On the Automorphisms of Incidence Algebras

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Communicated by Susan Montgomery

Received June 5, 2000

Given a locally finite partially ordered set, \( X \), a ring with identity, \( R \), and an automorphism, \( \phi \), of the incidence algebra of \( X \) over \( R \), it is determined when \( \phi \) is the composite of an inner automorphism, an automorphism of \( X \), and an induced automorphism of \( R \).

Let \( T_n(R) \) denote the ring of \( n \times n \) upper triangular matrices with entries from the ring with identity \( R \). In 1990, Barker and Kezlan [1] stated that for \( R \) an integral domain, any \( R \)-algebra automorphism of \( T_n(R) \) is inner. In 1990, Kezlan [4] showed that this conclusion remains true for any commutative ring \( R \). If \( k \) is a commutative ring with 1 and \( R \) a \( k \)-algebra, Jøndrup [3] finds sufficient conditions on \( R \) for any \( k \)-automorphism of \( T_n(R) \) to be the composition of an inner automorphism and a \( k \)-automorphism of \( R \) induced to \( T_n(R) \). If \( \alpha \) is a \( k \)-automorphism of \( R \), the induced automorphism, \( \hat{\alpha} \), on the matrix \( A = (a_{ij}) \) is given by \( (\hat{\alpha}(A))_{ij} = \alpha(a_{ij}) \). Coelho [2] determines other conditions on the \( k \)-algebra \( R \) which still give a similar conclusion. Koppinen [5] shows that for \( \Psi \) a \( k \)-automorphism of \( T_n(R) \), \( \Psi \) is the composition of an inner automorphism and an induced \( k \)-automorphism of \( R \) if and only if \( \Psi \) fixes the set of strictly upper triangular matrices. He shows that this result holds also for the ring of infinite matrices (indexed by \( \mathbb{Z}^+ \)). Furthermore, he proves that if \( R \) is an indecomposable \( k \)-algebra, then in the ring of matrices indexed by \( \mathbb{Z} \), any \( k \)-automorphism fixing the strictly upper triangular matrices is the composite of an inner automorphism, an induced \( k \)-automorphism of \( R \), and an index shift of matrix indices.

In this note we generalize both Koppinen’s and Kezlan’s results. We will consider automorphisms of the incidence algebra of a locally finite partially ordered set \( X \) over a ring with identity \( R \) and determine a necessary
and sufficient condition for an automorphism of this algebra to be the composite of an inner automorphism, an automorphism of $X$, and a generalized induced automorphism of $R$. This will extend Koppinen’s results. We then show that with suitable restrictions on $X$ and $R$, our condition is always satisfied. In particular, we will obtain a generalization of Kezlan’s result.

Recall the definition of the incidence algebra. Suppose $X$ is a locally finite (intervals are finite) partially ordered set and $R$ is a ring with identity. If $I(X, R) = \{ f: X \times X \to R \mid f(x, y) = 0 \text{ if } x \not\leq y \}$ with the operations

\[
(f + g)(x, y) = f(x, y) + g(x, y)
\]

\[
f g(x, y) = \sum_{x \leq z \leq y} f(x, z) g(z, y)
\]

\[
(r f)(x, y) = rf(x, y)
\]

for

\[
f, g \in I(X, R), r \in R, x, y, z \in X,
\]

then $I(X, R)$ is an $R$ algebra called the incidence algebra of $X$ over $R$.

We single out some elements and properties of $I(X, R)$. One can use [7] for information on incidence algebras, although results in that book are restricted to the case where $R$ is commutative. The necessary adjustments to the case where $R$ is non-commutative, of all the information that we will need, are straightforward. If $x, y \in X$ with $x \leq y$, let $\varepsilon_{x,y}$ be defined by $\varepsilon_{x,y}(u, v) = 1$ if $(u, v) = (x, y)$, and $\varepsilon_{x,y}(u, v) = 0$ otherwise. If $x = y$, write $e_x$ for $\varepsilon_{x,x}$. The identity element $\delta = \delta_X$ of $I(X, R)$ is the element defined by $\delta(u, v) = 1$ if $u = v$, and $\delta(u, v) = 0$ otherwise. The Jacobson radical, $J(I(X, R))$, of $I(X, R)$ consists of the set of all $f \in I(X, R)$ with the property that for each $x \in X$, $f(x, x)$ is in the Jacobson radical of $R$. The set $Z(I(X, R)) = \{ f \in I(X, R) \mid f(x, x) = 0 \text{ for all } x \in X \}$ is a two-sided ideal contained in $J(I(X, R))$. If $X$ can be ordered so that $x_i \leq x_j$ implies $i \leq j$, then the mapping $\Psi: I(X, R) \to T_{|X|}(R)$ given by $(\Psi(f))_{x_i, x_j} = f(x_i, x_j)$ is an $R$-algebra isomorphism of $I(X, R)$ and a subalgebra of $T_{|X|}(R)$. In particular, if $C_n$ is a chain of size $n$, then $I(C_n, R) = T_n(R)$.

We now consider some special types of automorphisms of $I(X, R)$. For notation, write $\text{Aut}(T)$ for the collection of automorphisms of the object $T$. An element $h \in I(X, R)$ is invertible precisely when $h(x, x)$ is a unit for each $x \in X$. To such an $h$, associate the inner automorphism $\tau_h$ of $I(X, R)$, where $\tau_h(f) = hfh^{-1}$. If $\gamma$ is an automorphism of the partially ordered set $X$, then $\gamma$ gives rise to an automorphism, $\tilde{\gamma}$, of $I(X, R)$, given by $\tilde{\gamma}(f)(x, y) = f(\gamma(x), \gamma(y))$. We refer to $\tilde{\gamma}$ as the automorphism of
LEMMA 1. Suppose \( \phi \in \text{Aut}(I(X, R)) \) is such that \( \phi(e_x) = e_x \), for each \( x \in X \). If \( X = \bigcup_{k \in K} X_k \) is the decomposition of \( X \) into its disjoint connected components, then, for each \( k \in K \), there are automorphisms \( \alpha_k \) of \( R \), with the property that \( \phi \) is the automorphism induced from \( \alpha_k \). Furthermore, \( \alpha_k(1) = 1 \), \( \alpha_k \) are invertible, and we may write \( \alpha_k = \alpha_k(x) \). The result now follows.

LEMMA 2. Suppose \( \phi \in \text{Aut}(I(X, R)) \) and that, for all \( x \in X \), \( (\phi(e_x))_D = e_x \). Then there exists an inner automorphism, \( \tau_x \in \text{Aut}(I(X, R)) \), having the property that \( \tau_x(\phi(e_x)) = e_x \), for each \( x \in X \).

Proof. For any \( x \leq y \in X \), let \( h \in I(X, R) \) be such that \( h(x, x) = 1 \) and \( h(x, y) = \phi(e_x)(x, y) \). Then \( h \) is invertible, and we may write \( h \) in the form \( h = \sum_{x \leq y} \phi(e_x)e_x \). Using the fact that \( \{e_x \mid x \in X \} \) and \( \{\phi(e_x) \mid x \in X \} \) are the disjoint connected components of \( X \). Here \( K \) is an index set. Given the collection of automorphisms \( \Theta = \{\alpha_k \in \text{Aut}(R) \mid k \in K \} \) of \( R \), as \( I(X, R) = \prod_{k \in K} I(X_k, R) \), then \( \Theta \) gives rise to an automorphism \( \hat{\Theta} \) of \( I(X, R) \) defined by \( \hat{\Theta}(f(x, y)) = \alpha_k(f(x, y)) \), for \( x, y \in X_k \). Call \( \hat{\Theta} \) the automorphism induced from \( \Theta = \{\alpha_k \mid k \in K \} \). The following lemma shows when an automorphism of \( I(X, R) \) is of this last type.

\[ \text{LEMMA 2.} \] Suppose \( \phi \in \text{Aut}(I(X, R)) \) and that, for all \( x \in X \), \( (\phi(e_x))_D = e_x \). Then there exists an inner automorphism, \( \tau_x \in \text{Aut}(I(X, R)) \), having the property that \( \tau_x(\phi(e_x)) = e_x \), for each \( x \in X \).
$X$) are each sets of pairwise orthogonal idempotents, we obtain that \( he_x = \phi(e_x)e_x \) and \( \phi(e_x)h = \phi(e_x)e_x \). Hence \( he_x = \phi(e_x)h = \tau_x \phi(e_x) = e_x \). 

Now define a class of automorphism that includes those considered in the previous two lemmas. Call the automorphism \( \phi \in \text{Aut}(I(X, R)) \) a type 1 automorphism if, for each \( x \in X \), \( \phi(e_x) - a_{\phi}(x) e_{\nu_{\phi}(x)} \in Z(I(X, R)) \), where \( a_{\phi}(x) \in R \) and \( \nu_{\phi}(x) \in X \). Associated to such an automorphism are the functions \( a_{\phi}: X \to R \) and \( \nu_{\phi}: X \to X \). If it is clear which type 1 automorphism we are considering, we will write \( a(x) \) for \( a_{\phi}(x) \) and \( \nu(x) \) for \( \nu_{\phi}(x) \). Call the type 1 automorphism \( \phi \in \text{Aut}(I(X, R)) \) special, if \( \nu \) is a bijective mapping of \( X \) and \( a(x) = 1 \), for all \( x \in X \). The previous lemmas have considered certain special automorphisms. Our next lemma includes the result that every type 1 automorphism is special when \( X \) is finite.

**Lemma 3.** Assume \( \phi \in \text{Aut}(I(X, R)) \) is a type 1 automorphism.

(i) \( \nu_{\phi} \) is surjective.

(ii) If \( X \) is a finite set then \( \phi \) is a special automorphism.

**Proof.** (i) Looking for a contradiction, we suppose that \( \nu_{\phi} \) is not onto. Then there exists \( z \in X \) with \( z \not\in \nu(X) \). Let \( f \in I(X, R) \) be such that \( \phi(f) = e_z \). As \( e_z \) is an idempotent, the same holds for \( f \). It follows that \( f_D \neq 0 \), and so there exists \( x \in X \) with \( f(x, x) \) a non-zero idempotent of \( R \). Hence \( (fe_x)f(x, x) = f(x, x) \), and, in particular, \( fe_x \neq 0 \). Now consider the image of this element, \( e_x \phi(e_x)e_x \), under the automorphism \( \phi \). We have \( e_x \phi(e_x)e_x = \phi(e_x)e_x = a(x)e_{\nu_{\phi}(x)}(z, z) = 0 \), the last equality holding since \( z \) is not in the range of \( \nu \). This is the desired contradiction.

(ii) Suppose that \( X \) is finite. Since \( \delta = \sum x e_x \), and \( \delta_D = \delta \), it follows that \( \delta = \phi(\delta)_D = \sum_x a(x)e_{\nu_{\phi}(x)} \). By (i) we know that \( \nu \) is surjective and, as \( X \) is finite, bijective. It follows that \( a(x) = 1 \) for each \( x \in X \). This shows that \( \phi \) is special.

Not all type 1 automorphisms are special, as the following example shows. Suppose \( R \) is a ring with identity having the property that \( R \times R \cong R \) and let \( X = \{x_1, x_2, \ldots \} \) be a countable antichain. Let \( \lambda \) be an isomorphism from \( R \times R \) onto \( R \) and write \( S \) for \( R \times R = \{(a, b) \mid a, b \in R\} \). Let \( f \in I(X, R) \) and define \( \phi_1: I(X, S) \to I(X, S) \) by

\[
\phi_1(f)(x_i, x_i) = (\lambda(f(x_{2i-1}, x_{2i-1})), \lambda(f(x_{2i}, x_{2i}))), \quad \text{for } i = 1, 2, \ldots .
\]

It is easy to check that \( \phi_1 \in \text{Aut}(I(X, S)) \) and that \( \phi_1 \) is a type 1 automorphism. However, \( \nu_{\phi_1} \) is not injective and \( a_{\phi_1} \) is not the function 1. Thus \( \phi_1 \) is not special. Our next lemma shows that if \( \phi \) is a type 1 automorphism...
which we wish to show special, then the needed properties of $a_\phi$ and $\nu_\phi$
are related.

**Lemma 4.** Suppose $\phi \in \text{Aut}(I(X, R))$ is a type 1 automorphism having
the property that $\phi(e_x) - a(x)e_{\nu(x)} \in Z(I(X, R))$, for each $x \in X$. The
following are equivalent:

(i) $a(x) = 1$, for all $x \in X$.
(ii) $\nu : X \to X$ is injective.
(iii) $\phi$ is a special automorphism.

**Proof.** By the definition of a special automorphism, (iii) implies
both (i) and (ii). Suppose (i) holds. If $\nu$ is not injective, there exists
distinct $x_1, x_2 \in X$ with $\nu(x_1) = \nu(x_2)$. Then $e_{x_1}e_{x_2} = 0$ while
$\phi(e_{x_1})\phi(e_{x_2})(\nu(x_1), \nu(x_1)) = 1$. This contradiction shows that (i) implies (ii).

We now check that (ii) implies (iii). Suppose that $\phi$ is not special. By
Lemma 3, $\nu$ is bijective, and thus there exists a $z \in X$ with $a(z) \neq 1$.
Then $a(z)$ is a non-zero idempotent of $R$. Let $J(I(X, R))$ denote the
Jacobson radical of $I(X, R)$. So $Z(I(X, R)) \subseteq J(I(X, R))$. Now, let $\phi'$ be
the automorphism induced by $\phi$ on $I(X, R)/J(I(X, R))$. This factor ring is
isomorphic to $\prod_{x \in X} R/J(R)$ (see [7]). Furthermore, $T = \{e_x +
J(I(X, R) \mid x \in X\}$ is a set of pairwise orthogonal idempotents in
$I(X, R)/J(I(X, R))$ which is not properly contained in a larger collection
of pairwise orthogonal non-zero idempotents. The same then holds for
$\phi'(T)$. Since $\phi'(e_x + J(I(X, R))) = a(z)e_{\nu(z)} + J(I(X, R))$, the idempo-
tent $(1-a(z))e_{\nu(z)} + J(I(X, R))$ is orthogonal with each of the idempo-
tents of $\phi'(T)$. This contradicts the maximality of the set $\phi'(T)$. We thus
conclude that $\phi$ is special, which establishes the lemma.

We now have all the ingredients to generalize Koppinen’s theorem.

**Theorem 1.** Let $X = \bigcup_{k \in K} X_k$ be a locally finite partially ordered set
written as the union of its disjointed connected components, let $R$ be a ring with
identity, and let $\phi \in \text{Aut}(I(X, R))$. Then there exist $\gamma \in \text{Aut}(X)$,
$\tau_k$ an inner automorphism of $I(X, R)$, and $\Theta = \{\alpha_k \in \text{Aut}(R) \mid k \in K\}$ such that
$\phi = \gamma \tau_k \Theta$ if and only if $\phi$ is a special automorphism.

**Proof.** If $\phi = \gamma \tau_k \Theta$ then it is straightforward to check that $\phi$
is special. We verify the converse. Suppose that $\phi$ is special. Then, for each
$x \in X$, $\phi(e_x) = e_{\nu(x)} \in Z(I(X, R))$, and $\nu : X \to X$ is a bijective mapping.
Note that $\bigcap_{n=1}^{\infty} Z(I(X, R))^n = \{0\}$. Then, by Stanley’s lemma [8], for any
\[ x, y \in X, \ e_{\nu(x)} I(X, R) e_{\nu(y)} = 0 \Leftrightarrow \phi(e_{\nu(x)}) I(X, R) \phi(e_{\nu(y)}) = 0. \] It now follows that

\[
x \leq y \Leftrightarrow e_{\nu(x)} I(X, R) e_{\nu(y)} \neq 0
\]
\[
\Leftrightarrow \phi(e_{\nu(x)}) I(X, R) \phi(e_{\nu(y)}) \neq 0
\]
\[
\Leftrightarrow e_{\nu(x)} I(X, R) e_{\nu(y)} \neq 0
\]
\[
\Leftrightarrow \nu(x) \leq \nu(y).
\]

We conclude that \( \nu \in \text{Aut}(X) \). Let \( \phi_1 = \nu^{-1} \phi \). Then, for each \( x \in X \),
\( \phi_1(e_x) - e_x \in Z(I(X, R)) \). By Lemma 2, there is an inner automorphism \( \tau_\phi \) of \( I(X, R) \) such that \( (\tau_\phi \phi_1) e_x = e_x \). From Lemma 1, for each \( k \in K \), there are then automorphisms \( \alpha_k \) of \( R \) such that \( \tau_\phi \phi_1 = \Theta_k \), where \( \Theta_k \) is the automorphism induced from \( \Theta = \{ \alpha_k \mid k \in K \} \). Hence \( \phi = \nu \tau_\phi \Theta_k \), completing the proof.

From Theorem 1 it is easy to see that the inverse of a special automorphism is again special. Furthermore, \( \phi(Z(I(X, R))) = Z(I(X, R)) \), when \( \phi \) is special. The example following Lemma 3 shows that there are non-special automorphisms fixing \( Z(I(X, R)) \). In our terminology, Koppinen [5] showed that when \( R \) is an indecomposable algebra over a commutative ring with identity \( k \), \( X = Z \), and \( \phi \) is a \( k \)-automorphism of \( I(X, R) \), then \( \phi \) is special if and only if \( \phi(Z(I(X, R))) = Z(I(X, R)) \). Furthermore, he gave an example to show that the condition of indecomposability is necessary. The following theorem shows that this type of result holds for any locally finite partially ordered set.

**Theorem 2.** Suppose \( X \) is a locally finite partially ordered set and \( R \) is an indecomposable ring with identity. Let \( \phi \in \text{Aut}(I(X, R)) \). Then \( \phi \) is special if and only if \( \phi(Z(I(X, R))) = Z(I(X, R)) \).

**Proof.** If \( \phi \) is special then we have already noted that \( \phi(Z(I(X, R))) = Z(I(X, R)) \). Suppose, then, that \( R \) is indecomposable and \( \phi(Z(I(X, R))) = Z(I(X, R)) \). It follows that \( \phi \) induces an automorphism, \( \phi' \), of \( I(X, R)/Z(I(X, R)) \). This latter ring is isomorphic to \( I(X', R) = \prod_{x \in X} R \), where \( X' \) is the set \( X \) regarded as an antichain. We can then consider \( \phi' \in \text{Aut}(I(X', R)) \). In particular, \( \phi' \) maps the center of \( I(X', R) \) to itself. This latter ring is isomorphic to \( \prod_{x \in X} C(R) \) (see [7]). Here \( C(R) \) denotes the center of the ring \( R \). But, in \( C(I(X', R)) \), as \( R \) is indecomposable, \( \{ e_x \mid x \in X' \} \) is the unique maximal set of primitive orthogonal idempotents. Hence, for \( x \in X \), \( \phi'(e_x) = e_{\nu(x)} \), where \( \nu \) is a permutation of \( X' \). We conclude that \( \phi(e_x) - e_{\nu(x)} \in Z(I(X, R)) \), and so \( \phi \) is special.
Koppinen [5] also showed that if $R$ is an algebra over the commutative ring $k$, then any $k$-automorphism, $\phi$, of $T_n(R)$ ($n$ finite or infinite), which fixes the strictly upper triangular matrices, is special. We give a proof of this theorem which utilizes our Theorem 1.

**THEOREM 3.** Suppose $R$ is a ring with identity, $1 \leq n \leq \infty$, and $T_n(R)$ is the ring of $n \times n$ upper triangular matrices with entries from $R$. Let $\phi \in \text{Aut}(T_n(R))$ and suppose $\phi$ fixes the ideal of strictly upper triangular matrices. Then there is an $\alpha \in \text{Aut}(R)$ and an inner automorphism $\tau_\alpha$ of $T_n(R)$ with $\phi = \tau_\alpha \alpha$.

**Proof.** If $n = \infty$ let $X = Z^+$, while if $n < \infty$, let $X$ be the chain of positive integers which are at most $n$. Then $I(X, R) = T_n(R)$. The strictly upper triangular matrices correspond with the ideal $Z(I(X, R))$. We may suppose, then, that $\phi \in \text{Aut}(I(X, R))$ and $\phi(Z(I(X, R))) = Z(I(X, R))$. For notation, let $\text{Ann}_e(T)$ and $\text{Ann}_l(T)$ denote the right and left annihilators of the ring $T$. Then $\text{Ann}_e(Z(I(X, R))) = \{f \in I(X, R) \mid f(u, v) = 0 \text{ for } u \neq 1\}$. A left identity, $f$, in $\text{Ann}_e(Z(I(X, R)))$ then satisfies $f(1, 1) = 1$. Since $e_1$ is a left identity of this set, it follows that $\phi(e_1) - e_1 \in Z(I(X, R))$. Now consider $\text{Ann}_e((Z(I(X, R))^2) \cap \text{Ann}_l(\text{Ann}_e(Z(I(X, R))))$ and the collection of left identities in this set. Such an identity, $f$, satisfies $f(2, 2) = 1$ and $f(u, v) = 0$ if $u \neq 2$. Since $e_2$ is a left identity of this set, it follows that $\phi(e_2) - e_2 \in Z(I(X, R))$. Inductively, by considering the left identities of $\text{Ann}_e((Z(I(X, R))^2) \cap \text{Ann}_l(\text{Ann}_e(Z(I(X, R))))$ we obtain that $\phi(e_r) - e_r \in Z(I(X, R))$ for all $r$. Hence $\phi$ is special, and, from Theorem 1, $\phi$ is the product of an automorphism of $X$, an inner automorphism, and an induced automorphism of $R$. Since $X$ has no non-trivial automorphism, the result follows.

We give an example showing that we cannot eliminate the condition that $R$ is an indecomposable ring in Theorem 2, even when $X$ is a finite connected set. Suppose $R$ is a ring with identity and $S = R \times R$. Let $X = \{z, x, y\}$ be the partially ordered set with $z \leq x, z \leq y$. For $f \in I(X, S)$, write

$$f = (r_1, r_2)e_z + (a_1, b_1)e_x + (a_2, b_2)e_y + (c_1, d_1)e_{zx} + (c_2, d_2)e_{zy}.$$ 

Let $\phi: I(X, S) \to I(X, S)$ be the mapping given by

$$\phi(f) = (r_1, r_2)e_z + (a_1, b_2)e_x + (a_2, b_1)e_y + (c_1, d_1)e_{zx} + (c_2, d_1)e_{zy}.$$ 

It is straightforward to verify that $\phi \in \text{Aut}(I(X, S))$ and that $\phi(Z(I(X, S))) = Z(I(X, S))$. Furthermore, $\phi(e_z) = (1, 0)e_z + (0, 1)e_x$, so that $\phi$ is not a special automorphism. Thus $\phi$ cannot be written in the form of the special automorphisms of Theorem 1.
To give an application of the above results we will make use of the following theorem. This theorem has been shown for the case where $R$ is a commutative ring with identity in [6] (or see [7, pp. 265–270]), although the result in these references is not stated in precisely the same way. The extension to the case where $R$ is not necessarily commutative is straightforward and will be omitted. An automorphism $\phi$ of $I(X, R)$ having the property that $\phi rf = r \phi(f)$, for $r \in R$ and $f \in I(X, R)$, will be called an $R$-automorphism.

**THEOREM 4.** Suppose $R$ is a ring with identity and $X$ is an at most countable locally finite partially ordered set. Let $\phi$ be an $R$-automorphism of $I(X, R)$ and $x \in X$. If $y \in X$ is such that $\phi(e_x)(y, y) \neq 0$, then there exists an automorphism $\gamma \in \text{Aut}(X)$ with $\gamma(x) = y$.

Generalizations of Kezlan’s Theorem can now be given.

**THEOREM 5.** Suppose $R$ is a ring with identity and $X$ is an at most countable locally finite partially ordered set. Let $\phi$ be an $R$-automorphism of $I(X, R)$ and assume that $\text{Aut}(X) = \{1\}$. Then $\phi$ is inner.

**Proof.** Since the automorphism group of $X$ is trivial, it follows from Theorem 4 that $\phi$ is a type 1 automorphism and that, for each $x \in X$, $\phi(e_x) = \alpha(x)e_x \in Z(I(X, R))$. Here $\alpha(x)$ is an idempotent of $R$. Hence $\phi((1 - \alpha(x))e_x) \in Z(I(X, R))$. Since $(1 - \alpha(x))e_x$ is an idempotent, and 0 is the only idempotent in $Z(I(X, R))$, we must have that $\alpha(x) = 1$, for each $x \in X$, and so $\phi$ is a special automorphism. Now, using Lemma 2 and its proof, there is an inner automorphism, $\tau_k$, with the properties that $\tau_k \phi$ is an $R$-automorphism of $I(X, R)$ and $\tau_k \phi(e_x) = e_x$, for each $x \in X$. Then, if $X = \bigcup_{k \in K} X_k$ is the decomposition of $X$ into its disjoint connected components, by Lemma 1, there are automorphisms $\alpha_k$ of $R$ such that $\tau_k \phi$ is the automorphism induced from $\{\alpha_k \mid k \in K\}$. Since $\tau_k \phi$ is an $R$-automorphism of $I(X, R)$, from the proof of Lemma 1, we see that, for each $k \in K$, $\alpha_k$ is the identity automorphism. The result then follows.

The following corollary includes the case of upper triangular matrices over a commutative ring. We write $C(R)$ for the center of the ring $R$.

**COROLLARY 1.** Let $X$ be an at most countable, connected, locally finite partially ordered set, and let $R$ be a ring with identity. Suppose that $\text{Aut}(X) = \{1\}$ and $\phi \in \text{Aut}(I(X, R))$ is such that $\phi$ keeps the subalgebra $I(X, C(R))$ invariant. Then $\phi$ is the composite of an inner automorphism and an induced automorphism of $R$.

**Proof.** The centralizer of $I(X, C(R))$ in $I(X, R)$ consists of scalar multiples of $\delta_X$ and is, thus, invariant under $\phi$. For $r \in R$, if we write $\phi(r \delta_X) = \alpha(r)\delta_X$ then $\alpha : R \to R$ is easily seen to be an automorphism of
Let $\phi'$ be the composite of $\phi$ and the automorphism induced from $\alpha^{-1}$. Then $\phi'$ is an $R$-automorphism of $I(X, R)$ and, from Theorem 5, $\phi'$ is an inner automorphism. The result now follows.

**ACKNOWLEDGMENT**

The author thanks the referee for useful suggestions, including the current revised version of Corollary 1.

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