

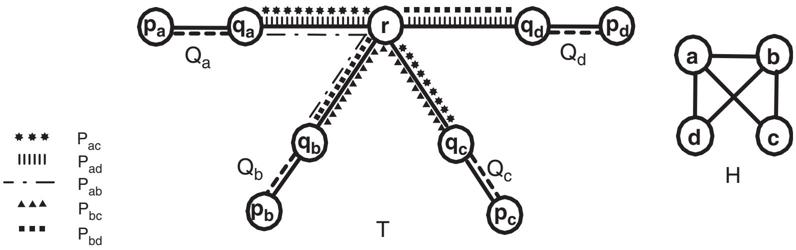

Fig. 2. Example from the proof of Theorem 17.

(4) Suppose $P_x \subset (S_2 \cup S_3)$, $P_y \subset (S_2 \cup S_3)$ and $0 < |P_x \cap L_2|, |P_y \cap L_2|, |P_x \cap L_3|, |P_y \cap L_3| < k$. Both $P_x$ and $P_y$ do not share an edge with $S_1$. In addition, $L_2, L_3 \subset P_z$, since $z$ is in the clique $C$ and $|P_z \cap P_y| \geqslant k$. Hence, $|P_z \cap P_x| \geqslant k$. Contradiction!

(5) Suppose $P_x \subset S_1$ and $P_y \subset (S_2 \cup S_3)$ and $0 < |P_y \cap L_2|, |P_y \cap L_3| < k$. Then $P_u \subset (S_2 \cup S_3)$, since $u$ is adjacent to $y$. Therefore, $|P_u \cap P_x| = 0$. Contradiction!

(6) Suppose $L_1 \subseteq P_x$ and $P_y \subset (S_2 \cup S_3)$ and $0 < |P_y \cap L_2|, |P_y \cap L_3| < k$. Then $P_u \subset (S_2 \cup S_3)$, since $u$ is adjacent to $y$. The path $P_x$ intersects with at most one of $L_2, L_3$ and on less than $k$ edges. Hence, $|P_u \cap P_x| < k$. Contradiction!

Since all the cases cover all possibilities, the theorem holds. □

**Corollary 14.** *Let $C$ be a maximal clique of a $k$-EPT graph $G$, then the branch graph $B(G/C)$ can be* 3*-colored.*

**Theorem 15** (*Golumbic [10]*). *Let $G$ be an undirected graph. The following statements are equivalent*:

(i) *$G$ is both a VPT graph and a* 1*-EPT graph.*
(ii) *$G$ is a VPT graph and for any maximal clique $C$ of $G$, the branch graph $B(G/C)$ is* 3*-colorable.*
(iii) *$G$ has a VPT representation on a degree* 3 *tree.*
(iv) *$G$ has a* 1*-EPT representation on a degree* 3 *tree.*

**Corollary 16.** *Let $G$ be a VPT graph. Then, $G$ is a* 1*-EPT graph if and only if $G$ is a $k$-EPT graph.*

**Proof.** ($\Rightarrow$) If $G$ is a VPT and 1-EPT graph, then by Lemma 3, $G$ is a VPT and $k$-EPT graph.

($\Leftarrow$) If $G$ is a VPT and $k$-EPT graph, then by Corollary 14, $G$ is a VPT graph and for any maximal clique $C$ of $G$, the branch graph $B(G/C)$ is 3-colorable. Hence, $G$ is a VPT and 1-EPT graph due to Theorem 15. □

**Theorem 17.** *It is an* NP*-complete problem to decide whether a VPT graph is a $k$-EPT graph for any fixed $k \geqslant 1$.*

**Proof.** We will prove NP-completeness by demonstrating that an arbitrary undirected graph $H$ is 3-colorable if and only if a certain VPT graph $G = (V, E)$ is a $k$-EPT graph. The decision problem is solved by checking every branch graph for being 3-colorable. We may assume that every vertex of $H$ has degree at least 3 (since vertices with degree $\leqslant$ 2 may be repeatedly stripped off without affecting 3-colorability).

Let $1, \ldots, n$ denote the vertices of $H$. Following [10], we construct $T$ and $\mathscr{P}$ as follows. The tree $T$ has edges $(p_i, q_i)$ and $(q_i, r)$ for $i = 1, \ldots, n$. For each edge $(i, j)$ in $H$, let $P_{i,j}$ be the path in $T$ from $q_i$ (through $r$) to $q_j$, and for each vertex $i$ in $H$, let $Q_i$ be the edge $(p_i, q_i)$ in $T$, see example in Fig. 2. We set $\mathscr{P} = \{P_{i,j}\} \cup \{Q_i\}$. The paths $\{P_{i,j}\}$ all share central vertex $r$ and correspond to a maximal clique $C$ of the VPT graph $G = \Omega(\mathscr{P})$. Furthermore, the branch graph $B(G/C)$ is isomorphic to $H$. The remaining branch graphs of $G$ over other maximal cliques, one for each $i$, are just stable sets.

If $G$ is a VPT and $k$-EPT graph, then $H$ is 3-colorable due to Corollary 14. If $H$ is 3-colorable and $G$ is a VPT graph, then $G$ is a VPT and 1-EPT graph due to Theorem 15, hence $G$ is a $k$-EPT graph according to Lemma 3. Therefore, $G$ is a VPT and $k$-EPT graph if and only if $G$ is a VPT graph and $H$ is 3-colorable, and hence the theorem follows. □

**Corollary 18.** *Recognizing whether an arbitrary graph is a $k$-EPT graph is an* NP-*complete problem.*

It is a simple exercise to modify the proof of Theorem 17 to obtain the following.

**Proposition 19.** *It is an* NP-*complete problem to decide whether a VPT graph is an EPT* graph.*

**Corollary 20.** *Recognizing whether an arbitrary graph is an EPT* graph is an* NP-*complete problem.*

## 5. Coloring of $k$-EPT graphs

**Definition 21.** Let $H$ be a multigraph. The line graph $L(H)$ of $H$ has vertices corresponding to the edges of $H$ with two vertices adjacent in $L(H)$ if the corresponding edges of $H$ share an endpoint.

**Theorem 22** (*Golumbic [11]*). *The following statements are equivalent*:

(i) *$G$ has a* 1-*EPT representation, where all paths share a common vertex.*
(ii) *$G$ is a line graph of a multigraph.*

**Corollary 23.** *The problem of finding a minimum coloring of a $k$-EPT graph is* NP-*complete.*

**Proof.** Clearly, if $G$ has a 1-EPT representation, where all paths share a common vertex, then one can obtain a $k$-EPT representation of $G$, where all paths share a common vertex. This can be done by adding $k-1$ dummy vertices to each edge in the 1-EPT representation, thus dividing the edge into $k$ edges.

Holyer [13] showed that the minimum coloring problem on line graphs of multigraphs is NP-complete. Therefore, the minimum coloring problem is NP-complete on $k$-EPT graphs due to Theorem 22. □

## 6. Hierarchy

In this section, we investigate the relationship between VPT, EPT and $k$-EPT ($k \geqslant 2$) graphs. Golumbic and Jamison [10] proved that the family of VPT is incomparable with the family of EPT graphs, but when restricted to degree 3 trees they coincide.

The class of chordal graphs corresponds to the class of vertex intersection graphs of subtrees in a tree [4,7,19]. Since the VPT graphs are the class of vertex intersection graphs of paths in a tree, the VPT graphs are chordal.

**Theorem 24.** *The family of VPT graphs is incomparable with the family of $k$-EPT graphs for any fixed $k \geqslant 1$. When restricted to degree* 3 *trees, the family of VPT graphs is strictly contained in the family of $k$-EPT graphs, $k \geqslant 2$.*

**Proof.** By [10], there exists a VPT graph which is not a 1-EPT graph, and therefore is not a $k$-EPT graph ($k \geqslant 2$) according to Corollary 16. The graph in Fig. 3 is a $k$-EPT graph for every fixed $k \geqslant 1$, but is not a VPT graph since it is not chordal. Hence, the families of VPT and $k$-EPT are incomparable.

When restricted to degree 3 trees, the family of VPT graphs coincides with the family of 1-EPT graphs, and therefore is contained in the family of $k$-EPT graphs ($k \geqslant 2$) according to Lemma 3. The containment is strict, since the graph in Fig. 3 has a $k$-EPT representation on a degree 3 tree for every fixed $k \geqslant 2$ and is not a VPT graph. □

**Conjecture 25** (*Jamison [14]*). *The family of $k$-EPT graphs $\subseteq$ the family of $k'$-EPT graphs, where $2 \leqslant k \leqslant k'$.*

Jamison and Mulder prove in [16] that Conjecture 25 holds for $k = 2$ and $k = 3$ and for all $k' \geqslant k^2 - 4k + 6$.

**Theorem 26.** *The family of* 1-*EPT graphs is strictly contained in the family of $k$-EPT graphs for any fixed $k \geqslant 2$. The containment is also strict, when restricted to a degree* 3 *tree.*

**Proof.** The family of 1-EPT graphs is contained in the family of $k$-EPT graphs by Lemma 3. The graph in Fig. 4 is not 1-EPT as proved in [10], but it has a $k$-EPT representation on a degree 3 tree, for $k \geqslant 2$, as illustrated. □

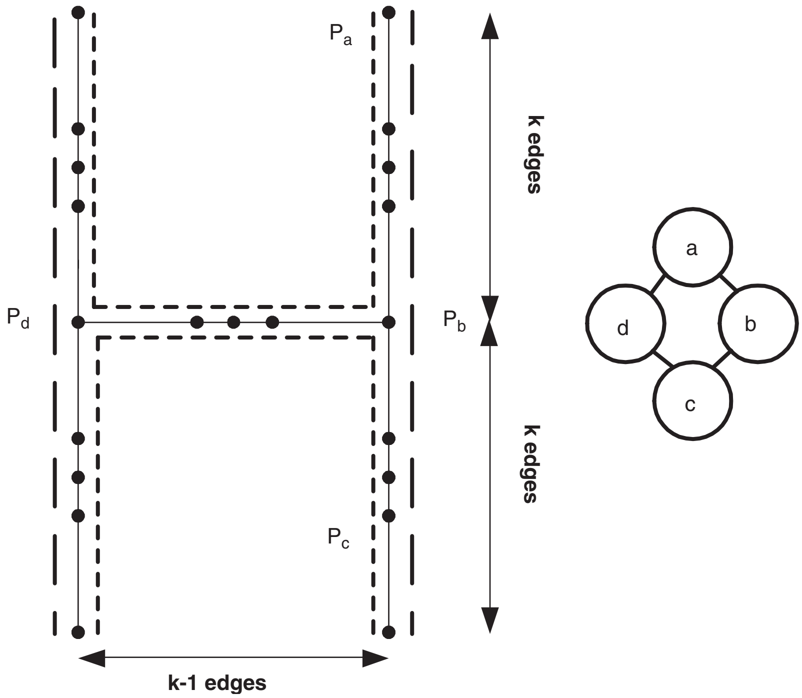


Fig. 3. A $k$-EPT graph for $k \geqslant 1$, which is not a VPT graph.

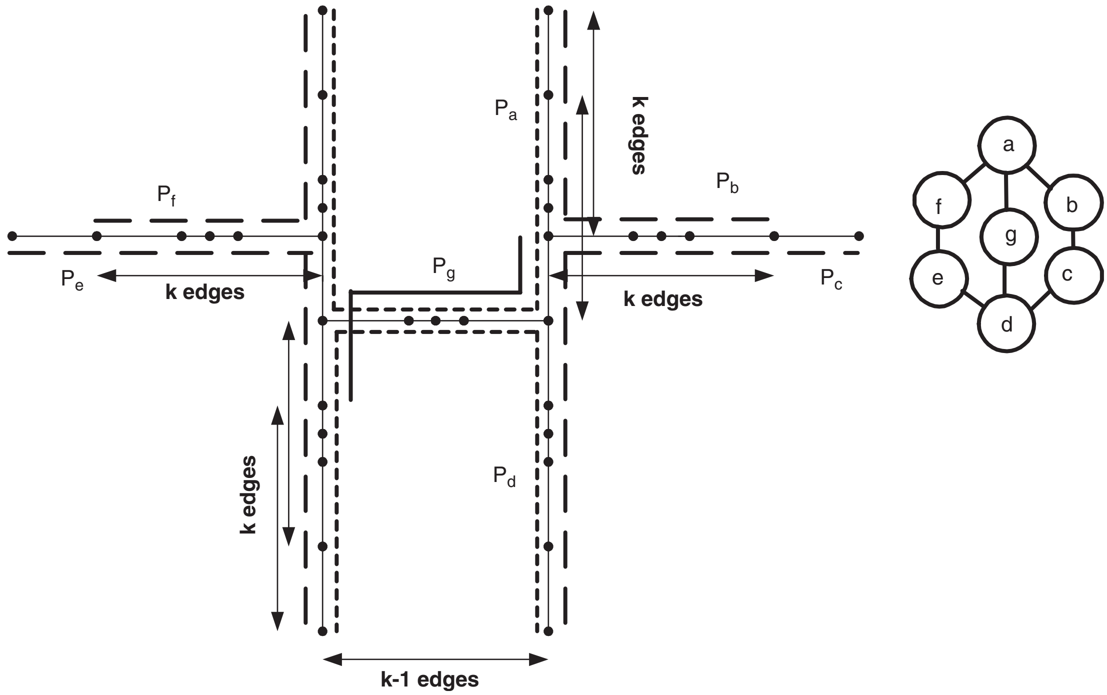


Fig. 4. A $k$-EPT graph for $k \geqslant 2$, which is not a 1-EPT graph.

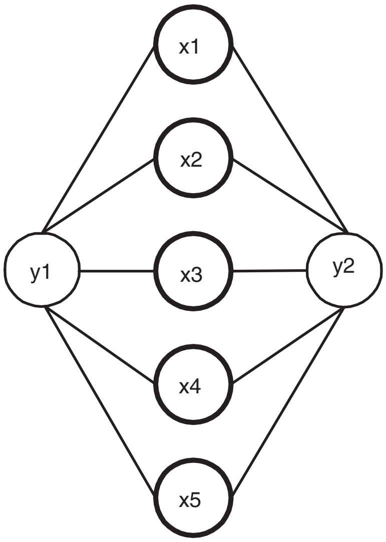


Fig. 5. Not a $k$-EPT graph for any $k \geqslant 1$.

**Theorem 27.** *The graph $G$ in Fig. 5 is not a $k$-EPT graph for any $k \geqslant 1$, that is, $G \notin EPT^*$.*

**Proof.** Suppose that the graph does have a $k$-EPT representation for some $k \geqslant 2$. We will handle the following cases separately.

*Case* 1: Suppose $P_{x_i} \cap P_{x_j} \neq \emptyset$, for every $i, j \in \{1, \ldots, 5\}$. Then, by Helly number 2, there exists a path $P$ such that $\bigcap_{i=1,\ldots,5} P_{x_i} = P$. Obviously $|P| < k$, since otherwise $\{x_1, \ldots, x_5\}$ would form a clique.

Neither $P_{y_1}$ nor $P_{y_2}$ is contained in $P$, since otherwise $|P_{y_1}| < k$ or $|P_{y_2}| < k$. Therefore, each of $P_{y_1}$, $P_{y_2}$ intersects with at least one subtree obtained by removing $P$. Each of $P_{y_1}$, $P_{y_2}$ intersects with more than one subtree obtained by removing $P$, since otherwise every $P_{x_i}$ contains $P$ and intersects with this subtree and there is an edge common to all $P_{x_i}$ in this subtree. If $P_{y_1}$, $P_{y_2}$ intersect on the same subtree, then every $P_{x_i}$ contains $P$ and intersects with this subtree, starting to intersect with $P_{y_1}$ and $P_{y_2}$ at the same point and continue intersecting with both of them on $k$ edges. This is a contradiction, since $|P_{y_1} \cap P_{y_2}| < k$.

Thus, each one of $P_{y_1}$ and $P_{y_2}$ intersects with two different subtrees. Every $P_{x_i}$ intersects with $P_{y_1}$ and $P_{y_2}$ on $k$ edges. Thus, every $P_{x_i}$ intersects with $P_{y_1}$ on some subtree, contains $P$ and intersects with $P_{y_2}$ on another subtree. If two of $P_{x_i}$ intersect with $P_{y_1}$ or $P_{y_2}$ on the same subtree, then they start to intersect with it at the same point and continue intersecting on at least $k$ edges, and therefore intersecting each other on $k$ edges. Hence, every $P_{x_i}$ intersects with $P_{y_1}$ and $P_{y_2}$ on a different subtree. This is a contradiction, since the five paths $\{P_{x_i}\}_{i=1,\ldots,5}$ cannot intersect on different subtrees.

*Case* 2: Suppose, without loss of generality, $P_{x_1} \cap P_{x_2} = \emptyset$. Then there exists a unique simple path $Q$ from $P_{x_1}$ to $P_{x_2}$, since $T$ is a tree.

Both $P_{y_1}$ and $P_{y_2}$ intersect with $P_{x_1}$ and $P_{x_2}$. There is only one unique path $Q$ from $P_{x_1}$ to $P_{x_2}$, therefore both $P_{y_1}$ and $P_{y_2}$ contain $Q$ and each intersect with more than one subtree obtained by removing $Q$. Moreover, $|Q| < k$.

**Claim 1.** *The paths $P_{y_1}$ and $P_{y_2}$ intersect on different subtrees obtained by removing $Q$.*

**Proof.** Let $v_1$ be the vertex on $P_{x_1}$, which intersects with $Q$, and let $v_2$ be the vertex on $P_{x_2}$, which intersects with $Q$. Both $P_{y_1}$ and $P_{y_2}$ intersect on at least $k$ edges with $P_{x_1}$ and both contain $Q$. Since $|P_{y_1} \cap P_{y_2}| < k$, $P_{y_1} \cap P_{x_1} \cap P_{y_2} = v_1$. Symmetrically, $P_{y_1} \cap P_{x_2} \cap P_{y_2} = v_2$. Therefore, $P_{y_1}$ and $P_{y_2}$ intersect in different subtrees obtained by removing $Q$. This proves Claim 1. □

**Claim 2.** *Each of the paths $P_{x_3}$, $P_{x_4}$, $P_{x_5}$ intersect with $Q$ on at least one vertex.*

**Proof.** Suppose $P_{x_3}$ does not intersect with $Q$. Then $P_{x_3}$ is contained in exactly one subtree $S$, obtained by removing $Q$. The paths $P_{y_1}$ and $P_{y_2}$ intersect with $P_{x_3}$ and therefore both intersect with $S$. This contradicts Claim 1 and therefore $P_{x_3}$ intersects with $Q$ on at least one vertex. The same argument follows for paths $P_{x_4}$ and $P_{x_5}$. This proves Claim 2. □

**Claim 3.** *The paths $P_{x_3}$, $P_{x_4}$, $P_{x_5}$ contain $Q$.*

**Proof.** Suppose $P_{x_3}$ does not contain $Q$, but intersects on at least one vertex with $Q$ due to Claim 2. Since both $P_{y_1}$ and $P_{y_2}$ intersect on at least $k$ edges with $P_{x_3}$, $P_{x_3}$ intersects with $P_{y_1}$ on exactly one subtree $S_1$ obtained by removing $Q$, and $P_{x_3}$ intersects with $P_{y_2}$ on exactly one subtree $S_2$ obtained by removing $Q$. Due to Claim 1, $S_1 \neq S_2$. Since $P_{y_1}$ and $P_{y_2}$ contain $Q$, the subtrees $S_1$ and $S_2$ intersect with $Q$ on exactly one vertex $v$. Therefore, $P_{x_3} \cap Q = \{v\}$.

Suppose, without loss of generality, $P_{x_1}$ contains $v$. Each of $S_1$ and $S_2$ intersects with $P_{x_1}$ on at least $k$ edges. Therefore, $P_{x_3}$ intersects with $P_{x_1}$ on at least $k$ edges. Contradiction!

The same argument follows for paths $P_{x_4}$ and $P_{x_5}$. This proves Claim 3. □

Let $S_1^1$, $S_1^2$ be subtrees intersecting with $P_{y_1}$ after removing $Q$. Let $S_2^1$, $S_2^2$ be subtrees intersecting with $P_{y_2}$ after removing $Q$.

Each of $P_{x_3}$, $P_{x_4}$, $P_{x_5}$ intersect on either $S_1^1$ or $S_1^2$ and on either $S_2^1$ or $S_2^2$ due to Claim 3. Two of $P_{x_3}$, $P_{x_4}$, $P_{x_5}$ cannot intersect on the same subtree among $S_2^1$, $S_1^2$, $S_2^1$, $S_2^2$, since otherwise they start to intersect with $P_{y_1}$ or $P_{y_2}$ at the same point and continue intersecting on $k$ edges, thus intersecting each other on at least $k$ edges. This is a contradiction, since three paths cannot intersect with these four subtrees.

Thus, the graph does not have a $k$-EPT representation for $k \geqslant 2$. Finally, Lemma 3 now implies that the graph is not 1-EPT. □

## 7. Further research

An interesting open problem is the relationship between the family of $k$-EPT graphs and $(k+1)$-EPT graphs. It is possible that there exists a value $t$, such that for every $k > t$, the families of $k$-EPT and $(k+1)$-EPT graphs coincide. In that case, we want to find a transformation from $k$-EPT representation into $(k+1)$-EPT representation. It is also possible that there exists a graph which is not a $k$-EPT graph, but is a $(k+1)$-EPT graph for any fixed $k$. In that case, the families of $k$-EPT graphs differ from each other for different values of $k$. A related problem is to characterize the class of EPT* graphs. Fig. 5 shows one example of a graph which is not EPT*.

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