Segments and Hilbert schemes of points

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\textbf{A B S T R A C T}

Using results obtained from a study of homogeneous ideals sharing the same initial ideal with respect to some term order, we prove the singularity of the point corresponding to a segment ideal with respect to a degrevlex term order (as, for example, the degrevlex order) in the Hilbert scheme of points in $\mathbb{P}^n$. In this context, we look into the properties of several types of “segment” ideals that we define and compare. This study also leads us to focus on the connections between the shape of generators of Borel ideals and the related Hilbert polynomial, thus providing an algorithm for computing all saturated Borel ideals with a given Hilbert polynomial.

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0. Introduction

The Hilbert scheme can be covered by some particular affine schemes [3,9,22,26,15] that have been called \textit{Gröbner strata} in [15] because they are computed from a monomial ideal by Gröbner basis techniques. The behavior of Gröbner strata can provide interesting information on the Hilbert scheme itself. Very recently, in [25,5], Roggero et al. showed that an open covering of the Hilbert scheme can be constructed from Borel ideals by avoiding introduction of any term order, which is instead needed for Gröbner strata. This fact gives us further reasons to investigate Borel ideals and their particular features.

Among Borel ideals, there are several types of “segment” ideals whose definitions are already well known or arise from some interesting properties of Gröbner strata studied in [15] (Definitions 3.1 and 3.7). In Section 3 we characterize, for some cases, the existence of these kinds of ideals in terms of the corresponding Hilbert polynomial. In this context, we also need to focus our attention on the shape of admissible polynomials.

In [12], the coefficients of Hilbert polynomials are completely characterized by the numbers of components of certain subschemes defined by particular ideals called \textit{right fans}. In [23], these numbers of components are described by the shape of the minimal generators of Borel ideals. Although the geometric meaning is contained in the fans, in Section 4, we observe that this connection between the coefficients of Hilbert polynomials and the minimal generators of Borel ideals can be directly described without using fans by the combinatorial properties of the Borel ideals themselves. This study led us to conceive an algorithm for computing all saturated Borel ideals with a given Hilbert polynomial. In Section 5, we describe this procedure that have been implemented by the second author in a software with an applet available at http://www.dm.unito.it/dottorato/dottorandi/lella/HSC/borelGenerator.html.

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1. General setting

Let \( K \) be an algebraically closed field of characteristic 0, where \( S := K[x_0, \ldots, x_n] \) is the ring of polynomials over \( K \) in \( n + 1 \) variables such that \( x_0 < x_1 < \cdots < x_n \), and \( \mathbb{P}^n = \text{Proj} S \) is the n-dimensional projective space over \( K \).

A term of \( S \) is a power product \( x^\alpha := x_0^{\alpha_0}x_1^{\alpha_1}\cdots x_n^{\alpha_n} \), where \( \alpha_0, \alpha_1, \ldots, \alpha_n \) are non-negative integers. We set \( \min(x^\alpha) := \min\{i : \alpha_i \neq 0\} \) and \( \max(x^\alpha) := \max\{i : \alpha_i \neq 0\} \). We also let \( \mathbb{T} := \{x_0^{\alpha_0}x_1^{\alpha_1}\cdots x_n^{\alpha_n} \mid (\alpha_0, \alpha_1, \ldots, \alpha_n) \in \mathbb{N}^{n+1}\} \) be the monoid of all terms of \( S \) and \( \mathbb{T}(n) := \mathbb{T} \cap K[x_0, \ldots, x_{n-1}] \).

A graded structure on \( S \) is defined by assigning a weight-vector \( w = (w_0, \ldots, w_n) \in \mathbb{R}_{++}^{n+1} \) and imposing \( v_w(x^\alpha) := \sum_{i=0}^n w_i\alpha_i \). For each non-negative integer \( t, S_t \) is the \( K \)-vector space spanned by \( \{x^\alpha \in \mathbb{T} : v_w(x^\alpha) = t\} \). The standard grading corresponds to \( w = (1, \ldots, 1) \), and we will use it, unless otherwise specified.

For any \( N \subseteq \mathbb{T}, N_t \) is the set of the \( t \)-degree elements of \( N \), and \( \lambda_{t,N}(N) := \{x^\alpha \in N_t : i \leq \min(x^\alpha)\} \) denotes the cardinality of the subset of terms of \( N_t \) that are not divisible by \( x_0, \ldots, x_{t-1} \). For any homogeneous ideal \( I \subseteq S, I_t \) is the vector space of the homogeneous polynomials in \( I \) of degree \( t \), and \( I_{\leq t} \) and \( I_{> t} \) are the ideals generated by the homogeneous polynomials of \( I \) of degree \( \leq t \) and \( > t \), respectively.

Given any term-order \( \preceq \) on \( \mathbb{T} \), each \( f \in S \) has a unique ordered representation \( f = \sum_{i=1}^k c_i (f_i, \tau_i) \), where \( c_i \in K^*, \tau_i \in \mathbb{T}, \tau_1 \succ \cdots \succ \tau_k \) and \( T(f) := \tau_1 \) is the maximal term of \( f \). For any \( F \subseteq S, T(F) := \{T(f) : f \in F\}, T(F) := \{T(f) : f \in F, \tau \in \mathbb{T}\} \) and \( \mathcal{N}(F) := \mathbb{T} \setminus T(F) \). For any ideal \( I \subseteq S, T(I) = T(I) \) and \( \mathcal{N}(I) \) is an order ideal, often called the sous-escalier or Gröbner-escalier of \( I \). A subset \( G \subseteq I \) is a Gröbner-basis of \( I \) if \( T(G) = T(I) \) (see, for instance, [21]).

For a monomial ideal \( I, G(I) \) denotes the unique set the minimal generators of \( I \) consisting of terms.

**Definition 1.1.** (1) In our setting, we consider mainly the (standard) graded term orders on \( \mathbb{T} \). In particular, given two terms \( x^\alpha \) and \( x^\beta \) of \( \mathbb{T} \) of the same degree \( t \), we say that \( x^\alpha \prec x^\beta \) with respect to the following:

(i) the deglex order if \( \alpha_k < \beta_k \), where \( k = \max\{i \in \{0, \ldots, n\} : \alpha_i \neq \beta_i\} \);

(ii) the degrevlex order if \( \alpha_h > \beta_h \), where \( h = \min\{i \in \{0, \ldots, n\} : \alpha_i \neq \beta_i\} \);

(iii) a degreverse order if \( \alpha_0 > \beta_0 \) or \( \alpha_0 = \beta_0 \) and \( \frac{x^\alpha}{x_0^{\alpha_0}} \preceq \frac{x^\beta}{x_0^{\beta_0}} \), where \(\preceq \) is any graded term order on \( \mathbb{T} \cap K[x_1, \ldots, x_n] \).

[14, Definition 4.4.1]. Recall that a degreverse order is well suited for the homogenization of a Gröbner basis [7], and that the degrevlex is a particular deglex order.

(2) Fixing any term order \( \preceq \) on \( \mathbb{T} \) and any weight vector \( w \), the weighted term order \( \preceq_w \) is defined as follows:

\[ x^\alpha \prec_w x^\beta \quad \text{iff} \quad v_w(x^\alpha) < v_w(x^\beta) \quad \text{or} \quad v_w(x^\alpha) = v_w(x^\beta) \quad \text{and} \quad x^\alpha \prec x^\beta. \]

When speaking of \( w \)-term order, we understand \( \preceq \) to be the deglex order.

Let \( I \subseteq S \) be any homogeneous ideal. Then, \( H_{S/I}(t) \) denotes the Hilbert function of the graded algebra \( S/I \). It is well known that there is a polynomial \( p_{S/I}(z) \in \mathbb{Q}[z] \), called the Hilbert polynomial, and positive integers \( \rho_I := \min \{t \in \mathbb{N} \mid H_{S/I}(t') = H_{S/I}(t'+1)\} \) and \( \omega_I := \min \{t \in \mathbb{N} \mid H_{S/I}(t) < H_{S/I}(t+1)\} \), which are called, respectively, the regularity of the Hilbert function \( H \) and the initial degree of \( H \) (or also of \( I \)). For convenience, we will also say that either \( p_{S/I}(z) \) is the Hilbert polynomial for \( I \) or \( I \) is an ideal with Hilbert polynomial \( p_{S/I}(z) \). If \( I \) is not Artinian, set \( \Delta H_{S/I}(t) := H_{S/I}(t) - H_{S/I}(t-1) \) for \( t > 0 \) and \( \Delta H_{S/I}(0) := 1 \); we use an analogous notation for Hilbert polynomials. If \( h \) is a linear form that is general on \( S/I \), then it is easy to prove that \( p_{S_I(t,h)} = \Delta p_{S/I} \).

The polynomials \( p(z) \in \mathbb{Q}[z] \) that are Hilbert polynomials of projective subschemes are called admissible and are completely characterized in [12] by the fact that they can always be written in a unique form of the following type (see [12,16]), where \( \ell \) is the degree of \( p(z) \) and \( m_0 \geq m_1 \geq \cdots \geq m_\ell \geq 0 \) are integers:

\[ p(z) = \sum_{i=0}^\ell \left( z + i \right) - \left( z + i - m_i \right). \]

The saturation of a homogeneous ideal \( I \subseteq S \) is \( I^{sat} := \{f \in S \mid \forall i \in \{0, \ldots, n\}, \exists k_i : x_i^{k_i}f \in I\} = \bigcup_{\ell \geq 0} (I : m^\ell) \), where \( m = (x_0, \ldots, x_n) \), and \( I \) is saturated if \( I = I^{sat} \).

If \( X \subseteq \mathbb{P}_K^n \) is a projective subscheme, \( \text{reg}(X) \) is its Castelnuovo–Mumford regularity, i.e., \( \text{reg}(X) = \min \{t \in \mathbb{N} \mid H^t(J_X(t' - i)) = 0, \forall t' \geq t\} \).
An ideal $I \subseteq S$ is $m$-regular if the $i$th syzygy module of $I$ is generated in degree $\leq m + i$, and the regularity $\text{reg}(I)$ of $I$ is the smallest integer $m$ for which $I$ is $m$-regular. If $I$ is saturated and defines a scheme $X$, then $\text{reg}(I) = \text{reg}(X)$, and we set $H_{S}(t) := H_{S/J}(t)$ and $p_{X}(z) := p_{S/J}(z)$.

For an admissible polynomial $p(z)$, the Gotzmann number $r$ is the best upper bound for the Castelnuovo–Mumford regularity of a scheme having $p(z)$ as its Hilbert polynomial and is computable by using the following unique form of an admissible polynomial:

$$p(z) = \left(\frac{z + a_{1}}{a_{1}}\right) + \left(\frac{z + a_{2} - 1}{a_{2}}\right) + \cdots + \left(\frac{z + a_{r} - (r - 1)}{a_{r}}\right),$$

where $a_{1} \geq a_{2} \geq \cdots \geq a_{r} \geq 0$. We refer to [10] for an overview of these topics.

**Example 1.2.** If $p(z) = dz + 1 - g$ is an admissible polynomial, then its Gotzmann number is $r = \left(\frac{d}{2}\right) + 1 - g$. Indeed, we obtain

$$p(z) = \left(\frac{z + 1}{1}\right) + \cdots + \left(\frac{z + 1 - (d - 2)}{1}\right) + \left(\frac{z + 0 - (d - 1)}{0}\right) + \cdots + \left(\frac{z + 0 - \left(\frac{d - 2}{2}\right) + g}{0}\right).$$

2. Borel ideals and Gröbner strata

**Definition 2.1.** (1) For any $x^{\alpha} \in T$ such that $a_{j} > 0$, the terms obtained from $x^{\alpha}$ via a $j$th elementary move are:

(i) $e_{+}^{j}(x^{\alpha}) := x_{0}^{a_{0}} \cdots x_{j-1}^{a_{j-1}} x_{j+1}^{a_{j+1}} \cdots x_{n}^{a_{n}}$, for any $j \in \{0, \ldots, n - 1\}$ and

(ii) $e_{-}^{j}(x^{\alpha}) := x_{0}^{a_{0}} \cdots x_{j-1}^{a_{j-1}} x_{j}^{a_{j}} \cdots x_{n}^{a_{n}}$, for any $j \in \{1, \ldots, n\}$,

and for each positive integer $a < a_{j}$, we will denote the corresponding elementary move applied $a$ times by $(e_{-}^{j})^{a}$ and $(e_{+}^{j})^{a}$.

(2) For any positive integer $t$, $<_{B}$ denotes the partial order on $T_{t}$ given by the transitive closure of the relation $e_{+}^{j}(x^{\beta}) < x^{\beta}$, i.e. $x^{\alpha} <_{B} x^{\beta}$ if $x^{\alpha}$ is obtained from $x^{\beta}$ via a finite sequence of elementary moves $e_{-}^{j}$.

(3) A set $B \subseteq T_{t}$ is a Borel set if, for every $x^{\alpha}$ of $B$ and $x^{\beta}$ of $T_{t}$, $x^{\alpha} <_{B} x^{\beta}$ implies that $x^{\beta}$ belongs to $B$.

(4) A monomial ideal $J \subseteq S$ is a Borel ideal if, for every degree $t$, $J \cap T_{t}$ is a Borel set.

The combinatorial property by which Borel ideals are defined is also called strong stability. It was first introduced in [11] and later discussed in [12], where the ideals that satisfy it are called balanced. In characteristic 0, it is equivalent to the property of an ideal $J$ being fixed by lower triangular matrices, from which the name “Borel ideals” is derived from.

From the definition, it immediately follows that if $B \subseteq T_{t}$ is a Borel set, then the set $N := T_{t} \setminus B$ has the property that for every $x^{\alpha} \in N$ and $x^{\beta} \in T_{t}$, with $x^{\alpha} <_{B} x^{\beta}$, $x^{\beta}$ belongs to $N$, that is, $N$ is closed w.r.t. elementary moves $e_{-}^{j}$. In particular, if $J$ is a Borel ideal, then for every integer $t \geq 0$ and $J_{t}$ is closed w.r.t. elementary moves $e_{-}^{j}$, and $J_{t}$ is closed w.r.t. elementary moves $e_{+}^{j}$.

**Remark 2.2.** Note that, for every term order $\prec$, if $x^{\alpha}, x^{\beta} \in T_{t}$ satisfy $x^{\alpha} <_{B} x^{\beta}$, then $x^{\alpha} \prec x^{\beta}$. Namely, because $x^{\alpha} <_{B} x^{\beta}$ means that there is a finite number of elementary moves $e_{j}$ connecting $x^{\beta}$ to $x^{\alpha}$, assuming that $x_{j} | x^{\beta}$ for a suitable $0 \leq j \leq n$, we can verify our contention for $x^{\alpha} = e_{j}^{-1}(x^{\beta})$. Setting $\tau := x_{j}^{-1}$ and writing $x^{\alpha} = e_{j}^{-1}(x^{\beta}) = x_{j}^{-1} \tau$ and $x^{\beta} = x_{\tau}$, we get $x^{\alpha} < x^{\beta}$ because $x_{j-1} < x_{j}$.

**Proposition 2.3.** For a Borel ideal $J \subseteq S$,

(i) in our notation, $J^{\text{sat}}$ is obtained by setting $x_{0} = 1$ in the minimal generators of $J$,

(ii) the Krull dimension of $S/J$ is equal to $\min \{ \max(x^{\alpha}) : x^{\alpha} \in J \} = \min \{ i \in \{0, \ldots, n\} : x_{i} \in J \}$ for some $t$, and

(iii) the regularity of $J$ is equal to the maximum degree of its minimal generators.

**Proof.** (i) For example, see [23, Property 2].

(ii) This result follows straightforwardly from Lemma 3.1(a) of [13] or from Corollary 4, Section 5, chapter 9 of [7]. Thus, if $J$ is saturated and $t$ is the degree of the Hilbert polynomial of $J$, we get $t = \min \{ i \in \{0, \ldots, n\} : x_{i} \in J \}$ for some $t$.

(iii) See [2, Proposition 2.9].

**Remark 2.4.** Let $B \subseteq T_{t}$ be a non-empty Borel set, $N := T_{t} \setminus B$ and $J = (B)$ is the Borel ideal generated by the terms of $B$; so, $\mathcal{N}(J)_{t} = N$. Thus,

$$\mathcal{N}(J)_{t+1} = x_{0}N \cup x_{1}\{x^{\alpha} \in N : 1 \leq \min(x^{\alpha})\} \cup x_{2}\{x^{\alpha} \in N : 2 \leq \min(x^{\alpha})\} \cup \cdots \cup x_{n-1}\{x^{\alpha} \in N : n - 1 \leq \min(x^{\alpha})\}.$$
and \( \mathcal{T}_{t+1} \setminus \mathcal{N}(J)_{t+1} \) is a Borel set. In particular, if \( J \) is a Borel ideal and \( \mathcal{N} := \mathcal{N}(J)_t \), for every degree \( t \) we have the following (see [18, Theorem 3.7]):

\[
\mathcal{N}(J)_{t+1} \subseteq \mathcal{N}(J)_{t+1} = x_0 N \cup x_1 \{x^\alpha \in N : 1 \leq \min(x^\alpha)\}
\]
\[
\cup x_2 \{x^\alpha \in N : 2 \leq \min(x^\alpha)\} \cup \ldots \cup x_{n-1} \{x^\alpha \in N : n-1 \leq \min(x^\alpha)\},
\]
from which \( |\mathcal{N}(J)_{t+1}| \leq |\mathcal{N}(J)_{t+1}| = \sum_{i=0}^{n-1} \lambda_{i,t}(\mathcal{N}(J)_{t+1}) \) and \( G(J)_{t+1} = \mathcal{N}(J)_{t+1} \setminus \mathcal{N}(J)_{t+1} \).

**Definition 2.5.** For each Borel subset \( A \subseteq \mathcal{T}_t \) the minimal elements of \( A \) w.r.t. \( \prec_\beta \) are the terms \( x^\alpha \in A \) such that \( e_j^\alpha(x^\alpha) \not\in A \) for every \( j > 0 \) with \( \alpha_j > 0 \), and the maximal elements outside \( A \) w.r.t. \( \prec_\beta \) are the terms \( x^\beta \not\in A \) such that \( e_j^\beta(x^\beta) \in A \) for every \( j > 0 \) with \( \beta_j > 0 \).

**Remark 2.6.** Let \( B \subseteq \mathcal{T}_t \) be a Borel subset, if \( x^\alpha \in B \) and \( x^\beta \not\in B \) are respectively a minimal element of \( B \) and a maximal element outside \( B \) w.r.t. \( \prec_\beta \), then both \( B \setminus \{x^\alpha\} \) and \( B \cup \{x^\beta\} \) are Borel subsets of \( \mathcal{T}_t \) as, by definition, both are closed w.r.t. elementary moves \( e_j^\alpha \).

**Proposition 2.7.** Let \( p(z) \) be an admissible polynomial with Gotzmann number \( r \), and let \( J \subseteq S \) be a Borel ideal with \( p(z) \) as the Hilbert polynomial. Then, for each \( s > r \), a minimal term of \( J_\beta \) w.r.t. \( \prec_\beta \) is divisible by \( x_0 \).

**Proof.** As \( s > r \geq \text{reg}(J) \), for each \( x^\alpha \in J_\beta \), we get \( x^\alpha = \tau \cdot x^\alpha \) for some \( x^\alpha \in G(J) \) and \( \tau \in \mathcal{T}_t \), with \( \text{deg} \tau > 0 \). If \( x^\alpha \) is not divisible by \( x_0 \), let \( j > 0 \) be such that \( \tau \) is divisible by \( x_j \). Then, \( x^\alpha = \tau' \cdot x^\alpha \), with \( \tau' = \frac{\tau}{x^\beta} \), satisfies \( x^\alpha \in J_j \) and \( x^\beta \prec_\beta x^\alpha \), thus contradicting the minimality of \( x^\alpha \). \( \square \)

Given an admissible polynomial \( p(z) \), a term order \( \preceq \) and a monomial ideal \( J \) with \( p(z) \) as the Hilbert polynomial, the Gröbner stratum \( \mathcal{S}(J, \preceq) \) in the Hilbert scheme \( \mathcal{H}_{\text{Adm}}(p(z)) \) of subschemes of \( \mathbb{P}^n \) with Hilbert polynomial \( p(z) \) is an affine variety that parameterizes the family of ideals having the same initial ideal \( J \) with respect to \( \preceq \) [26,9,15,32]. When only homogeneous ideals are concerned, we write \( \mathcal{S}(J, \preceq) \). Now, we recall briefly the construction of \( \mathcal{S}(J, \preceq) \), but it is noteworthy to point out an unexpected feature of Gröbner strata, which is that they are homogeneous varieties with respect to some non-standard graduation [9,15]. Thus, the embedding dimension of \( \mathcal{S}(J, \preceq) \), denoted by \( \text{ed} \mathcal{S}(J, \preceq) \), is the dimension of the Zariski tangent space of the stratum at the origin and can be computed by the same procedure that produces Gröbner strata. In fact, the ideal \( \mathcal{L}(J) \) generated in \( K[\mathbf{c}_{\alpha\beta}] \) by the linear components of the generators of \( \mathcal{A}(J) \), as computed above, defines the Zariski tangent space of the stratum at the origin (Theorems 3.6(ii) and 4.3 of [15]). This fact offers a new tool for studying the singularities of Hilbert schemes.

## 3. Segments

**Definition 3.1.** A set \( B \subseteq \mathcal{T}_t \) is a segment w.r.t. a term order \( \preceq \) on \( \mathcal{T} \) if whenever a term \( \tau \) belongs to \( B \), all the \( t \)-degree terms greater than \( \tau \) belong to \( B \). A monomial ideal \( I \) is a segment ideal w.r.t. \( \preceq \), if \( I \cap \mathcal{T}_t \) is a segment w.r.t. \( \preceq \) for every \( t \geq 0 \).

**Lemma 3.2.** Let \( I \subseteq S \) be a saturated Borel ideal, \( \preceq \) any term order on \( \mathcal{T} \) and \( p > q \) integers. If \( I_p \) is a segment, then \( I_q \) is a segment too.

**Proof.** Let \( x^\alpha \) be a term of \( I_q \) and \( x^\beta \) a term of \( I_n \) such that \( x^{\alpha'} \preceq x^\beta \); hence \( x_0^{\beta'-\alpha} x^\beta \preceq x_0^{\beta'-\alpha} x^\beta \), and because \( I_p \) is a segment, \( x_0^{\beta'-\alpha} x^\beta \) belongs to \( I_p \). Recalling that \( I \) is saturated, \( x^\beta \) belongs to \( I_q \), and we are done. \( \square \)

**Remark 3.3.** A segment is a Borel set, and a segment ideal is a Borel ideal. Indeed, \( x_i x^\alpha \preceq x_i x^\beta \) if \( i < h \); so, \( x^\alpha \preceq x^\beta \) implies that \( x^\alpha \preceq x^\beta \) for any term order \( \preceq \). In particular, if \( \preceq \) is the deglex order and \( I \) is a monomial ideal generated in degree \( \leq q \) such that \( I_q \) is a segment w.r.t. \( \preceq \), then \( I_p \) is a segment too for every \( p > q \).

**Remark 3.4.** (1) Each admissible polynomial \( p(z) \) of degree \( 0 \leq \ell \leq n \) corresponds to a unique saturated segment ideal \( L(p(z)) \) w.r.t. deglex order (see [1,16]). In particular for a constant polynomial \( p(z) = d \) we have the following

\[
L(d) = (x_0, x_{n-1}, \ldots, x_2, x_1),
\]
\[
\mathcal{N}(L(d)) = \left\{ x_i^{d+1}, x_0 x_i^{d+1}, \ldots, x_0^{d+1} x_i \right\} \quad \text{if } 0 \leq j < d
\]
\[
\mathcal{N}(L(d)) = \left\{ x_i^{d+i}, x_0 x_i^{d+i}, \ldots, x_0^{d+i} x_i \right\} \quad \text{if } j = d + i, \forall i \geq 0.
\]

(2) A segment ideal w.r.t. the deglex order exists if and only if the Hilbert polynomial is constant and the Hilbert function \( H \) is non-increasing, i.e., \( \Delta H(t) \leq 0 \) for every \( t > \alpha_t = \min \{ t \in \mathbb{N} : H(t) < \binom{t+n}{n} \} \) [8,19]. We let \( A_{\omega} := A_{\omega,j} \) be the set of the \( \omega \) smallest terms of \( T_j \) w.r.t. deglex order.

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(3) The same reasoning of [8,19] shows that, in general, a segment ideal $I$ w.r.t. a degreewise term order exists if and only if the Hilbert polynomial is constant and the Hilbert function $H$ is non-increasing. Namely, if $a_0$ is the initial degree and $x_1^{a_0} \in \mathcal{N}(J)$, then $x_1^{a_0 + 1}$ belongs to $J$, or otherwise, letting $r > x_1^{a_0}$ be the smallest degree $a_H$ term in $J$, $x_1^{a_H + 1}$ would belong to $\mathcal{N}(J)$ with $x_1^{a_H + 1} > x_0^r \in J$.

**Proposition 3.5.** If an ideal $I \subset S$ of initial degree $a_H$ has the property that there exists an integer $t \geq a_H$ and four terms $x^a, x^b \in \mathcal{N}(J)$, then $J$ is not a segment ideal w.r.t. any term order $\preceq$.

**Proof.** If $J$ were a segment ideal w.r.t. some $\preceq$, by the given assumptions, we would have in particular both $\mathcal{N}(J)_t \ni x^d < x^a \in J$ and $\mathcal{N}(J)_t \ni x^e < x^b \in J$. From these, we would obtain $x^a + r = x^e + r$, contradicting $x^a + r = x^e + r$. 

**Example 3.6.** (1) The (saturated) Borel ideal $J = (x_2^3, x_1^2x_2, x_1^2x_3, x_1x_2x_3) \subset K[x_0, x_1, x_2]$ is not a segment ideal w.r.t. any term order because it satisfies the conditions of Proposition 3.5. Namely, its initial degree is 3, and for $t = 3 \geq 3$, we have $J_3 \ni x_0^3, x_1^3$ and $x_0^3, x_1^3 \in \mathcal{N}(J)_3$ with $x_0^3 \preceq x_1^3 \preceq x_0^3$. Hence, $J$ is not a segment w.r.t. any term order $\preceq$, as if it were, $x_1^2x_2 \in J_3$ and $x_0^3 \in J_3$, we would get $x_0^3 \subsetneq x_1^3$, contradicting $(x_0^3x_2)^2 \in J_4$. Note also that $J^\text{id} = (x_1^2, x_1^3)$ is the saturated lex segment.

**Definition 3.7.** Let $I \subset S$ be a non-null saturated Borel ideal and $\preceq$ a term order on $\mathbb{T}$.

(a) [15] $I$ is a hilb-segment ideal if $I$ is a segment, where $r$ is the Gotzmann number of the Hilbert polynomial of $I$;

(b) $I$ is a reg-segment ideal if $I$ is a segment, where $\delta$ is the regularity of $I$; and

(c) $I$ is a gen-segment ideal if, for every integer $s$, $G(I)$, consists of the greatest terms among the $s$-degree terms not in $(I_{s-1})$.

**Remark 3.8.** The criterion given by Proposition 3.5 can also be adapted to hilb-segment ideals and to reg-segment ideals $I$ by simply verifying it at degree $r = \text{Gotzmann}$ and at degree $\delta = \text{reg}(I)$, respectively. Computational evidence suggests that this criterion is also necessary for reg-segment and hilb-segment ideals.

The following results about Gröbner strata motivate the definitions of the reg-segment ideal and of the hilb-segment ideal, respectively.

**Proposition 3.9.** (i) Let $I \subset S$ be a Borel saturated ideal. If $x_1$ does not appear in any monomial of degree $m + 1$ in the monomial basis of $I$, then $\delta_{t_0}(I_{r-1}) \preceq \delta_{t_0}(I_{r-2})$ (Theorem 4.7 (iii) of [15]).

(ii) An isolated irreducible component of $\mathcal{H}_{\text{hilb}}^n$ contains a smooth point corresponding to a hilb-segment ideal is rational (Corollary 6.10 of [15]).

**Proposition 3.10.** Let $I \subset S$ be a saturated Borel ideal and $\leq$ a term order on $\mathbb{T}$. Then,

(i) $I$ segment ideal $\Rightarrow \text{hilb-segment ideal} \Rightarrow \text{reg-segment ideal} \Rightarrow \text{gen-segment ideal};$

(ii) $\leq$ is the deglex order $\Rightarrow$ conditions in (i) are equivalent, for every ideal $I$; and

(iii) If the projective scheme defined by $I$ is 0-dimensional, then $I$ segment ideal $\iff$ hilb-segment ideal $\iff$ reg-segment ideal.

**Proof.** (i) The first implication is obvious. For the second one, we need to only apply Lemma 3.2 because $r \geq \delta$. For the third implication, recall that $I$ is generated in degrees $\leq \delta$, by definition. Moreover, if $I$ is a reg-segment ideal, by Lemma 3.2, $I_t$ contains the greatest terms of degree $t$ for $t \leq \delta$.

(ii) First, suppose that $\leq$ is the deglex order. Then, by (i), it is enough to show that a gen-segment ideal is also a segment ideal. Indeed, by induction on the degree $s$ of terms and with $s = 0$ as the base of induction, for $s > 0$, suppose that $I_{s-1}$ is a segment. Thus, by Remark 3.3, we know that $(I_{s-1})_s$ is a segment, and because the possible minimal generators are always the greatest possible, we are done.

Vice versa, if $\leq$ is not the deglex order, let $s$ be the minimum degree at which the terms are ordered in a different way from the deglex order. Therefore, there exist two terms $x^a$ and $x^b$ with maximum variables $x_t$ and $x_s$, respectively, such that $x^d < x^b$ and $x^d > x^a$. The ideal $I = (x_0, \ldots, x_n)$ is a gen-segment ideal but not a segment ideal because $x^d$ belongs to $I$ and $x^d$ does not.

(iii) We need only show that, in the 0-dimensional case, a reg-segment ideal $I$ is also a segment ideal. By induction on the degree $s$, if $s \leq \delta$, then the thesis follows by the hypothesis and by Lemma 3.2. Suppose that $s > \delta$ and that $I_{s-1}$ is a segment. At degree $s$, there are no minimal generators for $I$; so, a term of $I_t$ is always of type $x^a x_t$ with $x^a$ in $I_{s-1}$. Let $x^a$ be a term of degree $s$ such that $x^a > x^b x_t$; so, $x^a > x^b x_0$. By Proposition 2.3, we have that $(x_1, \ldots, x_n)^{\text{id}} \subseteq I$. So, if $x^b$ is not divisible by $x_0$, then $x^a$ belongs to $I_t$; otherwise, there exists a term $x^a$ such that $x^a = x^b x_0$. Thus, $x^a > x^b$, and by induction, $x^d$ belongs to $I_{s-1}$; so, $x^d = x^d x_0$ belongs to $I_t$.

**Example 3.11.** Let $\preceq$ be the degrevlex order.

(1) The ideal $I = (x_0, x_2) \subset K[x_0, x_1, x_2]$ is a hilb-segment ideal, but it is not a segment ideal. In this case, the Hilbert polynomial is $p(z) = z + 2$ with Gotzmann number 2 and $\text{reg}(I) = 2$. We have $x_1^3 \in \mathcal{N}(I)$ and $x_0x_1x_2 \in I$ with $x_1^3 > x_0x_1x_2$. 

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(2) $I' = (x_1^2, x_2 x_3, x_2^2 x_3) \subseteq K[x_0, x_1, x_2]$ is a reg-segment ideal but not a hilb-segment ideal. In this case, the Hilbert polynomial is $p(z) = z + 4$ with Gotzmann number 4 and $\text{reg}(I') = 3$. We obtain $x_0 x_2^2 \in I'$ with $x_0 x_2^2 \not\in I'$.

(3) $I'' = (x_1^2, x_3 x_4, x_2^2) \subseteq K[x_0, \ldots, x_4]$ is a gen-segment ideal but not a reg-segment ideal. In this case, the Hilbert polynomial is $p(z) = 2z^2 + 2z + 1$ with Gotzmann number 12 and $\text{reg}(I'') = 3$. We obtain $x_0 x_2^2 \in I''$ with $x_0 x_2^2 \not\in I''$.

**Remark 3.12.** If $I$ is a saturated Borel ideal and is also an “almost revlex segment ideal”, as defined in [8], then it is a gen-segment ideal w.r.t. degrevlex order.

**Theorem 3.13.** Let $J$ be the ideal generated by a Borel set $B \subseteq \mathbb{T}_d$ consisting of all but $d$ terms of degree $d$. Then, $J^{\text{sat}}$ defines a projective scheme with Hilbert polynomial $p(z) = d$.

**Proof.** By the Borel condition, $x_d^4$ belongs to $J$ or otherwise, $|\mathcal{N}(J)_d| \geq d + 1 > d$ so, by Remark 2.4, we have $|\mathcal{N}(J)_t| = d$, for every $t \geq d$. The ideal $I = J^{\text{sat}}$ is the saturated ideal of a projective subscheme with Hilbert polynomial $p(z) = d$. □

**Remark 3.14.** (1) For every positive integer $d$ and any term order $\preceq$ on $\mathbb{T}$, there exists a unique saturated segment ideal $I \subseteq S$ with Hilbert polynomial $p(z) = d$. This is a straightforward consequence of Theorem 3.13; it is enough to take the ideal $J$ generated by all but the last $d$ terms of degree $d$.

(2) In [19] for the degrevlex order and in [6] for each term order, it is shown that the generic initial ideal of a set $X$ of $d$ general points in $\mathbb{P}^3$ is a segment ideal with Hilbert polynomial $p(z) = d$. As the Hilbert function of $X$ is the maximum possible, that is, $H_X(t) = \binom{t+n}{n}$, we deduce that this is the Hilbert function of the saturated segment ideal of $X$.

(3) For the case of degreverse term orders, it is possible to give a direct and constructive proof of (1). If $I \subseteq S$ is a segment ideal w.r.t. a degreverse order with Hilbert polynomial $p(z) = d$, its Hilbert function must be non-increasing by Remark 3.4(2) and also strictly increasing until it reaches the value $d$, after which it remains equal to $d$ because $J$ is a saturated ideal of Krull dimension 1. Thus, $H_{S/J}(t)$ must be the maximum and we deduce the following:

(i) $\alpha_H = \rho_H + 1$; so $J = \langle x_1, \ldots, x_n \rangle^{\rho_H}$;

(ii) $\alpha_H = \rho_H$; so $J$ is generated only in degrees $\alpha_H$ and $\alpha_H + 1$. More precisely, the minimal generators of degree $\alpha_H$ are the greatest $\left(\alpha_H + \frac{\alpha_H}{\rho_H}\right)$ terms of $T_{\rho_H}$ (so, in $\mathcal{N}(J)$, there are $d - \left(\frac{\alpha_H + 1}{\rho_H} - 1\right)$ terms $x_\alpha$ with $\alpha < x_4$, $\alpha_H$), and the minimal generators of degree $\alpha_H + 1$ are all the terms $\tau > x_4$ that are not multiples of terms in $J_{\alpha_H}$ (these terms are at least $d - \left(\frac{\alpha_H + 1}{\rho_H} - 1\right)$, by Remark 2.4).

It follows that in case (i) we have $|G(J)| = \binom{\rho_H + n}{n-1}$, and in case (ii), $|G(J)| = \binom{\rho_H + n}{n-1}$, so $I = \langle x_{1,1}, \ldots, x_n \rangle$ is the segment ideal for $p(z)$. Moreover, we have already observed that a segment ideal always exists for a constant polynomial $p(z) = d$.

**3.1. On hilb-segment ideals**

Let $\preceq$ be any term order and $p(z)$ be an admissible polynomial with Gotzmann number $r$. We want to see under what conditions there exists a hilb-segment ideal for $p(z)$ in this context. It is immediately clear that if $r = 1$, then $p(z) = \binom{z+\ell}{\ell}$, where $\ell < n$ is the degree of $p(z)$; so $I = \langle x_{1,1}, \ldots, x_n \rangle$ is the hilb-segment ideal for $p(z)$. Moreover, we have already observed that a hilb-segment ideal always exists for a constant polynomial $p(z) = d$.

**Example 3.15.** The following saturated Borel ideals are not hilb-segment ideals for any term order:

(1) $J = (x_1^2, x_2 x_3, x_3^2) \subseteq K[x_0, x_1, x_2]$, as for $H = (1, 3, 5, 7, \ldots, p(z) = 7, \ldots)$, we have $r = 7$; so, if $J$ were a hilb-segment ideal w.r.t. some $\preceq$, at degree 7, we should have $\mathcal{N}(J)_{2} \ni x_0^2 x_1 x_2 < x_0^2 x_2^2 \in J_7$ and $\mathcal{N}(J)_{7} \ni x_0^2 x_1 x_2 < x_0^2 x_2^2 \in J_7$, contradicting $x_0^2 x_1 x_2^2 = x_0^2 x_2^2 < x_0^2 x_2^2$.

(2) $J = (x_2^3, x_1 x_3^2, x_2^2 x_3^2) \subseteq K[x_0, x_1, x_2]$, as for $H = (1, 3, 6, 7, \ldots, p(z) = 3 + 4, \ldots)$, we have $\text{reg}(J) = 3$ and $r = 4$; so, we can repeat the same reasoning of (1) with $x_0 x_2 x_3^2 \in J_4$ and $x_1^2 x_0^2 x_2^2 \in \mathcal{N}(J)_4$.

**Proposition 3.16.** In $K[x_0, x_1, x_2]$, every saturated Borel ideal with Hilbert polynomial $p(z) = d \leq 6$ is a hilb-segment ideal. Whereas, for every $p(z) = d \geq 7$, a saturated Borel ideal, which is not a hilb-segment for any term order always exists.

**Proof.** We give a direct constructive proof of the result, based in part on the characterization of the Borel subsets in three variables of [17].

- $d = 2$ there exists a unique saturated Borel ideal $(x_2, x_1^1)$, which is the hilb-segment ideal w.r.t. degrevlex order;
- $d = 3$ there are only two saturated Borel ideals: the hilb-segment ideals $(x_2, x_1^1)$ (w.r.t. deglex) and $(x_2^2, x_1 x_2, x_1^3)$ (w.r.t. degrevlex);
- $d = 4$ there are only two saturated Borel ideals: the hilb-segment ideals $(x_2, x_1^1)$ (w.r.t. deglex) and $(x_2^2, x_1 x_2, x_1^3)$ (w.r.t. degrevlex);
Proposition 2.3

Example 3.15

Proof. (so,

\[ j \geq 2, \text{ if } \deg f \geq 1. \]

We have already observed that a \( \text{hilb-segment ideal} \) always exists and do as \( \text{gen-segment ideal} \) for a

\[ \frac{d}{x^a} \cdot \left( \frac{x}{d} \right)^n \leq \frac{2}{x^a} \cdot \left( \frac{x}{d} \right)^n \]

Proposition 3.17. Let \( \leq \) be any degreverse term order and \( p(z) \) be an admissible polynomial of positive degree with Gotzmann number \( r \).

(1) If \( p(r) = \binom{r-1+n}{n} \), then the hilb-segment ideal for \( p(z) \) does not exist.

(2) If \( p(z) = dx + 1 - g \), then the hilb-segment ideal \( I \) for \( p(z) \) exists if and only if

\[ \begin{align*}
& (i) \ r = d \text{ or } r = d + 1, \text{ when } n = 2, \text{ and} \\
& (ii) \ r = d, \text{ when } n > 2. \\
\end{align*} \]

Proof. (1) By the hypothesis, we have that \( x_i \) belongs to the ideal; so, the Krull dimension must be 1 by Proposition 2.3, and we are done.

(2) In this case, the hilb-segment ideal \( I \) exists if and only if \( p(r) = \binom{n+1-r}{n} + d \). Indeed, the sous-escalier of \( (J, x_0) \), contains the least \( d \) terms not divisible by \( x_0 \), and because the term order is degreverse and \( r \geq d \), the sous-escalier of \( J \) must also contain the same least \( d \) terms not divisible by \( x_0 \). Hence, by the Borel property, all the terms divisible by \( x_0 \) must also belong to the sous-escalier of \( J \). Thus, because \( r = \left( \frac{d}{2} \right) + 1 - g \) by Example 1.2, we obtain the following:

\[ dr + r - \left( \frac{d}{2} \right) = n + r - 1 \]

\[ \Rightarrow d = \frac{1}{2} \left( 2r - 1 \pm \sqrt{8 \frac{r+1}{2} - 8 \frac{r+n-1}{n} + 1} \right); \]

so, \( J \) exists if and only if the argument \( \Delta \) under the square root is not-negative. By an easy calculation, we obtain the thesis. \( \square \)

3.2. On gen-segment ideals for degrelex order

We describe some procedures to construct gen-segment ideals w.r.t. degrelex order with a given admissible polynomial \( p(z) \) of degree 1. We have already observed that a hilb-segment ideal always exists and so does a gen-segment ideal for a constant polynomial \( p(z) = d \).

Lemma 3.18. If \( p(z) = dx + 1 - g \) is an admissible polynomial with Gotzmann number \( r \), there exist two integers \( n \geq 2 \) and \( j(n) > 0 \) such that \( \binom{j(n)-1+n}{n} \leq p(j(n) - 1) \) and \( p(j(n) + h) < \binom{j(n)+h+n}{n} \) for every \( h \geq 0 \).

Proof. Any projective scheme of dimension 1 with Hilbert polynomial \( p(z) \) has regularity \( \leq r \); so, \( p(r) < \binom{r+n}{n} \) for any \( n \geq 2 \). Now, it is enough to show that there exist integers \( n \geq 2 \) and \( t < r \) such that \( p(t) \geq \frac{r+n}{n} \). In the plane, i.e., for \( n = 2 \), it holds that \( g \leq \frac{1}{4}(d-1)(d-2) \). Therefore, \( p(t) = dt + 1 - g \geq dt + 1 - \frac{1}{2}(d-1)(d-2) \), and for \( t = d-1 \), we have \( d(d-1) + 1 - \frac{1}{2}(d-1)(d-2) = \frac{d-1+2}{2} \). Thus, \( n = 2, d \leq j(n) \leq r \). \( \square \)

Proposition 3.19. Let \( p(z) = dx + 1 - g \) be an admissible polynomial. For any \( n \geq 2 \), there exists a gen-segment ideal \( I(n) \subset S \) w.r.t. degrelex order with Hilbert polynomial \( p(z) \).

Proof. By Lemma 3.18, we can take an integer \( n \geq 2 \) for which there exists \( j(n) > 0 \) such that \( \binom{j(n)-1+n}{n} \leq p(j(n) - 1) \) and \( p(j(n) + h) < \binom{j(n)+h+n}{n} \) for every \( h \geq 0 \). First, we prove the thesis in this case.

Under the given assumptions, we have \( p(j(n)) - \binom{j(n)-1+n}{n} \geq d = p(j(n) - 1) - p(j(n) - 1) \); thus, by Remark 3.4(3), in \( A_{p(j(n)),j(n)} \), there are at least \( d \) terms \( \Delta \) such that \( \min(\Delta) \geq 1 \), and we let \( t_1 < \cdots < t_d \) be the least among them w.r.t. degrelex order. We also set for every \( 0 \leq t < j(n) \), \( N(t) \coloneqq T_t, N(j(n)) \coloneqq A_{p(j(n)),j(n)} \).

\[ x_0 \cdot N_{-1}(x^t) \cdot \{ t_1, \ldots, t_d \}, \]
for every \( t = j(n) + h, h \geq 1, \) and \( N := \bigcup_{t \geq 0} N(t). \) By construction, \( N \subset T \) is such that \( N_t = N(t), \) for each \( t \geq 0, \) and \( |N_t| = p(t), \) for every \( t \geq j(n). \) Thus, the monomial ideal \( I(n) \subset S \) such that \( \mathcal{N}(I(n)) = N \) is, by construction, a gen–segment ideal with Hilbert polynomial \( p(z). \) Moreover, \( G(I(n)) = \emptyset, \) for \( t < j(n) \) and \( t > j(n) + 1; \) so, \( \deg(I(n)) \leq j(n) + 1 \leq r. \)

Now, suppose that \( n \) is such that \( p(t) < \binom{1 + n}{n} \) for every \( t \geq 0, \) and let \( n_0 := \max(n' \mid \exists I(n') : \binom{j(n') + 1}{n'} \leq p(j(n') - 1) \) and \( p(j(n')) < \binom{j(n')+1}{n'} \). Above, we proved that for such an \( n_0, \) there exists a gen–segment ideal \( I(n_0) \subset K[x_0, \ldots, x_{n_0}] \) w.r.t. degrevlex order with Hilbert polynomial \( p(z). \) Now, it is enough to observe that \( I(n) := (I(n_0), x_{n_0+1}, \ldots, x_n) \subset S \) is a gen–segment ideal w.r.t. degrevlex order, as claimed. □

**Remark 3.20.** Given an admissible polynomial \( p(z) = dz + 1 - g, \) let \( N := A_{p(0),1} \) be the set of the lower \( p \) terms of degree \( I \) w.r.t. degrevlex order. If \( n > 2 \) is such that \( p(t) < \binom{1 + n}{n} \) for every \( t \geq 0 \) and there exists \( I(n) := \min(L \in N : A_{x_1}^1(N_t) \geq d) \), by a similar procedure, we can construct a gen–segment ideal \( I(n) \subset S \) w.r.t. degrevlex order with Hilbert polynomial \( p(z) \) that is different from those coming from the smaller \( n \)’s as in the proof of Proposition 3.19. Indeed, under the given assumptions, \( A_{p(n_0),n_0}(n_0) \subset T(n_0) \) no longer contains at least \( d \) terms \( x^e \) with \( \min(x^e) \geq 1, \) but its expansion in degree \( I(n) + 1 \) does, and we let \( T_1 < \cdots < T_d \) be the least one of them w.r.t. degrevlex order. Similarly as before, we take \( M(t) := T_1 \) for every \( 0 \leq t < I(n), \) \( M(I(n)) := A_{p(n_0),n_0}(n_0)(n_0 + 1) := x_0 \cdot M(I(n)) \cap \{ T_1, \ldots, T_d \} \) and \( M(t) := x_0 \cdot M(t - 1) \cap x^e_{I-I(n)-1} \{ T_1, \ldots, T_d \} \) for every \( t > I(n) + 1. \) We finally let \( j(n) \) be the gen–segment ideal such that \( \mathcal{N}(j(n)) = M := \bigcup_{t \geq 0} M(t) \) and note that it has \( p(z) \) as its Hilbert polynomial and regularity \( \leq I(n) + 2. \)

**Example 3.21.** The Gotzmann number of the admissible polynomial \( p(z) = 6z - 3 \) is 12, and we obtain the following gen–segment ideals:

(i) \( n = 2, \) we can apply the procedure described in the proof of Proposition 3.19 with \( j(2) = 9 \) and construct the ideal \( I(2) = (x_0^2, x_1^5, x_2^2 x_3, x_3^2) \);

(ii) \( n = 3, \) there is not a \( j(3), \) yet we can apply the procedure described in Remark 3.20 with \( l(3) = 2 \) because \( p(t) < \binom{1 + 3}{3} \) for every \( t > 0, \) obtaining \( j(3) = (x_3^2, x_3 x_3 x_3, x_4^2) \) besides \( I(2), x_3, \) and

(iii) \( n \geq 4, \) neither \( j(n) \) nor \( I(n) \) exists, and we have only \((I(2), x_3, x_4, \ldots, x_n) \) and \((I(3), x_4, \ldots, x_n). \)

**Proposition 3.22.** The saturated segment ideal \( L(p(z)) \subset S \) w.r.t. deglex order with Hilbert polynomial \( p(z) \) is a gen–segment ideal w.r.t. the deglex order if and only if \( \deg(p(Z)) \leq 1 \) or there are only two generators of degree \( > 1. \)

**Proof.** In Section 1, we recalled that given an admissible polynomial \( p(z) \) of degree \( \ell, \) there exist unique integers \( m_0 \geq m_1 \geq \cdots \geq m_\ell \geq 0 \) such that \( p(z) = \sum_{i=0}^{\ell} \binom{z+i}{i+1} - \binom{z+i-m_i}{i+1} \) \( 16, 1, 1, 1 \). Let \( a_i := m_i, a_{i-1} := m_{i-1} - m_i, a_0 := m_0 - m_1. \) Note that \( L(p(z)) \subset S \) has the \( n+1-\ell \geq 2 \) greatest variables as the generators of degree 1, i.e., \( \mathcal{N}(L(p(z))) = \{ x_{0}, \ldots, x_{\ell+1} \}. \) Thus, for every \( j \leq a_\ell, \) the greatest term of \( \mathcal{N}(L(p(z))) \) is \( x_{\ell+1}^{a_{\ell+1}} \) w.r.t. both deglex and degrevlex orders (namely, \( \mathcal{N}(L(p(z))) = T_{j} \cap K[x_{0}, \ldots, x_{\ell+1}] \). In degree \( a_\ell + 1, \) the ideal \( L(p(z)) \) has a new generator \( x_{\ell+1}^{a_{\ell+1}} \), so, \( \mathcal{N}(L(p(z)))_{a_{\ell}+1} = \{ (T_{a_\ell+1} \cap K[x_0, \ldots, x_{\ell+1}]) \} \). Therefore, its greatest term w.r.t. both deglex and degrevlex orders, is \( x_{\ell+1} x_{\ell+1}^{a_{\ell+1}} \). At this point, the greatest term of \( \mathcal{N}(L(p(z)))_{a_{\ell}+a_{\ell+1}} \) is \( x_{\ell} x_{\ell+1}^{a_{\ell}+a_{\ell+1}} \) w.r.t. deglex order and \( x_{\ell} x_{\ell+1}^{a_{\ell}+a_{\ell+1}} \) w.r.t. degrevlex order (similarly for the case in parentheses). Moreover, because the new generator of \( L(p(z)) \) at degree \( a_{\ell} + a_{\ell+1} + 1 \) is \( x_{\ell} x_{\ell+1}^{a_{\ell}+a_{\ell+1}} \), if \( a_{\ell} = 2, \) the third generator of degree \( > 1 \) is \( x_{\ell}^{2} x_{\ell}^{2} x_{\ell}^{2} \), and if \( a_{\ell} = 3, \) it is \( x_{\ell}^{2} x_{\ell}^{2} x_{\ell+1}^{2} \), it is not the greatest term w.r.t. deglex order. □

**Example 3.23.** (i) The ideal \( L(p(z)) = (x_4, x_5, x_6^2, x_3^2 x_4, x_5^2 x_4, x_6^2) \) is the saturated segment ideal w.r.t. deglex in \( K[x_0, \ldots, x_4], \) with Hilbert polynomial \( p(z) = 2z^2 + 2z + 1 \) and Gotzmann number 12, but it is not a gen–segment ideal w.r.t. deglex order.

(ii) The ideal \( L(p(z)) = (x_5, x_6^2, x_3^2 x_4) \) is the saturated segment ideal w.r.t. deglex in \( K[x_0, \ldots, x_5], \) with Hilbert polynomial \( p(z) = 2/3z^3 + 2z^2 - 11/3z + 10 \) and Gotzmann number 6, and it is also a gen–segment ideal w.r.t. deglex order.

4. Saturations of Borel ideals and Hilbert polynomial

Let \( J \subset S \) be a Borel ideal. Recall that in our notation, the (Borel) ideal \( J \) is obtained by setting \( x_0 = 1 \) in each minimal generator of \( J \) (Proposition 2.3(i)). In this section, we let \( J_{\text{sat}} := J^{\text{sat}} \) and denote as \( J_{x_0} \) the Borel ideal obtained by setting \( x_0 = x_1 = 1 \) in the minimal generators of \( J \). We call \( J_{x_0} \) the \( x_1 \)-saturated of \( J \) and say that \( J \) is \( x_1 \)-saturated if \( J = J_{x_0} \). Hence, an ideal \( J \) that is \( x_1 \)-saturated is also saturated.

**Remark 4.1.** An ideal \( J \subset S \) that is \( x_1 \)-saturated and has Hilbert polynomial \( p(z) = p_{J}(z) \) has the same minimal generators as the saturated Borel ideal \( J \cap K[x_1, \ldots, x_n] \subset K[x_1, \ldots, x_n] \) for which the Hilbert polynomial is \( \Delta(p) \).
The following result is analogous to Theorem 3 of [23], where the notion of “fan” is used. Here, we apply only the combinatorial properties of Borel ideals.

**Proposition 4.2.** Let \( J \subset S \) be a saturated Borel ideal with Hilbert polynomial \( p(z) \) and Gotzmann number \( r \). Let \( l = L_{0x1} \) be the \( x_1 \)-saturation of \( J \), and let \( q := \dim_k l - \dim_k J \). Then,

(i) \( p_{5_{/S}}(z) = p(z) - q \), and 
(ii) \( q \) is equal to the sum of the exponents of \( x_1 \) in the minimal generators of \( J \).

**Proof.** (i) We show that if \( q = \dim_k l - \dim_k J \) then \( q = \dim_k l_{s+1} - \dim_k J_{s+1} \) for every \( s \geq r \). Let \( x^{\beta_1}, \ldots, x^{\beta_k} \) be the terms of \( I \) not divisible by \( x_1 \). Thus, \( x_0 x^{\beta_1}, \ldots, x_0 x^{\beta_k} \) are some terms of \( l_{s+1} \setminus J_{s+1} \), and so, \( \dim_k l_{s+1} - \dim_k J_{s+1} \geq q \) because \( x_0 x^{\beta_i} \) belongs to \( J_{s+1} \) if and only if \( x^{\beta_i} \) belongs to \( J_{s+1} \), as \( J \) is saturated. Now, to obtain the opposite inequality, it is enough to show that every term of \( l_{s+1} \setminus J_{s+1} \) is divisible by \( x_0 \). Let \( \beta \in \mathbb{N} \). Hence, for every \( x^{\beta} \) of degree \( s = 1 - |\alpha| \) and with \( \min(x^{\beta}) \geq 1 \), by the Borel property, \( x^{\beta} \) belongs to \( J_{s+1} \). In particular, \( x^{\beta} \in J_{s+1} \).

(ii) Let \( x^{\beta_1}, \ldots, x^{\beta_k} \) be the minimal generators of \( J \) with \( x^{\beta_i} \) not divisible by \( x_1 \), for every \( 1 \leq i \leq h \). Because \( \sum s_i \) terms \( x^{\beta_i} \) is \( H \) minimal in \( J \), one has \( \min(x^{\beta}) \geq 2 \) and \( u < s_i \). Let \( s \) be the minimum non-negative integer such that \( x^{\beta_i} \) is \( H \). Then, there exists an \( s \) such that \( x^{\beta_i} \in x^{\beta_i} \), i.e., \( x^{\beta_i} \) and \( s_i \leq s \). By the definition of \( s \), we obtain \( s_i = s \), and there exists \( x^{\beta_i} \) with \( \min(x^{\beta}) \geq 2 \) such that \( x^{\beta} \) is \( x^{\beta_i} \). Because \( x^{\beta} \) does not belong to \( J \), we have \( |\beta| < s_i = s \), or \( x^{\beta_i} \) and \( s_i \leq s \). Hence, by the Borel property, \( x^{\beta} \) should belong to \( J \). Now we can take \( x^{\beta_i} \) and observe that this term belongs to \( J \) because it follows \( x^{\beta_i} \) in the Borel relation. Thus, \( s \leq s - |\beta| \); so, \( r = 0 \), i.e., \( x^{\beta} = x^{\beta_i} \) as claimed. \( \Box \)

**Proposition 4.3.** Let \( J \subset S \) be a saturated Borel ideal of Hilbert polynomial \( p(z) \) and Gotzmann number \( r \). Let \( x^{\beta_1} \) be a term of \( J \) of degree \( s \geq r \) that is minimal in \( J \) w.r.t. \( \preceq_b \). Then, the ideal \( H := (G(J_1)) \setminus (x^{\beta_1}x_0) \) is Borel, and \( p_{5_{/H}}(z) = p(z) + 1 \).

**Proof.** First, note that \( H \) is closed w.r.t. \( \preceq_b \) by Remark 2.6. We show that, for every \( t \geq 0 \), \( x^{\beta_1} \) is the unique term in \( J_{s+1} \setminus H_{s+1} \). If \( t = 0 \), we have the hypothesis. For \( t > 0 \), note that \( x^{\beta_1} \) cannot belong to \( J \). On the contrary, there would be a term \( x^{\beta} \in H \) such that \( x^{\beta} \neq x^{\beta_1} \) and \( x^{\beta} \ | x^{\beta_1} \). But every degree \( s \) factor of \( x^{\beta_1} \) different from \( x^{\beta_1} \) is lower w.r.t. \( \preceq_b \), and so, \( x^{\beta_1} \setminus H_{s+1} \). Then, \( x^{\beta_1} \setminus H_{s+1} \). If \( x^{\beta_1} \) is a term of \( J_{s+1} \setminus H_{s+1} \), there exists a term of \( J_{s+1} \setminus H_{s+1} \) that divides \( x^{\beta_1} \). By induction, this term is \( x^{\beta_1} \), and the thesis follows from the fact that every\( \sum s_i \). \( \Box \)

**Proposition 4.4.** Let \( I \) and \( J \) be Borel ideals of \( S \). If, for every \( s \gg 0 \), we have \( I \subset J \) and \( p_{5_{/I}}(z) = p_{5_{/J}}(z) + a \), with \( a \in \mathbb{N} \), then \( I \) and \( J \) have the same \( x_1 \)-saturation.

**Proof.** Let \( s \geq \max(\text{reg}(I), \text{reg}(J)) \). In the case \( 1 \), there exists a unique term in \( J_{s+1} \setminus I_{s+1} \) for every \( t \geq 0 \). Let \( x^{\beta_1} \) be the unique term in \( J_{s+1} \setminus I_{s+1} \). Then, both \( x^{\beta_1} \) and \( x^{\beta_1} \) belong to \( J_{s+1} \). By the Borel property, \( x^{\beta_1} \) must be in \( I_{s+1} \), and so, the unique term in \( J_{s+1} \setminus I_{s+1} \) belongs to \( I_{s+1} \). This is enough to say that \( I \) and \( J \) have the same \( x_1 \)-saturation. If \( a > 1 \), the thesis follows by induction and by applying Proposition 4.3. \( \Box \)

**Corollary 4.5.** Let \( p(z) \) be an admissible polynomial of degree \( h \leq n \) and \( P := \{ q(z) = p(z) + u \ | \ u \in \mathbb{Z} \text{ and } q(z) \text{admissible} \} \) be the set of all admissible polynomials of degree \( h \) differing from \( p(z) \) for only an integer. Then,

(i) there is a polynomial \( \hat{p}(z) \) in \( P \) such that for every \( q(z) \) in \( P \), \( q(z) = \hat{p}(z) + c \) with \( c \geq 0 \), and 
(ii) every saturated Borel ideal \( I \) with Hilbert polynomial \( p_{5_{/I}} = \hat{p}(z) \) is \( x_1 \)-saturated.

**Proof.** (i) Every admissible polynomial \( p(z) \) has a unique saturated lex segment ideal \( L(p(z)) \). If \( H \) is the saturated lex segment ideal of \( p(z) + u \), then we have \( H \subset L(p(z)) \) if \( u > 0 \) and \( L(p(z)) \subset H \) if \( u < 0 \). Thus, we can apply Proposition 4.4, finding that \( L(p(z)) \) and \( H \) have the same \( x_1 \)-saturation, \( I \). We claim that \( \hat{p}(z) \) is the Hilbert polynomial of \( I \). Indeed, by Proposition 4.2, the Hilbert polynomial of \( I \) is of type \( p(z) - q \). If \( \hat{p}(z) = p(z) - q - t \) with \( t > 0 \), then the saturated lex segment ideal of \( p(z) \) should have \( I \) as the \( x_1 \)-saturation and should be contained in \( I \), which is possible only if \( t = 0 \).

(ii) Let \( J \) be a Borel ideal with \( \hat{p}(z) \) as the Hilbert polynomial. If \( J \) were not \( x_1 \)-saturated, by Proposition 4.2, the \( x_1 \)-saturation of \( J \) should have a Hilbert polynomial of type \( p(z) - q \), with \( q > 0 \), against the definition of \( \hat{p}(z) \).

**Definition 4.6.** The polynomial \( \hat{p}(z) \) of Corollary 4.5 is called the minimal polynomial.

**Remark 4.7.** An alternative proof of the previous statement can be obtained by following the construction of the Gotzmann number.
By Proposition 4.2, a Borel ideal with a minimal Hilbert polynomial is $x_1$-saturated. The opposite is not true. For example, the ideal $I = (x_1^2, x_2x_3, x_4^2) \subset K[x_0, x_1, x_2, x_3]$ is $x_1$-saturated and is a reg-segment ideal w.r.t. the degrevlex order. The corresponding Hilbert polynomial is $p_{3/(1)}(z) = 3z + 1$, which is not minimal because the Borel ideal $(x_3, x_2^2)$ has Hilbert polynomial $3z$.

**Remark 4.9.** From the proof of Corollary 4.5, we deduce the following fact. Let $I \subset K[x_0, \ldots, x_n]$ be a Borel ideal with Hilbert polynomial $p(z)$. If $I = I_n \cdot K[x_0, \ldots, x_n]$, where $I_n \subset K[x_1, \ldots, x_n]$ is the hilb-segment ideal w.r.t. deglex order with Hilbert polynomial $\Delta p(z)$, then $p(z) = \tilde{p}(z)$.

### 5. An algorithm to compute saturated Borel ideals

In this section, by exploiting the results of Section 4, we describe an algorithm for computing all the saturated Borel ideals with a given Hilbert polynomial $p(z)$. We first give an efficient strategy to find the minimal elements in a Borel set $B$, which consists of representing $B$ by a connected planar graph in which the nodes are the terms of $B$ and the edges are the elementary moves connecting the terms. In Fig. 1, we give some examples showing that it is easy to single out the minimal terms by looking at these graphs.

Let $0 \leq k \leq n$ be an integer. Recall that, if $I \subset K[x_0, \ldots, x_n]$ is a saturated Borel ideal that has a non-null Hilbert polynomial $p(z)$ with Gotzmann number $r$, then $J \coloneqq \left(\frac{(I, x_k)}{(x_k)}\right) \subset K[x_{k+1}, \ldots, x_n]$ has Hilbert polynomial $\Delta p(z)$ with $x_k$ as a non-zero-divisor on $\frac{K[x_k, \ldots, x_n]}{(x_k)}$.

This fact shows that every saturated Borel ideal $I \subset K[x_k, \ldots, x_n]$ with Hilbert polynomial $p(z)$ “comes from” a Borel ideal $J \subset K[x_{k+1}, \ldots, x_n]$ with Hilbert polynomial $\Delta p(z)$ and generated in degrees $\leq r$. So, our idea to construct all saturated Borel ideals with a given Hilbert polynomial $p(z)$ consists of applying a recursion on the number of variables. By the hypothesis we know all of the Borel ideals $J$ in the $n - k$ variables generated in degrees $\leq r$ with Hilbert polynomial $\Delta p(z)$. Then, we construct the saturated Borel ideals $I$ in $n - k + 1$ variables such that $J := \left(\frac{(I, x_k)}{(x_k)}\right)$ for some of the ideals $J$.

Let $J \subset K[x_{k+1}, \ldots, x_n]$ be a Borel ideal with Hilbert polynomial $\Delta p(z)$ and $\tilde{J} := (J^{\text{rev}} \cdot K[x_k, \ldots, x_n])_r$, where $r$ is the Gotzmann number of $p(z)$. Let $\overline{N}$ be the set of terms $x^\alpha$ of $K[x_k, \ldots, x_n]$ such that there exists a composition $F$ of elementary moves of type $e_{\tau}$ and a term $\tau$ of $\mathcal{N}(J)$ such that $F(\tau) = x^\alpha$. Hence, by construction, the terms of $\overline{N} \setminus \mathcal{N}(J)$ are not maximal, and $\overline{N}$ is contained in the sous-escalier of any ideal of $K[x_k, \ldots, x_n]$ having $J$ as the hyperplane section. Note that the Gotzmann number of $\Delta^{k+1} p(z)$ is not higher than the Gotzmann number of $\Delta^k p(z)$.

**Lemma 5.1.** $\mathcal{N}(\tilde{J}) = \overline{N}$.
The theses follow from the results presented in Section I.

**Proof.** It is enough to show that $K[x_1, \ldots, x_n] = (\overline{I}, \overline{N})$ (Fig. 2). Indeed, let $x^{r'} = x^{r_1}_1 \cdots x^{r_n}_n$ be in $K[x_1, \ldots, x_n]$. The term $x^\delta = (e^+_{i_k})^{r_k}(x^{r'})$ belongs to $K[x_{k+1}, \ldots, x_n]$, and is hence in $I_r$ or in $\mathcal{N}(J)_r$. If $x^\delta$ is in $J_r$, then $x^\delta \cdot x^{r_1}_1 \cdots x^{r_n}_n$ belongs to $J^\text{sat}_r$ and thus to $I$; otherwise, $x^{r'} = (e^+_{i_k+1})^{r_k}(x^\delta)$ is in $\overline{N}$. □

**Proposition 5.2.** With the above notation, the Hilbert polynomial $\overline{p}(z)$ for $\overline{I}$ differs from $p(z)$ only by a constant. If $\overline{q} = p(r) - \overline{p}(r) > 0$, execute the following instruction $\overline{q}$ times: select a minimal term $\tau$ in $I_r$, and set $\overline{I} := (\overline{G}((\overline{I}_r) \setminus \{\tau\}))$. After these $\overline{q}$ steps, the new ideal obtained has Hilbert polynomial $p(z)$.

**Proof.** The theses follow from the results presented in Section 4. □

**Proposition 5.2** suggests the design of the following two routines BorelGenerator and Remove, which have been implemented by the second author in a software with an applet available at [http://www.dm.unito.it/dottorato/dottorandi/lella/HSC/borelGenerator.html](http://www.dm.unito.it/dottorato/dottorandi/lella/HSC/borelGenerator.html).

**procedure** BorelGenerator($n, p(z), r, k$) $\rightarrow \mathcal{F}$

if $p(z) = 0$ then
  return $\{1\}$
else
  $\mathcal{E} \leftarrow$ BorelGenerator($n, \Delta p(z), r, k + 1$);
  $\mathcal{F} \leftarrow \emptyset$;
  for all $I \in \mathcal{E}$ do
    $\overline{I} \leftarrow I \cdot k[x_1, \ldots, x_n]$;
    $q \leftarrow p(r) - \text{dim}_k k[x_1, \ldots, x_n]$ + $\text{dim}_k \overline{I}$;
    if $q > 0$ then
      $\mathcal{F} \leftarrow \mathcal{F} \cup \text{Remove}(n, k, r, \overline{I}, q)$;
    end if
  end for
  return $\mathcal{F}$;
end if

**procedure** Remove($n, k, r, \overline{I}, q$) $\rightarrow \mathcal{E}$

$\mathcal{E} \leftarrow \emptyset$;
if $q = 0$ then
  return $\mathcal{E} \cup \mathcal{I}^\text{sat}$;
else
  $\mathcal{F} \leftarrow \text{MinimalElements}(\overline{I}, r)$;
  for all $x^\delta \in \mathcal{F}$ do
    $\mathcal{E} \leftarrow \mathcal{E} \cup \text{Remove}(n, k, r, (G((\overline{I}_r) \setminus \{x^\delta\}), q - 1)$;
  end for
  return $\mathcal{E}$;
end if

**Remark 5.3.** The terms removed by our strategy are minimal in $\overline{I}$. An alternative strategy consists of adding to $J_r K[x_1, \ldots, x_n]$ the maximal terms of $\mathcal{I}_r \setminus J$. In this case, because we want to have $\text{dim}_k I_r = \binom{n-k+1}{r} - p(r)$ and we already have $\binom{n-k+1}{r} - \Delta p(r)$ terms of $J$, we should add...
With the above notation, we obtain that $d$ can be recovered from $\Delta p(z)$ by the Gotzmann decomposition in the following way. If
\[
\Delta p(z) = \left( z + b_1 \right) + \left( z + b_2 - 1 \right) + \cdots + \left( z + b_t - (t-1) \right)
\]
with $b_1 \geq b_2 \geq \cdots \geq b_t \geq 0$, then
\[
\hat{p}(z) = \left( z + a_1 \right) + \left( z + a_2 - 1 \right) + \cdots + \left( z + a_t - (t-1) \right),
\]
where $a_i = b_i + 1$. The Gotzmann number of $\Delta p(z)$ is also the Gotzmann number $\hat{r}$ of $\hat{p}(z)$. If $r$ is the Gotzmann number of $p(z)$, then $r - \hat{r} = p(0) - \hat{p}(0) \geq p(0) - \hat{p}(0) = \bar{q}$. We prove that $q' \geq \bar{q}$ by induction on $c = r - \hat{r}$. If $c = 0$, then we obtain $\bar{q} = 0$. If $c > 0$, by induction, we have that $q'(r-1, n-k, p(z)-1) \geq \bar{q} - 1$; hence,
\[
q'(r, n-k, p(z)) = \left( \frac{r-1+n-k}{n-k} \right) - p(r-1)
\]
\[
= \left( \frac{r-2+n-k}{n-k} \right) + \left( \frac{r-1+n-k-1}{n-k-1} \right) - p(r-1) + p(r-2) - p(r-2)
\]
\[
geq q'(r-1, n-k, p(z)-1) + \left( \frac{r-1+n-k-1}{n-k-1} \right) - \Delta p(r-1) - 1
\]
\[
\geq \bar{q} + \left( \frac{r-1+n-k-1}{n-k-1} \right) - \Delta p(r-1) - 2,
\]
and $\left( \frac{r-1+n-k-1}{n-k-1} \right) - \Delta p(r-1) - 2$ because $r-1$ is an upper bound of the Gotzmann number of $\Delta p(z)$ and $J$ is not a hypersurface because $n-k-1 > \deg(\Delta p(z)) + 1$.

**Example 5.4.** (a) If $p(z) = d$, then $r = d$ and $\hat{r} = 0$, so $\bar{q} = d$ and $q' = \left( \frac{d-1+n}{n} \right) - d$. Moreover, if $n = \deg(p(z)) + 1$, then $q' = 0$.

(b) The Gotzmann number of $p(z) = 3z + 1$ is $r = 4$, and if $n = 3$ and $k = 0$, then $q'(r, n, p(z)) = \left( \frac{r-1+n}{n} \right) - p(r-1) = 20 - 10 = 10$ and $r - \hat{r} = 1$. If $J_d = (x_3, x_2^2)$, we obtain $|\bar{N}_r| = 12$ and $\bar{q} = 1$; meanwhile, if $J_d = (x_3^2, x_2x_3, x_2^2)$, we obtain $|\bar{N}_r| = 13$ and $\bar{q} = 0$.

6. Degreese points

In this section, by exploiting results of [15], we study the points corresponding to the hilb-segment ideals in the Hilbert scheme $\mathcal{H}ilb^n_S$ of subschemes of $\mathbb{P}^n$ with Hilbert polynomial $p(z) = d$, where $d$ is a fixed positive integer. Recall that for $p(z) = d$, the Gotzmann number is $d$ itself.

From now on, $J \subset S$ is a hilb-segment ideal with respect to some term order $\prec$ and with Hilbert polynomial $p(z) = d$, and let $\mathcal{B} := \{x^\beta \in \mathcal{N}(J)_d : x_1x^\beta \in J\}$. Recall that $G(J)$ denotes the set of minimal generators of $J$, and $ed(S_{th}(J, \leq))$ is the embedding dimension of the Gröbner stratum $S_{th}(J, \leq)$.

**Lemma 6.1.** With the above notation, we obtain that $ed(S_{th}(J, \leq)) \geq |G(J)| \cdot |\mathcal{B}|$.

**Proof.** Using the same notation introduced in Section 2, by Theorem 4.7(i) of [15], it is enough to look at the variables $c_{a\beta}$ appearing in the polynomials $F_a$ such that $x^\beta = x^{\alpha} x^\beta_{\alpha \beta}$, where $x^\alpha$ belongs to $G(J)$. More precisely, we need to count the number of such variables that do not correspond to a pivot in a Gauss reduction of the generators of $L(J)$ (see also Proposition 4.3 and Definition 4.4 of [15]).

First, we note that in every S-polynomial involving such an $F_a$, the polynomial $F_a$ itself is multiplied by a term in which at least a variable $x_0$ appears with $h > 0$ (otherwise the other polynomial involved in the S-polynomial should have $x_0^d$ as its initial term). It is enough to investigate the terms $x^\beta x_1$, where $x^\beta$ belongs to $\mathcal{B}$ because if $x^\beta x_1$ belongs to $J_{h_0}$, then $x^\beta x_0$ belongs to $J_{h_0}$ for any $h > 0$. Because $J$ is a hilb-segment ideal, every term $x^\beta$ of $\mathcal{B}$ is less than $x^\alpha$. By the definition of $\mathcal{B}$, every term $x^\beta$ of $\mathcal{B}$ is always involved in a reduction step so it does not appear in any generator of $L(J)$ (see Criterion 4.6 of [15]). The number of such terms is at least $|G(J)| \cdot |\mathcal{B}|$, and we are done. □
Theorem 6.2. If, for the hilb-segment ideal J, we have |G(J)| · |B| > nd, then J corresponds to a singular point in $\mathcal{H}ilb^n_2$.

Proof. Let $H_{BS}$ be the unique irreducible component of $\mathcal{H}ilb^n_2$ containing the lexicographic point [24]. Recall that $H_{BS}$ has a dimension equal to $nd$ and that every Borel ideal belongs to $H_{BS}$ [23]. Because J is a hilb-segment ideal w.r.t. $\prec$, the Gröbner stratum $St_\alpha(J, \prec)$ is an open subset of $H_{BS}$ (Corollary 6.9 of [15]), and hence, dim $St_\alpha(J, \prec) = nd$. Thus, the point J is smooth for $\mathcal{H}ilb^n_2$ if and only if it is smooth for the Gröbner stratum $St_\beta(J, \prec)$ (see Corollary 4.5 of [15]). In particular, J is smooth if and only if ed($St_\alpha(J, \prec)$) = nd. By Lemma 6.1, the thesis is proved.

In the following, it is important to keep in mind that if $\prec$ is a degreverse term order and $\alpha$ is the initial degree of a degreverse segment ideal J for $p(z) = d$, then $St_\beta(J) = St_\beta(J_{\geq \alpha}) \cong St_\beta(J_{\geq \alpha + k})$ for every $k > 0$, by Proposition 3.9(i) and Remark 3.14.

Example 6.3. The generic initial ideal of 7 general points in $\mathbb{P}^3$ w.r.t. a degreverse term order, i.e., the (saturated) hilb-segment ideal with Hilbert polynomial $p(z) = 7$, can be one of the two following saturated Borel ideals with a maximal Hilbert function: the hilb-segment ideal I w.r.t. degrevlex or I' = $(x_1^2, x_2 x_3, x_3^2, x_4 x_5, x_5^2, x_6 x_7, x_7^2)$. Theorem 6.4. For every $d > n \geq 3$, the hilb-segment ideal J w.r.t. a degreverse order corresponds to a singular point in $\mathcal{H}ilb^n_2$.

Proof. In Remark 3.14, we observed that J must have a maximal Hilbert function; so, the regularity $\rho_H$ of its Hilbert function is the integer such that $(\rho_H^{-1} + n) < d \leq (\rho_H^{-1} + n)$. Moreover, if $d = (\rho_H^{-1} + n)$, then $|G(J)| = (\rho_H^{-1} + n)$; otherwise, $|G(J)| \geq (\rho_H^{-1} + n)$.

If $d = n + 1$, then $\rho_H = 1$ and $J = (x_1, \ldots, x_n)^2$; so, $|G(J)| = (\frac{2 + n - 1}{n - 1}) = (\frac{n + 1}{2})$. Moreover, B consists of the terms of type $x_i^{n - 1} x_1$ with $i > 0$; thus, $|B| = n$, and the statement is true because $\left(\frac{n + 1}{2}\right) \cdot n > n(n + 1)$ for every $n \geq 3$.

If $d \geq n + 2$, then $\rho_H \geq 2$.

If $d = (\rho_H^{-1} + n)$, we show that $|B| > \rho_H + 1$. If we multiply every term of degree $\rho_H$ in the variables $x_1, \ldots, x_n$ by $x_0^{-\rho_H}$, we obtain terms of degree $d$ that multiplied by $x_1$ give $(\rho_H^{-1} + n)$ terms that belong to B. Thus, $|B| \geq (\rho_H^{-1} + n) > \rho_H + 1$, and $|G(J)| \cdot |B| > \frac{d n}{\rho_H + 1} \cdot (\rho_H + 1) = dn$.

If $d < (\rho_H^{-1} + n)$ and $\rho_H \geq 3$, we show that $|B| \geq \rho_H + n$. Let $x^d$ be any of the $(\rho_H^{-1} + n)$ terms of degree $\rho_H - 1$ in the variables $x_1, \ldots, x_n$. Thus, if $x^d x_0^{-\rho_H - 1}$ belongs to B; otherwise, if $x^d x_1$ does not belong to J, then $x^d x_0^{-\rho_H - 1}$ belongs to B. Either way, the term $x^d x_1$ belongs to J because it is not divisible by $x_0$ and has degree $\rho_H + 1$, and the terms of $N(J)$ are all divisible by $x_0$. Such terms are all distinct; so, $|B| \geq (\rho_H^{-1} + n)$. Now it is easy to check that $(\rho_H^{-1} + n) \geq \rho_H + n$ for every $\rho_H \geq 3$ and $n \geq 3$. Thus, $|G(J)| \cdot |B| \geq (\rho_H^{-1} + n) \cdot (\rho_H + n) > nd$, by Remark 3.14(3).

It remains to study the case $\rho_H = 2$ in which $|G(J)| \geq (\frac{n + 1}{2})$ and $|B| \geq n$ because of the above results with $\left(\frac{n + 1}{n}\right) < d < \left(\frac{2 + n}{2}\right)$. If $d < (\frac{n + 1}{2})$, then we immediately obtain $|G(J)| \cdot |B| > nd$. If $d > (\frac{n + 1}{2}) \geq (\frac{n + 2}{2})$, all the d terms of $N(J)$ are in B except for most the n + 1 terms divisible by $x_0^{-1}$. Thus, in this case, $|B| \geq d - (n + 1)$, which is $\geq n + 2$ except for $n = 3$ and $d = 7, 8$. Thus, these last two cases have been already studied in Example 6.3.

Example 6.5. We can list all the saturated Borel ideals in $K[x_0, x_1, \ldots, x_3]$ with constant Hilbert polynomial $d$ for $d = 4, 5, 6, 7$, e.g., by the implementation BORELGENERATOR of the algorithm described in Section 5. Then, we see if these Borel ideals correspond to smooth or singular points in $\mathcal{H}ilb^n_2$ by a direct computation, as in Example 6.3.

For $d = 4$, the segment ideal J w.r.t. degrevlex is the unique Borel ideal that corresponds to a singular point of $\mathcal{H}ilb^n_2$. More precisely, the singular locus of $\mathcal{H}ilb^n_2$ is determined by the 3-dimensional orbit of J under the action of the projective linear group on $\mathbb{P}^3$. 

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Also, for $d = 5$, the segment ideal $J$ w.r.t. degrevlex is the unique Borel ideal that corresponds to a singular point of $\mathcal{J}(\mathcal{I}(b))$. In this case, the singular locus of $\mathcal{J}(\mathcal{I}(b))$ is 6-dimensional and contains the orbit of the ideals $J(\lambda) = (x^2, x_3x_2, x^2_2, x_3x_1, x_2x_1, x_2^2(x_1 + \lambda x_3))$ (a linear family depending on the parameter $\lambda \in K$) under the action of the projective linear group on $\mathbb{P}^4$. For $\lambda = 0$, we get the ideal $J$ itself, while for $\lambda \neq 0$, the subscheme defined by $J(\lambda) = (x^2, x_3x_2, x^2_2, x_3x_1, x_2x_1, x_2^2(x_1 + \lambda x_3))$ is the union of the degree 4 non-reduced scheme given by the segment ideal $J$ w.r.t. the degrevlex of degree 4 and one more point.

For $d = 6$, there is one more Borel ideal $J' = (x^2, x_3x_2, x^2_2, x_3x_1, x_2x_1, x_2^2(x_1 + \lambda x_3))$ that corresponds to a singular point of the Hilbert scheme in addition to the segment ideal $J$ w.r.t. degrevlex. Both of them are naturally related to the segment ideal $J$ w.r.t. the degrevlex of degree 4; indeed $J$ and $J'$ are the initial ideals w.r.t. degrevlex of the ideal defining the union of the degree 4 non-reduced scheme on the point $O$ (given by the degree 4 segment ideal w.r.t. degrevlex) and of two more points $P, Q$. We obtain $J$ if $O, P, Q$ span a plane, and we obtain $J'$ if $O, P, Q$ span a line.

Finally, for $d = 7$, there are four Borel ideals corresponding to singular points on $\mathcal{J}(\mathcal{I}(b))$. Two of them are segment ideals w.r.t. degrevlex term orders (see Example 6.3), and two of them are not.

For $d = 4, 5, 6, 7$, $\mathcal{J}(\mathcal{I}(b))$ only has one component (see [4]), and we can see this for $d = 4, 5$ directly by our computations.

By observing the following,

(i) a segment ideal w.r.t. degrevlex order gives rise to a singular point in $\mathcal{J}(\mathcal{I}(b))$ and defines a scheme not contained in any hyperplane and

(ii) a segment ideal w.r.t. deglex order gives rise to a smooth point in $\mathcal{J}(\mathcal{I}(b))$ and defines a scheme contained in some hyperplane,

one might guess that there is a relationship between the smoothness of a point in $\mathcal{J}(\mathcal{I}(b))$ corresponding to a (saturated) monomial ideal and the presence of linear forms in the ideal. But, the next example (for which we are indebted to G. Floystad) shows that this is not the case.

**Example 6.6.** (i) Let $I = (x_1^2, \ldots, x_n^2)$ be a (saturated monomial) complete intersection ideal defining a 0-dimensional scheme $\mathbb{V}(I)$ of degree $d = \prod a_i$ in $\mathbb{P}^n$, and let $z_j$ denote the corresponding point of $\mathcal{J}(\mathcal{I}(b))$. As $I$ is a monomial ideal, $z_j$ lies in the closure of the lexicographic point component of $\mathcal{J}(\mathcal{I}(b))$ (see, for example, Corollary 18.30 of [20]). Using the normal sheaf to $\mathbb{V}(I)$, we get that the dimension of the tangent space to $\mathcal{J}(\mathcal{I}(b))$ at $z_j$ is $nd$, coinciding with that of the lexicographic point component. Thus, $I$ gives an example of a monomial ideal that does not contain linear forms and that corresponds to a smooth point in $\mathcal{J}(\mathcal{I}(b))$.

(ii) Let $J \subset K[x_1, \ldots, x_n]$ be a saturated monomial ideal giving a singular point $z_j$ of $\mathcal{J}(\mathcal{I}(b))^{-1}$; so, the dimension of the tangent space to $\mathcal{J}(\mathcal{I}(b))^{-1}$ in $z_j$ is $\alpha > (n - 1)d$. Taking $J = ((x_0) + J) \subset K[x_0, \ldots, x_n]$, the dimension of the tangent space to $\mathcal{J}(\mathcal{I}(b))$ in $z_j$ is $\alpha + d > (n - 1)d + d = nd$, and hence, $z_j$ is singular too.

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**References**


