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Second method of Lyapunov and comparison principle for systems with impulse effect *

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Abstract: In the present paper questions of stability and boundedness of the solutions of systems with impulse effect at fixed moments with respect to a manifold are considered. The investigations are carried out by means of piecewise continuous vector-valued functions which are analogues of Lyapunov's functions. By means of a vector comparison equation and differential inequalities for piecewise continuous functions, theorems of stability and boundedness of the solutions of systems with impulses with respect to a manifold have been obtained.

Keywords: Second method of Lyapunov, systems with impulse effect.

1. Introduction

Many processes studied by physics, chemistry, biology, etc. are characterized by the fact that at certain moments they change their state by jumps. These processes during their evolution are subject to short-time perturbations whose duration is negligible in comparison with the duration of the process. That is why we can assume that these perturbations are carried out "instantly", in the form of impulses. Adequate mathematical models of such processes are the systems of differential equations with impulses.

In recent years, the mathematical theory of systems with impulses developes intensively in relation to their numerous applications in radio engineering, control theory, biotechnologies, etc. [1–11].

In the present paper questions of stability and boundedness of the solutions of systems with impulse effect at fixed moments with respect to a manifold are considered. The investigations are carried out by means of piecewise continuous vector-valued functions which are analogues of Lyapunov's functions [10]. Using a vector comparison equation and differential inequalities for piecewise continuous functions, we have obtained sufficient conditions for various types of stability and boundedness of the solutions of systems with impulses with respect to a manifold.

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2. Preliminary notes

Let $\mathbb{R}^n$ be an n-dimensional Euclidean space with norm $\|\cdot\|$ and let $I=[0, \infty)$. Consider the following system of differential equations with impulse effect at fixed moments:

$$\begin{cases} \mathrm{d}x/\mathrm{d}t = f(t, x), & t \neq t_i, \\ \Delta x|_{t=t_i} = I_i(x(t_i)), & i=1, 2, \ldots, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $f: I \times \mathbb{R}^n \to \mathbb{R}^n$, $I_i: \mathbb{R}^n \to \mathbb{R}^n$, $\Delta x|_{t=t_i} = x(t_i+0) - x(t_i-0)$ and the moments $\{t_i\}$ of impulse effect form a strictly increasing sequence

$$0 < t_1 < t_2 < \ldots < t_i < \ldots \quad \text{and} \quad \lim_{i\to\infty} t_i = \infty.$$

Let $(t_0, x_0) \in I \times \mathbb{R}^n$. Introduce the notation $x(t; t_0, x_0)$ for the solution of system (1) which satisfies the initial condition $x(t_0+0; t_0, x_0) = x_0$ and by $\mathscr{I}^+ = \mathscr{I}^+(t_0, x_0)$ denote the maximal interval of the form (t_0, ω) in which the solution $x(t; t_0, x_0)$ is defined.

The systems with impulse effect of the type (1) are characterized in the following way:

(i) For $t \neq t_i$ the solution of system (1) is determined by the system $\mathrm{d}x/\mathrm{d}t = f(t, x)$ and the mapping point $(t, x(t))$ moves along some of the integral curves of this system.

(ii) At the moment $t = t_i$ system (1) is subject to an impulse effect and the mapping point is transferred instantly from the position $(t_i, x(t_i))$ into the position $(t_i, x(t_i) + I_i(x(t_i)))$. In the interval $(t_i, t_{i+1}]$ the solution $x(t)$ of system (1) coincides with the solution $y(t)$ of the system $\mathrm{d}y/\mathrm{d}t = f(t, y)$ for which $y(t_i) = x(t_i) + I_i(x(t_i))$.

(iii) Each solution $x(t)$ of system (1) is a piecewise continuous function with points of discontinuity $\{t_i\}$ of first type at which it is left continuous, i.e. the following relations hold

$$x(t_i - 0) = x(t_i); \qquad x(t_i+0) = x(t_i) + \Delta x(t_i) = x(t_i) + I_i(x(t_i)).$$

Let $g: I \times \mathbb{R}^n \to \mathbb{R}^k$ $(k \leqslant n)$ be a function defined in $I \times \mathbb{R}^n$. For $t \in I$ introduce the sets

$$M_t(n-k) = \{x \in \mathbb{R}^n: g(t, x) = 0\},$$

$$M_t(n-k)(\alpha) = \{x \in \mathbb{R}^n: \|g(t, x)\| < \alpha\},$$

$$M_t(n-k)(\bar{\alpha}) = \{x \in \mathbb{R}^n: \|g(t, x)\| \leqslant \alpha\} \quad (\alpha > 0).$$

We shall say that conditions (A) are satisfied if the following conditions hold:

(A1) The function $f(t, x)$ is continuous in $I \times \mathbb{R}^n$ and satisfies the Lipschitz condition with respect to x uniformly on $t \in I$ with a constant $L > 0$, i.e.

$$\|f(t, x_1) - f(t, x_2)\| \leqslant L \|x_1 - x_2\|, \qquad t \in I, \quad x_1, x_2 \in \mathbb{R}^n.$$

(A2) $f(t, 0) = 0$ for $t \in I$.

(A3) The functions $I_i(x)$, $i = 1, 2, \ldots$ are continuous in $\mathbb{R}^n$ and $I_i(0) = 0$.

(A4) The numbers t_i, $i = 1, 2, \ldots$ form a strictly increasing sequence

$$0 < t_1 < t_2 < \ldots < t_i < \ldots \quad \text{and} \quad \lim_{i\to\infty} t_i = \infty.$$

(A5) The function $g(t, x)$ is continuous in $I \times \mathbb{R}^n$ and the set $M_t(n-k)$ is an $(n-k)$-dimensional manifold in $\mathbb{R}^n$.

We shall say that condition (B) is satisfied if the following condition holds:

(B) Each solution $x(t;\, t_0,\, x_0)$ of system (1) satisfying the estimate

$$\| g(t,\, x(t;\, t_0,\, x_0)) \| \leqslant h < \infty \quad \text{for } t \in \mathscr{J}^+(t_0,\, x_0)$$

is defined in the interval $(t_0,\, \infty)$.

We shall give definitions of stability of the zero solution of system (1) with respect to the function $g(t,\, x)$ which correspond to the definitions given in [12].

Definition 1. The zero solution of system (1) is called:

(a) *Stable* with respect to the function $g(t,\, x)$ if for any $t_0 \in I$ and any $\epsilon > 0$ there exists a positive function $\delta = \delta(t_0,\, \epsilon)$ which is continuous at t_0 for any $\epsilon > 0$ fixed and such that if $x_0 \in M_{t_0}(n-k)(\bar{\delta})$ and $t \in \mathscr{J}^+(t_0,\, x_0)$, then $x(t;\, t_0,\, x_0 \in M_t(n-k)(\epsilon)$.

(b) *Uniformly stable* with respect to the function $g(t,\, x)$ if the function δ from (a) does not depend on t_0.

(c) *Globally equi-attractive* with respect to the function $g(t,\, x)$ if for any $t_0 \in I$, $\alpha > 0$ and $\epsilon > 0$ there exists a positive number $T = T(t_0,\, \alpha,\, \epsilon)$ such that if $x_0 \in M_{t_0}(n-k)(\bar{\alpha})$, then $t_0 + T \in \mathscr{J}^+(t_0,\, x_0)$ and $x(t;\, t_0,\, x_0) \in M_t(n-k)(\epsilon)$ for $t \geqslant t_0 + T$, $t \in \mathscr{J}^+(t_0,\, x_0)$.

(d) *Uniformly globally attractive* with respect to the function $g(t,\, x)$ if the number T from (c) does not depend on t_0.

(e) *Globally equi-asymptotically stable* with respect to the function $g(t,\, x)$ if it is stable and globally equi-attractive with respect to the function $g(t,\, x)$.

(f) *Uniformly globally asymptotically stable* with respect to the function $g(t,\, x)$ if it is uniformly stable and uniformly globally attractive with respect to the function $g(t,\, x)$.

(g) *Unstable* with respect to the function $g(t,\, x)$ if there exist $\epsilon > 0$ and $t_0 \in I$ such that for any $\delta > 0$ we can choose $x_0 \in M_{t_0}(n-k)(\bar{\delta})$ and $t \in \mathscr{J}^+(t_0,\, x_0)$ so that the inequality $\| g(t,\, x(t;\, t_0,\, x_0)) \| \geqslant \epsilon$ should hold.

Definition 2. The solutions of system (1) are called:

(a) *Equi-bounded* with respect to the function $g(t,\, x)$ if for any $t_0 \in I$ and any $\alpha > 0$ there exists a positive function $\beta = \beta(t_0,\, \alpha)$ which is continuous in t_0 for any $\alpha > 0$ and is such that if $x_0 \in M_{t_0}(n-k)(\bar{\alpha})$, $t \in \mathscr{J}^+(t_0,\, x_0)$, then $x(t;\, t_0,\, x_0) \in M_t(n-k)(\beta)$.

(b) *Uniformly bounded* with respect to the function $g(t,\, x)$ if the function β from (a) does not depend on t_0.

(c) *Ultimately bounded* with respect to the function $g(t,\, x)$ for bound N if there exists a number $N > 0$ and for any $t_0 \in I$ and $\alpha > 0$ there exists a positive number $T = T(t_0,\, \alpha)$ such that if $x_0 \in M_{t_0}(n-k)(\bar{\alpha})$, then $t_0 + T \in \mathscr{J}^+(t_0,\, x_0)$ and $x(t;\, t_0,\, x_0) \in M_t(n-k)(N)$ for $t \geqslant t_0 + T$, $t \in \mathscr{J}^+(t_0,\, x_0)$.

(d) *Uniformly ultimately bounded* with respect to the function $g(t,\, x)$ for bound N if the number T from (c) does not depend on t_0.

Remark 1. If the zero solution of system (1) is stable with respect to the function $g(t,\, x)$ and if $x_o \in M_{t_0}(n-k)$, then $x(t;\, t_0,\, x_0) \in M_t(n-k)$ for $t \in \mathscr{J}^+(t_0,\, x_0)$. This shows that the subset $\{(t,\, x) : t \in I,\ x \in M_t(n-k)\}$ of $I \times \mathbb{R}^n$ is a positively invariant set of system (1).

Remark 2. If $n = k$ and $g(t, x) = x$, then Definition 1 is reduced to the definition of stability by Lyapunov of the zero solution of system (1) [9, Definition 2] and Definition 2 is reduced to the definition of boundedness of the solutions of system (1) [11, Definitions 4, 5, 7, 8].

Together with system (1) consider the following system

$$\begin{cases} du/dt = F(t, u), & t \neq t_i, \\ \Delta u|_{t=t_i} = B_i(u(t_i)), & i = 1, 2, \ldots, \end{cases} \tag{2}$$

where $u \in \mathbb{R}^m$, $F: I \times \Omega \to \mathbb{R}^m$, $B_i: \Omega \to \mathbb{R}^m$, Ω is an open subset of $\mathbb{R}^m$ containing the origin.

In order to formulate the main results of the comparison method we need a partial ordering in $\mathbb{R}^m$ which is introduced in the usual way.

Of the vectors $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$ and $v = (v_1, \ldots, v_m) \in \mathbb{R}^m$ we say that $u \geqslant v$ if $u_i \geqslant v_i$ for any $i \in \{1, 2, \ldots, m\}$ and $u > v$ if $u_i > v_i$ for any $i \in \{1, 2, \ldots, m\}$.

Definition 3. The function $\psi: \Omega \to \mathbb{R}^m$ is called *monotonely increasing* in Ω if $\psi(u) > \psi(v)$ for $u > v$ and $\psi(u) \geqslant \psi(v)$ for $u \geqslant v$, $u, v \in \Omega$.

Definition 4. The function $F: I \times \Omega \to \mathbb{R}^m$ is called *quasi-monotonely increasing* in $I \times \Omega$ if for any two points (t, u) and (t, v) of $I \times \Omega$ and for any $i \in \{1, 2, \ldots, m\}$ the inequality $F_i(t, u) \geqslant F_i(t, v)$ holds always when $u_i = v_i$ and $u \geqslant v$, i.e. if for any $t \in I$ fixed and any $i \in \{1, 2, \ldots, m\}$ the function $F_i(t, u)$ is non-decreasing with respect to $(u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_m)$.

Definition 5. The solution $u^+: (t_0, \omega) \to \mathbb{R}^m$ of system (2) such that $u^+(t_0 + 0) = u_0$ is called a *maximal solution* if any other solution $u: (t_0, \tilde{\omega}) \to \mathbb{R}^m$ of system (2) such that $u(t_0 + 0) = u_0$ satisfies the inequality $u^+(t) \geqslant u(t)$ for $t \in (t_0, \omega) \cap (t_0, \tilde{\omega})$.

A minimal solution $u^-(t)$ of system (2) is defined in an analogous way.

Let $e \in \mathbb{R}^m$ be the vector $(1, 1, \ldots, 1)$.

We shall say that condition (C) is satisfied if the following condition holds:

(C) If $\{u \in \mathbb{R}^m: 0 \leqslant u \leqslant he\} \subset \Omega$ and the maximal solution $u^+(t; t_0, u_0)$ of system (2) satisfies the estimate

$$u^+(t; t_0, u_0) \leqslant he \quad \text{for } t \in \mathscr{J}^+(t_0, u_0),$$

then this solution is defined in the interval (t_0, ∞).

Let $\Omega \supset \{u: 0 \leqslant u \leqslant e\}$. Further on we shall consider only such solutions $u(t)$ of system (2) for which $u(t) \geqslant 0$. That is why the following definitions of stability and boundedness of the solutions of system (2) are appropriate:

Definition 6. The zero solution of system (2) is called:

(a) *Stable* if for any $t_0 \in I$ and $\epsilon > 0$ there exists a positive function $\delta = \delta(t_0, \epsilon)$ which is continuous in t_0 for any $\epsilon > 0$ and is such that if $0 \leqslant u_0 \leqslant \delta e$ and $t \in \mathscr{J}^+(t_0, u_0)$, then $u^+(t; t_0, u_0) < \epsilon e$.

(b) *Uniformly stable* if the function δ from (a) does not depend on t_0.

(c) *Globally equi-attractive* if for any $t_0 \in I$, $\alpha > 0$ and $\epsilon > 0$ there exists a positive number $T = T(t_0, \alpha, \epsilon)$ such that if $0 \leqslant u_0 \leqslant \alpha e$, then $t_0 + T \in \mathscr{I}^+(t_0, u_0)$ and $u^+(t; t_0, u_0) < \epsilon e$ for $t \geqslant t_0 + T$, $t \in \mathscr{I}^+(t_0, u_0)$.

(d) *Uniformly globally attractive* if the number T from (c) does not depend on t_0.

(e) *Globally equi-asymptotically stable* if it is stable and globally equi-attractive.

(f) *Uniformly globally asymptotically stable* if it is uniformly stable and uniformly globally attractive.

(g) *Unstable* if there exist $\epsilon > 0$ and $t_0 \in I$ such that for any $\delta > 0$ one can find $u_0 \in \Omega$, $0 \leqslant u_0 \leqslant \delta e$ and $t > t_0$ so that the inequality $u^-(t; t_0, u_0) \nless \epsilon e$ should hold (We shall note that the symbol $\nless$ is not equivalent to the symbol $\geqslant$ and means that there exists $j \in \{1, 2, \ldots, m\}$ such that $u_j^-(t; t_0, u_0) \geqslant \epsilon$).

Definition 7. The solutions of system (2) are called:

(a) *Equi-bounded* if for any $t_0 \in I$ and $\alpha > 0$ there exists a positive function $\beta = \beta(t_0, \alpha)$ which is continuous in t_0 for any $\alpha > 0$ and is such that if $0 \leqslant u_0 \leqslant \alpha e$ and $t \in \mathscr{I}^+(t_0, u_0)$, then $u^+(t; t_0, u_0) \leqslant \beta e$.

(b) *Uniformly bounded* if the function β from (a) does not depend on t_0.

(c) *Ultimately bounded for bound* N if there exists a positive number $N > 0$ and for any $t_0 \in I$ and $\alpha > 0$ there exists a positive number $T = T(t_0, \alpha)$ such that if $0 \leqslant u_0 \leqslant \alpha e$, then $t_0^+ T \in \mathscr{I}^+(t_0, u_0)$ and $u^+(t; t_0, u_0) < Ne$ for $t \geqslant t_0 + T$, $t \in \mathscr{I}^+(t_0, u_0)$.

(d) Uniformly ultimately bounded for bound N if the number T from (c) does not depend on t_0.

In the further considerations we shall use piecewise continuous auxiliary functions which are an analogue of the classical Lyapunov's function [10].

Definition 8. We shall say that the vector-valued function $V: I \times \mathbb{R}^n \to \mathbb{R}^m$, $V = (V_1, \ldots, V_m)$ belongs to the class $\mathscr{V}_0$ if it satisfies the following conditions:

(i) The function V is continuous in each of the sets

$$G_i = \{(t, x) \in I \times \mathbb{R}^n : t_{i-1} < t < t_i\}, \quad i = 1, 2, \ldots, t_0 = 0$$

and $V(t, 0) = 0$ for $t \in I$.

(ii) The function V is locally Lipschitz continuous on x in each of the sets G_i.

(iii) For each $x_0 \in \mathbb{R}^n$ and for any $i = 1, 2, \ldots$ there exist the finite limits

$$V(t_i - 0, x_0) = \lim_{\substack{(t,x) \to (t_i, x_0) \\ (t,x) \in G_i}} V(t, x); \qquad V(t_i + 0, x_0) = \lim_{\substack{(t,x) \to (t_i, x_0) \\ (t,x) \in G_{i+1}}} V(t, x)$$

and the following equality holds

$$V(t_i - 0, x_0) = V(t_i, x_0).$$

Let $V \in \mathscr{V}_0$. For $(t, x) \in \bigcup_{i=1}^{\infty} G_i$ we define the function

$$\dot{V}_{(1)}(t, x) = \limsup_{h \to 0^+} h^{-1}[V(t + h, x + hf(t, x)) - V(t, x)].$$

It is immediately verified that if $x(t)$ is any solution of system (1), then $\dot{V}_{(1)}(t, x(t)) = \mathrm{D}^+ V(t, x(t))$, $t \neq t_i$, where $\mathrm{D}^+ V(t, x(t))$ is the upper right Dini derivative of the function $V(t, x(t))$.

Further on we shall denote by $\mathcal{K}$ the class of all continuous and strictly increasing functions $a\colon I \to I$ such that $a(0) = 0$.

3. Main results

In the proof of the main theorems we shall use the following lemmas:

Lemma 1 [10, Lemma 2]. *Let the following conditions be fulfilled*:

(i) *The function* $F\colon I \times \Omega \to \mathbb{R}^m$ *is continuous and quasi-monotonely increasing in* $I \times \Omega$.

(ii) *The functions* $B_i\colon \Omega \to \mathbb{R}^m$, $i = 1, 2, \ldots$, *are such that the functions* $\psi_i(u) = u + B_i(u)$ *are monotonely increasing in* Ω.

(iii) *The function* $u^+\colon (t_0, \omega) \to \mathbb{R}^m$ *is the maximal solution of system* (2) *for which* $u^+(t_0 + 0) = u_0$, $(t_0, u_0) \in I \times \Omega$ *and* $u^+(t_i + 0) \in \Omega$ *if* $t_i \in (t_0, \omega)$.

(iv) *The function* $w\colon (t_0, \tilde{\omega}) \to \Omega$ $(\tilde{\omega} \leqslant \omega)$ *is piecewise continuous with points of discontinuity of first type* $\{t_i\}$ *at which it it left continuous and such that*

$$w(t_i + 0) \in \Omega \quad \text{if } t_i \in (t_0, \tilde{\omega}),$$

$$w(t_0 + 0) \leqslant u_0$$

$$\mathrm{D}w(t) \leqslant F(t, w(t)) \quad \text{for } t \in (t_0, \tilde{\omega}), \quad t \neq t_i,$$

where $\mathrm{D}w(t)$ *is any of the Dini derivatives of the function* $w(t)$,

$$w(t_i + 0) \leqslant \psi_i(w(t_i)) \quad \text{if } t_i \in (t_0, \tilde{\omega}).$$

Then $w(t) \leqslant u^+(t)$ for $t \in (t_0, \tilde{\omega})$.

Lemma 2 [10, Lemma 3]. *Let the following conditions be fulfilled*:

(i) *Conditions* (i) *and* (ii) *of Lemma* 1 *hold.*

(ii) *The function* $u^-\colon (t_0, \omega) \to \mathbb{R}^m$ *is the minimal solution of system* (2) *for which* $u^-(t_0 + 0) = u_0$, $(t_0, u_0) \in I \times \Omega$ *and* $u^-(t_i + 0) \in \Omega$ *if* $t_i \in (t_0, \omega)$.

(iii) *The function* $w\colon (t_0, \tilde{\omega}) \to \Omega$ $(\tilde{\omega} \leqslant \omega)$ *is piecewise continuous with points of discontinuity of first type* $\{t_i\}$ *at which it is left continuous and such that*

$$w(t_i + 0) \in \Omega \quad \text{if } t_i \in (t_0, \tilde{\omega})$$

$$w(t_0 + 0) \geqslant u_0$$

$$\mathrm{D}w(t) \geqslant F(t, w(t)) \quad \text{for } t \in (t_0, \tilde{\omega}), \quad t \neq t_i$$

$$w(t_i + 0) \geqslant \psi_i(w(t_i)) \quad \text{for } t_i \in (t_0, \tilde{\omega}).$$

Then $w(t) \geqslant u^-(t)$ *for* $t \in (t_0, \tilde{\omega})$.

Lemma 3. *Let the following conditions be fulfilled*:

(i) *Conditions* (i), (ii) *and* (iii) *of Lemma* 1 *hold.*

(ii) *The functions* $k\colon I \to (0, \infty)$ *and* $w\colon (t_0, \tilde{\omega}) \to \mathbb{R}^m$ $(\tilde{\omega} \leqslant \omega)$ *are piecewise continuous with points of discontinuity* $\{t_i\}$ *at which they are left continuous and are such that* $k(t)w(t) \in \Omega$ *for* $t \in (t_0, \tilde{\omega})$ *and* $k(t_i + 0)w(t_i + 0) \in \Omega$ *if* $t_i \in (t_0, \tilde{\omega})$.

(iii) *The following inequalities hold*

$$k(t_0+0)w(t_0+0) \leqslant u_0,$$

$$\mathrm{D}k(t)w(t) \leqslant F(t,\, k(t)w(t))$$

for $t \in (t_0,\, \tilde{\omega})$, $t \neq t_i$.

$$k(t_i+0)w(t_i+0) \leqslant \psi_i(k(t_i)w(t_i)) \quad \text{if } t_i \in (t_0,\, \tilde{\omega}).$$

Then $k(t)w(t) \leqslant u^+(t)$ *for* $t \in (t_0,\, \tilde{\omega})$.

The proof of Lemma 3 is analogous to the proof of Lemma 1. Lemma 3.2 from [12] is used.

Theorem 1. *Let the following conditions be fulfilled*:

(i) *Conditions* (A), (B) *and* (C) *hold.*

(ii) *There exists a function* $V: I \times \mathbb{R}^n \to \mathbb{R}^m$, $V \in \mathscr{V}_0$ *and*

$$\sup_{I \times \mathbb{R}^n} \| V(t,\, x) \| < K \leqslant \infty, \qquad \Omega = \{ u \in \mathbb{R}^m: \|u\| < K \}.$$

(iii) *There exists a function* $F: I \times \Omega \to \mathbb{R}^m$ *which is continuous and quasi-monotonely increasing in* $I \times \Omega$ *and* $F(t, 0) = 0$ *for* $t \in I$.

(iv) *There exist functions* $B_i: \Omega \to \mathbb{R}^m$, $i = 1, 2, \ldots$ *such that the functions* $\psi_i(u) = u + B_i(u)$ *are monotonely increasing in* Ω *and* $B_i(0) = 0$.

(v) *The following inequalities hold*:

(a) $a(\| g(t,\, x) \|)e \leqslant V(t,\, x) \leqslant \gamma(t)b(\| g(t,\, x) \|)e$ *for* $(t,\, x) \in I \times \mathbb{R}^n$ *where* $a,\, b \in \mathscr{K}$ *and the function* $\gamma(t) \geqslant 1$ *is defined and continuous for* $t \in I$.

(b) $\dot{V}_{(1)}(t,\, x) \leqslant F(t,\, V(t,\, x))$ *for* $(t,\, x) \in \bigcup_{i=1}^{\infty} G_i$.

(c) $V(t_i+0,\, x+I_i(x)) \leqslant \psi_i(V(t_i,\, x))$ *for* $x \in \mathbb{R}^n$, $i = 1, 2, \ldots$.

Then:

(A) *If the zero solution of system* (2) *is stable, then the zero solution of system* (1) *is stable with respect to the function* $g(t,\, x)$.

(B) *If the zero solution of system* (2) *is globally equi-attractive, then the zero solution of system* (1) *is globally equi-attractive with respect to the function* $g(t,\, x)$.

Proof. (A) Let $t_0 \in I$ and $\epsilon > 0$ ($a(\epsilon) < K$) be given. From the stability of the zero solution of system (2) it follows that there exists a positive function $\delta^* = \delta^*(t_0,\, \epsilon)$ which is continuous in t_0 for any $\epsilon > 0$ and such that if $0 \leqslant u_0 \leqslant \delta^* e$ and $t \in \mathscr{I}^+(t_0,\, u_0)$, then $u^+(t;\, t_0,\, u_0) < a(\epsilon)e$. Then from condition (C) it follows that $\mathscr{I}^+(t_0,\, u_0) = (t_0,\, \infty)$.

Set

$$\delta = \delta(t_0,\, \epsilon) = b^{-1}(\delta^*(t_0,\, \epsilon)/\gamma(t_0)).$$

Let $x_0 \in M_{t_0}(n-k)(\bar{\delta})$. This shows that $\gamma(t_0)b(\| g(t_0,\, x_0) \|) \leqslant \delta^*$. Then from condition (a) of Theorem 1 we obtain

$$V(t_0+0,\, x_0) \leqslant \gamma(t_0)b(\| g(t_0,\, x_0) \|)e \leqslant \delta^* e.$$

Hence $u^+(t;\, t_0,\, V(t_0+0,\, x_0)) < a(\epsilon)e$ for $t > t_0$.

On the other hand, if $x(t) = x(t;\, t_0,\, x_0)$ is a solution of system (1), then from the conditions of Theorem 1 it follows that the function $w(t) = V(t,\, x(t))$ satisfies the conditions of Lemma 1.

Hence $V(t, x(t)) \leqslant u^+(t; t_0, V(t_0+0, x_0))$ for $t \in \mathscr{I}^+(t_0, x_0)$. Then, in view of (a), we obtain the inequalities

$$a(\| g(t, x(t)) \|)e \leqslant V(t, x(t)) \leqslant u^+(t; t_0, V(t_0+0, x_0)) < \alpha(\epsilon)e$$

for $t \in \mathscr{I}^+(t_0, x_0)$. Hence $\| g(t, x(t)) \| < \epsilon$ for $t > t_0$.

(B) Let $t_0 \in I$, $\alpha > 0$, $\epsilon > 0$ $(\alpha(\epsilon) < K$, $\alpha < b^{-1}(K/\gamma(t_0)))$. Set $\alpha^* = \gamma(t_0)b(\alpha)$. From the equi-attraction of the zero solution of system (2) it follows that there exists a positive number $T = T(t_0, \alpha, \epsilon)$ such that if $0 \leqslant u_0 \leqslant \alpha^* e$, then $t_0 + T \in \mathscr{I}^+(t_0, u_0)$ and $u^+(t; t_0, u_0) < a(\epsilon)e$ for $t \geqslant t_0 + T$, $t \in \mathscr{I}^+(t_0, u_0)$. Hence $\mathscr{I}^+(t_0, u_0) = (t_0, \infty)$.

Let $x_0 \in M_{t_0}(n-k)(\bar{\alpha})$. This means that $\gamma(t_0)b(\| g(t_0, x_0) \|) \leqslant \alpha^*$. Then from (a) it follows that $V(t_0+0, x_0) \leqslant \gamma(t_0)b(\| g(t_0, x_0) \|)e \leqslant \alpha^* e$. Hence $u^+(t; t_0, V(t_0+0, x_0)) < a(\epsilon)e$ for $t \geqslant t_0 + T$.

Moreover, if $x(t) = x(t; t_0, x_0)$ is a solution of system (1), then from Lemma 1 it follows that $V(t, x(t)) \leqslant u^+(t; t_0, V(t_0+0, x_0))$ for $t > t_0$. Then, in view of condition (a) of Theorem 1, we obtain the inequalities

$$a(\| g(t, x(t)) \|)e \leqslant V(t, x(t)) \leqslant u^+(t; t_0, V(t_0+0, x_0)) < a(\epsilon)e$$

for $t \geqslant t_0 + T$. Hence $\| g(t, x(t)) \| < \epsilon$ for $t \geqslant t_0 + T$.

This completes the proof of Theorem 1. □

Remark 3. It is known that the function $V \in \mathscr{V}_0$ is positively definite if there exists a function $a \in \mathscr{K}$ such that $a(\| x \|) \leqslant V(t, x)$ for $(t, x) \in I \times \mathbb{R}^n$. If $V(t, x)$ is positively definite, the inequality $V(t, x) \leqslant \gamma(t)b(\| x \|)$, $b \in \mathscr{K}$, holds as well. Hence in the case when $g(t, x) = x$, condition (a) of Theorem 1 is equivalent to the condition $V(t, x)$ to be positively definite. We shall note that in the general case we consider the inequality $a(\| g(t, x) \| \leqslant V(t, x)$ does not imply condition (a) [12].

Corollary 1. *Let the following conditions be fulfilled*:

(i) *Conditions* (A) *and* (B) *hold.*

(ii) *There exists a function* $V: I \times \mathbb{R}^n \to \mathbb{R}$, $V \in \mathscr{V}_0$, *functions* $a, b, c \in \mathscr{K}$ *and functions* $h(t) \geqslant 0$ *and* $\gamma(t) \geqslant 1$ *defined and continuous for* $t \in I$ *for which the following relations hold*:

(a) $a(\| g(t, x) \|) \leqslant V(t, x) \leqslant \gamma(t)b(\| g(t, x) \|$ *for* $(t, x) \in I \times \mathbb{R}^n$,

(b) $\dot{V}_{(1)}(t, x) \leqslant -h(t)c(\| g(t, x) \|)$ *for* $x \in \mathbb{R}^n$, $t \neq t_i$,

(c) $V(t_i+0, x+I_i(x)) \leqslant V(t_i, x)$, $i = 1, 2, \ldots, x \in \mathbb{R}^n$,

(d) $\int_0^\infty h(s)c[b^{-1}(\eta/\gamma(s))]\, \mathrm{d}s = \infty$ *for any sufficiently small* $\eta > 0$.

Then the zero solution of system (1) *is globally equi-asymptotically stable with respect to* $g(t, x)$.

Proof. Consider the scalar equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -h(t)c\left[b^{-1}\left(\frac{u}{\gamma(t)}\right)\right]$$

without impulses (i.e. $B_i(x) \equiv 0$).

From condition (d) it follows that the zero solution of this equation is globally equi-asymptotically stable.

Then from Theorem 1 it follows that the zero solution of system (1) is globally equi-asymptotically stable with respect to the function $g(t, x)$.

Theorem 2. *Let the following conditions be fulfilled*:

(i) *Conditions* (i)–(iv) *of Theorem* 1 *hold.*

(ii) *The inequalities that follow are satisfied*:

(a) $a(\|g(t, x)\|)e \leqslant V(t, x) \leqslant b(\|g(t, x)\|)e$ *for* $(t, x) \in I \times \mathbb{R}^n$, $a, b \in \mathscr{K}$.

(b) $\dot{V}_{(1)}(t, x) \leqslant F(t, V(t, x))$ *for* $(t, x) \in \bigcup_{i=1}^{\infty} G_i$

(c) $V(t_i + 0, x + I_i(x)) \leqslant \psi_i(V(t_i, x))$, $x \in \mathbb{R}^n$, $i = 1, 2, \ldots$.

Then:

(A) *If the zero solution of system* (2) *is uniformly stable, then the zero solution of system* (1) *is uniformly stable with respect to the function* $g(t, x)$.

(B) *If the zero solution of system* (2) *is uniformly globally attractive, then the zero solution of system* (1) *is uniformly globally attractive with respect to the function* $g(t, x)$.

The proof of Theorem 2 is analogous to the proof of Theorem 1. It suffices to note that in this case we can choose the function δ^* (hence the function δ as well) and the number T independent of t_0.

Theorem 3. *Let the conditions of Theorem* 1 *hold and let* $a(r) \to \infty$ *for* $r \to \infty$. *Then*:

(A) *If the solutions of system* (2) *are equi-bounded, then the solutions of system* (1) *are equi-bounded with respect to the function* $g(t, x)$.

(B) *If the solutions of system* (2) *are ultimately bounded for bound* N, *then the solutions of system* (1) *are ultimately bounded for bound* $a^{-1}(N)$ *with respect to the function* $g(t, x)$.

Proof. (A) Let $t_0 \in I$ and $\alpha > 0$ $(\alpha < b^{-1}(K/\gamma(t_0)))$ be given. Set $\alpha^* = \gamma(t_0)b(\alpha)$. Then from the condition $a(r) \to \infty$ for $r \to \infty$ it follows that $\alpha \to \infty$ for $\alpha^* \to \infty$.

From the equi-boundedness of the solutions of system (2) it follows that there exists a positive function $\beta_1 = \beta_1(t_0, \alpha)$ which is continuous in t_0 for any $\alpha > 0$ and is such that if $0 \leqslant u_0 \leqslant \alpha^* e$ and $t \in \mathscr{J}^+(t_0, u_0)$, then $u^+(t; t_0, u_0) < \beta_1 e$. Hence $\mathscr{J}^+(t_0, u_0) = (t_0, \infty)$.

Set $\beta = \beta(t_0, \alpha) = a^{-1}(\beta_1(t_0, \alpha))$.

Let $x_0 \in M_{t_0}(n-k)(\bar{\alpha})$. Then $\gamma(t_0)b(\|g(t_0, x_0)\|) \leqslant \alpha^*$ and since $V(t_0 + 0, x_0) \leqslant \gamma(t_0)b(\|g(t_0, x_0)\|)e$, then $V(t_0 + 0, x_0) \leqslant \alpha^* e$. Hence $u^+(t; t_0, V(t_0 + 0, x_0)) < \beta_1 e$ for $t > t_0$.

On the other hand, from Lemma 1 it follows that if $x(t) = x(t; t_0, x_0)$ is a solution of system (1), then $V(t, x(t)) \leqslant u^+(t; t_0, V(t_0 + 0, x_0))$ for $t > t_0$. Hence, in view of condition (a), we obtain the inequalities $a(\|g(t, x(t))\|)e \leqslant V(t, x(t)) \leqslant u^+(t; t_0, V(t_0 + 0, x_0)) < \beta_1 e$ for $t > t_0$, whence it follows that $\|g(t, x(t))\| < a^{-1}(\beta_1) = \beta$ for $t > t_0$.

(B) Let $t_0 \in I$ and $\alpha > 0$ $(\alpha < b^{-1}(K/\gamma(t_0)))$. Set $\alpha^* = \gamma(t_0)b(\alpha)$.

From the ultimate boundedness of the solutions of system (2) it follows that there exist positive numbers N and $T = T(t_0, \alpha)$ such that if $0 \leqslant u_0 \leqslant \alpha^* e$ and $t \geqslant t_0 + T$, then $u^+(t; t_0, u_0) < Ne$.

Let $x_0 \in M_{t_0}(n-k)(\bar{\alpha})$. Then $\gamma(t_0)b(\|g(t_0, x_0)\|) \leqslant \alpha^*$. In view of condition (a) we obtain $V(t_0 + 0, x_0) \leqslant \alpha^* e$. Hence $u^+(t; t_0, V(t_0 + 0, x_0)) < Ne$ for $t \geqslant t_0 + T$.

Let $x(t) = x(t; t_0, x_0)$ be a solution of system (1). From Lemma 1 we obtain that for $t > t_0$ the inequality $V(t, x(t)) \leqslant u^+(t; t_0, V(t_0 + 0, x_0))$ holds. Hence

$$a(\|g(t, x(t))\|)e \leqslant V(t, x(t)) \leqslant u^+(t; t_0, V(t_0 + 0, x_0)) < Ne$$

for $t \geqslant t_0 + T$, whence it follows that $\|g(t, x(t))\| < a^{-1}(N)$ for $t \geqslant t_0 + T$.

Theorem 3 is proved. □

Theorem 4. *Let the conditions of Theorem* 2 *hold and let* $a(r) \to \infty$ *for* $r \to \infty$. *Then*:

(A) *If the solutions of system* (2) *are uniformly bounded, then the solutions of system* (1) *are uniformly bounded with respect to the function* $g(t, x)$.

(B) *If the solutions of system* (2) *are uniformly ultimately bounded for bound* N, *then the solutions of system* (1) *are uniformly ultimately bounded for bound* $a^{-1}(N)$ *with respect to the function* $g(t, x)$.

The proof of Theorem 4 is analogous to the proof of Theorem 3. In this case the function β and the number T can be chosen independent of t_0.

Theorem 5. *Let the following conditions be fulfilled*:

(i) *Conditions* (i), (ii), (iii) *and* (iv) *of Theorem* 1 *hold.*

(ii) *There exists a positive function* $k: I \to (0, \infty)$ *which is piecewise continuous with points of discontinuity of first type* $\{t_i\}$ *at which it is left continuous,* $k(t) \to \infty$ *for* $t \to \infty$ *and* $k(t_i + 0) > 0$ *for* $i = 1, 2, \ldots$

(iii) *The following inequalities hold*:

(a) $a(\|g(t, x)\|)e \leqslant V(t, x) \leqslant \gamma(t)b(\|g(t, x)\|)e$

for $(t, x) \in I \times \mathbb{R}^n$, $a, b \in \mathscr{K}$ *and the function* $\gamma(t) \geqslant 1$ *is defined and continuous for* $t \in I$.

(b) $D^+x(t)V(t, x) \leqslant F(t, k(t)V(t, x))$

for $t \neq t_i$ *where*

$$D^+k(t)V(t, x) = \limsup_{h \to 0^+} h^{-1}[k(t+h)V(t+h, x+hf(t, x)) - k(t)V(t, x)]$$

(c) $k(t_i + 0)V(t_i + 0, x + I_i(x)) \leqslant \psi_i(k(t_i)V(t_i, x)), \quad i = 1, 2, \ldots$.

Then, if the zero solution of system (2) is stable, then the zero solution of system (1) is globally equi-asymptotically stable with respect to the function $g(t, x)$.

Proof. Let $\lambda = \inf_{t \geqslant 0} k(t)$. From condition (ii) of Theorem 5 it follows that $\lambda > 0$.

Let $t_0 \in I$ and $\epsilon > 0$ $(a(\epsilon) < K)$. From the stability of the zero solution of system (2) it follows that there exists a positive function $\delta^* = \delta^*(t_0, \epsilon)$ which is continuous in t_0 for any $\epsilon > 0$ and is such that if $0 \leqslant u_0 \leqslant \delta^* e$ and $t > t_0$, then $u^+(t; t_0, u_0) < \lambda a(\epsilon)e$. Set

$$\delta = \delta(t_0, \epsilon) = b^{-1}(\delta^*(t_0, \epsilon)/\gamma(t_0)k(t_0)).$$

Then, if $x_0 \in M_{t_0}(n-k)(\bar{\delta})$, then $k(t_0)\gamma(t_0)b(\|g(t_0, x_0)\|) \leqslant \delta^*$, whence, in view of condition (a) we obtain

$$k(t_0)V(t_0 + 0, x_0) \leqslant k(t_0)\gamma(t_0)b(\|g(t_0, x_0)\|)e \leqslant \delta^* e.$$

Hence $u^+(t; t_0, k(t_0)V(t_0 + 0, x_0)) < \lambda\alpha(\epsilon)e$ for $t > t_0$.

Let $x(t) = x(t; t_0, x_0)$ be a solution of system (1). From Lemma 3 it follows that the inequality $k(t)V(t, x(t)) \leqslant u^+(t; t_0, k(t_0)V(t_0 + 0, x_0))$ holds for all $t > t_0$.

Then for $t > t_0$ the following inequalities hold

$$\lambda a(\| g(t, x(t)) \|)e \leqslant k(t)V(t, x(t))$$
$$\leqslant u^+(t; t_0, k(t_0)V(t_0+0, x_0)) < \lambda a(\epsilon)e$$

whence it follows that $\| g(t, x(t)) \| < \epsilon$ for $t > t_0$, i.e. the zero solution of system (1) is stable.

We shall prove that the zero solution of system (1) is globally equi-attractive.

Let $t_0 \in I$, $\eta > 0$. From the stability of the zero solution of system (2) it follows that there exists a positive function $\delta^* = \delta^*(t_0, \eta)$ which is continuous in t_0 for any $\eta > 0$ and is such that if $0 \leqslant u_0 \leqslant \delta^* e$ and $t > t_0$, then $u^+(t; t_0, u_0) < \eta e$.

Let $t_0 \in I$, $\alpha > 0$ and $\epsilon > 0$ be given. Choose the number $\eta > 0$ so that the following equality should hold:

$$\alpha = b^{-1}(\delta^*(t_0, \eta)/k(t_0)\gamma(t_0)).$$

Then, if $x_0 \in M_{t_0}(n-k)(\bar{\alpha})$, then $k(t_0)\gamma(t_0)b(\| g(t_0, x_0) \|) \leqslant \delta^*$. On the other hand, from condition (a) of Theorem 5 we obtain

$$k(t_0)V(t_0+0, x_0) \leqslant k(t_0)\gamma(t_0)b(\| g(t_0, x_0) \|)e,$$

i.e. $k(t_0)V(t_0+0, x_0) \leqslant \delta^* e$. Hence $u^+(t; t_0, k(t_0)V(t_0+0, x_0)) < \eta e$ for $t > t_0$.

Let $x(t) = x(t; t_0, x_0)$ be a solution of system (1). Applying Lemma 3, we obtain that for $t > t_0$ the inequality $k(t)V(t, x(t)) \leqslant u^+(t; t_0, k(t_0)V(t_0+0, x_0))$ holds. Then, from inequality (a) we obtain

$$k(t)a(\| g(t, x(t)) \|)e \leqslant k(t)V(t, x(t)) \leqslant u^+(t; t_0, k(t_0)V(t_0+0, x_0)) < \eta e$$

for $t > t_0$, whence it follows that

$$\| g(t, x(t)) \| < a^{-1}(\eta/k(t)).$$

Since $k(t) \to \infty$ for $t \to \infty$, then $a^{-1}(\eta/k(t)) \to 0$ for $t \to \infty$, hence there exists a positive number $T^* = T^*(t_0, \alpha, \epsilon)$, such that if $t \geqslant T^*(t_0, \alpha, \epsilon)$, then $\| g(t, x(t)) \| < \epsilon$.

Let $T = T(t_0, \alpha, \epsilon) = T^*(t_0, \alpha, \epsilon) - t_0$. Then for $t \geqslant t_0 + T$ the inequality $\| g(t, x(t)) \| < \epsilon$ holds, i.e. the zero solution of system (1) is globally equi-attractive. Theorem 5 is proved. □

Theorem 6. *Let the conditions of Theorem* 5 *be fulfilled and let* $a(r) \to \infty$ *for* $r \to \infty$.

Then, if the solutions of system (2) *are equi-bounded, then the solutions of system* (1) *are equi-bounded and ultimately bounded with respect to the function* $g(t, x)$.

Proof. Let $\lambda = \inf_{t \geqslant 0} k(t)$. Then $\lambda > 0$.

Let $t_0 \in I$ and $\alpha > 0$. Set $\alpha^* = k(t_0)\gamma(t_0)b(\alpha)$. From the condition $a(r) \to \infty$ for $r \to \infty$ it follows that $\alpha \to \infty$ for $\alpha^* \to \infty$.

From the equi-boundedness of the solutions of system (2) it follows that there exists a positive function $\beta_1 = \beta_1(t_0, \alpha)$ which is continuous in t_0 for any $\alpha > 0$ and is such that if $0 \leqslant u_0 \leqslant \alpha^* e$ and $t > t_0$, then $u^+(t; t_0, u_0) < \lambda\beta_1 e$.

Set $\beta = \beta(t_0, \alpha) = a^{-1}(\beta_1(t_0, \alpha))$.

Let $x_0 \in M_{t_0}(n-k)(\bar{\alpha})$. This means that

$$k(t_0)\gamma(t_0)b(\| g(t_0, x_0) \|) \leqslant \alpha^*$$

and since $k(t_0)V(t_0+0, x_0) \leqslant k(t_0)\gamma(t_0)b(\| g(t_0, x_0) \|)e$, then $k(t_0)V(t_0+0, x_0) \leqslant \alpha^* e$. Hence $u^+(t; t_0, k(t_0)V(t_0+0, x_0)) < \lambda\beta_1 e$ for $t > t_0$.

On the other hand, if $x(t)=x(t;\ t_0,\ x_0)$ is a solution of system (1), then from Lemma 3 it follows that for $t>t_0$ the inequality $k(t)V(t,\ x(t))\leqslant u^+(t;\ t_0,\ k(t_0)V(t_0+0,\ x_0))$ holds.

Hence the inequalities

$$\lambda a(\| g(t,\ x(t))\|)e\leqslant k(t)V(t,\ x(t))\leqslant u^+(t;\ t_0,\ k(t_0)V(t_0+0,\ x_0))<\lambda\beta_1 e$$

are satisfied for any $t>t_0$, whence it follows that $\| g(t,\ x(t))\| <a^{-1}(\beta_1)=\beta$ for $t>t_0$, i.e. the solutions of system (1) are equi-bounded with respect to the function $g(t,\ x)$.

Now let $t_0\in I$ and $\alpha>0$ be given and let $\alpha^*=k(t_0)\gamma(t_0)b(\alpha)$. Choose the positive function $\beta^*=\beta^*(t_0,\ \alpha)$ so that if $0\leqslant u_0\leqslant\alpha^* e$ and $t>t_0$, then $u^+(t;\ t_0,\ u_0)<\beta^* e$.

Let $x_0\in M_{t_0}(n-k)(\bar{\alpha})$. This means that

$$k(t_0)\gamma(t_0)b(\| g(t_0,\ x_0)\|)\leqslant\alpha^*$$

and since $k(t_0)V(t_0+0,\ x_0)\leqslant k(t_0)\gamma(t_0)b(\| g(t_0,\ x_0)\|)e$, then $k(t_0)V(t_0+0,\ x_0)\leqslant\alpha^* e$. Hence $u^+(t;\ t_0,\ k(t_0)V(t_0+0,\ x_0))<\beta^* e$ for $t>t_0$.

Moreover, if $x(t)=x(t;\ t_0,\ x_0)$ is a solution of system (1), then from Lemma 3 it follows that the inequality $k(t)V(t,\ x(t))\leqslant u^+(t;\ t_0,\ k(t_0)V(t_0+0,\ x_0))$ holds for $t>t_0$.

Hence

$$k(t)a(\| g(t,\ x(t))\|)e\leqslant k(t)V(t,\ x(t))\leqslant u^+(t;\ t_0,\ k(t_0)V(t_0+0,\ x_0))<\beta^* e$$

for $t>t_0$, whence it follows that

$$\| g(t,\ x(t))\| <a^{-1}(\beta^*/k(t)).$$

From the condition $k(t)\to\infty$ for $t\to\infty$ it follows that $a^{-1}(\beta^*/k(t))\to 0$ for $t\to\infty$. Hence, if $N>0$ is given, then there exists a positive number $T^*=T^*(t_0,\ \alpha)$ such that for $t\geqslant T^*(t_0,\ \alpha)$ the inequality $\| g(t,\ x(t))\| <N$ should hold. Then, if we set $T=T(t_0,\ \alpha)=T^*(t_0,\ \alpha)-t_0$, we obtain that for $t\geqslant t_0+T$ we have $\| g(t,\ x(t))\| <N$, i.e. the solutions of system (1) are ultimately bounded with respect to the function $g(t,\ x)$.

Theorem 6 is proved. □

Theorem 7. *Let the following conditions be fulfilled*:

(i) *Conditions* (i), (ii), (iii) *and* (iv) *of Theorem* 1 *hold.*

(ii) *For any* $\delta>0$ *and* $t_0\in I$ *there exists* $x_0\in M_{t_0}(n-k)(\delta)$ *such that* $V(t_0,\ x_0)>0$.

(iii) *The following inequalities hold*:

(a) $V(t,\ x)\leqslant a(\| g(t,\ x)\|)e$ *for* $(t,\ x)\in I\times\mathbb{R}^n,\ a\in\mathscr{K}$,

(b) $\dot{V}_{(1)}(t,\ x)\geqslant F(t,\ V(t,\ x))$ *for* $x\in\mathbb{R}^n,\ t\neq t_i$,

(c) $V(t_i+0,\ x+I_i(x))\geqslant\psi_i(V(t_i,\ x)),\ i=1,2,\ldots$.

Then, if the zero solution of system (2) *is unstable, then the zero solution of system* (1) *is unstable with respect to the function* $g(t,\ x)$.

Proof. From the unstability of the zero solution of system (2) it follows that there exist $\epsilon^*>0$ and $t_0\in I$ such that for any $\delta^*>0$ there exist $u_0\in\Omega$: $0\leqslant u_0\leqslant\delta^* e$ and $t^*>t_0,\ t^*\in\mathscr{I}^+(t_0,\ u_0)$, for which the inequality $u^-(t^*;\ t_0,\ u_0)\not<\epsilon^* e$ holds.

Choose the number $\epsilon>0$ so that $a(\epsilon)<\epsilon^*$.

(i) Let $t_0\neq t_i,\ i=1,2,\ldots$ and let $\delta>0$ be given. From condition (ii) of Theorem 7 it follows that we can choose $x_0\in M_{t_0}(n-k)(\delta)$ such that $V(t_0,\ x_0)>0$. Let $\delta^*>0$ be chosen so that $0<\delta^* e\leqslant V(t_0,\ x_0)$. Then there exist $u_0,\ 0\leqslant u_0\leqslant\delta^* e$ and $t^*>t_0,\ t^*\in\mathscr{I}^+(t_0,\ u_0)$ such that

$$u^-(t^*;\ t_0,\ u_0)\not<\epsilon^* e. \tag{3}$$

Let $x(t)=x(t;\, t_0,\, x_0)$ be a solution of system (1). From Lemma 2 it follows that for $t \in \mathscr{I}^+(t_0,\, x_0) \cap \mathscr{I}^+(t_0,\, u_0)$ the following inequality holds

$$V(t,\, x(t)) \geqslant u^-(t;\, t_0,\, u_0). \tag{4}$$

Assume that for any $t \in \mathscr{I}^+(t_0,\, x_0)$ we have

$$\| g(t,\, x(t)) \| < \epsilon. \tag{5}$$

Then from condition (B) we obtain that $\mathscr{I}^+(t_0,\, x_0)=(t_0,\, \infty)$. Applying condition (a) of Theorem 7, (4) and (5), we obtain

$$\epsilon^* e > a(\epsilon) e > a(\| g(t^*,\, x(t^*)) \|) e \geqslant V(t^*,\, x(t^*)) \geqslant u^-(t^*;\, t_0,\, u_0)$$

which contradicts (3).

Hence $\| g(t,\, x(t)) \| \not< \epsilon$ for some $t \in \mathscr{I}^+(t_0,\, x_0)$.

(ii) Let $t_0=t_i$ for some $i \in \mathbb{N}$. We shall prove that in this case for any $\delta>0$ there exists $x_0 \in M_{t_0}(n-k)(\delta)$ such that $V(t_0+0,\, x_0)>0$.

Suppose that this is not true, i.e. that there exists $\delta>0$ such that for any $x_0 \in M_{t_0}(n-k)(\delta)$ and for some $s \in \{1, 2, \dots, m\}$ the following inequality holds

$$V_s(t_0+0,\, x_0) \leqslant 0. \tag{6}$$

From the continuity of the function $g(t_0,\, x)$ for $x=0$ and condition (A5) it follows that there exists a number δ_1 $(0<\delta_1<\delta)$ such that if $\| x+I_i(x) \| < \delta_1$, then

$$\| g(t_0,\, x+I_i(x)) \| < \delta. \tag{7}$$

On the other hand, from the continuity of the function $I_i(x)$ for $x=0$ and condition (A5) it follows that there exists a number δ_2 $(0<\delta_2<\delta_1)$ such that if $x \in M_{t_0}(n-k)(\delta_2)$, then the following inequality holds

$$\| x+I_i(x) \| < \delta_1. \tag{8}$$

Let $x_0 \in M_{t_0}(n-k)(\delta_2)$ be such that $V(t_0,\, x_0)>0$. Then, in view of (8), (7) and (6) and condition (c) of Theorem 7, we obtain the contradiction

$$0 \geqslant V_s(t_0+0,\, x_0+I_i(x_0)) \geqslant \psi_{is}(V(t_0,\, x_0))>0.$$

Further on we can carry out the proof as for item (i) with the only difference that we choose the number δ^* so that the inequality $0<\delta^* e \leqslant V(t_0+0,\, x_0)$ should hold.

This completes the proof of Theorem 7. □

4. Examples

Example 1. Consider the scalar equation with impulse effect at fixed moments

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t}=[\operatorname{sinlog}(t+1)+\operatorname{coslog}(t+1)-\alpha]x, & t \neq t_i, \\ \Delta x \mid_{t=t_i}=\beta x, & i=1, 2, \dots, \end{cases} \tag{9}$$

where $-1<\beta \leqslant 0$.

We shall make use of the function

$$V(t,\, x)=x^2 \exp[2(t+1)(\alpha-\operatorname{sinlog}(t+1))].$$

If $\alpha > 1$, then $V(t, x)$ is positively definite. Moreover, $\dot{V}_{(9)}(t, x) = 0$, $t \neq t_i$ and $V(t_i + 0, x + \beta x) = (1+\beta)^2 V(t_i, x) \leqslant V(t_i, x)$.

The zero solution of the equation

$$\begin{cases} du/dt = 0, & t \neq t_i, \\ \Delta u\,|_{t=t_i} = 0 \end{cases} \tag{10}$$

is stable. Then from Theorem 1 it follows that the zero solution of system (9) is stable with respect to the function $g(t, x) = x$.

The conditions of Theorem 5 are satisfied as well. Hence the zero solution of equation (9) is globally equi-asymptotically stable with respect to the function $g(t, x) = x$.

Example 2. Consider the system

$$\begin{cases} dx/dt = e^{-t}x + y \sin t - \left(x^3 + xy^2\right) \sin^2 t, & t \neq t_i \\ dy/dt = x \sin t + e^{-t}\, y - \left(x^2 y + y^3\right) \sin^2 t, & t \neq t_i \\ \Delta x\,|_{t=t_i} = ax + by, \qquad \Delta y\,|_{t=t_i} = bx + ay, \end{cases} \tag{11}$$

where

$$a = \tfrac{1}{2}\left(\sqrt{1+c_1} + \sqrt{1+c_2} - 2\right), \quad b = \tfrac{1}{2}\left(\sqrt{1+c_1} - \sqrt{1+c_2}\right),$$
$$-1 < c_1 \leqslant 0, \qquad -1 < c_2 \leqslant 0.$$

Consider the comparison system

$$\begin{cases} du/dt = (e^{-t} + \sin t)u, & t \neq t_i \\ dv/dt = (e^{-t} - \sin t)v, & t \neq t_i, \\ \Delta u\,|_{t=t_i} = c_1 u, \qquad \Delta v\,|_{t=t_i} = c_2 v. \end{cases} \tag{12}$$

We shall make use of the vector-valued function

$$V(t, x, y) = \left((x+y)^2, (x-y)^2\right)^{\mathrm{T}} \tag{13}$$

Straightforward calculations show that

$$\max_{i=1,2} V_i(t, x, y) = \begin{cases} (x+y)^2 & \text{for } xy > 0, \\ (x-y)^2 & \text{for } xy < 0, \\ x^2 & \text{for } y = 0, \\ y^2 & \text{for } x = 0. \end{cases}$$

Hence $V(t, x, y)$ is positively definite and $V(t, x, y) \to 0$ for $x^2 + y^2 \to 0$ uniformly on $t \in I$. Moreover, the inequality

$$\dot{V}_{(11)}(t, x, y) \leqslant \begin{pmatrix} e^{-t} + \sin t & 0 \\ 0 & e^{-t} - \sin t \end{pmatrix} V(t, x, y)$$

holds for $t \neq t_i$. We have

$$V(t_i+0,\ x+ax+by,\ y+bx+ay) = V(t_i,\ x,\ y) + \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} V(t_i,\ x,\ y).$$

Hence the conditions of Theorem 2 are satisfied and since the zero solution of system (12) is uniformly stable, then the zero solution of system (11) is uniformly stable with respect to the function $g(t,\ x,\ y) = \sqrt{x^2+y^2}$.

Moreover, the conditions of Theorem 4 are satisfied and since the solutions of system (12) are uniformly bounded, then the solutions of system (11) are uniformly bounded with respect to the function $g(t,\ x,\ y) = \sqrt{x^2+y^2}$.

Example 3. Consider the system

$$\begin{cases} \mathrm{d}x/\mathrm{d}t = x(2+y)\sin t, & t \neq t_i, \\ \mathrm{d}y/\mathrm{d}t = (2\alpha x + y)\sin t, & t \neq t_i, \\ \Delta x|_{t=t_i} = -\tfrac{1}{2}x, & \Delta y|_{t=t_i} = (1/\sqrt{2}-1)y. \end{cases} \tag{14}$$

Let $g(t,\ x,\ y) = y^2 - 4\alpha x$ and $V(t,\ x,\ y) = g^2 = (y^2-4\alpha x)^2$.

We shall make use of the comparison equation

$$\begin{cases} \mathrm{d}u/\mathrm{d}t = 4u\sin t, & t \neq t_i, \\ \Delta u|_{t=t_i} = -\tfrac{3}{4}u. \end{cases} \tag{15}$$

Straightforward calculations yield

$$\dot{V}_{(14)}(t,\ x,\ y) = 4V(t,\ x,\ y)\sin t \quad \text{for } t \neq t_i$$

and

$$V\left(t_i+0,\ x-\tfrac{1}{2}x,\ y+(1/\sqrt{2}-1)y\right) = \tfrac{1}{4}V(t_i,\ x,\ y) = V(t_i,\ x,\ y) - \tfrac{3}{4}V(t_i,\ x,\ y).$$

Moreover, the zero solution of equation (15) is stable. Applying Theorem 1(A), we obtain that the zero solution of system (14) is stable with respect to the function $g(t,\ x,\ y) = y^2 - 4\alpha x$.

The solutions of equation (15) are equi-bounded. Then by Theorem 3(A) the solutions of system (14) are equi-bounded with respect to the function $g(t,\ x,\ y) = y^2 - 4\alpha x$.

Example 4. Consider the system

$$\begin{cases} \mathrm{d}x/\mathrm{d}t = -2x\left[y^2-(2+\sin t)x\right]^2 - \dfrac{x\cos t}{2+\sin t}, & t \neq t_i, \\ \mathrm{d}y/\mathrm{d}t = -y\left[y^2-(2+\sin t)x\right]^2, & t \neq t_i, \\ \Delta x|_{t=t_i} = -\tfrac{1}{2}x,\ \Delta y|_{t=t_i} = (1/\sqrt{2}-1)y. \end{cases} \tag{16}$$

Let $g(t,\ x,\ y) = y^2 - (2+\sin t)x$ and $V(t,\ x,\ y) = g^2$.

Consider the equation

$$\begin{cases} \mathrm{d}u/\mathrm{d}t = -4u^2 & \text{for } t \neq t_i, \\ \Delta u|_{t=t_i} = -\frac{3}{4}u. \end{cases} \tag{17}$$

Straightforward calculations show that

$$\dot{V}_{(16)}(t, x, y) = -4V^2(t, x, y) \quad \text{for } t \neq t_i$$

and

$$V\left(t_i + 0, x - \tfrac{1}{2}x, y + \left(1/\sqrt{2} - 1\right)y\right) = V(t_i, x, y) - \tfrac{3}{4}V(t_i, x, y).$$

Moreover, the zero solution of equation (17) is uniformly globally asymptotically stable. Applying theorem 2, we obtain that the zero solution of system (16) is uniformly globally asymptotically stable with respect to the function g.

The solutions of equation (17) are uniformly bounded and uniformly ultimated bounded, Then from Theorem 4 it follows that the solutions of system (16) are uniformly bounded and uniformly ultimately bounded with respect to the function g.

Example 5. Consider the scalar equation with impulses

$$\begin{cases} \mathrm{d}x/\mathrm{d}t = -h(t)x, & t \neq t_i, \\ \Delta x|_{t=t_i} = c_i x, \end{cases} \tag{18}$$

where the function $h(t)$ is positive and continuous for $t \in I$, $-1 < c_i \leqslant 0$.

If $h(t) \to 0$ for $t \to \infty$ and $\int_0^\infty h(t)\mathrm{d}t = \infty$, then the zero solution of system (18) is not uniformly globally asymptotically stable.

But for the function $V(t, x) = \frac{1}{2}x^2$ the conditions of Corollary 1 are satisfied.

In fact,

$$\dot{V}_{(18)}(t, x) = -h(t)x^2 \quad \text{for } t \neq t_i,$$

$$V(t_i + 0, x + c_i x) = \tfrac{1}{2}(1 + c_i)^2 x^2 \leqslant \tfrac{1}{2}x^2 = V(t_i, x).$$

Hence the zero solution of system (18) is globally equi-asymptotically stable with respect to the function $g(t, x) = x$.

This example shows that the conditions of Corollary 1 do not imply uniform global asymptotic stability of the zero solution.

References

[1] S. Leela, Stability of differential systems with impulsive perturbations in terms of two measures, *Nonlinear Anal.* **1** (6) (1977) 667–677.

[2] T. Pavlidis, Stability of systems described by differential equations containing impulses, *IEEE Trans.* **AC-12** (1967) 43–45.

[3] S.G. Pandit, On the stability of impulsively perturbed differential systems, *Bull. Austral. Math. Soc.* **17** (1977) 423–432.

[4] M. Rama Mohana Rao and V. Sree Hari Rao, Stability of impulsively perturbed systems, *Bull. Austral. Math. Soc.* **16** (1977) 99–110.

[5] V.D. Mil'man and A.D. Myshkis, On the stability of motion in the presence of impulses (in Russian), *Sib. Math. J.* **1** (1960) 233–237.
[6] A.M. Samoilenko and N.A. Perestyuk, On the stability of the solutions of systems with impulse effect (in Russian), *Diff. Uravn.* **11** (1981) 1995–2001.
[7] G.K. Kulev and D.D. Bainov, Application of Lyapunov's direct method to the investigation of the global stability of the solutions of systems with impulse effect, *Appl. Anal.* **26** (1988) 255–270.
[8] G.K. Kulev and D.D. Bainov, Global stability of the solutions of systems with impulse effect, to appear.
[9] P.S. Simeonov and D.D. Bainov, The second method of Lyapunov for systems with impulse effect, *Tamkang J. Math.* **16** (4) (1985) 19–40.
[10] P.S. Simeonov and D.D. Bainov, Stability with respect to part of the variables in systems with impulse effect, *J. Math. Anal. Appl.* **117** (1) (1986) 247–263.
[11] S.G. Hristova and D.D. Bainov, Application of Lyapunov's functions for studying the boundedness of solutions of systems with impulses, *COMPEL* **5** (1) (1986) 23–40.
[12] P. Bhatia and V. Lakshmikantham, An extension of Lyapunov's direct method, *Mich. Math. J.* **12** (1965) 183–191.