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Segments and Hilbert schemes of points

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ABSTRACT

Using results obtained from a study of homogeneous ideals sharing the same initial ideal with respect to some term order, we prove the singularity of the point corresponding to a segment ideal with respect to a degreverse term order (as, for example, the degrevlex order) in the Hilbert scheme of points in $\mathbb{P}^n$. In this context, we look into the properties of several types of "segment" ideals that we define and compare. This study also leads us to focus on the connections between the shape of generators of Borel ideals and the related Hilbert polynomial, thus providing an algorithm for computing all saturated Borel ideals with a given Hilbert polynomial.

0. Introduction

The Hilbert scheme can be covered by some particular affine schemes [3,9,22,26,15] that have been called *Gröbner strata* in [15] because they are computed from a monomial ideal by Gröbner basis techniques. The behavior of Gröbner strata can provide interesting information on the Hilbert scheme itself. Very recently, in [25,5], Roggero et al. showed that an open covering of the Hilbert scheme can be constructed from Borel ideals by avoiding introduction of any term order, which is instead needed for Gröbner strata. This fact gives us further reasons to investigate Borel ideals and their particular features.

Among Borel ideals, there are several types of "segment" ideals whose definitions are already well known or arise from some interesting properties of Gröbner strata studied in [15] (Definitions 3.1 and 3.7). In Section 3 we characterize, for some cases, the existence of these kinds of ideals in terms of the corresponding Hilbert polynomial. In this context, we also need to focus our attention on the shape of admissible polynomials.

In [12], the coefficients of Hilbert polynomials are completely characterized by the numbers of components of certain subschemes defined by particular ideals called *tight fans*. In [23], these numbers of components are described by the shape of the minimal generators of Borel ideals. Although the geometric meaning is contained in the fans, in Section 4, we observe that this connection between the coefficients of Hilbert polynomials and the minimal generators of Borel ideals can be directly described without using fans by the combinatorial properties of the Borel ideals themselves. This study led us to conceive an algorithm for computing all saturated Borel ideals with a given Hilbert polynomial. In Section 5, we describe this procedure that have been implemented by the second author in a software with an applet available at http://www.dm.unito.it/dottorato/dottorandi/lella/HSC/borelGenerator.html.

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In [24] and in [20], the smoothness of the points of Hilbert schemes is studied by means of the dimension of the vector space of the global sections of the normal sheaf to the corresponding projective subscheme. In Section 6, by applying the results of [15] about Gröbner strata, we make some new considerations (Theorem 6.2) on the smoothness of the points in the Hilbert scheme $\mathcal{H}\mathrm{ilb}_d^n$ and, in particular, prove that the point of $\mathcal{H}\mathrm{ilb}_d^n$ corresponding to the segment ideal with respect to (w.r.t.) a degreverse term order (Definition 1.1(iii)) is singular (Theorem 6.4). Of course, this result cannot be generalized to Hilbert schemes with a Hilbert polynomial of a positive degree because in that more general case, the segment ideal with respect to a degreverse term order does not exist (Remark 3.4(2)(3)). In the literature, we have not found any proof of such a result.

1. General setting

Let K be an algebraically closed field of characteristic 0, where $S := K[x_0, \ldots, x_n]$ is the ring of polynomials over K in $n+1$ variables such that $x_0 < x_1 < \cdots < x_n$, and $\mathbb{P}_k^n = \operatorname{Proj} S$ is the n-dimensional projective space over K.

A *term* of S is a power product $x^\alpha := x_0^{\alpha_0} x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, where $\alpha_0, \alpha_1, \ldots, \alpha_n$ are non-negative integers. We set $\min(x^\alpha) := \min\{i : \alpha_i \neq 0\}$ and $\max(x^\alpha) := \max\{i : \alpha_i \neq 0\}$. We also let $\mathbb{T} := \{x_0^{\alpha_0} x_1^{\alpha_1} \ldots x_n^{\alpha_n} \mid (\alpha_0, \alpha_1 \ldots, \alpha_n) \in \mathbb{N}^{n+1}\}$ be the monoid of all terms of S and $\mathbb{T}(n) := \mathbb{T} \cap K[x_0, \ldots, x_{n-1}]$.

A *graded structure* on S is defined by assigning a *weight-vector* $w = (w_0, \ldots, w_n) \in \mathbb{R}_+^{n+1}$ and imposing $v_w(x^\alpha) = \sum_{i=0}^n w_i \alpha_i$. For each non-negative integer t, S_t is the K-vector space spanned by $\{x^\alpha \in \mathbb{T} : v_w(x^\alpha) = t\}$. The *standard grading* corresponds to $w = (1, \ldots, 1)$, and we will use it, unless otherwise specified.

For any $N \subseteq \mathbb{T}$, N_t is the set of the t-degree elements of N, and $\lambda_{i,t}(N) := |\{x^\alpha \in N_t : i \leq \min(x^\alpha)\}|$ denotes the cardinality of the subset of terms of N_t that are not divisible by $x_0, \ldots, x_{i-1}$. For any homogeneous ideal $I \subseteq S$, I_t is the vector space of the homogeneous polynomials in I of degree t, and $I_{\leq t}$ and $I_{\geq t}$ are the ideals generated by the homogeneous polynomials of I of degree $\leq t$ and $\geq t$, respectively.

Given any *term-order* $\preceq$ on $\mathbb{T}$, each $f \in S$ has a unique ordered representation $f = \sum_{i=1}^s c(f, \tau_i)\tau_i$, where $c(f, \tau_i) \in K^*$, $\tau_i \in \mathbb{T}$, $\tau_1 \succ \cdots \succ \tau_s$ and $T(f) := \tau_1$ is the *maximal term* of f. For any $F \subset S$, $T\{F\} := \{T(f) : f \in F\}$, $T(F) := \{\tau T(f) : f \in F, \tau \in \mathbb{T}\}$ and $\mathcal{N}(F) := \mathbb{T} \setminus T(F)$. For any ideal $I \subset S$, $T\{I\} = T(I)$ and $\mathcal{N}(I)$ is an *order ideal*, often called the *sous-escalier* or *Gröbner-escalier* of I. A subset $G \subset I$ is a *Gröbner-basis* of I if $T(G) = T(I)$ (see, for instance, [21]).

For a monomial ideal I, $G(I)$ denotes the unique set the minimal generators of I consisting of terms.

Definition 1.1. (1) In our setting, we consider mainly the (standard) graded term orders on $\mathbb{T}$. In particular, given two terms x^α and x^β of $\mathbb{T}$ of the same degree t, we say that x^α is less than x^β with respect to the following:

(i) the *deglex* order if $\alpha_k < \beta_k$, where $k = \max\{i \in \{0, \ldots, n\} : \alpha_i \neq \beta_i\}$;
(ii) the *degrevlex* order if $\alpha_h > \beta_h$, where $h = \min\{i \in \{0, \ldots, n\} : \alpha_i \neq \beta_i\}$;
(iii) a *degreverse* order if $\alpha_0 > \beta_0$ or $\alpha_0 = \beta_0$ and $\frac{x^\alpha}{x_0^{\alpha_0}} \preceq \frac{x^\beta}{x_0^{\beta_0}}$, where $\preceq$ is any graded term order on $\mathbb{T} \cap K[x_1, \ldots, x_n]$ [14, Definition 4.4.1]. Recall that a degreverse order is well suited for the homogenization of a Gröbner basis [7], and that the degrevlex is a particular degreverse order.

(2) Fixing any term order $\preceq$ on $\mathbb{T}$ and any weight vector w, the *weighted term order* $\preceq_w$ is defined as follows:

$$x^\alpha \prec_w x^\beta \quad \text{if } v_w(x^\alpha) < v_w(x^\beta) \quad \text{or} \quad v_w(x^\alpha) = v_w(x^\beta) \text{ and } \quad x^\alpha \prec x^\beta.$$

When speaking of w-term order, we understand $\preceq$ to be the deglex order.

Let $I \subset S$ be any homogeneous ideal. Then, $H_{S/I}(t)$ denotes the Hilbert function of the graded algebra S/I. It is well known that there is a polynomial $p_{S/I}(z) \in \mathbb{Q}[z]$, called the *Hilbert polynomial*, and positive integers $\rho_H := \min\{t \in \mathbb{N} \mid H_{S/I}(t') = p_{S/I}(t'), \forall\, t' \geq t\}$ and $\alpha_H := \min\{t \in \mathbb{N} \mid H_{S/I}(t) < \binom{n+t}{t}\}$, which are called, respectively, the *regularity of the Hilbert function* H and the *initial degree of* H (or also of I). For convenience, we will also say that either $p_{S/I}(z)$ is the Hilbert polynomial *for* I or I is an ideal *with* Hilbert polynomial $p_{S/I}(z)$. If I is not Artinian, set $\Delta H_{S/I}(t) := H_{S/I}(t) - H_{S/I}(t-1)$ for $t > 0$ and $\Delta H_{S/I}(0) := 1$; we use an analogous notation for Hilbert polynomials. If h is a linear form that is general on S/I, then it is easy to prove that $p_{S/(I,h)} = \Delta p_{S/I}$.

The polynomials $p(z) \in \mathbb{Q}[z]$ that are Hilbert polynomials of projective subschemes are called *admissible* and are completely characterized in [12] by the fact that they can always be written in a unique form of the following type (see [12,16]), where ℓ is the degree of $p(z)$ and $m_0 \geq m_1 \geq \cdots \geq m_\ell \geq 0$ are integers:

$$p(z) = \sum_{i=0}^{\ell} \binom{z+i}{i+1} - \binom{z+i-m_i}{i+1}.$$

The saturation of a homogeneous ideal $I \subset S$ is $I^{sat} := \{f \in S \mid \forall\, i \in 0, \ldots, n, \exists\, k_i : x_i^{k_i} f \in I\} = \cup_{h \geq 0}(I : \mathfrak{m}^h)$, where $\mathfrak{m} = (x_0, \ldots, x_n)$, and I is saturated if $I = I^{sat}$.

If $X \subset \mathbb{P}_K^n$ is a projective subscheme, $reg(X)$ is its *Castelnuovo–Mumford regularity*, i.e., $reg(X) = \min\{t \in \mathbb{N} \mid H^i(\mathcal{I}_X(t'-i)) = 0, \forall\, t' \geq t\}$.

An ideal $I \subset S$ is m-regular if the ith syzygy module of I is generated in degree $\leq m+i$, and the regularity $reg(I)$ of I is the smallest integer m for which I is m-regular. If I is saturated and defines a scheme X, then $reg(I) = reg(X)$, and we set $H_X(t) := H_{S/I}(t)$ and $p_X(z) := p_{S/I}(z)$.

For an admissible polynomial $p(z)$, the *Gotzmann number* r is the best upper bound for the Castelnuovo–Mumford regularity of a scheme having $p(z)$ as its Hilbert polynomial and is computable by using the following unique form of an admissible polynomial:

$$p(z) = \binom{z+a_1}{a_1} + \binom{z+a_2-1}{a_2} + \cdots + \binom{z+a_r-(r-1)}{a_r},$$

where $a_1 \geq a_2 \geq \cdots \geq a_r \geq 0$. We refer to [10] for an overview of these topics.

Example 1.2. If $p(z) = dz + 1 - g$ is an admissible polynomial, then its Gotzmann number is $r = \binom{d}{2} + 1 - g$. Indeed, we obtain

$$p(z) = \binom{z+1}{1} + \cdots + \binom{z+1-(d-2)}{1} + \binom{z+0-(d-1)}{0} + \cdots + \binom{z+0-\binom{d-2}{2}+g}{0}.$$

2. Borel ideals and Gröbner strata

Definition 2.1. (1) For any $x^\alpha \in \mathbb{T}$ such that $\alpha_j > 0$, the terms obtained from x^α via a jth *elementary move* are:

(i) $e_j^+(x^\alpha) := x_0^{\alpha_0} \ldots x_j^{\alpha_j - 1} x_{j+1}^{\alpha_{j+1}+1} \ldots x_n^{\alpha_n}$, for any $j \in \{0, \ldots, n-1\}$ and

(ii) $e_j^-(x^\alpha) := x_0^{\alpha_0} \ldots x_{j-1}^{\alpha_{j-1}+1} x_j^{\alpha_j - 1} \ldots x_n^{\alpha_n}$, for any $j \in \{1, \ldots, n\}$,

and for each positive integer $a < \alpha_j$, we will denote the corresponding elementary move applied a times by $(e_j^-)^a$ and $(e_j^+)^a$.

(2) For any positive integer t, $<_B$ denotes the partial order on $\mathbb{T}_t$ given by the transitive closure of the relation $e_j^-(x^\beta) < x^\beta$, i.e $x^\alpha <_B x^\beta$ if x^α is obtained from x^β via a finite sequence of elementary moves e_j^-.

(3) A set $B \subset \mathbb{T}_t$ is a *Borel set* if, for every x^α of B and x^β of $\mathbb{T}_t$, $x^\alpha <_B x^\beta$ implies that x^β belongs to B.

(4) A monomial ideal $J \subset S$ is a *Borel ideal* if, for every degree t, $J \cap \mathbb{T}_t$ is a Borel set.

The combinatorial property by which Borel ideals are defined is also called *strong stability*. It was first introduced in [11] and later discussed in [12], where the ideals that satisfy it are called *balanced*. In characteristic 0, it is equivalent to the property of an ideal J being fixed by lower triangular matrices, from which the name “Borel ideals” is derived from.

From the definition, it immediately follows that if $B \subset \mathbb{T}_t$ is a Borel set, then the set $N := \mathbb{T}_t \setminus B$ has the property that for every $x^\gamma \in N$ and $x^\delta \in \mathbb{T}_t$ with $x^\delta <_B x^\gamma$, x^δ belongs to N, that is, N is closed w.r.t. elementary moves e_j^-. In particular, if J is a Borel ideal, then for every integer $t \geq 0$, $\mathcal{N}(J)_t$ is closed w.r.t. elementary moves e_j^-, and J_t is closed w.r.t. elementary moves e_j^+.

Remark 2.2. Note that, for every term order $\preceq$, if $x^\alpha, x^\beta \in \mathbb{T}_t$ satisfy $x^\alpha <_B x^\beta$, then $x^\alpha \prec x^\beta$. Namely, because $x^\alpha <_B x^\beta$ means that there is a finite number of elementary moves e_j^- connecting x^β to x^α, assuming that $x_j \mid x^\beta$ for a suitable $0 \leq j \leq n$, we can verify our contention for $x^\alpha = e_j^-(x^\beta)$. Setting $\tau := \frac{x^\beta}{x_j}$ and writing $x^\alpha = e_j^-(x^\beta) = x_{j-1}\tau$ and $x^\beta = x_j\tau$, we get $x^\alpha \prec x^\beta$ because $x_{j-1} \prec x_j$.

Proposition 2.3. *For a Borel ideal $J \subset S$,*

(i) *in our notation, J^{sat} is obtained by setting $x_0 = 1$ in the minimal generators of J,*

(ii) *the Krull dimension of S/J is equal to* $\min\{\max(x^\alpha) : x^\alpha \in J\} = \min\{i \in \{0, \ldots, n\} : x_i^t \in J,$ *for some t}, and*

(iii) *the regularity of J is equal to the maximum degree of its minimal generators.*

Proof. (i) For example, see [23, Property 2].

(ii) This result follows straightforwardly from Lemma 3.1(a) of [13] or from Corollary 4, Section 5, chapter 9 of [7]. Thus, if J is saturated and ℓ is the degree of the Hilbert polynomial of J, we get $\ell = \min\{i \in \{0, \ldots, n\} : x_i^t \in J,$ for some $t\} - 1$.

(iii) See [2, Proposition 2.9]. □

Remark 2.4. Let $B \subset \mathbb{T}_t$ be a non-empty Borel set, $N := \mathbb{T}_t \setminus B$ and $J = (B)$ is the Borel ideal generated by the terms of B; so, $\mathcal{N}(J)_t = N$. Thus,

$$\begin{aligned}\mathcal{N}(J)_{t+1} &= x_0 N \sqcup x_1\{x^\alpha \in N : 1 \leq \min(x^\alpha)\} \\ &\sqcup\ x_2\{x^\alpha \in N : 2 \leq \min(x^\alpha)\} \sqcup \ldots \sqcup x_{n-1}\{x^\alpha \in N : n-1 \leq \min(x^\alpha)\},\end{aligned}$$

and $\mathbb{T}_{t+1} \setminus \mathcal{N}(J)_{t+1}$ is a Borel set. In particular, if J is a Borel ideal and $N := \mathcal{N}(J)_t$, for every degree t we have the following (see [18, Theorem 3.7]):

$$\begin{aligned}\mathcal{N}(J)_{t+1} \subseteq \mathcal{N}(J_{\leq t})_{t+1} &= x_0 N \sqcup x_1\{x^\alpha \in N : 1 \leq \min(x^\alpha)\} \\ &\sqcup\ x_2\{x^\alpha \in N : 2 \leq \min(x^\alpha)\} \sqcup \ldots \sqcup x_{n-1}\{x^\alpha \in N : n-1 \leq \min(x^\alpha)\},\end{aligned}$$

from which $|\mathcal{N}(J)_{t+1}| \leq |\mathcal{N}(J_{\leq t})_{t+1}| = \sum_{i=0}^{n-1} \lambda_{i,t}(N(J_{\leq t}))$ and $G(J)_{t+1} = \mathcal{N}(J_{\leq t})_{t+1} \setminus \mathcal{N}(J)_{t+1}$.

Definition 2.5. For each Borel subset $A \subset \mathbb{T}_t$ the *minimal* elements of A w.r.t. $<_B$ are the terms $x^\alpha \in A$ such that $\mathrm{e}_j^-(x^\alpha) \notin A$ for every $j > 0$ with $\alpha_j > 0$, and the *maximal* elements outside A w.r.t. $<_B$ are the terms $x^\beta \notin A$ such that $\mathrm{e}_j^+(x^\beta) \in A$ for every $j > 0$ with $\beta_j > 0$.

Remark 2.6. Let $B \subset \mathbb{T}_t$ be a Borel subset, if $x^\alpha \in B$ and $x^\beta \notin B$ are respectively a minimal element of B and a maximal element outside B w.r.t. $<_B$, then both $B \setminus \{x^\alpha\}$ and $B \cup \{x^\beta\}$ are Borel subsets of $\mathbb{T}_t$ as, by definition, both are closed w.r.t. elementary moves e_j^+.

Proposition 2.7. *Let $p(z)$ be an admissible polynomial with Gotzmann number r, and let $J \subset S$ be a Borel ideal with $p(z)$ as the Hilbert polynomial. Then, for each $s > r$, a minimal term of J_s w.r.t. $<_B$ is divisible by x_0.*

Proof. As $s > r \geq reg(J)$, for each $x^\alpha \in J_s$, we get $x^\alpha = \tau \cdot x^\gamma$ for some $x^\gamma \in G(J)$ and $\tau \in \mathbb{T}$, with $\deg(\tau) > 0$. If x^α is not divisible by x_0, let $j > 0$ be such that τ is divisible by x_j. Then, $x^{\alpha'} := \tau' \cdot x^\gamma$, with $\tau' = \frac{\tau \cdot x_0}{x_j}$, satisfies $x^{\alpha'} \in J_s$ and $x^{\alpha'} <_B x^\alpha$, thus contradicting the minimality of x^α. □

Given an admissible polynomial $p(z)$, a term order $\preceq$ and a monomial ideal J with $p(z)$ as the Hilbert polynomial, the *Gröbner stratum* $\mathcal{S}t(J, \preceq)$ in the Hilbert scheme $\mathcal{H}\mathrm{ilb}^n_{p(z)}$ of subschemes of $\mathbb{P}^n$ with Hilbert polynomial $p(z)$ is an affine variety that parameterizes the family of ideals having the same initial ideal J with respect to $\preceq$ [26,9,15,3,22]. When only homogeneous ideals are concerned, we write $\mathcal{S}t_h(J, \preceq)$. Now, we recall briefly the construction of $\mathcal{S}t(J, \preceq)$ and hence of $\mathcal{S}t_h(J, \preceq)$, referring to Definition 3.4 of [15], although here we omit many details that make the procedure more efficient.

For any term x^α of $G(J)$, we set $F_\alpha := x^\alpha + \sum_{\{x^\beta \in \mathcal{N}(J)\ :\ x^\beta < x^\alpha\}} c_{\alpha\beta} x^\beta$, considering $c_{\alpha\beta}$ as new variables. Then, we reduce all the S-polynomials $S(F_\alpha, F_{\alpha'})$ with respect to $\{F_\alpha\}_{x^\alpha \in J}$. The ideal $\mathcal{A}(J)$ generated in $K[c_{\alpha\beta}]$ by the coefficients of the monomials in the variables x of the reduced polynomials is the defining ideal of $\mathcal{S}t(J, \preceq)$ and does not depend on the reduction choices. In particular, if we set $F_\alpha := x^\alpha + \sum_{\{x^\beta \in \mathcal{N}(J)_t\ :\ x^\beta < x^\alpha\}} c_{\alpha\beta} x^\beta$, where t is the degree of x^α, then we obtain the ideal of $\mathcal{S}t_h(J, \preceq)$.

For the properties of Gröbner strata, we refer to [26,9,15], but it is noteworthy to point out an unexpected feature of Gröbner strata, which is that they are homogeneous varieties with respect to some non-standard graduation [9,15]. Thus, the embedding dimension of $\mathcal{S}t_h(J, \preceq)$, denoted by $ed(\mathcal{S}t_h(J, \preceq))$, is the dimension of the Zariski tangent space of the stratum at the origin and can be computed by the same procedure that produces Gröbner strata. In fact, the ideal $\mathcal{L}(J)$ generated in $K[c_{\alpha\beta}]$ by the linear components of the generators of $\mathcal{A}(J)$, as computed above, defines the Zariski tangent space of the stratum at the origin (Theorems 3.6(ii) and 4.3 of [15]). This fact offers a new tool for studying the singularities of Hilbert schemes.

3. Segments

Definition 3.1. A set $B \subset \mathbb{T}_t$ is a *segment* w.r.t. a term order $\preceq$ on $\mathbb{T}$ if whenever a term τ belongs to B, all the t-degree terms greater than τ belong to B. A monomial ideal I is a *segment ideal* w.r.t. $\preceq$, if $I \cap \mathbb{T}_t$ is a segment w.r.t. $\preceq$ for every $t \geq 0$.

Lemma 3.2. *Let $I \subset S$ be a saturated Borel ideal, $\preceq$ any term order on $\mathbb{T}$ and $p > q$ integers. If I_p is a segment, then I_q is a segment too.*

Proof. Let x^α be a term of I_q and x^β a term of $\mathbb{T}_q$ such that $x^\alpha \preceq x^\beta$; hence $x_0^{p-q} x^\alpha \preceq x_0^{p-q} x^\beta$, and because I_p is a segment, $x_0^{p-q} x^\beta$ belongs to I_p. Recalling that I is saturated, x^β belongs to I_q, and we are done. □

Remark 3.3. A segment is a Borel set, and a segment ideal is a Borel ideal. Indeed, $x_i x^\alpha \prec x_h x^\alpha$ if $i < h$; so, $x^\alpha <_B x^\beta$ implies that $x^\alpha \prec x^\beta$, for any term order $\preceq$. In particular, if $\preceq$ is the deglex order and I is a monomial ideal generated in degree $\leq q$ such that I_q is a segment w.r.t. $\preceq$, then I_p is a segment too for every $p > q$.

Remark 3.4. (1) Each admissible polynomial $p(z)$ of degree $0 \leq \ell \leq n$ corresponds to a unique saturated segment ideal $L(p(z))$ w.r.t. deglex order (see [1,16]). In particular for a constant polynomial $p(z) = d$ we have the following

$$L(d) = (x_n, x_{n-1}, \ldots, x_2, x_1^d),$$

$$\mathcal{N}(L(d))_j = \begin{cases} \mathbb{T}(2)_j & \text{if } 0 \leq j < d \\ \{x_0^{d+i}, x_0^{d+i-1}x_1, \ldots, x_0^{i+1}x_1^{d-1}\} & \text{if } j = d+i,\ \forall\, i \geq 0. \end{cases}$$

(2) A segment ideal w.r.t. the degrevlex order exists if and only if the Hilbert polynomial is constant and the Hilbert function H is non-increasing, i.e., $\Delta H(t) \leq 0$ for every $t > \alpha_H = \min\{t \in \mathbb{N} | H(t) < \binom{t+n}{n}\}$ [8,19]. We let $\Lambda_\omega := \Lambda_{\omega,j}$ be the set of the ω smallest terms of $\mathbb{T}_j$ w.r.t. degrevlex order.

(3) The same reasoning of [8,19] shows that, in general, a segment ideal J w.r.t. a degreverse term order exists if and only if the Hilbert polynomial is constant and the Hilbert function H is non-increasing. Namely, if α_H is the initial degree and $x_1^{\alpha_H} \in \mathcal{N}(J)$, then $x_1^{\alpha_H+1}$ belongs to J, or otherwise, letting $\tau \succ x_1^{\alpha_H}$ be the smallest degree α_H term in J, $x_1^{\alpha_H+1}$ would belong to $\mathcal{N}(J)$ with $x_1^{\alpha_H+1} \succ x_0\tau \in J$.

Proposition 3.5. *If an ideal $J \subset S$ of initial degree α_H has the property that there exists an integer $t \geq \alpha_H$ and four terms $x^\alpha, x^\beta \in \mathcal{N}(J)_t$, $x^\gamma, x^\delta \in J_t$ with $x^{\alpha+\beta} = x^{\gamma+\delta}$, then J is not a segment ideal w.r.t. any term order $\preceq$.*

Proof. If J were a segment ideal w.r.t. some $\preceq$, by the given assumptions, we would have in particular both $\mathcal{N}(J)_t \ni x^\beta \prec x^\delta \in J_t$ and $\mathcal{N}(J)_t \ni x^\alpha \prec x^\gamma \in J_t$. From these, we would obtain $x^{\alpha+\beta} \prec x^{\alpha+\delta} \prec x^{\gamma+\delta}$, contradicting $x^{\alpha+\beta} = x^{\gamma+\delta}$. □

Example 3.6. (1) The (saturated) Borel ideal $J = (x_2^3, x_1^3x_2^2, x_1^5x_2, x_1^6) \subset K[x_0, x_1, x_2]$ is not a segment ideal w.r.t. any term order because it satisfies the conditions of Proposition 3.5. Namely, its initial degree is 3, and for $t = 6 \geq 3$, we have $J_6 \ni x_0^3x_2^3, x_1^6$ and $x_0^2x_1^2x_2^2, x_0x_1^4x_2 \in \mathcal{N}(J)_6$ with $x_0^3x_2^3 \cdot x_1^6 = x_0^2x_1^2x_2^2 \cdot x_0x_1^4x_2$.

(2) Here, we show that Proposition 3.5 cannot be inverted. The Borel ideal $J = (x_2^3, x_1x_2^2, x_1^2x_2, x_0^2x_2^2, x_0^3x_1x_2, x_0^5x_2, x_1^7) \subset K[x_0, x_1, x_2]$ of [6, Example 5.8] has the property that J_3 is a segment w.r.t. degrevlex order, while J_t is a segment w.r.t. deglex order for every $t \geq 4$; so, at each degree, it does not satisfy the conditions of Proposition 3.5. Nevertheless, J is not a segment w.r.t. any term order $\preceq$, as if it were, from $x_1^2x_2 \in J_3$ and $x_0x_2^2 \in \mathcal{N}(J)_3$, we would get $x_0x_2 \prec x_1^2$, contradicting $(x_0x_2)^2 \in J_4, x_1^4 \in \mathcal{N}(J)_4$. Note also that $J^{sat} = (x_2, x_1^7)$ is the saturated lex segment.

Definition 3.7. Let $I \subset S$ be a non-null saturated Borel ideal and $\preceq$ a term order on $\mathbb{T}$.

(a) [15] I is a *hilb-segment ideal* if I_r is a segment, where r is the Gotzmann number of the Hilbert polynomial of I;
(b) I is a *reg-segment ideal* if I_δ is a segment, where δ is the regularity of I; and
(c) I is a *gen-segment ideal* if, for every integer s, $G(I)_s$ consists of the greatest terms among the s-degree terms not in $\langle I_{s-1}\rangle$.

Remark 3.8. The criterion given by Proposition 3.5 can also be adapted to hilb-segment ideals and to reg-segment ideals I by simply verifying it at degree $r =$ the Gotzmann number and at degree $\delta = reg(I)$, respectively. Computational evidence suggests that this criterion is also necessary for reg-segment and hilb-segment ideals.

The following results about Gröbner strata motivate the definitions of the reg-segment ideal and of the hilb-segment ideal, respectively.

Proposition 3.9. (i) *Let $I \subset S$ be a Borel saturated ideal. If x_1 does not appear in any monomial of degree $m+1$ in the monomial basis of I, then $\mathcal{S}t_h(I_{\geq m-1}) \cong \mathcal{S}t_h(I_{\geq m})$ (Theorem 4.7* (iii) *of* [15]*).*

(ii) *An isolated irreducible component of $\mathcal{H}\mathrm{ilb}^n_{\boldsymbol{p}(z)}$ that contains a smooth point corresponding to a hilb-segment ideal is rational (Corollary 6.10 of* [15]*).*

Proposition 3.10. *Let $I \subset S$ be a saturated Borel ideal and $\preceq$ a term order on $\mathbb{T}$. Then,*

(i) *I segment ideal $\Rightarrow$ I hilb-segment ideal $\Rightarrow$ I reg-segment ideal $\Rightarrow$ I gen-segment ideal;*
(ii) *$\preceq$ is the deglex order $\Leftrightarrow$ conditions in* (i) *are equivalent, for every ideal I; and*
(iii) *If the projective scheme defined by I is 0-dimensional, then I segment ideal $\Leftrightarrow$ I hilb-segment ideal $\Leftrightarrow$ I reg-segment ideal.*

Proof. (i) The first implication is obvious. For the second one, we need to only apply Lemma 3.2 because $r \geq \delta$. For the third implication, recall that I is generated in degrees $\leq \delta$, by definition. Moreover, if I is a reg-segment ideal, by Lemma 3.2, I_t contains the greatest terms of degree t for every $t \leq \delta$.

(ii) First, suppose that $\preceq$ is the deglex order. Then, by (i), it is enough to show that a gen-segment ideal is also a segment ideal. Indeed, by induction on the degree s of terms and with $s = 0$ as the base of induction, for $s > 0$, suppose that I_{s-1} is a segment. Thus, by Remark 3.3, we know that $\langle I_{s-1}\rangle_s$ is a segment, and because the possible minimal generators are always the greatest possible, we are done.
Vice versa, if $\preceq$ is not the deglex order, let s be the minimum degree at which the terms are ordered in a different way from the deglex one. Thus, there exist two terms x^α and x^β with maximum variables x_l and x_h, respectively, such that $x^\beta \prec x^\alpha$ but $x_h \succ x_l$. The ideal $I = (x_h, \ldots, x_n)$ is a gen-segment ideal but not a segment ideal because x^β belongs to I and x^α does not.

(iii) We need only show that, in the 0-dimensional case, a reg-segment ideal I is also a segment ideal. By induction on the degree s, if $s \leq \delta$, then the thesis follows by the hypothesis and by Lemma 3.2. Suppose that $s > \delta$ and that I_{s-1} is a segment. At degree s, there are no minimal generators for I; so, a term of I_s is always of type $x^\alpha x_h$ with x^α in I_{s-1}. Let x^β be a term of degree s such that $x^\beta \succ x^\alpha x_h$; so, $x^\beta \succ x^\alpha x_0$. By Proposition 2.3, we have that $(x_1, \ldots, x_n)^s \subseteq I$. So, if x^β is not divisible by x_0, then x^β belongs to I_s; otherwise, there exists a term x^γ such that $x^\beta = x^\gamma x_0$. Thus, $x^\gamma \succ x^\alpha$, and by induction, x^γ belongs to I_{s-1}; so, $x^\beta = x^\gamma x_0$ belongs to I_s. □

Example 3.11. Let $\preceq$ be the degrevlex order.

(1) The ideal $I = (x_2^2, x_1x_2) \subset K[x_0, x_1, x_2]$ is a hilb-segment ideal, but it is not a segment ideal. In this case, the Hilbert polynomial is $p(z) = z + 2$ with Gotzmann number 2 and $reg(I) = 2$. We have $x_1^3 \in \mathcal{N}(I)$ and $x_0x_1x_2 \in I$ with $x_1^3 \succeq x_0x_1x_2$.

(2) $I' = (x_2^3, x_1x_2^2, x_1^2x_2) \subset K[x_0, x_1, x_2]$ is a reg-segment ideal but not a hilb-segment ideal. In this case, the Hilbert polynomial is $p(z) = z + 4$ with Gotzmann number 4 and $reg(I') = 3$. We obtain $x_0x_2^3 \in I'$ with $x_0x_2^3 \preceq x_1^4 \notin I'$.
(3) $I'' = (x_4^2, x_3x_4, x_3^3) \subset K[x_0, \dots, x_4]$ is a gen-segment ideal but not a reg-segment ideal. In this case, the Hilbert polynomial is $p(z) = 2z^2 + 2z + 1$ with Gotzmann number 12 and $reg(I'') = 3$. We obtain $x_0x_4^2 \in I''$ with $x_0x_4^2 \preceq x_2^3 \notin I''$.

Remark 3.12. If I is a saturated Borel ideal and is also an "almost revlex segment ideal", as defined in [8], then it is a gen-segment ideal w.r.t. degrevlex order.

Theorem 3.13. *Let J be the ideal generated by a Borel set $B \subset \mathbb{T}_d$ consisting of all but d terms of degree d. Then, J^{sat} defines a projective scheme with Hilbert polynomial $p(z) = d$.*

Proof. By the Borel condition, x_1^d belongs to J or otherwise, $|\mathcal{N}(J)_d| \geq d + 1 > d$ so, by Remark 2.4, we have $|\mathcal{N}(J)_t| = d$, for every $t \geq d$. The ideal $I = J^{sat}$ is the saturated ideal of a projective subscheme with Hilbert polynomial $p(z) = d$. □

Remark 3.14. (1) For every positive integer d and any term order $\preceq$ on $\mathbb{T}$, there exists a unique saturated segment ideal $I \subset S$ with Hilbert polynomial $p(z) = d$. This is a straightforward consequence of Theorem 3.13; it is enough to take the ideal J generated by all but the least d terms of degree d.
(2) In [19] for the degrevlex order and in [6] for each term order, it is shown that the generic initial ideal of a set X of d general points in $\mathbb{P}^n$ is a segment ideal with Hilbert polynomial $p(z) = d$. As the Hilbert function of X is the maximum possible, that is, $H_X(t) = \min\{\binom{t+n}{t}, d\}$, we deduce that this is the Hilbert function of the saturated segment ideal of (1).
(3) For the case of degreverse term orders, it is possible to give a direct and constructive proof of (1). If $J \subset S$ is a segment ideal w.r.t. a degreverse order with Hilbert polynomial $p(z) = d$, its Hilbert function must to be non-increasing by Remark 3.4(2) and also strictly increasing until it reaches the value d, after which it remains equal to d because J is a saturated ideal of Krull dimension 1. Thus, $H_{S/J}(t)$ must be the maximum and we deduce the following:
 (i) $\alpha_H = \rho_H + 1$; so, $J = (x_1, \dots, x_n)^{\alpha_H}$;
 (ii) $\alpha_H = \rho_H$; so, J is generated only in degrees α_H and $\alpha_H + 1$. More precisely, the minimal generators of degree α_H are the greatest $\binom{\alpha_H+n}{\alpha_H} - d$ terms of $\mathbb{T}_{\alpha_H}$ (so, in $\mathcal{N}(J)_{\alpha_H}$, there are $d - \binom{\alpha_H+n-1}{n-1}$ terms x^β with $\min(x^\beta) \geq 1$), and the minimal generators of degree $\alpha_H + 1$ are all the terms $\tau \succeq x_1^{\alpha_H+1}$ that are not multiples of terms in J_{α_H} (these terms are at least $d - \binom{\alpha_H-1+n}{\alpha_H-1}$, by Remark 2.4).

It follows that in case (i) we have $|G(J)| = \binom{\rho_H+n}{n-1}$, and in case (ii), $|G(J)| \geq \binom{\rho_H+n}{n} - d + d - \binom{\rho_H+n-1}{n} = \binom{\rho_H+n-1}{n-1}$.

3.1. On hilb-segment ideals

Let $\preceq$ be any term order and $p(z)$ be an admissible polynomial with Gotzmann number r. We want to see under what conditions there exists a hilb-segment ideal for $p(z)$. In this context, it is immediately clear that if $r = 1$, then $p(z) = \binom{z+\ell}{\ell}$, where $\ell < n$ is the degree of $p(z)$; so, $I = (x_{\ell+1}, \dots, x_n)$ is the hilb-segment ideal for $p(z)$. Moreover, we have already observed that a hilb-segment ideal always exists for a constant polynomial $p(z) = d$.

Example 3.15. The following saturated Borel ideals are not hilb-segment ideals for any term order:

(1) $J = (x_2^2, x_1^3x_2, x_1^4) \subset K[x_0, x_1, x_2]$, (see [6]), as for $H = (1, 3, 5, 7, \dots, p(z) = 7, \dots)$, we have $r = 7$; so, if J were a hilb-segment ideal w.r.t. some $\preceq$, at degree 7, we should have $\mathcal{N}(J)_7 \ni x_0^4x_1^2x_2 \prec x_0^5x_2^2 \in J_7$ and $\mathcal{N}(J)_7 \ni x_0^4x_1^2x_2 \prec x_0^3x_1^4 \in J_7$, contradicting $(x_0^4x_1^2x_2)^2 = x_0^5x_2^2 \cdot x_0^3x_1^4$.
(2) $J = (x_2^3, x_1x_2^2, x_1^2x_2) \subset K[x_0, x_1, x_2]$, as for $H = (1, 3, 6, 7, \dots, p(z) = z + 4, \dots)$, we have $reg(J) = 3$ and $r = 4$; so, we can repeat the same reasoning of (1) with $x_0x_1^2x_2 \in J_4$ and $x_1^4, x_0^2x_2^2 \in \mathcal{N}(J)_4$.

Proposition 3.16. *In $K[x_0, x_1, x_2]$, every saturated Borel ideal with Hilbert polynomial $p(z) = d \leqslant 6$ is a hilb-segment ideal. Whereas, for every $p(z) = d \geqslant 7$, a saturated Borel ideal, which is not a hilb-segment for any term order always exists.*

Proof. We give a direct constructive proof of the result, based in part on the characterization of the Borel subsets in three variables of [17].

$d \leqslant 2$ there exists a unique saturated Borel ideal (x_2, x_1^d), which is the hilb-segment ideal w.r.t. deglex order;
$d = 3$ there are only two saturated Borel ideals: the hilb-segment ideals (x_2, x_1^3) (w.r.t. deglex) and (x_2^2, x_1x_2, x_1^2) (w.r.t. degrevlex);
$d = 4$ there are only two saturated Borel ideals: the hilb-segment ideals (x_2, x_1^4) (w.r.t. deglex) and (x_2^2, x_1x_2, x_1^3) (w.r.t. degrevlex);

$d = 5$ there are three saturated Borel ideals: the hilb-segment ideals (x_2, x_1^5) (w.r.t. deglex), (x_2^2, x_1x_2, x_1^4) (w.r.t. (4, 2, 1)-term order) and $(x_2^2, x_1^2x_2, x_1^3)$ (w.r.t. degrevlex);

$d = 6$ there are four saturated Borel ideals: the hilb-segment ideals (x_2, x_1^6) (w.r.t. deglex), (x_2^2, x_1x_2, x_1^5) (w.r.t. (5, 2, 1)-term order), $(x_2^2, x_1^2x_2, x_1^4)$ (w.r.t. (3, 2, 1)-term order) and $(x_2^3, x_1x_2^2, x_1^2x_2, x_1^3)$ (w.r.t. degrevlex);

$d \geqslant 7$ (i) Let us first consider the case $d = 2a + 1, a \geq 3$ and the ideal $J = (x_2^2, x_1^ax_2, x_1^{a+1})$. It has Hilbert polynomial $p(z) = 2a+1$, because in degree $2a+1$ the $2a+1$ monomials $\{x_0^{a+i}x_1^{a-i}x_2, x_0^{a+j+1}x_1^{a-j},\ i = 1, \ldots, a,\ j = 0, \ldots, a\}$ belong to the quotient. Moreover, $x_0^{2a-1}x_2^2, x_0^ax_1^{a+1} \in J$, and $x_0^{2a-2}x_1^2x_2, x_0^{a+1}x_1^{a-1}x_2 \notin J$, but $x_0^{2a-1}x_2^2 \cdot x_0^ax_1^{a+1} = x_0^{2a-2}x_1^2x_2 \cdot x_0^{a+1}x_1^{a-1}x_2$ (if $a = 3$, this is exactly the ideal of Example 3.15 (1)).

(ii) In the case $d = 2a, a \geq 4$, let us consider the ideal $J = (x_2^3, x_1x_2^2, x_1^2x_2, x_1^{2a-3})$. It has Hilbert polynomial $p(z) = 2a$, namely, $\mathcal{N}(J)_{2a} = \{x_0^{2a-2}x_2^2, x_0^{2a-2}x_1x_2, x_0^{2a-1}x_2, x_0^{2a-i}x_1^i,\ i = 0, \ldots, 2a-4\}$. Moreover, $x_0^{2a-3}x_1^2x_2 \in J_{2a}$, $x_0^{2a-2}x_2^2, x_0^{2a-4}x_1^4 \in \mathcal{N}(J)_{2a}$, and $(x_0^{2a-3}x_1^2x_2)^2 = x_0^{2a-2}x_2^2 \cdot x_0^{2a-4}x_1^4$. □

Proposition 3.17. *Let $\preceq$ be any degreverse term order and $p(z)$ be an admissible polynomial of positive degree with Gotzmann number r.*

(1) *If $p(r) \leq \binom{r-1+n}{n}$, then the hilb-segment ideal for $p(z)$ does not exist.*
(2) *If $p(z) = dz + 1 - g$, then the hilb-segment ideal J for $p(z)$ exists if and only if*
  (i) *$r = d$ or $r = d + 1$, when $n = 2$, and*
  (ii) *$r = d = 1$, when $n > 2$.*

Proof. (1) By the hypothesis, we have that x_1^r belongs to the ideal; so, the Krull dimension must be 1 by Proposition 2.3, and we are done.

(2) In this case, the hilb-segment ideal J exists if and only if $p(r) = \binom{n+r-1}{n} + d$. Indeed, the sous-escalier of $(J, x_0)_r$ contains the least d terms not divisible by x_0, and because the term order is degreverse and $r \geq d$, the sous-escalier of J_r must also contain the same least d terms not divisible by x_0. Hence, by the Borel property, all the terms divisible by x_0 must also belong to the sous-escalier of J_r. Thus, because $r = \binom{d}{2} + 1 - g$ by Example 1.2, we obtain the following:

$$dr + r - \binom{d}{2} = \binom{n+r-1}{n} + d \Leftrightarrow d = \frac{1}{2}\left(2r - 1 \pm \sqrt{8\binom{r+1}{2} - 8\binom{r+n-1}{n} + 1}\right);$$

so, J exists if and only if the argument Δ under the square root is not-negative. By an easy calculation, we obtain the thesis. □

3.2. On gen-segment ideals for degrevlex order

We describe some procedures to construct gen-segment ideals w.r.t. degrevlex order with a given admissible polynomial $p(z)$ of degree 1. We have already observed that a hilb-segment ideal always exists and so does a gen-segment ideal for a constant polynomial $p(z) = d$.

Lemma 3.18. *If $p(z) = dz + 1 - g$ is an admissible polynomial with Gotzmann number r, there exist two integers $n \geq 2$ and $j(n) > 0$ such that $\binom{j(n)-1+n}{n} \leq p(j(n) - 1)$ and $p(j(n) + h) < \binom{j(n)+h+n}{n}$ for every $h \geq 0$.*

Proof. Any projective scheme of dimension 1 with Hilbert polynomial $p(z)$ has regularity $\leq r$; so, $p(r) < \binom{r+n}{n}$ for any $n \geq 2$. Now, it is enough to show that there exist integers $n \geq 2$ and $t < r$ such that $p(t) \geq \binom{t+n}{n}$. In the plane, i.e., for $n = 2$, it holds that $g \leq \frac{1}{2}(d-1)(d-2)$. Therefore, $p(t) = dt + 1 - g \geq dt + 1 - \frac{1}{2}(d-1)(d-2)$, and for $t = d - 1$, we have $d(d-1) + 1 - \frac{1}{2}(d-1)(d-2) = \binom{d-1+2}{2}$. Thus, $n = 2, d \leq j(n) \leq r$. □

Proposition 3.19. *Let $p(z) = dz + 1 - g$ be an admissible polynomial. For any $n \geq 2$, there exists a gen-segment ideal $I(n) \subset S$ w.r.t. degrevlex order with Hilbert polynomial $p(z)$.*

Proof. By Lemma 3.18, we can take an integer $n \geq 2$ for which there exists $j(n) > 0$ such that $\binom{j(n)-1+n}{n} \leq p(j(n) - 1)$ and $p(j(n) + h) < \binom{j(n)+h+n}{n}$ for every $h \geq 0$. First, we prove the thesis in this case.

Under the given assumptions, we have $p(j(n)) - \binom{j(n)-1+n}{n} \geq d = p(j(n)) - p(j(n) - 1)$; thus, by Remark 3.4(3), in $\Lambda_{p(j(n)),j(n)}$, there are at least d terms x^α such that $\min(x^\alpha) \geq 1$, and we let $\tau_1 < \cdots < \tau_d$ be the least among them w.r.t. degrevlex order. We also set for every $0 \leq t < j(n), N(t) := \mathbb{T}_t, N(j(n)) := \Lambda_{p(j(n)),j(n)}, N(t) := x_0 \cdot N_{t-1} \sqcup x_1^h \cdot \{\tau_1, \ldots, \tau_d\}$, for every $t = j(n) + h, h \geq 1$, and $N := \sqcup_{t\geq 0} N(t)$. By construction, $N \subset \mathbb{T}$ is such that $N_t = N(t)$, for each $t \geq 0$, and $| N_t | = p(t)$, for every $t \geq j(n)$. Thus, the monomial ideal $I(n) \subset S$ such that $\mathcal{N}(I(n)) = N$ is, by construction, a gen-segment ideal with Hilbert polynomial $p(z)$. Moreover, $G(I(n))_t = \emptyset$, for $t < j(n)$ and $t > j(n) + 1$; so, $reg(I(n)) \leq j(n) + 1 \leq r$.

Now, suppose that n is such that $p(t) < \binom{t+n}{n}$ for every $t \geq 0$, and let $n_0 := \max\{n' \mid \exists j(n') : \binom{j(n')-1+n'}{n'} \leq p(j(n') - 1)$ and $p(j(n')) < \binom{j(n')+n'}{n'}\}$. Above, we proved that for such an n_0, there exists a gen-segment ideal $I(n_0) \subset K[x_0, \dots, x_{n_0}]$ w.r.t. degrevlex order with Hilbert polynomial $p(z)$. Now, it is enough to observe that $I(n) := (I(n_0), x_{n_0+1}, \dots, x_n) \subset S$ is a gen-segment ideal w.r.t. degrevlex order, as claimed. □

Remark 3.20. Given an admissible polynomial $p(z) = dz+1-g$, let $N_l := \Lambda_{p(l),l}$ be the set of the lower $p(l)$ terms of degree l w.r.t. degrevlex order. If $n > 2$ is such that $p(t) < \binom{t+n}{n}$ for every $t \geq 0$ and *there exists* $l(n) := \min\{l \in \mathbb{N} : \lambda_{1,l}(N_l) \geq d\}$, by a similar procedure, we can construct a gen-segment ideal $J(n) \subset S$ w.r.t. degrevlex order with Hilbert polynomial $p(z)$ that is different from those coming from the smaller n''s as in the proof of Proposition 3.19. Indeed, under the given assumptions, $\Lambda_{p(l(n)),l(n)} \subset \mathbb{T}_{l(n)}$ no longer contains at least d terms x^α with $\min(x^\alpha) \geq 1$, but its expansion in degree $l(n) + 1$ does, and we let $\bar{\tau}_1 < \cdots < \bar{\tau}_d$ be the least one of them w.r.t. degrevlex order. Similarly as before, we take $M(t) := \mathbb{T}_t$ for every $0 \leq t < l(n), M(l(n)) := \Lambda_{p(l(n)),l(n)}, M(l(n)+1) := x_0 \cdot M(l(n)) \sqcup \{\bar{\tau}_1, \dots, \bar{\tau}_d\}$ and $M(t) := x_0 \cdot M(t-1) \sqcup x_1^{t-l(n)-1}\{\bar{\tau}_1, \dots, \bar{\tau}_d\}$ for every $t > l(n) + 1$. We finally let $J(n)$ be the gen-segment ideal such that $\mathcal{N}(J(n)) = M := \sqcup_{t\geq 0} M(t)$ and note that it has $p(z)$ as its Hilbert polynomial and regularity $\leq l(n) + 2$.

Example 3.21. The Gotzmann number of the admissible polynomial $p(z) = 6z - 3$ is 12, and we obtain the following gen-segment ideals:

(i) if $n = 2$, we can apply the procedure described in the proof of Proposition 3.19 with $j(2) = 9$ and construct the ideal $I(2) = (x_2^9, x_1x_2^8, x_1^2x_2^7, x_1^3x_2^6)$;
(ii) if $n = 3$, there is not a $j(3)$, yet we can apply the procedure described in Remark 3.20 with $l(3) = 2$ because $p(t) < \binom{3+t}{t}$ for every $t > 0$, obtaining $J(3) = (x_3^2, x_2^2x_3, x_2^4)$ besides $(I(2), x_3)$; and
(iii) if $n \geq 4$, neither $j(n)$ nor $l(n)$ exists, and we have only $(I(2), x_3, \dots, x_n)$ and $(J(3), x_4, \dots, x_n)$.

Proposition 3.22. *The saturated segment ideal $L(p(z)) \subset S$ w.r.t. deglex order with Hilbert polynomial $p(z)$ is a gen-segment ideal w.r.t. the degrevlex order if and only if $\deg(p(Z)) \leq 1$ or there are only two generators of degree > 1.*

Proof. In Section 1, we recalled that given an admissible polynomial $p(z)$ of degree ℓ, there exist unique integers $m_0 \geq m_1 \geq \cdots \geq m_\ell \geq 0$ such that $p(z) = \sum_{i=0}^{\ell} \binom{z+i}{i+1} - \binom{z+i-m_i}{i+1}$ [16,12,1]. Let $a_\ell := m_\ell, a_{\ell-1} := m_{\ell-1} - m_\ell, \dots, a_0 := m_0 - m_1$. Note that $L(p(z)) \subset S$ has the $n+1-\ell-2$ greatest variables as the generators of degree 1, i.e., $\mathcal{N}(L(p(z)))_1 = \{x_0, \dots, x_{\ell+1}\}$. Thus, for every $j \leq a_\ell$, the greatest term of $\mathcal{N}(L(p(z)))_j$ is $x_{\ell+1}^j$ w.r.t. both deglex and degrevlex orders (namely, $\mathcal{N}(L(p(z)))_j = \mathbb{T}_j \cap K[x_0, \dots, x_{\ell+1}]$). In degree $a_\ell + 1$, the ideal $L(p(z))$ has a new generator $x_{\ell+1}^{a_\ell+1}$; so, $\mathcal{N}(L(p(z)))_{a_\ell+1} = (\mathbb{T}_{a_\ell+1} \cap K[x_0, \dots, x_{\ell+1}]) \setminus \{x_{\ell+1}^{a_\ell+1}\}$. Therefore, its greatest term w.r.t. both deglex and degrevlex orders, is $x_\ell x_{\ell+1}^{a_\ell}$ and so on, until there is a new generator in degree $a_\ell + a_{\ell-1} + 1$ if $a_{l-2} \neq 0$, which is $x_\ell^{a_{\ell-1}+1}x_{\ell+1}^{a_\ell}$ (or, if $a_{\ell-2} = 0$, the new generator is $x_\ell^{a_{\ell-1}}x_{\ell+1}^{a_\ell}$). At this point, the greatest term in $\mathcal{N}(L(p(z)))_{a_\ell+a_{\ell-1}+1}$ is $x_\ell^{a_{\ell-1}+2}x_{\ell+1}^{a_\ell-1}$ w.r.t. degrevlex order and $x_{\ell-1}x_\ell^{a_{\ell-1}}x_{\ell+1}^{a_\ell}$ w.r.t. deglex order (similarly for the case in parentheses). Moreover, because the new generator of $L(p(z))$ at degree $a_\ell + a_{\ell-1} + a_{\ell-2} + 1$ is $x_{\ell-1}^{a_{\ell-2}+1}x_\ell^{a_{\ell-1}}x_{\ell+1}^{a_\ell}$ (if $\ell = 2$, the third generator of degree > 1 is $x_1^{a_0}x_2^{a_1}x_3^{a_2}$, and if $a_{\ell-3} = 0$, it is $x_{\ell-1}^{a_{\ell-2}}x_\ell^{a_{\ell-1}}x_{\ell+1}^{a_\ell}$), it is not the greatest term w.r.t. degrevlex order. □

Example 3.23. (i) The ideal $L(p(z)) = (x_4, x_3^5, x_2^3x_3^4, x_1^6x_2^2x_3^4)$ is the saturated segment ideal w.r.t. deglex in $K[x_0, \dots, x_4]$, with Hilbert polynomial $p(z) = 2z^2+2z+1$ and Gotzmann number 12, but it is not a gen-segment ideal w.r.t. degrevlex order.
(ii) The ideal $L(p(z)) = (x_5, x_4^5, x_3^2x_4^4)$ is the saturated segment ideal w.r.t. deglex in $K[x_0, \dots, x_5]$, with Hilbert polynomial $p(z) = 2/3z^3 + 2z^2 - 11/3z + 10$ and Gotzmann number 6, and it is also a gen-segment ideal w.r.t. degrevlex order.

4. Saturations of Borel ideals and Hilbert polynomial

Let $J \subset S$ be a Borel ideal. Recall that in our notation, the (Borel) ideal J^{sat} is obtained by setting $x_0 = 1$ in each minimal generator of J (Proposition 2.3(i)). In this section, we let $J_{x_0} := J^{sat}$ and denote as $J_{x_0x_1}$ the Borel ideal obtained by setting $x_0 = x_1 = 1$ in the minimal generators of J. We call $J_{x_0x_1}$ the x_1-*saturation* of J and say that J is x_1-*saturated* if $J = J_{x_0x_1}$. Hence, an ideal J that is x_1-saturated is also saturated.

Remark 4.1. An ideal $J \subset S$ that is x_1-saturated and has Hilbert polynomial $p(z) := p_{S/J}(z)$ has the same minimal generators as the saturated Borel ideal $J \cap K[x_1, \dots, x_n] \subset K[x_1, \dots, x_n]$ for which the Hilbert polynomial is $\Delta p(z)$.

The following result is analogous to Theorem 3 of [23], where the notion of "fan" is used. Here, we apply only the combinatorial properties of Borel ideals.

Proposition 4.2. *Let $J \subset S$ be a saturated Borel ideal with Hilbert polynomial $p(z)$ and Gotzmann number r. Let $I = J_{x_0x_1}$ be the x_1-saturation of J, and let $q := \dim_K I_r - \dim_K J_r$. Then,*

(i) *$p_{S/I}(z) = p(z) - q$, and*
(ii) *q is equal to the sum of the exponents of x_1 in the minimal generators of J.*

Proof. (i) We show that if $q = \dim_K I_s - \dim_K J_s$ then $q = \dim_K I_{s+1} - \dim_K J_{s+1}$ for every $s \geq r$. Let $x^{\beta_1}, \dots, x^{\beta_q}$ be the terms of $I_s \setminus J_s$. Thus, $x_0x^{\beta_1}, \dots, x_0x^{\beta_q}$ are some terms of $I_{s+1} \setminus J_{s+1}$, and so, $\dim_K I_{s+1} - \dim_K J_{s+1} \geq q$ because $x_0x^{\beta_i}$ belongs to J_{s+1} if and only if x^{β_i} belongs to J_s, as J is saturated. Now, to obtain the opposite inequality, it is enough to show that every term of $I_{s+1} \setminus J_{s+1}$ is divisible by x_0. Let $x^\gamma \in I_{s+1}$ be such that $\min(x^\gamma) \geq 1$, and let x^α be a minimal generator of I such that $x^\gamma = x^\alpha x^\delta$. Because J is saturated and I is the x_1-saturation of J, $x^\alpha x_1^a$ is a minimal generator of J for some non-negative integer a. Hence, for every $x^{\delta'}$ of degree $s+1-|\alpha|$ and with $\min(x^{\delta'}) \geq 1$, by the Borel property, $x^\alpha x^{\delta'}$ belongs to J_{s+1}. In particular, $x^\gamma \in J_{s+1}$.
(ii) Let $x^{\alpha_1}x_1^{s_1}, \dots, x^{\alpha_h}x_1^{s_h}$ be the minimal generators of J with x^{α_i} not divisible by x_1, for every $1 \leq i \leq h$. Because the $\sum s_i$ terms $x^{\alpha_i}x_1^{s_i-t}x_0^{r-|\alpha_i|-s_i+t}$, $1 \leq t \leq s_i$, are in $I_r \setminus J_r$, one has $q \geq \sum s_i$. Vice versa, we show that each term x^δ in $I_r \setminus J_r$ is of the previous type. We can write $x^\delta = x^\beta x_0^{r-|\beta|-u}x_1^u$ with $\min(x^\beta) \geq 2$ and $u < s_i$. Let s be the minimum non-negative integer such that $x^\beta x_1^s$ is in J. Then, there exists an i such that $x^{\alpha_i}x_1^{s_i} | x^\beta x_1^s$, i.e., $x^{\alpha_i} | x^\beta$ and $s_i \leq s$. By the definition of s, we obtain $s_i = s$, and there exists x^γ with $\min(x^\gamma) \geq 2$ such that $x^\beta = x^{\alpha_i}x^\gamma$. Because x^β does not belong to J, we have $|\gamma| < s_i = s$, or otherwise, $x^{\alpha_i}x_1^{|\gamma|}$, and hence, by the Borel property, $x^\beta = x^{\alpha_i}x^\gamma$ should belong to J. Now we can take $x^\beta x_1^{s-|\gamma|}$ and observe that this term belongs to J because it follows $x^{\alpha_i}x_1^s$ in the Borel relation. Thus, $s \leq s - |\gamma|$; so, $\gamma = 0$, i.e. $x^\beta = x^{\alpha_i}$ as claimed. □

Proposition 4.3. *Let $J \subset S$ be a saturated Borel ideal with Hilbert polynomial $p(z)$ and Gotzmann number r. Let $x^\beta x_0$ be a term of J of degree $s \geq r$ that is minimal in J w.r.t. $<_B$. Then, the ideal $H := (G((J_s)) \setminus \{x^\beta x_0\})$ is Borel, and $p_{S/H}(z) = p(z) + 1$.*

Proof. First, note that H_s is closed w.r.t. $<_B$ by Remark 2.6. We show that, for every $t \geq 0$, $x^\beta x_0^{1+t}$ is the unique term in $J_{s+t} \setminus H_{s+t}$. For $t = 0$, we have the hypothesis. For $t > 0$, note that $x^\beta x_0^{1+t}$ cannot belong to H. On the contrary, there would be a term $x^\gamma \in H_s$ such that $x^\gamma \neq x^\beta x_0$ and $x^\gamma \mid x^\beta x_0^{1+t}$. But every degree s factor of $x^\beta x_0^{1+t}$ different from $x^\beta x_0$ is lower w.r.t. $<_B$, and so, it cannot belong to H_s. Then, $x^\beta x_0^{1+t} \notin H_{s+t}$. If x^α is a term of $J_{s+t} \setminus H_{s+t}$, there exists a term of $J_{s+t-1} \setminus H_{s+t-1}$ that divides x^α. By induction, this term is $x^\beta x_0^t$, and the thesis follows from the fact that every multiple of degree $s+t$ of $x^\beta x_0^t$ different from $x^\beta x_0^{1+t}$ belongs to H_{s+t}. □

Proposition 4.4. *Let I and J be Borel ideals of S. If, for every $s \gg 0$, we have $I_s \subset J_s$ and $p_{S/I}(z) = p_{S/J}(z) + a$, with $a \in \mathbb{N}$, then I and J have the same x_1-saturation.*

Proof. Let $s \geq \max\{reg(I), reg(J)\}$. In the $a = 1$ case, there exists a unique term in $J_{s+t} \setminus I_{s+t}$ for every $t \geq 0$. Let x^α be the unique term in $J_s \setminus I_s$. Then, both $x^\alpha x_0$ and $x^\alpha x_1$ belong to J_{s+1}. By the Borel property, $x^\alpha x_1$ must be in I_{s+1}, and so, the unique term in $J_{s+t} \setminus I_{s+t}$ is $x^\alpha {x_0}^t$. This is enough to say that I and J have the same x_1-saturation. If $a > 1$, the thesis follows by induction and by applying Proposition 4.3. □

Corollary 4.5. *Let $p(z)$ be an admissible polynomial of degree $h \leq n$ and $P := \{q(z) = p(z) + u \mid u \in \mathbb{Z} \text{ and } q(z) \text{admissible}\}$ be the set of all admissible polynomials of degree h differing from $p(z)$ only for an integer. Then,*

(i) *there is a polynomial $\hat{p}(z)$ in P such that for every $q(z)$ in P, $q(z) = \hat{p}(z) + c$ with $c \geq 0$, and*
(ii) *every saturated Borel ideal I with Hilbert polynomial $p_{S/I} = \hat{p}(z)$ is x_1-saturated.*

Proof. (i) Every admissible polynomial $p(z)$ has a unique saturated lex segment ideal $L(p(z))$. If H is the saturated lex segment ideal of $p(z)+u$, then we have $H \subset L(p(z))$ if $u > 0$ and $L(p(z)) \subset H$ if $u < 0$. Thus, we can apply Proposition 4.4, finding that $L(p(z))$ and H have the same x_1-saturation, I. We claim that $\hat{p}(z)$ is the Hilbert polynomial of I. Indeed, by Proposition 4.2, the Hilbert polynomial of I is of type $p(z) - q$. If $\hat{p}(z) = p(z) - q - t$ with $t \geq 0$, then the saturated lex segment ideal of $\hat{p}(z)$ should have I as the x_1-saturation and should be contained in I, which is possible only if $t = 0$.
(ii) Let J be a Borel ideal with $\hat{p}(z)$ as the Hilbert polynomial. If J were not x_1-saturated, by Proposition 4.2, the x_1-saturation of J should have a Hilbert polynomial of type $\hat{p}(z) - q$, with $q > 0$, against the definition of $\hat{p}(z)$.

Definition 4.6. The polynomial $\hat{p}(z)$ of Corollary 4.5 is called the *minimal polynomial.*

Remark 4.7. An alternative proof of the previous statement can be obtained by following the construction of the Gotzmann number.

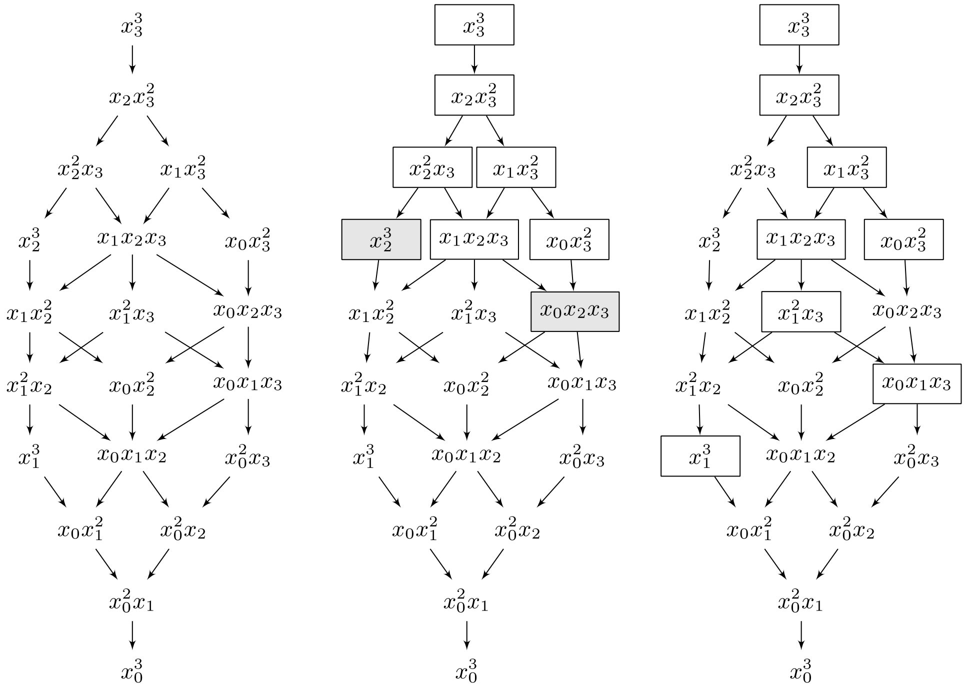

Fig. 1. To the left is the graph of $K[x_0, \ldots, x_3]_3$, the center is the graph of $(x_3^2, x_2x_3, x_2^3)_3$, in which we colored the minimal elements, and the right is the graph of $(x_3^2, x_1x_3, x_1^3)_3$, which is not Borel (the terms in the ideal are the boxed ones).

Example 4.8. By Proposition 4.2, a Borel ideal with a minimal Hilbert polynomial is x_1-saturated. The opposite is not true. For example, the ideal $I = (x_3^2, x_2x_3, x_2^2) \subset K[x_0, x_1, x_2, x_3]$ is x_1-saturated and is a reg-segment ideal w.r.t. the degrevlex order. The corresponding Hilbert polynomial is $p_{S/I}(z) = 3z + 1$, which is not minimal because the Borel ideal (x_3, x_2^3) has Hilbert polynomial $3z$.

Remark 4.9. From the proof of Corollary 4.5, we deduce the following fact. Let $I \subset K[x_0, \ldots, x_n]$ be a Borel ideal with Hilbert polynomial $p(z)$. If $I = I_{x_1} \cdot K[x_0, \ldots, x_n]$, where $I_{x_1} \subset K[x_1, \ldots, x_n]$ is the hilb-segment ideal w.r.t. deglex order with Hilbert polynomial $\Delta p(z)$, then $p(z) = \hat{p}(z)$.

5. An algorithm to compute saturated Borel ideals

In this section, by exploiting the results of Section 4, we describe an algorithm for computing all the saturated Borel ideals with a given Hilbert polynomial $p(z)$. We first give an efficient strategy to find the minimal elements in a Borel set B, which consists of representing B by a connected planar graph in which the nodes are the terms of B and the edges are the elementary moves connecting the terms. In Fig. 1, we give some examples showing that it is easy to single out the minimal terms by looking at these graphs.

Let $0 \leq k < n$ be an integer. Recall that, if $I \subset K[x_k, \ldots, x_n]$ is a saturated Borel ideal that has a non-null Hilbert polynomial $p(z)$ with Gotzmann number r, then $J := \frac{(I, x_k)}{(x_k)} \subset K[x_{k+1}, \ldots, x_n]$ has Hilbert polynomial $\Delta p(z)$ with x_k as a non-zero-divisor on $\frac{K[x_k, \ldots, x_n]}{I}$.

This fact shows that every saturated Borel ideal $I \subset K[x_k, \ldots, x_n]$ with Hilbert polynomial $p(z)$ "comes from" a Borel ideal $J \subset K[x_{k+1}, \ldots, x_n]$ with Hilbert polynomial $\Delta p(z)$ and generated in degrees $\leq r$. So, our idea to construct all saturated Borel ideals with a given Hilbert polynomial $p(z)$ consists of applying a recursion on the number of variables. By the hypothesis we know all of the Borel ideals J in the $n - k$ variables generated in degrees $\leq r$ with Hilbert polynomial $\Delta p(z)$. Then, we construct the saturated Borel ideals I in $n - k + 1$ variables such that $J := \frac{(I, x_k)}{(x_k)}$ for some of the ideals J.

Let $J \subset K[x_{k+1}, \ldots, x_n]$ be a Borel ideal with Hilbert polynomial $\Delta p(z)$ and $\bar{I} := (J^{sat} \cdot K[x_k, \ldots, x_n])_r$, where r is the Gotzmann number of $p(z)$. Let $\overline{N}$ be the set of terms x^α of $K[x_k, \ldots, x_n]_r$ such that there exists a composition F of elementary moves of type e_j^- and a term τ of $\mathcal{N}(J)_r$ such that $F(\tau) = x^\alpha$. Hence, by construction, the terms of $\overline{N} \setminus \mathcal{N}(J)$ are not maximal, and $\overline{N}$ is contained in the *sous-escalier* of any ideal of $K[x_k, \ldots, x_n]$ having J as the hyperplane section. Note that the Gotzmann number of $\Delta^{k+1}p(z)$ is not higher than the Gotzmann number of $\Delta^k p(z)$.

Lemma 5.1. $\mathcal{N}(\bar{I})_r = \overline{N}$.

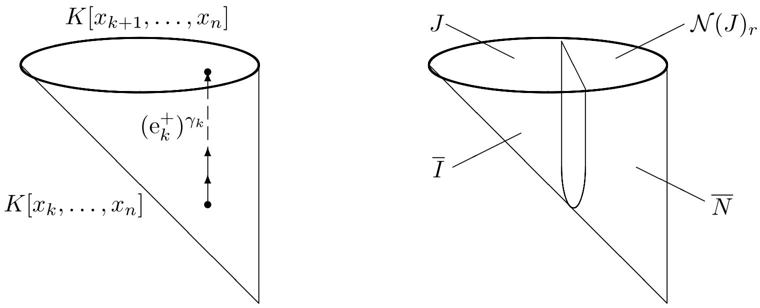

Fig. 2. Partition of $K[x_k,\ldots,x_n]$.

Proof. It is enough to show that $K[x_k,\ldots,x_n]_r = (\bar{I},\overline{N})$ (Fig. 2). Indeed, let $x^\gamma = x_k^{\gamma_k}\cdots x_n^{\gamma_n}$ be in $K[x_k,\ldots,x_n]_r$. The term $x^\beta = (e_k^+)^{\gamma_k}(x^\gamma)$ belongs to $K[x_{k+1},\ldots,x_n]_r$ and is hence in J_r or in $\mathcal{N}(J)_r$. If x^β is in J_r, then $x_{k+2}^{\gamma_{k+2}}\cdots x_n^{\gamma_n}$ belongs to J_r^{sat} and thus to $\bar{I}$; otherwise, $x^\gamma = (e_{k+1}^-)^{\gamma_k}(x^\beta)$ is in $\overline{N}$. □

Proposition 5.2. *With the above notation, the Hilbert polynomial $\bar{p}(z)$ for $\bar{I}$ differs from $p(z)$ only by a constant. If $\bar{q} = p(r) - \bar{p}(r) > 0$, execute the following instruction $\bar{q}$ times: select a minimal term τ in $\bar{I}_r$, and set $\bar{I} := (G((\bar{I}_r)) \setminus \{\tau\})$. After these $\bar{q}$ steps, the new ideal obtained has Hilbert polynomial $p(z)$.*

Proof. The theses follow from the results presented in Section 4. □

Proposition 5.2 suggests the design of the following two routines BorelGenerator and Remove, which have been implemented by the second author in a software with an applet available at http://www.dm.unito.it/dottorato/dottorandi/lella/HSC/borelGenerator.html.

procedure BorelGenerator(n,$p(z)$,r,k) $\rightarrow \mathcal{F}$
  if $p(z) = 0$ **then**
    return $\{(1)\}$;
  else
    $\mathcal{E} \leftarrow$ BorelGenerator(n,$\Delta p(z)$,r,$k+1$);
    $\mathcal{F} \leftarrow \emptyset$;
    for all $J \in \mathcal{E}$ **do**
      $\bar{I} \leftarrow J \cdot k[x_k,\ldots,x_n]$;
      $q \leftarrow p(r) - \dim_k k[x_k,\ldots,x_n]_r + \dim_k \bar{I}_r$;
      if $q \geqslant 0$ **then**
        $\mathcal{F} \leftarrow \mathcal{F} \cup$ Remove(n,k,r,$\bar{I}$,q);
      end if
    end for
    return $\mathcal{F}$;
  end if
end procedure
procedure Remove(n,k,r,$\bar{I}$,q) $\rightarrow \mathcal{E}$
  $\mathcal{E} \leftarrow \emptyset$;
  if $q = 0$ **then**
    return $\mathcal{E} \cup \bar{I}^{sat}$;
  else
    $\mathcal{F} \leftarrow$ MinimalElements($\bar{I}$,r)
    for all $x^\alpha \in \mathcal{F}$ **do**
      $\mathcal{E} \leftarrow \mathcal{E} \cup$ Remove(n,k,r,$(G((\bar{I}_r)) \setminus x^\alpha)$,$q-1$);
    end for
    return $\mathcal{E}$;
  end if
end procedure

Remark 5.3. The terms removed by our strategy are minimal in $\bar{I}$. An alternative strategy consists of adding to $J_r K[x_k,\ldots,x_n]$ the maximal terms of $\bar{I}_r \setminus J$. In this case, because we want to have $\dim_K I_r = \binom{n-k+r}{r} - p(r)$ and we already have $\binom{n-(k+1)+r}{r} - \Delta p(r)$ terms of J, we should add

$$q' = \binom{n-k+r}{r} - p(r) - \binom{n-(k+1)+r}{r} + \Delta p(r) = \binom{n-k-1+r}{r-1} - p(r-1)$$

terms for any J, where q' depends only on r, $n-k$ and $p(z)$; hence, we will write $q'(r, n-k, p(z))$ instead of q'. However, the value of $\overline{q} = p(r) - |\overline{N}_r| = p(0) - \overline{p}(0)$ depends on J. Note that $q' + \overline{q} = \dim_K \overline{I}_r - \dim_K J_r$. Observe that if $n-k > \deg(p(z))+1$, then $q' \geq \overline{q}$. The minimal polynomial $\hat{p}(z)$ of Definition 4.6 can be recovered from $\Delta p(z)$ by the Gotzmann decomposition in the following way. If

$$\Delta p(z) = \binom{z+b_1}{b_1} + \binom{z+b_2-1}{b_2} + \cdots + \binom{z+b_t-(t-1)}{b_t}$$

with $b_1 \geq b_2 \geq \cdots \geq b_t \geq 0$, then

$$\hat{p}(z) = \binom{z+a_1}{a_1} + \binom{z+a_2-1}{a_2} + \cdots + \binom{z+a_t-(t-1)}{a_t},$$

where $a_i = b_i + 1$. The Gotzmann number of $\Delta p(z)$ is also the Gotzmann number $\hat{r}$ of $\hat{p}(z)$. If r is the Gotzmann number of $p(z)$, then $r - \hat{r} = p(0) - \hat{p}(0) \geq p(0) - \overline{p}(0) = \overline{q}$. We prove that $q' \geq \overline{q}$ by an induction on $c = r - \hat{r}$. If $c = 0$, then we obtain $\overline{q} = 0$. If $c > 0$, by induction, we have that $q'(r-1, n-k, p(z)-1) \geq \overline{q} - 1$; hence,

$$\begin{aligned} q'(r, n-k, p(z)) &= \binom{r-1+n-k}{n-k} - p(r-1) \\ &= \binom{r-2+n-k}{n-k} + \binom{r-1+n-k-1}{n-k-1} - p(r-1) + p(r-2) - p(r-2) \\ &= q'(r-1, n-k, p(z)-1) + \binom{r-1+n-k-1}{n-k-1} - \Delta p(r-1) - 1 \\ &\geq \overline{q} + \binom{r-1+n-k-1}{n-k-1} - \Delta p(r-1) - 2, \end{aligned}$$

and $\binom{r-1+n-k-1}{n-k-1} - \Delta p(r-1) \geq 2$ because $r-1$ is an upper bound of the Gotzmann number of $\Delta p(z)$ and J is not a hypersurface because $n-k-1 > deg(\Delta p(z)) + 1$.

Example 5.4. (a) If $p(z) = d$, then $r = d$ and $\hat{r} = 0$, so $\overline{q} = d$ and $q' = \binom{d-1+n}{n} - d$. Moreover, if $n = \deg(p(z)) + 1$, then $q' = 0$.

(b) The Gotzmann number of $p(z) = 3z + 1$ is $r = 4$, and if $n = 3$ and $k = 0$, then $q'(r, n, p(z)) = \binom{r-1+n}{n} - p(r-1) = 20 - 10 = 10$ and $r - \hat{r} = 1$. If $J_4 = (x_3, x_2^3)_4$, we obtain $|\overline{N}_r| = 12$ and $\overline{q} = 1$; meanwhile, if $J_4 = (x_3^2, x_2x_3, x_2^2)$, we obtain $|\overline{N}_r| = 13$ and $\overline{q} = 0$.

6. Degreverse points

In this section, by exploiting results of [15], we study the points corresponding to the hilb-segment ideals in the Hilbert scheme $\mathcal{H}\mathrm{ilb}_d^n$ of subschemes of $\mathbb{P}^n$ with Hilbert polynomial $p(z) = d$, where d is a fixed positive integer. Recall that for $p(z) = d$, the Gotzmann number is d itself.

From now, $J \subset S$ is a hilb-segment ideal with respect to some term order $\preceq$ and with Hilbert polynomial $p(z) = d$, and let $\mathcal{B} := \{x^\beta \in \mathcal{N}(J)_d \ : \ x_1 x^\beta \in J\}$. Recall that $G(J)$ denotes the set of minimal generators of J, and $ed(St_h(J, \preceq))$ is the embedding dimension of the Gröbner stratum $St_h(J, \preceq)$.

Lemma 6.1. *With the above notation, we obtain that* $ed(St_h(J, \preceq)) \geq |G(J)| \cdot |\mathcal{B}|$.

Proof. With the same notation introduced in Section 2, by Theorem 4.7(i) of [15], it is enough to look at the variables $c_{\alpha\beta}$ appearing in the polynomials F_α such that $x^\alpha = x^\gamma x_0^{d-|\gamma|}$, where x^γ belongs to $G(J)$. More precisely, we need to count the number of such variables that do not correspond to a pivot in a Gauss reduction of the generators of $L(J)$ (see also Proposition 4.3 and Definition 4.4 of [15]).

First, we note that in every S-polynomial involving such an F_α, the polynomial F_α itself is multiplied by a term in which at least a variable x_h appears with $h > 0$ (otherwise the other polynomial involved in the S-polynomial should have x_0^d as its initial term). It is enough to investigate the terms $x^\beta x_1$, where x^β belongs to $\mathcal{B}$ because if $x^\beta x_1$ belongs to J_{d+1}, then $x^\beta x_h$ belongs to J_{d+1} for any $h > 0$. Because J is a hilb-segment ideal, every term x^β of $\mathcal{B}$ is less than x^α. By the definition of $\mathcal{B}$, every term x^β of $\mathcal{B}$ is always involved in a reduction step so it does not appear in any generator of $L(J)$ (see Criterion 4.6 of [15]). The number of such terms is at least $|G(J)| \cdot |\mathcal{B}|$, and we are done. $\square$

Theorem 6.2. *If, for the hilb-segment ideal J, we have $|G(J)| \cdot |\mathcal{B}| > nd$, then J corresponds to a singular point in $\mathcal{H}\mathrm{ilb}^n_d$.*

Proof. Let H_{RS} be the unique irreducible component of $\mathcal{H}\mathrm{ilb}^n_d$ containing the lexicographic point [24]. Recall that H_{RS} has a dimension equal to nd and that every Borel ideal belongs to H_{RS} [23]. Because J is a hilb-segment ideal w.r.t. $\preceq$, the Gröbner stratum $St_h(J, \preceq)$ is an open subset of H_{RS} (Corollary 6.9 of [15]), and hence, dim $St_h(J, \preceq) = nd$. Thus, the point J is smooth for $\mathcal{H}\mathrm{ilb}^n_d$ if and only if it is smooth for the Gröbner stratum $St_h(J, \preceq)$ (see Corollary 4.5 of [15]). In particular, J is smooth if and only if $ed(St_h(J, \preceq)) = nd$. By Lemma 6.1, the thesis is proved. □

In the following, it is important to keep in mind that if $\preceq$ is a degreverse term order and α is the initial degree of a degreverse segment ideal J for $p(z) = d$, then $\mathcal{S}t_h(J) = \mathcal{S}t_h(J_{\geq\alpha}) \cong \mathcal{S}t_h(J_{\geq\alpha+k})$ for every $k > 0$, by Proposition 3.9(i) and Remark 3.14.

Example 6.3. The generic initial ideal of 7 general points in $\mathbb{P}^3$ w.r.t. a degreverse term order, i.e., the (saturated) hilb-segment ideal with Hilbert polynomial $p(z) = 7$, can be one of the two following saturated Borel ideals with a maximal Hilbert function: the hilb-segment ideal I w.r.t. degrevlex or $I' = (x_3^2, x_3x_2, x_3x_1, x_2^3, x_2^2x_1, x_2x_1^2, x_1^3)$.

We obtain $\mathcal{N}(I)_7 = \{x_0^7, x_0^6x_1, x_0^6x_2, x_0^6x_3, x_0^5x_1^2, x_0^5x_1x_2, x_0^5x_1x_3\}$, and $\mathcal{B} = \{x_0^5x_1^2, x_0^5x_1x_2, x_0^5x_1x_3\}$. Thus, $|G(I)| \cdot |\mathcal{B}| = 6 \cdot 3 = 18 < nd = 21$. But, as it is shown in [15], we can compute directly the Gröbner stratum of $I_{\geq 7}$ showing that its embedding dimension is $27 > nd = 21$. Actually, in [15], the authors construct the stratum of $I_{\geq 3}$, which is isomorphic to the stratum of $I_{\geq 7}$, obtaining a significant improvement in the computation.

For the ideal I', we obtain $\mathcal{N}(I')_7 = \{x_0^7, x_0^6x_1, x_0^6x_2, x_0^6x_3, x_0^5x_1^2, x_0^5x_1x_2, x_0^5x_2^2\}$ so $\mathcal{B}' = \{x_0^5x_1^2, x_0^5x_1x_2, x_0^5x_2^2\}$ with $|\mathcal{B}'| = 3$ and $|G(I')| = 7$. Hence, we obtain $|G(I')| \cdot |\mathcal{B}'| = 7 \cdot 3 = 21 = nd = 21$. But, also in this case, we can compute directly the Gröbner stratum and its embedding dimension, which is $29 > 21$.

For 8 points in $\mathbb{P}^3$, the (saturated) hilb-segment ideal I w.r.t. degrevlex with the Hilbert polynomial $p(z) = 8$ is the unique Borel ideal with a maximal Hilbert function. We obtain $\mathcal{N}(I)_8 = \{x_0^8, x_0^7x_1, x_0^7x_2, x_0^7x_3, x_0^6x_1^2, x_0^6x_1x_2, x_0^6x_1x_3, x_0^6x_2^2\}$ and $\mathcal{B} = \{x_0^6x_1^2, x_0^6x_1x_2, x_0^6x_1x_3, x_0^6x_2^2\}$ with $|\mathcal{B}| = 4$. Because $|G(I)| = 7$, we get $|G(I)| \cdot |\mathcal{B}| = 7 \cdot 4 = 28 > 3 \cdot 8 = 24$.

Theorem 6.4. *For every $d > n \geq 3$, the hilb-segment ideal J w.r.t. a degreverse order corresponds to a singular point in $\mathcal{H}\mathrm{ilb}^n_d$.*

Proof. In Remark 3.14, we observed that J must have a maximal Hilbert function; so, the regularity ρ_H of its Hilbert function is the integer such that $\binom{\rho_H-1+n}{n} < d \leq \binom{\rho_H+n}{n}$. Moreover, if $d = \binom{\rho_H+n}{n}$, then $|G(J)| = \binom{\rho_h+n}{n-1}$; otherwise, $|G(J)| \geq \binom{\rho_H+n-1}{n-1}$.

If $d = n+1$, then $\rho_H = 1$ and $J = (x_1, \ldots, x_n)^2$; so, $|G(J)| = \binom{2+n-1}{n-1} = \binom{n+1}{2}$. Moreover, $\mathcal{B}$ consists of the terms of type $x_0^{d-1}x_i$ with $i > 0$; thus, $|\mathcal{B}| = n$, and the statement is true because $\binom{n+1}{2} \cdot n > n(n+1)$ for every $n \geq 3$.

If $d \geq n+2$, then $\rho_H \geq 2$.

If $d = \binom{\rho_H+n}{n}$, we show that $|\mathcal{B}| > \rho_H + 1$. If we multiply every term of degree ρ_H in the variables $x_1, \ldots, x_n$ by $x_0^{d-\rho_H}$, we obtain terms of degree d that multiplied by x_1 give $\binom{\rho_H+n-1}{n-1}$ terms that belong to $\mathcal{B}$. Thus, $|\mathcal{B}| \geq \binom{\rho_H+n-1}{n-1} > \rho_H + 1$, and $|G(J)| \cdot |\mathcal{B}| > \frac{\mathrm{d}n}{\rho_H+1} \cdot (\rho_H + 1) = \mathrm{d}n$.

If $d < \binom{\rho_H+n}{n}$ and $\rho_H \geq 3$, we show that $|\mathcal{B}| \geq \rho_H + n$. Let x^β be any of the $\binom{\rho_H+n-2}{n-1}$ terms of degree $\rho_H - 1$ in the variables $x_1, \ldots, x_n$. Thus, if $x^\beta x_1$ belongs to J, then $x^\beta x_0^{d-\rho_H+1}$ belongs to $\mathcal{B}$; otherwise, if $x^\beta x_1$ does not belong to J, then $x^\beta x_0^{d-\rho_H} x_1$ belongs to $\mathcal{B}$. Either way, the term $x^\beta x_1^2$ belongs to J because it is not divisible by x_0 and has degree $\rho_H + 1$, and the terms of $\mathcal{N}(J)_{\rho_H+1}$ are all divisible by x_0. Such terms are all distinct; so, $|\mathcal{B}| \geq \binom{\rho_H+n-2}{n-1}$. Now it is easy to check that $\binom{\rho_H+n-2}{n-1} \geq \rho_H + n$ for every $\rho_H \geq 3$ and $n \geq 3$. Thus, $|G(J)| \cdot |\mathcal{B}| \geq \binom{\rho_H+n-1}{n-1} \cdot (\rho_H + n) > nd$, by Remark 3.14(3).

It remains to study the case $\rho_H = 2$ in which $|G(J)| \geq \binom{n+1}{2}$ and $|\mathcal{B}| \geq n$ because of the above results with $\binom{n+1}{n} < d < \binom{2+n}{n}$. If $d < \binom{n+1}{2}$, then we immediately obtain $|G(J)| \cdot |\mathcal{B}| > nd$. If $\binom{n+1}{2} < d < \binom{n+2}{2}$, all the d terms of $\mathcal{N}(J)_d$ are in $\mathcal{B}$ except for most the $n+1$ terms divisible by x_0^{d-1}. Thus, in this case, $|\mathcal{B}| \geq d - (n+1)$, which is $\geq n+2$ except for $n = 3$ and $d = 7, 8$. These last two cases have been directly studied in Example 6.3. □

Example 6.5. We can list all the saturated Borel ideals in $K[x_0, \ldots, x_3]$ with constant Hilbert polynomial d for $d = 4, 5, 6, 7$, e.g., by the implementation BorelGenerator of the algorithm described in Section 5. Then, we see if these Borel ideals correspond to smooth or singular points in $\mathcal{H}\mathrm{ilb}^3_d$ by a direct computation, as in Example 6.3.

For $d = 4$, the segment ideal J w.r.t. degrevlex is the unique Borel ideal that corresponds to a singular point of $\mathcal{H}\mathrm{ilb}^3_4$. More precisely, the singular locus of $\mathcal{H}\mathrm{ilb}^3_4$ is determined by the 3-dimensional orbit of J under the action of the projective linear group on $\mathbb{P}^3$.

Also, for $d = 5$, the segment ideal J w.r.t. degrevlex is the unique Borel ideal that corresponds to a singular point of $\mathcal{H}\mathrm{ilb}_5^3$. In this case, the singular locus of $\mathcal{H}\mathrm{ilb}_5^3$ is 6-dimensional and contains the orbit of the ideals $J(\lambda) = (x_3^2, x_3x_2, x_2^2, x_3x_1, x_2x_1, x_1^2(x_1 + \lambda x_0))$ (a linear family depending on the parameter $\lambda \in K$) under the action of the projective linear group on $\mathbb{P}^3$. For $\lambda = 0$, we get the ideal J itself, while for $\lambda \neq 0$, the subscheme defined by $J(\lambda) = (x_3^2, x_3x_2, x_2^2, x_3x_1, x_2x_1, x_1^2) \cap (x_3, x_2, x_1 + \lambda x_0)$ is the union of the degree 4 non-reduced scheme given by the segment ideal w.r.t. the degrevlex of degree 4 and one more point.

For $d = 6$, there is one more Borel ideal $J' = (x_3^2, x_3x_2, x_2^2, x_3x_1, x_2x_1, x_1^4)$ that corresponds to a singular point of the Hilbert scheme in addition to the segment ideal J w.r.t. degrevlex. Both of them are naturally related to the segment ideal w.r.t. the degrevlex of degree 4; indeed J and J' are the initial ideals w.r.t. degrevlex of the ideal defining the union of the degree 4 non-reduced scheme on the point O (given by the degree 4 segment ideal w.r.t. degrevlex) and of two more points P, Q. We obtain J if O, P, Q span a plane, and we obtain J' if O, P, Q span a line.

Finally, for $d = 7$, there are four Borel ideals corresponding to singular points on $\mathcal{H}\mathrm{ilb}_7^3$. Two of them are segments ideals w.r.t. degreverse term orders (see Example 6.3), and two of them are not.

For $d = 4, 5, 6, 7$, $\mathcal{H}\mathrm{ilb}_d^3$ only has one component (see [4]), and we can see this for $d = 4, 5$ directly by our computations.

By observing the following,

(i) a segment ideal w.r.t. degrevlex order gives rise to a singular point in $\mathcal{H}\mathrm{ilb}_d^n$ and defines a scheme not contained in any hyperplane and
(ii) a segment ideal w.r.t. deglex order gives rise to a smooth point in $\mathcal{H}\mathrm{ilb}_d^n$ and defines a scheme contained in some hyperplane,

one might guess that there is a relationship between the smoothness of a point in $\mathcal{H}\mathrm{ilb}_d^n$ corresponding to a (saturated) monomial ideal and the presence of linear forms in the ideal. But, the next example (for which we are indebted to G. Floystad) shows that this is not the case.

Example 6.6. (i) Let $I = (x_1^{a_1}, \dots, x_i^{a_i}, \dots, x_n^{a_n})$ be a (saturated monomial) complete intersection ideal defining a 0-dimensional scheme $\mathbb{X}$ of degree $d = \prod_i a_i$ in $\mathbb{P}^n$, and let z_I denote the corresponding point of $\mathcal{H}\mathrm{ilb}_d^n$. As I is a monomial ideal, z_I lies in the closure of the lexicographic point component of $\mathcal{H}\mathrm{ilb}_d^n$ (see, for example, Corollary 18.30 of [20]). Using the normal sheaf to $\mathbb{X}$, we get that the dimension of the tangent space to $\mathcal{H}\mathrm{ilb}_d^n$ at z_I is nd, coinciding with that of the lexicographic point component. Thus, I gives an example of a monomial ideal that does not contain linear forms and that corresponds to a smooth point in $\mathcal{H}\mathrm{ilb}_d^n$.

(ii) Let $J \subset K[x_1, \dots, x_n]$ be a saturated monomial ideal giving a singular point z_J of $\mathcal{H}\mathrm{ilb}_d^{n-1}$; so, the dimension of the tangent space to $\mathcal{H}\mathrm{ilb}_d^{n-1}$ in z_J is $\alpha > (n-1)d$. Taking $\widetilde{J} = ((x_0) + J) \subset K[x_0, \dots, x_n]$, the dimension of the tangent space to $\mathcal{H}\mathrm{ilb}_d^n$ in $z_{\widetilde{J}}$ is $\alpha + d > (n-1)d + d = nd$, and hence, $z_{\widetilde{J}}$ is singular too.

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References

[1] D. Bayer, The division algorithm and the Hilbert schemes, Ph.D. Thesis, Harvard University, 1982.
[2] D. Bayer, M. Stillman, A criterion for detecting m-regularity, Invent. Math. 87 (1987) 1–11.
[3] G. Carrà Ferro, Gröbner bases and Hilbert schemes. I, J. Symbolic Comput. 6 (1988) 219–230. Computational aspects of commutative algebra.
[4] D. Cartwright, D. Erman, M. Velasco, B. Viray, Hilbert schemes of 8 points, Algebra Number Theory 3 (2009) 763–795.
[5] F. Cioffi, M. Roggero, Flat families by strongly stable ideals and a generalization of Gröbner bases, 2011, Available at http://arxiv.org/abs/1101.2866.
[6] A. Conca, J. Sidman, Generic initial ideals of points and curves, J. Symbolic Comput. 40 (2005) 1023–1038.
[7] D. Cox, J. Little, D. O'Shea, Ideals, varieties, and algorithms, in: Undergraduate Texts in Mathematics, third ed., Springer, New York, 2007, An introduction to computational algebraic geometry and commutative algebra.
[8] T. Deery, Rev-lex segment ideals and minimal Betti numbers, in: The Curves Seminar at Queen's, vol. X, Kingston, ON, (1995), volume 102 of Queen's Papers in Pure and Appl. Math., Queen's Univ., Kingston, ON, 1996, pp. 193–219.
[9] G. Ferrarese, M. Roggero, Homogeneous varieties for Hilbert schemes, Int. J. Algebra 3 (2009) 547–557.
[10] M.L. Green, Generic initial ideals, in: Six lectures on commutative algebra (Bellaterra), 1996, in: Progr. Math., vol 166, Birkhäuser, Basel, 1998, pp. 119–186.
[11] N. Gunther, Sur les modules des formes algébriques, Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] 9 (1941) 97–206.
[12] R. Hartshorne, Connectedness of the Hilbert scheme, Inst. Hautes Études Sci. Publ. Math. (1966) 5–48.
[13] J. Herzog, H. Srinivasan, Bounds for multiplicities, Trans. Amer. Math. Soc. 350 (1998) 2879–2902.
[14] M. Kreuzer, L. Robbiano, Computational Commutative Algebra. 2, Springer-Verlag, Berlin, 2005.
[15] P. Lella, M. Roggero, Rational components of Hilbert schemes, Rend. Semin. Mat. Univ. Padova (2009). Available at http://arxiv.org/abs/0903.1029 (in press).
[16] F.S. Macaulay, Some properties of enumeration in the theory of modular systems, Proc. London Math. Soc. (1926) 531–555.
[17] M.G. Marinari, On Borel ideals, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. 8 (4) (2001) 207–237.
[18] M.G. Marinari, L. Ramella, Borel ideals in three variables, Beiträge Algebra Geom. 47 (2006) 195–209.

[19] M.G. Marinari, L. Ramella, Some properties of Borel ideals, J. Pure Appl. Algebra 139 (1999) 183–200. Effective methods in algebraic geometry (Saint–Malo), 1998.
[20] E. Miller, B. Sturmfels, Combinatorial commutative algebra, in: Graduate Texts in Mathematics, vol. 227, Springer-Verlag, New York, 2005.
[21] T. Mora, Solving polynomial equation systems. II, in: Encyclopedia of Mathematics and its Applications, vol. 99, Cambridge University Press, Cambridge, 2005, Macaulay's paradigm and Gröbner technology.
[22] R. Notari, M.L. Spreafico, A stratification of Hilbert schemes by initial ideals and applications, Manuscripta Math. 101 (2000) 429–448.
[23] A.A. Reeves, The radius of the Hilbert scheme, J. Algebraic Geom. 4 (1995) 639–657.
[24] A. Reeves, M. Stillman, Smoothness of the lexicographic point, J. Algebraic Geom. 6 (1997) 235–246.
[25] M. Roggero, Borel open covering of Hilbert schemes, 2009, Available at arxiv:0909.2184]http://arxiv.org/abs/0909.2184, Preprint.
[26] M. Roggero, L. Terracini, Ideals with an assigned initial ideal, Int. Math. Forum 5 (2010) 2731–2750.