 Remarks on some fixed point theorems of Dhage

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Abstract

In this paper, we define P-Lipschitzian maps and focus our attention on some fixed point theorems of Dhage on a Banach algebra. It is shown that these results can be proved under weaker conditions. Our claim is also illustrated with some examples.

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1. Introduction

It is well-known that the important fixed point theorem due to Krasnoselskii [1] which combines the metric fixed point theorem of Banach with the topological fixed point theorem of Schauder [2] in a Banach space has many applications to nonlinear integral equations. Many results have been obtained to improve and weaken the hypotheses of Krasnoselskii’s fixed point theorem due to several authors (see [3] and the references therein). The study of the nonlinear integral equations in Banach algebras was initiated by Dhage [4] via fixed point theorems.

In this paper, we define a new concept of P-Lipschitzian maps which is weaker than D-Lipschitzian maps and prove some fixed point theorems of Dhage under some weaker conditions. In the sequel, we apply our main result to the existence of solution of some nonlinear integral equations. Finally, we give some important remarks related to fixed point theorems of Dhage which were also used by many authors in their papers.

2. Preliminaries

We first recall the following. In 1969, Boyd and Wong [5] introduced, without nomenclature, the concept of nonlinear contraction (see also, [4]).

Definition 2.1 ([5]). A mapping T on a Banach space X with norm ∥ · ∥ is said to be nonlinear contraction if it satisfies

\[ \| Tx - Ty \| \leq \phi(\| x - y \|) \]  

(1)

for all x, y ∈ X, where \( \phi \) is a real continuous function such that \( \phi(r) < r \), \( r > 0 \).

Definition 2.2 ([6]). A mapping T on a Banach space X is called D-Lipschitzian if there exists a continuous and nondecreasing function \( \phi : R^+ \rightarrow R^+ \) such that

\[ \| Tx - Ty \| \leq \phi(\| x - y \|) \]  

(2)

for all x, y ∈ X, where \( \phi(0) = 0 \).

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It is shown in [6] that every Lipschitzian mapping is a D-Lipschitzian map, but the converse may not be true. Again, \( T \) is called a compact operator if \( T(X) \) is a compact subset of \( X \). \( T : X \to X \) is called totally bounded if for any bounded subset \( S \) of \( X \), \( T(S) \) is a totally bounded set of \( X \). Further, \( T \) is called completely continuous if it is continuous and totally bounded. Note that every compact operator is totally bounded, but the converse may not be true; however, two notions are equivalent on a bounded subset of \( X \).

The famous Krasnoselskii [1] fixed point theorem is stated as follows.

**Theorem 2.1** ([1]). Let \( S \) be a closed, convex, and bounded subset of a Banach algebra \( X \) and let \( A, B : S \to X \) be two operators such that

(a) \( A \) is a contraction;
(b) \( B \) is completely continuous;
(c) \( Ax + By \in S, \forall x, y \in S \).

Then the operator equation

\[ Ax + Bx = x \]  

has a solution in \( S \).

It has been mentioned in [3, Theorem 2] that hypothesis (c) of Theorem 2.1 is very strong and can be replaced with a mild one. Indeed, he proved the following modification of Krasnoselskii’s fixed-point theorem.

**Theorem 2.2** ([3]). Let \( S \) be a closed, convex, and bounded subset of a Banach algebra \( X \) and let \( A : X \to X \) and \( B : S \to X \) be two operators such that

(a) \( A \) is a contraction;
(b) \( B \) is completely continuous;
(c) \( x = Ax + By, \forall y \in S \Rightarrow x \in S \).

Then the operator equation (3) has a solution.

In 2003, Dhage [6, Theorem 2.3] proved the following fixed point theorem.

**Theorem 2.3** ([6]). Let \( S \) be a closed, convex, and bounded subset of a Banach algebra \( X \) and let \( A : X \to X \) and \( B : S \to X \) be two operators such that

(a) \( A \) is D-Lipschitzian;
(b) \( (I/A)^{-1} \) exists on \( B(S) \), \( I \) being the identity operator on \( X \);
(c) \( B \) is completely continuous;
(d) \( x = AxBy \Rightarrow x \in S, \forall y \in S \).

Then the operator equation

\[ AxBy = x, \tag{4} \]

has a solution, whenever \( M\phi(r) < r, r > 0 \), where \( M = B(S) \).

In 2005, Dhage [7, Theorem 2.1] improved Theorem 2.3 in the following way.

**Theorem 2.4** ([7]). Let \( S \) be a closed, convex, and bounded subset of a Banach algebra \( X \) and let \( A : X \to X \) and \( B : S \to X \) be two operators such that

(a) \( A \) is D-Lipschitzian;
(b) \( B \) is completely continuous;
(c) \( x = AxBy \Rightarrow x \in S, \forall y \in S \).

Then the operator equation (4) has a solution, whenever \( M\phi(r) < r, r > 0 \), where \( M = B(S) \).

Note that the proof of Theorems 2.3 and 2.4 does not realize the continuity of the function \( \phi \) involved in the definition of D-Lipschitzian maps. Therefore, it is of interest to prove the improved version of these results under some weaker conditions. Further, it may be remarked that Dhage ([4,6,7] and some references therein) cites the following form of Boyd–Wong’s fixed point theorem.

**Theorem 2.5** ([4]). Let \( T : X \to X \) be a nonlinear contraction on a Banach space \( X \). Then \( T \) has a unique fixed point.
But Boyd–Wong’s fixed point theorems in its original form are as follows.

**Theorem 2.6** ([5]). Let \((X, \rho)\) be a complete metric space, and let \(T : X \rightarrow X\) satisfies
\[
\rho(T(x), T(y)) \leq \psi(\rho(x, y)), \quad \text{for } x, y \in X, \tag{5}
\]
where \(\psi : \overline{P} \rightarrow \mathbb{R}^+\) is upper semicontinuous from right on \(\overline{P}\), and satisfies \(\psi(t) < t\) for \(t \in \overline{P} \setminus \{0\}\), where \(P = \{\rho(x, y) : x, y \in X\}\) and \(\overline{P}\) denotes the closure of \(P\). Then \(T\) has a fixed point \(x_0 \in X\) and \(T^n(x) \rightarrow x_0\) for each \(x \in X\).

**Theorem 2.7** ([5]). Suppose that \((X, \rho)\) is a completely metrically convex metric space and that \(T : X \rightarrow X\) satisfies
\[
\rho(T(x), T(y)) \leq \psi(\rho(x, y)), \quad \text{for } x, y \in X, \tag{6}
\]
where \(\psi : \overline{P} \rightarrow \mathbb{R}^+\) satisfies \(\psi(t) < t\) for \(t \in \overline{P} \setminus \{0\}\), where \(P = \{\rho(x, y) : x, y \in X\}\) and \(\overline{P}\) denotes the closure of \(P\). Then \(T\) has a fixed point \(x_0 \in X\) and \(T^n(x) \rightarrow x_0\) for each \(x \in X\).

As observed by Boyd and Wong that for a metrically convex space, the condition of semi continuity may be dropped. Since every Banach space is metrically convex, it is just sufficient to consider only the second version of Boyd–Wong’s theorem i.e., **Theorem 2.7** in the setting of Banach algebra i.e., there is no need to take \(\psi\) to be continuous as Dhage considered in his papers (see, for instance, [4,6,7]).

### 3. \(\mathcal{P}\)-Lipschitzian maps

We now introduce a new concept of \(\mathcal{P}\)-Lipschitzian mapping which is weaker than the concept of \(\mathcal{D}\)-Lipschitzian mapping.

**Definition 3.1.** A mapping \(T\) on a Banach space \(X\) is called \(\mathcal{P}\)-Lipschitzian if there exists a nondecreasing function \(\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+\) such that
\[
\|Tx - Ty\| \leq \phi(\|x - y\|),\tag{7}
\]
for all \(x, y \in X\).

Sometimes we call the function \(\phi\) a \(\mathcal{P}\) function of \(T\) on \(X\). Note that every \(\mathcal{D}\)-Lipschitzian mapping is a \(\mathcal{P}\)-Lipschitzian mapping, but the converse need not be true.

**Example 3.1.** Let \(X = \mathbb{R}, f : X \rightarrow X\) defined by
\[
f(x) = \begin{cases} 
sin x, & \text{for } x \geq 0; \\
\frac{1}{1 + |x|}, & \text{for } x < 0
\end{cases}
\]
and \(\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+\) defined by
\[
\phi(t) = \begin{cases} 
e^t, & \text{for } t > 0; \\
2, & \text{for } t = 0.
\end{cases}
\]

We consider the following two cases.

Case 1. When \(x \geq 0\),
\[
|fx - fy| = |\sin x - \sin y| \leq |x - y| \leq e^{|x-y|} \leq \phi(|x - y|).
\]

Case 2. When \(x < 0\),
\[
|fx - fy| = \left| \frac{1}{1 + |x|} - \frac{1}{1 + |y|} \right| \leq |x - y| \leq e^{|x-y|} \leq \phi(|x - y|).
\]

Thus, we conclude that \(\|fx - fy\| \leq \phi(\|x - y\|)\) \(\forall x, y \in X\). We also observe that
\[
\bullet \text{ } \phi \text{ is not continuous at } t = 0, \\
\bullet \text{ } \phi \text{ is nondecreasing}, \\
\bullet \phi(0) \neq 0.
\]

Thus, \(f\) is a \(\mathcal{P}\)-Lipschitzian mapping but not \(\mathcal{D}\)-Lipschitzian. Hence, every \(\mathcal{D}\) Lipschitzian mapping is \(\mathcal{P}\)-Lipschitzian map, but the converse need not be true.

**Remark 3.1.** Note that from **Definition 3.1** and **Example 3.1**, it is clear that the reverse implications in the following diagram need not be true.

\[
\begin{array}{ccc}
\text{Contraction mappings} & \Downarrow & \text{Lipschitzian mappings} \\
\mathcal{P}\text{-Lipschitzian mappings} & \iff & \mathcal{D}\text{-Lipschitzian mappings}
\end{array}
\]
4. Main results

In this section, by relaxing the hypothesis in the main theorem of Dhage [8], we prove the following fixed point theorem involving three operators on a Banach algebra.

**Theorem 4.1.** Let $S$ be a closed, convex, and bounded subset of a Banach algebra $X$ and let $A, C : X \to X$ and $B : S \to X$ be three operators such that

(a) $A$ and $C$ are $P$-Lipschitzian with $P$ function $\phi_A$ and $\phi_C$;
(b) $B$ is completely continuous;
(c) if $x = AxBy + Cx$ then $x \in S$. $\forall y \in S$.

Then the operator equation $AxBy + Cx = x$ has a solution, whenever $M\phi_A(r) + \phi_C(r) < r, r > 0$, where $M = B(S)$.

**Proof.** Let $y \in S$ and define a mapping $A_y : X \to X$ by

$$A_y(x) = AxBy + Cx, \quad x \in X.$$

Then we have

$$\|A_yx_1 - A_yx_2\| \leq \|Ax_1 - Ax_2\|\|By\| + \|Cx_1 - Cx_2\|,

\leq M\phi_A(\|x_1 - x_2\|) + \phi_C(\|x_1 - x_2\|), \quad x_1, x_2 \in X.$$

This shows that $A_y$ is a nonlinear contraction on $X$, since $M\phi_A(r) + \phi_C(r) < r, r > 0$. Hence, by Theorem 2.7, there is a unique point $x^* \in X$ such that

$$A_y(x^*) = Ax^*By + Cx^* = x^*.$$

Therefore by (c), we have that $x^* \in S$. Define a mapping $N : S \to S$ by

$$Ny = z,$$

where $z \in X$ is the unique solution of the equation

$$z = AzBy + Cz, \quad y \in S.$$

We show that $N$ is continuous. Let $\{y_n\}$ be a sequence in $S$ converging to a point $y$. Since, $S$ is closed, $y \in S$. Now

$$\|Ny_n - Ny\| = \|AN(y_n)By_n - AN(y)By\| + \|C(Ny_n) - C(Ny)\|

\leq \|AN(y_n)By_n - AN(y)By_n\| + \|AN(y)By_n - AN(y)By\| + \|C(Ny_n) - C(Ny)\|

\leq \|AN(y_n) - AN(y)\|\|By_n - By\| + \|AN(y)\|\|By_n - By\| + \|C(Ny_n) - C(Ny)\|

\leq M\phi_A(\|Ny_n - Ny\|) + \|AN\|\|By_n - By\| + \phi_C(\|Ny_n - Ny\|).$$

Since, $M\phi_A(r) + \phi_C(r) < r, r > 0$, there exists $k \in (0, 1)$ such that $M\phi_A(r) + \phi_C(r) = kr$ and

$$\|Ny_n - Ny\| \leq k(\|Ny_n - Ny\|) + \|AN\|\|By_n - By\|.$$

Taking the limit superior as $n \to \infty$ on both sides, we obtain

$$\limsup_{n \to \infty} \|Ny_n - Ny\| \leq k \limsup_{n \to \infty}(\|Ny_n - Ny\|) + \|AN\|\|\limsup_{n \to \infty}\|By_n - By\||.$$

This shows that $\lim_{n \to \infty} \|Ny_n - Ny\| = 0$ and consequently $N$ is continuous on $S$. Next we show that $N$ is a compact operator on $S$. Now for any $z \in S$ we have

$$\|Az\| \leq ||Aa|| + ||Az - Aa||

\leq ||Aa|| + \alpha \|z - a\|,

\leq c,$$

where $c = ||Aa|| + \text{diam}(S)$ for some fixed $a \in S$.

Let $\epsilon > 0$ be given. Since, $B$ is completely continuous, $B(S)$ is totally bounded. Hence there is a set $Y = \{y_1, y_2, \ldots, y_n\}$ in $S$ such that

$$B(S) \subseteq \bigcup_{i=1}^{n} B_\delta(w_i),$$

where $w_i = B(y_i), \delta = \left(\frac{1 - (\alpha M + \beta)}{c}\right)\epsilon$ and $B_\delta(w_i)$ is an open ball in $X$ centered at $w_i$ of radius $\delta$. Therefore, for any $y \in S$ we have a $y_k \in Y$ such that

$$\|By - By_k\| < \left(\frac{1 - (\alpha M + \beta)}{c}\right)\epsilon.$$
Also, we have
\[
\|Ny - Ny_k\| \leq \|AzBy - Az_kBy_k\| + \|Cz - Cz_k\|
\leq \|AzBy - Az_kBy\| + \|Az_kBy - Az_kBy_k\| + \|Cz - Cz_k\|
\leq \|Az - Az_k\| \|By\| + \|Az_k\| \|By - By_k\| + \|Cz - Cz_k\|
\leq (\alpha M + \beta) \|z - z_k\| + \|Az\| \|By - By_k\|
< \epsilon.
\]

This is true for every \(y \in S\) and hence
\[
N(S) \subset \bigcup_{i=1}^{n} B_\epsilon(z_i),
\]
where \(z_i = N(y_i)\). As a result \(N(S)\) is totally bounded. Since \(N\) is continuous, it is a compact operator on \(S\). Now an application of Schauder's fixed point theorem yields that \(N\) has a fixed point in \(S\). Then by the definition of \(N\)
\[
x = Nx = A(Nx)Bx + C(Nx) = AxBy + Cx.
\]
and so, the operator equation \(x = AxBy + Cx\) has a solution in \(S\).

Similarly, we prove the following fixed point theorems involving two operators on a Banach algebra relaxing the hypothesis of Dhage [6, Theorem 2.3] and Dhage [7, Theorem 2.1] respectively.

**Theorem 4.2.** Let \(S\) be a closed, convex, and bounded subset of a Banach algebra \(X\) and let \(A : X \to X\) and \(B : S \to X\) be two operators such that

(a) \(A\) is \(P\)-Lipschitzian;
(b) \((I/A)^{-1}\) exists on \(B(S)\), \(I\) being the identity operator on \(X\);
(c) \(B\) is completely continuous;
(d) \(x = AxBy \Rightarrow x \in S, \forall y \in S\).

Then the operator equation (4) has a solution, whenever \(M\phi(r) < r, r > 0\), where \(M = B(S)\).

**Proof.** Let \(y \in S\) and define a mapping \(A_y : X \to X\) by
\[
A_y(x) = AxBy, \quad x \in X.
\]
Then we have
\[
\|A_yx_1 - A_yx_2\| \leq \|Ax_1 - Ax_2\| \|By\|,
\leq M\phi(\|x_1 - x_2\|), \quad x_1, x_2 \in X.
\]
Hence, by Theorem 2.7, there is a unique point \(x^* \in X\) such that
\[
A_y(x^*) = x^*.
\]
Remaining proof of the theorem is similar to the proof of Theorem 2.3 [6] and Theorem 4.1. So we omit the details.

**Theorem 4.3.** Let \(S\) be a closed, convex, and bounded subset of a Banach algebra \(X\) and let \(A : X \to X\) and \(B : S \to X\) be two operators such that

(a) \(A\) is \(P\)-Lipschitzian;
(b) \(B\) is completely continuous;
(c) \(x = AxBy \Rightarrow x \in S, \forall y \in S\).

Then the operator equation (4) has a solution, whenever \(M\phi(r) < r, r > 0\), where \(M = B(S)\).

**Proof.** Let \(y \in S\) and define a mapping \(A_y : X \to X\) by
\[
A_y(x) = AxBy, \quad x \in X.
\]
Following Theorem 4.2, we can show that there is a unique point \(x^* \in X\) such that
\[
A_y(x^*) = x^*.
\]
The rest of the proof follows on the lines of the proof furnished in Theorem 4.1 (see also Theorem 2.1 of Dhage [7]).
Remark 4.4. Since every Lipschitzian and D-Lipschitzian mappings are P-Lipschitzian, we obtain the fixed point theorems studied in [6–8] as a particular case of Theorems 4.1–4.3, which are useful to obtain the solutions of some nonlinear differential and integral equations.

The following sufficient condition guarantees the hypothesis (c) of Theorem 4.1.

Proposition 4.5. Let S be a closed, convex, and bounded subset of a Banach Algebra X such that S = \{y ∈ X : ∥y∥ ≤ r\} for some real number r > 0. Let A, C : X → X, B : S → S be two operators satisfying hypothesis (a)–(b) of Theorem 4.1. Further, if

\[ ∥x∥ ≤ \left( \frac{1 - C}{A} \right) x, \]

for all x ∈ X, then x ∈ S.

Proof. The proof follows on the lines of the proof of Proposition 2.1 of [6]. □

5. Applications to nonlinear integral equations

To illustrate our Theorem 4.1, we consider the following example of nonlinear integral equation.

Example 5.1. Given a closed and bounded interval J = [0, 1] in \( \mathbb{R} \), the set of all real numbers, consider the nonlinear integral equation (in short IE)

\[ x(t) = p(t, x(t)) + q(t, x(t))\left( \lambda(t) + \int_0^t f(s, x(s))ds \right), \]

for all t ∈ J, where \( \lambda : J → \mathbb{R}, f : J × \mathbb{R} → \mathbb{R} \) are continuous and p : J × \( \mathbb{R} \) → \( \mathbb{R} \) is given by

\[ p(t, x) = \frac{2}{3}x, \]

\[ q : J × \mathbb{R} → \mathbb{R} \] is given by

\[ q(t, x) = \begin{cases} 
\frac{1}{3(1+x)}, & \text{if } x ≥ 0; \\
\frac{1}{6}, & \text{if } x < 0.
\end{cases} \]

By the solution of the integral equation (9), we mean a continuous function \( x : J → \mathbb{R} \) that satisfies (9) on J. Let \( X = C(J, \mathbb{R}) \) be a Banach algebra of all continuous real valued functions on J with the norm

\[ ∥x∥ = \sup_{t ∈ J} |x(t)|. \]

We shall obtain the solution of (9), under some suitable conditions, suppose that the function f satisfies the following condition:

\[ |f(t, x)| ≤ 1 - ∥\lambda∥, \quad ∥\lambda∥ < 1, \]

for all t ∈ J and x ∈ \( \mathbb{R} \).

Define a subset S of X by

\[ S = \{y ∈ X : ∥y∥ ≤ 1\}. \]

Consider two mappings A, B : X → X defined by

\[ Ax(t) = q(t, x(t)), \quad t ∈ J, \]

\[ Bx(t) = \lambda(t) + \int_0^t f(s, x(s))ds, \quad t ∈ J, \]

\[ Cx(t) = p(t, x(t)), \quad t ∈ J. \]

We shall show that operators A, B and C satisfies all the conditions of Theorem 4.1.

First, we show that A is a Lipschitzian map on X. Let x, y ∈ X. Then

\[ |Ax(t) - Ay(t)| = |q(t, x(t)) - q(t, y(t))| \leq \frac{1}{3} ∥x - y∥, \]

which shows that A is a Lipschitzian map. Similarly, C is a Lipschitzian map.
Now, it is an easy exercise to prove that $B$ is completely continuous on $S$. We show that $B : S \rightarrow S$. Let $x \in S$. Then by (11) and (14),

$$|Bx(t)| = \left| \lambda(t) + \int_0^t f(t, x(t)) \, ds \right|$$

$$\leq |\lambda(t)| + \int_0^t |f(t, x(t))| \, ds$$

$$< |\lambda(t)| + \int_0^t (1 - \|\lambda\|) \, ds.$$

Since $Bx \in C(J, \mathbb{R})$, there is a point $t^* \in J$ such that

$$\|Bx\| = |Bx(t^*)| = \max_{t \in J} |Bx(t)|.$$

Therefore, we have

$$\|Bx\| = |Bx(t^*)|$$

$$< |\lambda(t^*)| + \int_0^{t^*} (1 - \|\lambda\|) \, ds$$

$$\leq \|\lambda\| + \int_0^1 (1 - \|\lambda\|) \, ds$$

$$= 1,$$

i.e. $\|Bx\| < 1$. As a result, $B : S \rightarrow S$. Finally, we show that condition (8) of Proposition 4.5 holds. Now, for any $x \in X$, we have

$$\left\| \left( \frac{I - C}{A} \right) x \right\| = \sup_{t \in J} \left| x(t) - Cx(t) \right| / Ax(t)$$

$$\geq \|x\|.$$

Thus, operators $A$, $B$ and $C$ satisfy all the conditions of Theorem 4.1. Hence, the integral equation (9) has a solution on $J$.

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