Nil Algebras and Unipotent Groups of Finite Width

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INTRODUCTION

One of the most ambitious and successful projects in the theory of pro-p-groups and pronipotent groups was the study of groups of finite width (see [KLP], [S], [SZ]).

Let $G$ be a group and let $G = G_1 \triangleright G_2 \triangleright \cdots$ be its lower central series.

**Definition.**

(a) A residually $p$-group is said to be of finite width if all factors $G_i/G_{i+1}$ are finite groups and orders $|G_i/G_{i+1}|$ are uniformly bounded.

(b) Let $G$ be residually (torsion free) nilpotent. Then $G$ is of finite width if all Betty numbers $b_i = \dim_{\mathbb{Q}}(G_i/G_{i+1} \otimes \mathbb{Q})$ are uniformly bounded.

Let $K$ be $\mathbb{Z}/p\mathbb{Z}$ in the case (a) or $\mathbb{Q}$ in the case (b), respectively. Consider the Lie algebra $L = \bigoplus_{i \geq 1} L_i$, $L_i = G_i/G_{i+1} \otimes K$ with the bracket $[a_{G_i}, b_{G_{i+1}}] = (a_i, b_j) G_i \otimes G_{i+1}$, where $(a_i, b_j) = a_i^{-1} b_j^{-1} a_i b_j$ stands for the group commutator. Clearly, $L = L_1 + L_2 + \cdots$ is a $\mathbb{Z}$-gradation on $L$. If the group $G$ has finite width, then the dimensions of all homogeneous components $L_i$ are uniformly bounded. In particular, the Gelfand–Kirillov dimension of $L$ is $\leq 1$.

In [G] R. Grigorchuk constructed for an arbitrary prime number $p$ a remarkable finitely generated infinite residually finite group $G$ with the following properties:

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(1) $G$ is a $p$-group, i.e., an arbitrary element of $G$ has a finite order which is a power of $p$. In particular, $G$ is a counterexample to the General Burnside Problem,

(2) $G$ has intermediate growth, thus providing an answer to the question by J. Milnor (see [Mi]). Moreover, Rozhkov (see [R]) and Bartholdi and Grigorchuck (see [BG]) proved that all factors of the lower central series of $G$ have orders $p$ or $p^2$. Hence,

(3) $G$ is a group of finite width.

Let $L = \bigoplus_{i \geq 1} L_i$, $L_i = G_i/G_{i+1} \otimes \mathbb{Z}/p\mathbb{Z}$ be the Lie algebra of the group $G$. From (1) it follows (see [VL]) that for an arbitrary homogeneous element $a \in L_i$ its adjoint operator $\text{ad}(a): L \to L; \text{ad}(a): x \mapsto [x, a]$ is nilpotent (we say that the element $a$ is ad-nilpotent). However, the algebra $L$ is not nilpotent. This should be compared with the theory of associative algebras of Gelfand–Kirillov dimension 1 (see [SSW]).

Informally speaking, in this paper we prove that there are no analogues of the Grigorchuk group in zero characteristic.

An algebra is said to be locally nilpotent if every finitely generated subalgebra of it is nilpotent.

**Theorem 1.** Let $L = \bigoplus_{a \in \Gamma} L_a$ be a Lie algebra over a field $F$ of zero characteristic graded by an abelian group $\Gamma$. Suppose that

1. there exists $d > 0$ such that $\dim_F L_a \leq d$ for all $a \in \Gamma$,
2. every homogeneous element $a \in L_a$ is ad-nilpotent.

Then the Lie algebra $L$ is locally nilpotent.

As we have seen above this assertion is not true if the characteristic of $F$ is prime.

Let $V$ be an $F$-vector space which is a module over a group $G$. We say that the action of $G$ is unipotent if for an arbitrary element $g \in G$ there exists a number $n = n(g)$ such that $V(1 - g)^n = (0)$.

A $G$-module $V$ is said to be residually finite if there exists a family $\mathcal{P}$ of $G$-submodules such that each $V' \in \mathcal{P}$ has finite codimension in $V$ and $\bigcap_{V' \in \mathcal{P}} V' = (0)$.

**Theorem 2.** Let $G$ be a group. Suppose that all Betty numbers $h_i = \dim_{Q}(G_i/G_{i+1} \otimes \mathbb{Q})$, $i \geq 1$, are uniformly bounded. Then every finitely generated residually finite unipotent module over $G$ is finite dimensional.
1. PROOF OF THEOREM 1

Let \( A \) be an algebra (not necessarily associative) generated by a subset \( X \). For elements \( x_1, \ldots, x_k \in X \) and for integers \( l_1 \geq 1, \ldots, l_k \geq 1 \) let \( (x_1^{l_1}, \ldots, x_k^{l_k}) \) denote the \( F \)-linear span of all products in \( x_1, \ldots, x_k \) (with all possible arrangements of brackets), involving \( l_1 \) elements \( x_1 \), \( l_2 \) elements \( x_2 \), \ldots, \( l_k \) elements \( x_k \).

**Definition 1.1.** We say that the pair \((A, X)\) satisfies the condition \(C_d\) if for arbitrary elements \( x_1, \ldots, x_k \in X \), arbitrary integers \( l_1 \geq 1, \ldots, l_k \geq 1 \) we have

1. \( \dim_F (x_1, \ldots, x_k) \leq d \),
2. every element from \( (x_1, \ldots, x_k) \) is nilpotent.

If \( A \) is an associative (Jordan) algebra then nilpotency of an element is understood in the usual sense. If \( A \) is a Lie algebra we are speaking about ad-nilpotency.

Let \( L = \bigoplus_{x \in \Gamma} L_x \) be a Lie algebra such that \( \dim_F L_x \leq d \) for all \( \Gamma \) and all homogeneous elements are nilpotent. If \( X \) is a generating set of \( L \) consisting of homogeneous elements then the pair \((L, X)\) satisfies \(C_d\).

Recall that the Baer radical of an associative algebra \( A \) is the smallest ideal \( B(A) \) of \( A \) such that the quotient \( A/B(A) \) does not contain nonzero nilpotent ideals (see [J1]).

**Lemma 1.1.** Let \( A \) be an associative algebra generated by a subset \( X \subseteq A \). If \((A, X)\) satisfies \( C_d \) then \( A = B(A) \).

**Proof.** We will prove the lemma by induction on \( d \). Let \( d = 1 \).

Choose an arbitrary word \( v \) in \( X \) and an arbitrary homogeneous (with respect to all variables) expression \( w \) in \( X \) (in other words, \( w \) lies in the subspace \( (x_1^{l_1}, \ldots, x_k^{l_k}) \), for some \( x_1, \ldots, x_k \in X; l_1, \ldots, l_k \geq 1 \)). Since \( vw \) and \( wv \) lie in a 1-dimensional subspace of \( A \) it follows that \( vw \) lies in the ideal generated by \( v^2 \).

A sequence \( a_1, a_2, \ldots \in A \) is called an \( m \)-sequence if for an arbitrary \( i \geq 1 \) we have \( a_{i+1} \in a_i A a_i \). It is known (see [J1]) that an element \( a \) lies in the Baer radical \( B(A) \) if and only if every \( m \)-sequence starting with \( a \) vanishes.

In view of what we said above, if an \( m \)-sequence \( a_1, a_2, \ldots \) starts with the word \( v \) then for an arbitrary \( k \geq 1 \), \( a_k \) lies in the ideal generated by \( v^2 \). Since \( v \) is a nilpotent element it follows that every \( m \)-sequence starting with \( v \) vanishes. Hence \( v \in B(A) \).
Now suppose that \( d \geq 1 \) and for all pairs \((A, X)\) satisfying the condition \( C_{d-1}\), the assertion is true.

Without loss of generality we will assume that \( B(A) = (0) \).

If \( A \neq (0) \) then there exists a nonzero word \( a \) in \( X \) such that \( a^2 = 0 \).

Consider the left ideal \( L = Aa \). Let \( W \) be the multiplicative semigroup generated by the set \( X \). The algebra \( L \) is spanned by \( Wa \). The subset

\[
L = L / r_a(a) \cap L, \quad : L \to L
\]

be the natural homomorphism.

We claim that the pair \((L, Wa)\) satisfies the condition \( C_{d-1}\). Indeed, choose arbitrary elements \( v_1, ..., v_r \in W \) and arbitrary integers \( l_1 \geq 1, ..., l_r \geq 1 \). Choose \( d \) arbitrary elements \( u_1, ..., u_d \in (\{a, ..., a^r\}) \). We need to prove that the elements \( au_1, ..., au_d \) are linearly dependent over \( F \).

We have \( u_i = u'_ia \), where elements \( u'_i \) are homogeneous expressions in \( X \).

The elements \( au'_1, ..., au'_d, u_1, ..., u_d \) lie in a \( d \)-dimensional space. Hence, they are linearly dependent. \( \sum_{i=1}^{d} \xi_i au'_i + \sum_{j=1}^{r} \beta_j u_j = 0 \); the coefficients \( \xi_1, ..., \xi_d, \beta_1, ..., \beta_d \) lie in \( F \) and not all of them are equal to zero.

If at least one coefficient \( \beta_j \) is \( \neq 0 \) then multiplying the linear dependence relation above by \( a \) on the left we get \( \sum_{j=1}^{d} \beta_j au_j = 0 \).

If all coefficients \( \beta_j \) are equal to 0 then \( \sum_{i=1}^{d} \xi_i au'_i = 0 \) and therefore \( \sum_{i=1}^{d} \xi_i au_i = 0 \). The claim is verified.

By the induction assumption we have \( L = B(L) \), which implies \( a \in B(A) \), the contradiction. The lemma is proved.

Recall that a linear transformation \(* : A \to A\) of an associative algebra \( A \) is called an involution if for arbitrary elements \( a, b \in A \) we have \((a*b)^* = a^*b^*a \). Let \( H(A, *) = \{a \in A \mid a^* = a\} \) be the subspace of hermitian elements.

**Lemma 1.2.** Let \( A \) be an associative algebra with an involution \(* : A \to A\), which is generated by a subset \( X \subseteq H(A, *) \). Suppose that for an arbitrary \( k \geq 1 \), arbitrary elements \( x_1, ..., x_k \in X \), arbitrary integers \( l_1 \geq 1, ..., l_k \geq 1 \), we have \( \dim(x_1, ..., x_k) \leq d \) and all elements from \((x_1, ..., x_k) \cap H(A, *)\) are nilpotent. Then \( A = B(A) \).

**Proof.** Suppose that \( B(A) = (0) \) but \( A \neq (0) \). Then there exists an element \( a \) (a power of one of the generators from \( X \)) such that \( a \neq 0 \), \( a^2 = 0 \), \( a^* = a \). Let \( w \) be a homogeneous expression in \( X \).

We claim that the element \( wa \) is nilpotent. Indeed, the element \( wa + aw^* \) is nilpotent. If \( (wa + aw^*)^k = 0 \) then \( a(wa + aw^*)^k = a(wa)^k = 0 \), which implies \((wa)^{k+1} = 0 \).
If \( W \) is the set of all words in \( X \) then the pair \((L = Aa, Wa)\) satisfies \( C_d \) (in fact, even \( C_{d-1} \) as we have shown in the proof of Lemma 1.1). Hence, by Lemma 1.1, \( L = B(L) \). Therefore, \( a \in B(A) \), the contradiction. The lemma is proved.

A linear \( F \)-algebra \( J \) is said to be a Jordan algebra (see [J3, and ZSSS]) if it satisfies the identities

\[
\begin{align*}
(1) & \quad xy = yx, \\
(2) & \quad (x^2y)x = x^2(yx).
\end{align*}
\]

**Example 1.** If \( A \) is an associative algebra then the vector space of \( A \) with the new operation \( a \cdot b = \frac{1}{2}(ab + ba) \) is a Jordan algebra. This Jordan algebra is denoted as \( A^{(+)} \).

**Example 2.** If \( A \) is an associative algebra with an involution \( *: A \to A \) then the space of hermitian elements \( H(A, *) \) is a subalgebra of the Jordan algebra \( A^{(+)} \).

**Example 3.** Let \( K \) be a field extension of \( F \) and let \( V \) be a vector space over \( K \) with a symmetric bilinear form \( \langle \cdot, \cdot \rangle: V \times V \to K \). Then the direct sum \( K \cdot 1 + V \) with the multiplication defined by \( v \cdot w = \langle v, w \rangle 1 \) for \( v, w \in V \) is a Jordan algebra. It is called a Jordan algebra of a symmetric bilinear form \( \langle \cdot, \cdot \rangle \).

A Jordan algebra \( J \) is called special if it is embeddable into \( A^{(+)} \) for some associative algebra \( A \). The algebras of Examples 1–3 are special. If \( J \subseteq A^{(+)} \) and the algebra \( A \) is generated by the subspace \( J \) then \( A \) is said to be an associative enveloping algebra of \( J \). Every special Jordan algebra \( J \) has the universal associative enveloping algebra \( U(J) \) with the canonical involution \( *: U(J) \to U(J) \) such that \( J \subseteq H(U(J), *) \) (see [J3]).

For elements \( x, y, z \) of a Jordan algebra \( J \) define their Jordan triple product as \( [x, y, z] = (x \cdot y) \cdot z + x \cdot (y \cdot z) - y \cdot (x \cdot z) \). If \( J \subseteq A^{(+)} \) with juxtaposition denoting the associative product in \( A \) then \( [x, y, z] = \frac{1}{2}(xyz + zyx) \).

An element \( a \) of a Jordan algebra \( J \) is called an absolute zero divisor if \( a^2 = 0 \) and \( \{a, J, a\} = (0) \). An algebra is called nondegenerate if it does not contain nonzero absolute zero divisors. The smallest ideal \( M(J) \) of \( J \) such that the quotient algebra \( J/M(J) \) is nondegenerate is called the McCrimmon radical of \( J \).

A Jordan algebra \( J \) is said to be PI (see [J3]) if there exists an element \( f(x_1, x_2, \ldots, x_m) \) of the free Jordan algebra such that

\[
\begin{align*}
(1) & \quad f \text{ is not identically zero on all special Jordan algebras,} \\
(2) & \quad f \text{ is identically zero on } J, \text{ that is, } f(a_1, \ldots, a_m) = 0 \text{ for arbitrary elements } a_1, \ldots, a_m \in J.
\end{align*}
\]
A (nonassociative) algebra is said to be **prime** if for any two nonzero ideals their product is also nonzero.

**Lemma 1.3.** Let $J$ be a Jordan algebra which is PI and has a basis consisting of nilpotent elements. Then $J = M(J)$.

**Proof.** If $J \not= M(J)$ then $J$ has a nonzero homomorphic image $J'$ which is prime and nondegenerate (see [Z2]). In [Z1] it was proved that in a prime nondegenerate PI-algebra $J'$ the center $Z$ of $J'$ is not equal to $(0)$ and the ring of fractions $(Z \setminus \{0\})^{-1} J'$ is either a simple central finite dimensional algebra over the field $K = (Z \setminus \{0\})^{-1} Z$ or a Jordan algebra of a symmetric bilinear nondegenerate form in an infinite dimensional vector space over $K$.

According to the Jordan analogue of Wedderburn’s theorem (see [J3]) a finite dimensional Jordan algebra, which has a basis consisting of nilpotent elements, is nilpotent. Hence, the first case is impossible. A Jordan algebra of a bilinear form $K + V$ also cannot have a basis consisting of nilpotent elements because all nilpotent elements of $K + V$ lie in $V$. This contradiction proves the lemma.

**Lemma 1.4.** Let $J$ be a Jordan algebra generated by a subset $X \subseteq J$ such that $(J, X)$ satisfies $C_d$. Then $J = M(J)$.

**Proof.** Let us first show that the algebra $J$ is locally nilpotent. In doing this we can assume that the set $X$ is finite, $X = \{x_1, \ldots, x_m\}$.

If the algebra $J$ is not nilpotent then (see [Z2]) $J$ has a nonzero prime nondegenerate homomorphic image that does not contain nonzero locally nilpotent ideals. Hence, without loss of generality we can assume that $J$ is prime, nondegenerate and does not contain nonzero locally nilpotent ideals. By Lemma 1.3 the algebra $J$ is not PI. In [Z1] it was shown that a prime nondegenerate Jordan algebra that is not PI is special.

Let $R = U(J)$ be the universal associative enveloping algebra of $J$, $J \subseteq R^+$. $R$ is generated by $x_1, \ldots, x_m$. Moreover, the algebra $R$ is equipped with the involution $*$ such that $J \subseteq H(R, *)$.

Let $(x_1, \ldots, x_k)$ denote the linear span of all Jordan products in $x_1, \ldots, x_m$ involving $l_1$ elements $x_1, l_2$ elements $x_2, \ldots, l_m$ elements $x_m$. The notation $(x_1, \ldots, x_k)$ is still reserved for associative products.

From the results of V. G. Skosirskii [SK] it follows that

\[
\mathbf{x}_1, \ldots, \mathbf{x}_k \subseteq \sum \left( \frac{x_1, \ldots, x_{i_1}, \ldots, x_{i_k}, \ldots, x_m}{l_1, \ldots, l_{i_1-1}, \ldots, l_{i_k-1}, \ldots, l_m/} \right) \mathbf{x}_{i_1} \cdots \mathbf{x}_{i_k},
\]

where the summation goes over all subsets of $\{1, 2, \ldots, m\}, i_1 < \cdots < i_k$. 

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Hence $\dim_F \langle t_1, \ldots, t_k \rangle \leq d 2^m$.

Let $T$ be the tetrad eater ideal of the free Jordan algebra constructed in [Z1]. Let $T(J)$ be the ideal of $J$ consisting of values of elements from $T$ on $J$. We will need the following property of the ideal $T$ (see [Z1]): for arbitrary elements $a_1, \ldots, a_k \in T(J)$ the element $a_1 \cdots a_k + a_k \cdots a_1$ again lies in $T(J)$ (here the juxtaposition denotes the multiplication in $R$).

Let $t_i = t_i(x_1, \ldots, x_m)$, $1 \leq i \leq k$, be homogeneous (in all variables) expressions from $T$. As we have seen above, $\dim_F \langle t_1, \ldots, t_k \rangle \leq d 2^m$ for arbitrary $l_i \geq 0$, $\ldots$, $l_k \geq 0$. Let $\langle t_1, \ldots, t_k \rangle$ be the associative subalgebra of $R$ generated by $t_1, \ldots, t_k$. If a homogeneous expression $h(t_1, \ldots, t_k)$ lies in $H(\langle t_1, \ldots, t_k \rangle, \ast)$ then $h$ is a homogeneous Jordan polynomial in $x_1, \ldots, x_m$. Hence, $h$ is nilpotent. By Lemma 2 the associative algebra $\langle t_1, \ldots, t_k \rangle$ is nilpotent. Hence, the ideal $T(J)$ of $J$ is locally nilpotent. Since the algebra $J$ is not PI we have $T(J) \neq \langle 0 \rangle$. This contradicts the assumption that the algebra $J$ does not contain nonzero locally nilpotent ideals.

We have proved that the algebra $J$ is locally nilpotent. In order to prove that $J = M(J)$ we will drop the assumption that $X$ is finite. However, we will still assume that $J$ is prime, nondegenerate, and is not PI. As above, $T(J) \neq \langle 0 \rangle$, the algebra $J$ is special, and $R$ is the universal associative enveloping algebra of $J$.

Let $\langle T(J) \rangle$ be the subalgebra of $R$ generated by the set $T(J)$. Let $B(\langle T(J) \rangle)$ be the Baer radical of $\langle T(J) \rangle$.

If for an arbitrary finite collection of elements $t_1, \ldots, t_k \in T(J)$ we have $\langle t_1, \ldots, t_k \rangle^{2d+1} \subseteq B(\langle T(J) \rangle)$ then $\langle T(J) \rangle^{2d+1} \subseteq B(\langle T(J) \rangle)$ and therefore $\langle T(J) \rangle = B(\langle T(J) \rangle)$. This easily implies that $T(J) \subseteq M(J)$, the contradiction.

Therefore, there exist homogeneous (in $X$) elements $t_1, \ldots, t_k \in T(J)$ such that $\langle t_1, \ldots, t_k \rangle^{2d+1} \not\subseteq B(\langle T(J) \rangle)$. Since the algebra $\langle t_1, \ldots, t_k \rangle$ is nilpotent (see [ZSSS]) there exists the maximal integer $q$ such that $t_1, \ldots, t_q \not\in \langle T(J) \rangle$ for some indexes $1 \leq i_1, \ldots, i_q \leq k$. Clearly, $q \geq 2d + 1$.

Let $u$ be a product of homogeneous (in $X$) elements from $T(J)$. We claim that $t_{i_0} \cdots t_{i_q} u t_{i_q} \cdots t_{i_0} \in B(\langle T(J) \rangle)$. Indeed, consider the elements

$u_0 = t_{i_0} \cdots t_{i_q} = u t_{i_0} \cdots t_{i_q} + t_{i_q} \cdots t_{i_0} u^*$,

$u_1 = t_{i_1} u t_{i_0} \cdots t_{i_{q-1}} = t_q u t_{i_0} \cdots t_{i_{q-1}} + t_{i_{q-1}} \cdots t_1 u^* t_{i_q}$,

$\ldots$

$u_d = t_{i_{q-d+1}} \cdots t_{i_q} u t_{i_q} \cdots t_{i_{q-d+1}}$

$= t_{i_{q-d+1}} \cdots t_{i_q} u t_{i_q} \cdots t_{i_{q-d+1}} + t_{i_{q-d+1}} \cdots t_1 u^* t_{i_q} \cdots t_{i_{q-d+1}}$.

These elements lie in a $d$-dimensional subspace of $T(J)$, hence they are linearly dependent.
Hence there exists $p$, $0 \leq p \leq d$, such that

$$u_p = x_{p+1} u_{p+1} + \cdots + x_d u_d, \quad x_j \in F.$$ 

Substituting

$$t_{ q-p} \cdots t_q u_{ t_1} \cdots t_q = -t_{ q-p} \cdots t_q u_{ t_1} \cdots t_q + \sum_{j=p+1}^d x_j u_j,$$

we get $t_{ q-p} \cdots t_q u_{ t_1} \cdots t_q \in B(\langle H \rangle)$, because after the substitution every summand will contain a product of $q + 1$ elements $t_i$.

Hence, $t_{ q-p} \cdots t_q \in B(\langle J \rangle)$. Hence, $t_{ q-p} \cdots t_q \in B(\langle J \rangle)$, the contradiction. The lemma is proved.

A pair of vector spaces $P = (P^-, P^+)$ with a pair of trilinear operations

$$\{\cdot, \cdot, \cdot\}: P^\times P^\times P^- \to P^-,$$

$$\{\cdot, \cdot, \cdot\}: P^\times P^\times P^+ \to P^+$$

is called a Jordan pair (see [L], [M]) if the following identities are satisfied:

$$(P.1) \quad \{x^\sigma, y^\sigma, \{x^\sigma, z^\sigma, x^\sigma\}\} = \{x^\sigma, \{y^\sigma, x^\sigma, z^\sigma\}, x^\sigma\},$$

$$(P.2) \quad \{\{x^\sigma, y^\sigma, x^\sigma\}, y^\sigma, u^\sigma\} = \{x^\sigma, \{y^\sigma, x^\sigma, y^\sigma\}, u^\sigma\},$$

$$(P.3) \quad \{\{x^\sigma, y^\sigma, x^\sigma\}, z^\sigma, \{x^\sigma, y^\sigma, x^\sigma\}\} = \{x^\sigma, \{y^\sigma, x^\sigma, z^\sigma\}, x^\sigma\},$$

for every $x^\sigma, u^\sigma \in P^\sigma, y^\sigma, z^\sigma \in P^{-\sigma}, \sigma = +/-$.

Fix an element $u \in P^{-\sigma}, \sigma = +/-$ - The operation $x \circ y = \{x, y, z\}$ for $x, y \in P^\sigma$, defines a structure of a Jordan algebra on $P^\sigma$. We will denote this Jordan algebra as $P^{(u)}$.

An element $a \in P^\sigma$ is called an absolute zero divisor of the pair $P$ if $[a, P^{-\sigma}, a] = (0)$. A Jordan pair is said to be nondegenerate if it does not contain nonzero absolute zero divisors.

The smallest ideal $M(P)$ of the pair $P$ whose quotient is nondegenerate is called these McCrimmon radical of $P$.

A sequence of elements $a_1, a_2, \ldots \in P$ is called an $m$-sequence if $a_{i+1} \in [a_i, P^{-\sigma}, a_i]$ for $i \geq 1$. In [Z2] it was proved that an element lies in $M(P)$ if and only if every $m$-sequence starting with it vanishes. This implies another elementwise characterization of $M(P)$: an element $u \in P^{-\sigma}$ lies in $M(P)$ if and only if $P^{(u)} = M(P^{(u)})$.

Let $L$ be a Lie algebra over a field $F$ of characteristic $\geq 5$ or 0. Let $a, b \in L, ad(a)^3 = ad(b)^3 = 0$. Then the pair of subspaces

$$P^- = Fa + [[L, a], a], \quad P^+ = Fb + [[L, B], b]$$
with the operations
\[ \{x^\sigma, y^{-\sigma}, z^\tau\} = [[x^\sigma, y^{-\sigma}], z^\tau] \in P^\sigma; \quad x^\sigma, z^\tau \in P^\tau; \quad y^{-\sigma} \in P^{-\sigma}, \quad \sigma = +/-. \]

is a Jordan pair (see \([L]\)).

**Lemma 1.5.** Let \( L \) be a Lie algebra generated by a subset \( X \), such that \((L, X)\) satisfies \( C_\delta \). Let \( a_1, b_1 \) be homogeneous \((in X)\) elements such that
\[
ad(a_1)^3 = \ad(b_1)^3 = 0. \]
Then any sequence of elements \( a_1, a_2 = [[[b_1, a_1], a_1], ... a_i+1 \in [[[L, a_1], a_1], a_1] \) vanishes.

**Proof.** Consider the Jordan pair
\[
P^- = Fa_1 + [[[L, a_1], a_1], a_1], \quad P^+ = Fb_1 + [[[L, b_1], b_1].
\]

We claim that there exists an \( m \)-sequence \( b_1, b_2, ... \in P^+ \) (starting with the element \( b_1 \)) such that for an arbitrary \( i \geq 1 \) we have \( a_{i+1} = [[[b_1, a_1], a_1], a_1]. \)
Indeed, for \( i = 1 \) the equality is true by one of the assumptions of the lemma. Suppose that there exist elements \( b_1, ..., b_k \in P^+ \) such that \( a_{i+1} = [[[b_i, a_1], a_1], a_1] \) for \( 1 \leq i \leq k. \)

We know that \( a_{k+2} = [[[c, a_{k+1}], a_{k+1}], a_1] \) for some element \( c \in L. \) Since \( a_{k+1} = [[[b_k, a_1], a_1], a_1] \) it follows that \( \ad(a_{k+1})^2 = \ad(a_1)^2 \ad(b_k)^2 \ad(a_1)^2. \)
Now it remains to define \( b_{k+1} = c \ad(a_1)^2 \ad(b_k)^2 \in P^+. \)

To prove that the sequence \( a_1, a_2, ... \) vanishes it is sufficient to prove that the \( m \)-sequence \( b_1, b_2, ... \) vanishes. We will show that \( P = M(P). \)

Let \( W \) be the set of all commutators in elements from \( X \) (elements of \( X \) themselves are viewed as commutators of length 1). The space \( P^- \) is spanned by the set \([[[W, a_1], a_1] \cup \{a_1\}\) whereas the space \( P^+ \) is spanned by \([[[W, b_1], b_1] \cup \{b_1\}\). For an arbitrary element \( u \in [[[W, b_1], b_1] \cup \{b_1\] the Jordan algebra \( P^{\omega_u} \) and its generating set \([[[W, a_1], a_1] \cup \{a_1\}\) satisfy the condition \( C_\delta. \) By Lemma 1.4, \( P^{\omega_u} = M(P^{\omega_u}). \) Hence, \( u \in M(P) \) and therefore \( P = M(P). \) The lemma is proved.

**Lemma 1.6.** Let \( L \) be a Lie algebra generated by a subset \( X \) such that \((L, X)\) satisfies \( C_\delta. \) Let \( a \) be a homogeneous \((in X)\) element of \( L \) such that \( \ad(a)^3 = 0. \) Then for an arbitrary sequence of homogeneous \((in X)\) elements \( b_1, b_2, ..., \) the sequence \( a_1 = a, ..., a_{i+1} = [[[b_i, a_1], a_1] \) vanishes.

**Proof.** Consider the set \( Y = X \cup \{y_0, y_1, y_2, ...\}. \) We will define a Lie algebra \( L \) by the generating set \( Y \) and a set of relators \( R. \) Let \( L(Y) \) be the free Lie algebra over \( F \) generated by the set \( Y. \) Consider the homomorphism \( \pi: L(Y) \rightarrow L \) such that \( \pi(x) = x \) for an arbitrary element \( x \in X, \pi(y_0) = a, \pi(y_i) = b_i \) for \( i \geq 1. \)
Let $W$ be the set of all commutators in $Y$. For an arbitrary element $u \in W$ we let

$$u \text{ ad}(y_0)^3 \in R. \quad (1)$$

For an arbitrary commutator $w \in W$ find an integer $n(w)$ such that the element $w$ is ad-nilpotent of degree $\leq n(w)$. Then

$$u \text{ ad}(w)^{n(w)} \in R, \quad (2)$$

for arbitrary $u, w \in W$.

Finally, for arbitrary elements $x_1, ..., x_r \in X$, an arbitrary $q \geq 1$, and arbitrary integers $l_1 \geq 0, ..., l_r \geq 0, t_0 \geq 0, ..., t_q \geq 0$ consider the finite set $M$ of all commutators in $x_1, ..., x_r, y_0, y_1, ..., y_q$ having degrees $l_1, ..., l_r, d_0, ..., d_q$, respectively.

For an arbitrary $(d+1)$-tuple $(\rho_1, ..., \rho_{d+1}) \in M^{d+1}$ choose coefficients $\alpha_1, ..., \alpha_{d+1} \in F$, not all of them equal to 0, such that

$$\alpha_1 \pi(\rho_1) + \cdots + \alpha_{d+1} \pi(\rho_{d+1}) = 0.$$

We let

$$\sum_{i=1}^{d+1} \alpha_i \rho_i \in R \quad (3)$$

Let $\bar{L} = \langle Y | R = (0) \rangle$. Clearly, the pair $(\bar{L}, Y)$ satisfies $C_d$ and there exists a homomorphism $\varphi: \bar{L} \to L$ which is a factor of $\pi$. Moreover, we have $\bar{L} \text{ ad}(y_0)^3 = (0)$.

Remark that since all relators in $R$ are homogeneous with respect to all variables the algebra $\bar{L}$ is graded.

To prove the lemma it is sufficient to prove that the sequence $z_1 = y_0, ..., z_{i+1} = \left[ [y_i, z_i], z_i \right], i \geq 1$, vanishes in $\bar{L}$.

Let $I$ be the ideal of $\bar{L}$ generated by $\bar{L} \text{ ad}(y_1)^3$. By Lemma 1.5 there exists $k \geq 1$ such that $z_k \in I$.

Since the algebra $\bar{L}$ is graded it follows that

$$z_k = \sum_{z} u_z \text{ ad}(y_1)^3 \text{ ad}(v_{z_1}) \cdots \text{ ad}(v_{z_{k-1}}),$$

where $u_z, v_{z_1}, ..., v_{z_{k-1}}$ are commutators in $y_0, ..., y_{k-1}$ and each summand on the right hand side has the same degree with respect to each $y_j$ as $z_k$.

For an arbitrary commutator $\rho$ let $\deg_{d}(\rho)$ denote the degree of $\rho$ with respect to $y_0$ and let $\deg_{v}(\rho)$ denote the total degree of $\rho$ with respect to all other variables. Let $\deg(\rho) = \deg_{d}(\rho) + \deg_{v}(\rho)$. Clearly, $\deg(z_k) = 1$. In [K1] A. I. Kostrikin proved that $\deg(\rho) \geq 2$ implies $\rho = 0$. 

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For each \( x \) we have
\[
def(u_x) + \cdots + \def(v_x) = \def(z_x) + 3 = 4.
\]
If \( \def(u_x) \geq 2 \) then \( u_x = 0 \). Suppose that \( \def(u_x) \leq 1 \). Then \( \def(v_x) + \cdots + \def(v_x) \geq 3 \).

Making \( v = u_x \text{ad}(y_1)^3 \) a new variable we see that
\[
def(v_x) \cdots \text{ad}(v_x) \geq 2,
\]
hence \( \text{ad}(v_x) \cdots \text{ad}(v_x) = 0 \). The lemma is proved.

**Lemma 1.7** (see [P]). *If a Lie algebra \( L \) is generated by a set \( X \) and all commutators in \( X \) are ad-nilpotent, then there exists a subset \( Y \subseteq X \) such that the ideal \( \text{id}_L(Y) \) of \( L \) generated by \( Y \) is locally nilpotent and for every subset \( Y' \) of \( X \) properly containing \( Y \) the quotient \( \text{id}_L(Y')/\text{id}_L(Y) \) is not locally nilpotent.*

**Lemma 1.8** (see [K1]). *Let \( L \) be a Lie algebra over a field \( F \) of characteristic \( > n \) or 0. Let \( a \) be an element of \( L \) such that \( \text{ad}(a)^n = 0 \) and let \( n \geq 4 \). Then for an arbitrary element \( b \in L \) we have \( \text{ad}(b \text{ad}(a)^{n-1})^{n-1} = 0 \).*

**Lemma 1.9.** *Let \( L \) be a Lie algebra, generated by a subset \( X \). If the pair \( (L, X) \) satisfies \( C_d \) then the algebra \( L \) is locally nilpotent.*

**Proof.** Without loss of generality we will assume that the set \( X \) is closed under commutation. By Lemma 1.7 we can also assume that no nonempty and nonzero subset of \( X \) generates a locally nilpotent ideal of \( L \).

An element \( a \) of a Lie algebra \( L \) is called a sandwich (see [K2]) if \( [[L, a], a] = (0) \). A. N. Grishkov [Gr] proved that in a Lie algebra over a field of characteristic 0 a sandwich generates a locally nilpotent ideal. Hence no nonzero element of \( X \) is a sandwich.

If \( L \neq (0) \) then by A. I. Kostrikin’s Lemma the set \( X \) contains a nonzero element \( a \) such that \( \text{ad}(a)^4 = 0 \). Since the element \( a \) is not a sandwich there exists an element \( b_1 \in X \) such that \( [[b_1, a], a] = a_2 \neq 0 \). Since the element \( a_2 \) is not a sandwich there exists an element \( b_2 \in X \) such that \( [[b_2, a_2], a_2] = a_3 \neq 0 \) and so on. The resulting nonvanishing sequence \( a_1 = a, a_2, \ldots \) contradicts Lemma 1.6. The lemma is proved.

**Proof of Theorem 1.** Let \( L = \bigoplus_{x \in \Gamma} L_x \) be a \( \Gamma \)-graded Lie algebra such that \( \dim_{\Gamma} L_x \leq d \) for all \( x \in \Gamma \) and an arbitrary homogeneous element is ad-nilpotent.

Let \( X = \bigcup_{x \in \Gamma} L_x \). Clearly, the pair \( (L, X) \) satisfies the condition \( C_d \). By Lemma 1.9 the algebra \( L \) is locally nilpotent. This finishes the proof of Theorem 1.
2. PROOF OF THEOREM 2

Let $G$ be a group of finite width, that is, there exists an integer $d \geq 1$ such that $\dim_{d}(G/G_{i+1}) \leq d$ for $i \geq 1$. Being of finite width means that for arbitrary elements $g_{1}, \ldots, g_{d+1} \in G$ there exist integers $k_{1}, \ldots, k_{d+1}$, not all of them equal to 0, such that $g^{k_{1}}_{1} \cdots g^{k_{d+1}}_{d+1} \in G_{i+1}$. In particular, this implies that a homomorphic image of a group of finite width has finite width.

Throughout this chapter $F$ is field of characteristic 0. Let $V$ be a vector space over $F$ which is a finitely generated module over $G$. Let $\rho$ denote the corresponding representation $\rho: G \rightarrow GL(V)$. We assume further that the module $V$ is residually finite, that is, there exists a family of $G$-submodules $\{V_{\alpha}\}$ such that $\cap_{\alpha} V_{\alpha} = (0)$ and $\dim_{F} V/V_{\alpha} < \infty$ for an arbitrary $\alpha$. Denote as $\rho_{\alpha}$ the representation $\rho_{\alpha}: G \rightarrow GL(V/V_{\alpha})$.

Suppose that the representation $\rho$ is unipotent. This means that for an arbitrary group element $g \in G$ there exists an integer $n(g) \geq 1$ such that $V(g-1)^{n(g)} = (0)$. Then by the celebrated Kolchin's theorem (see [Hu]) the representations $\rho_{\alpha}$ are unitriangular in some bases.

Let $G_{i}'$ be the set of elements $g \in G$ such that for an arbitrary submodule $V_{\alpha} \in \mathcal{P}$ the element $\rho_{\alpha}(g)$ is periodic modulo $\rho_{\alpha}(G_{i})$. In other words, for an arbitrary submodule $V_{\alpha} \in \mathcal{P}$ there exists an integer $m_{\alpha} \geq 1$ and an element $g_{\alpha} \in G$, such that $\rho_{\alpha}(g_{\alpha})^{m_{\alpha}} = \rho_{\alpha}(g_{i})$.

**Lemma 2.1** (see [H]). Let $\Gamma$ be a nilpotent group with a lower central series $\Gamma_{1} = \Gamma_{2} = \cdots$

1. if elements $g, h \in \Gamma$ are periodic then their product $gh$ is periodic as well.
2. if elements $g, h \in \Gamma$ are periodic modulo $\Gamma_{i}, \Gamma_{j}$, respectively, then the commutator $(g, h)$ is periodic modulo $\Gamma_{i+j}$.

**Lemma 2.2.**

1. $G_{i}'$ is a subgroup of $G_{i}'$.
2. $G_{i}' \supseteq G_{i}' \supseteq \cdots$ is a filtration, that is, $(G_{i}', G_{i}') \subseteq G_{i+j}$ for arbitrary $i, j \geq 1$.
3. all factors $G_{i}'/G_{i+1}'$ are torsion free.

**Proof.** Let $g, h \in G_{i}'$. For an arbitrary submodule $V_{\alpha} \in \mathcal{P}$ the elements $\rho_{\alpha}(g), \rho_{\alpha}(h)$ are periodic modulo the subgroup $\rho_{\alpha}(G_{1})$ of $\rho_{\alpha}(G)$. The quotient group $\rho_{\alpha}(G)/\rho_{\alpha}(G_{i})$ is nilpotent. By Lemma 2.1 the element $\rho_{\alpha}(gh)$ is periodic modulo $\rho_{\alpha}(G_{i})$.

Let $g, h \in G_{i}'$. For an arbitrary submodule $V_{\alpha} \in \mathcal{P}$ the elements $\rho_{\alpha}(g), \rho_{\alpha}(h)$ are periodic modulo the subgroups $\rho_{\alpha}(G_{i})$, $\rho_{\alpha}(G_{i+j})$, respectively. By Lemma 2.1(2) applied to the nilpotent group $\rho_{\alpha}(G)/\rho_{\alpha}(G_{i+j})$ the commutator $(\rho_{\alpha}(g), \rho_{\alpha}(h))$ is periodic modulo $\rho_{\alpha}(G_{i+j})$. 


The assertion (3) is obvious. The lemma is proved.

The filtration \( G = G_1 \supseteq G_2 \supseteq \cdots \) gives rise to the Lie ring \( L = \bigoplus_{i \geq 1} G_i / G_{i+1} \). [ \( g, G_{i+1}, g' G_{i+1} \{ = (g, g) G_{i+1} \). Remark that the ring \( L \) does not have additive torsion.

**Lemma 2.3.** Let \( g \in G_i \setminus G_{i+1} \). Suppose that \( V(1 - g)^m = (0), m \geq 1 \). Then \( L \operatorname{ad}(g G_{i+1}^2) = (0) \).

**Proof.** Fix an element \( h \in G_j \). We have to show that the commutator \( u = (\cdots ((h, g), g), \cdots, g) \) lies in \( G_{j+(2m-1)i+1} \). In other words, we have to show that for an arbitrary submodule \( V \in \mathcal{P} \) the element \( \rho_s(u) \) is periodic modulo \( \rho_s(G_{j+(2m-1)i+1}) \).

Let \( a = \log \rho_s(g), b = \log \rho_s(h) \), so that \( g = \exp(a), h = \exp(b) \).

Let \( L(k) \) be the \( Q \)-linear span of all commutators in \( a, b \) of total length \( \geq k \) which involve at least one element \( b \) and at least \( 2m-1 \) elements \( a \), thus \( k \geq 2m \).

If \( s = \dim_F V / V_E \), then any product of elements \( a, b \) of length \( s \) is equal to 0, hence \( L(s) = (0) \).

We will show by induction on \( k, 2m+1 \leq k \leq s \), that there exists an integer \( d_k \geq 1 \) and an element \( g_k \in G_{j+(2m-1)i+1} \) such that \( \rho_s(u)^{d_k} \rho_s(g_k) \in \exp L(k) \). For \( k = s \) that will imply \( \rho_s(u)^s = \rho_s(g_s^{-1}) \in \rho_s(G_{j+(2m-1)i+1}) \).

By the Campbell–Hausdorff formula (see [12]) we have

\[
\rho_s(u) = \left( \cdots \left( \exp(b), \exp(a) \right), \exp(a) \right), \ldots, \exp(a) \right) \in \exp\left( \cdots \left( \left[ b, a \right], a \right), \ldots, a \right) + L(2m+1).
\]

Since

\[
\left[ \cdots \left[ b, a \right], a \right), \ldots, a \right) = 0,
\]

we can choose \( d_{2m+1} = 1, g_{2m+1} = 1 \).

Suppose that an integer \( d_k \) and an element \( g_k \) have been found. Let \( c_1, \ldots, c_t \) be commutators in \( a, b \) of length \( k \) which \( Q \)-span \( L(k) \) modulo \( L(k+1) \). We have

\[
\rho_s(u)^{d_k} \rho_s(g_k) = \exp(\pi_1 c_1 + \cdots + \pi_t c_t + c') \text{ for some rational numbers } \pi_1, \ldots, \pi_t.
\]

Let \( r \) be an integer such that \( \pi_i \in \mathbb{Z}, 1 \leq i \leq t \).
Let $\tilde{c}_i$ be the group commutator in $g, h$ of the same structure as $c_i$. Then $p_d(\tilde{c}_i) \in \exp(c_i + L(k + 1))$.

Now

$$(p_d(u^h g_k))^r \tilde{c}_i^{-rn} \cdots \tilde{c}_1^{-rn} \in \exp L(k + 1).$$

We have $(u^h g_k)^r = u^h g_k^{r_m} \text{ mod } G_{j + (2m - 1)i + 1}$.

Since $k \geq 2m + 1$ it follows that the elements $\tilde{c}_1, \ldots, \tilde{c}_i$ also lie in $G_{j + (2m - 1)i + 1}$. Hence, we can define $d_{k + 1} = dkr$. The lemma is proved.

**Lemma 2.4.** \(\dim_{\mathbb{Q}}(G_i/G_{i+1} \otimes_{\mathbb{Q}} \mathbb{Q}) \leq d\) for \(i \geq 1\).

**Proof.** Choose arbitrary elements $g_1, \ldots, g_{d+1} \in G_{i}$. We have to find integers $k_1, \ldots, k_{d+1}, n$ not all equal to 0, such that $g_1^{k_1} \cdots g_{d+1}^{k_{d+1}} \in G_{i+1}$.

For a $G$-module $V_{a} \in \mathcal{P}$ there exists an integer $m \geq 1$ such that $p_d(g_1)^m, \ldots, p_d(g_{d+1})^m \in \rho_d(G_i)$. Since $\dim_{\mathbb{Q}}(G_i/G_{i+1} \otimes_{\mathbb{Q}} \mathbb{Q}) \leq d$ it follows that there exist integers $m_1, \ldots, m_{d+1}, n$, not all equal to 0, such that $p_d(g_1^{m_1} \cdots g_{d+1}^{m_{d+1}}) \in \rho_d(G_i)$.

Thus, for an arbitrary $V_a \in \mathcal{P}$ the set

$$P(V_a) = \{ (r_1, \ldots, r_{d+1}) \in \mathbb{Z}^{d+1} \mid p_d(g_1^{r_1}) \cdots p_d(g_{d+1}^{r_{d+1}}) \in \rho_d(G_{i+1}) \}$$

is nonzero. Let $Q(V_a)$ be the $\mathbb{Q}$-linear span of $P(V_a)$.

If $V_\beta \subseteq V_a$ then $\text{Ker} \rho_a \subseteq \text{Ker} \rho_\beta$. This implies that $P(V_\beta) \subseteq P(V_a)$ and therefore $Q(V_\beta) \subseteq Q(V_a)$.

Choose a submodule $V_\alpha \in \mathcal{P}$ such that the dimension $\dim_{\mathbb{Q}} Q(V_\alpha)$ is minimal. Then for an arbitrary submodule $V_\beta \in \mathcal{P}$ contained in $V_\alpha$ we have $Q(V_\beta) = Q(V_\alpha)$.

Let $0 \neq (n_1, \ldots, n_{d+1}) \in \mathbb{Z}^{d+1} \cap Q(V_a)$.

We claim that $g_1^{n_1} \cdots g_{d+1}^{n_{d+1}} \in G_{i+1}$.

Indeed, it is sufficient to show that $p_d(g_1^{n_1} \cdots g_{d+1}^{n_{d+1}})$ is periodic modulo $\rho_d(G_{i+1})$ for all submodules $V_\beta \subseteq V_a$. Then $(m_1, \ldots, m_{d+1}) \in Q(V_\beta)$.

Let $(r_1, \ldots, r_{d+1}) \in \mathbb{Z}^{d+1} \cap Q(V_a)$. Then $(m_1, \ldots, m_{d+1}) = \sum_{i=1}^{s} k_i (r_1, \ldots, r_{d+1})$, where $k_i, 1 \leq i \leq s$, are integers. Since the group $G_i/G_{i+1}$ is abelian it follows that

$$(g_1^{n_1} \cdots g_{d+1}^{n_{d+1}})^{k_i} = (g_1^{n_1} \cdots g_{d+1}^{n_{d+1}})^{k_1} \cdots (g_1^{n_1} \cdots g_{d+1}^{n_{d+1}})^{k_i} \cdots (g_1^{n_1} \cdots g_{d+1}^{n_{d+1}})^{k_s} \cdots (g_1^{n_1} \cdots g_{d+1}^{n_{d+1}})^{k_{s+1}} \cdots (g_1^{n_1} \cdots g_{d+1}^{n_{d+1}})^{k_s},$$

where $g' \in G_{i+1}$. The elements

$$p_d(g_1) \cdots g_1^{n_{d+1}}, \ldots, p_d(g_1) \cdots g_{d+1}^{n_{d+1}}, p_d(g')$$
are periodic modulo $\rho_d(G_{i+1})$. By Lemma 2.1(1) the element $\rho_d(g_1^n \cdots g_d^n)$ is also periodic modulo $\rho_d(G_{i+1})$. The lemma is proved.

**Lemma 2.5.** The group $\rho(G)$ is nilpotent.

*Proof.* As above consider the Lie ring $L = \sum_{i \geq 1} L_i$, $L_i = G_i/G_{i+1}$, and the Lie algebra $\bar{L} = \sum_{i \geq 1} \bar{L}_i$; $\bar{L}_i = L_i \otimes_{\mathbb{Q}} \mathbb{Q}$. By Lemmas 2.3 and 2.4 every homogeneous element of $\bar{L}$ is ad-nilpotent and for an arbitrary $i \geq 1$ we have $\dim_{\mathbb{Q}} \bar{L}_i \leq d$.

Hence, by Theorem 1 the subalgebra of $\bar{L}$ generated by $\bar{L}_i$ is nilpotent. This means that there exists $s \geq 1$ such that $G_k \leq G_{k+1}$ for all $k \geq s$. We will show that $\rho(G_s) = (1)$. To do this it is sufficient to prove that $\rho_d(G_s) = (1)$ for all $V_s \in \mathcal{V}$. Suppose, on the contrary, that $\rho_d(G_s) \neq (1)$. Since the group $\rho_d(G_s)$ is nilpotent there exists $k \geq s$ such that $\rho_d(G_k) \neq (1)$ but $\rho_d(G_{k+1}) = (1)$.

An arbitrary element from $\rho_d(G_{k+1})$ is periodic modulo $\rho_d(G_{k+1}) = (1)$, hence periodic.

Since the characteristic of the ground field $F$ is 0, a nonidentical unipotent linear transformation cannot be periodic. Hence, $\rho_d(G_{k+1}) = (1)$. From the inclusion $G_k \leq G_{k+1}$ it follows that $\rho_d(G_k) = (1)$ contrary to our assumption. The Lemma is proved.

We say that a group $G$ is periodic over a subgroup $H \leq G$ (which is not necessarily normal) if for an arbitrary element $g \in G$ there exists an integer $n = n(g) \geq 1$, such that $g^n \in H$.

**Lemma 2.6.** Let $\Gamma$ be a nilpotent group with the lower central series $\Gamma = \Gamma_1 > \Gamma_2 > \cdots > \Gamma_s = (1)$. If the vector spaces $\Gamma_i/\Gamma_{i+1} \otimes_{\mathbb{Q}} \mathbb{Q}$, $1 \leq i \leq s - 1$, are finite dimensional then there exists a finitely generated subgroup $H \leq \Gamma$ such that $\Gamma$ is periodic over $H$.

*Proof.* Let $H'$ be a finitely generated subgroup of the abelian group $\Gamma_{s-1}$ that spans $\Gamma_{s-1} \otimes_{\mathbb{Q}} \mathbb{Q}$ over $\mathbb{Q}$. Then $\Gamma_{s-1}/H'$ is a periodic group. We will use induction on $s$. By the induction assumption there exists a finitely generated subgroup $H''/\Gamma_{s-1}$ of $\Gamma/\Gamma_{s-1}$ such that $\Gamma$ is periodic over $H''$.

Suppose that the quotient group $H''/\Gamma_{s-1}$ is generated by cosets $h_1 \Gamma_{s-1}, \ldots, h_k \Gamma_{s-1}$. Let $\langle h_1, \ldots, h_k \rangle$ be the subgroup of $\Gamma$ generated by $h_1, \ldots, h_k$. For an arbitrary element $g \in \Gamma$ there exists an integer $m \geq 1$ and elements $h \in \langle h_1, \ldots, h_k \rangle$, $g' \in \Gamma_{s-1}$ such that $g^m = hg'$. Since the group $\Gamma_{s-1}/H'$ is periodic, there exists an integer $n$ such that $g'^n \in H'$. Now the element $g'^m = (hg')^n = h^ng^n$ lies in the subgroup $H$ of $\Gamma$ generated by $h_1, \ldots, h_k$ and by $H'$. The lemma is proved.

Now we are ready to prove that the vector space of $V$ is finite dimensional.

If $\varphi$ is a unipotent linear transformation of the vector space $V$ then for an arbitrary element $v \in V$ and for an arbitrary integer $n \geq 1$ we have
Indeed, if \( \varphi = \text{Id}_V + a, a^n = 0 \), then \( \tau F[\varphi^{-1}, \varphi] = \tau F[a] \). We have \( \varphi^n = \text{Id}_V + (\begin{array}{c} n \\ \end{array}) a^2 + \cdots + a^n \). Since \( \text{char} F = 0 \) it is easy to see that the element \( na + (\begin{array}{c} n \\ \end{array}) a^2 + \cdots + a^n \) generates \( F[a] \). This implies the assertion.

By Lemma 2.6 there exists a finitely generated subgroup \( H \) of \( \rho(G) \) such that \( \rho(G) \) is periodic over \( H \). From what we proved above it follows that for an arbitrary element \( v \in V \) we have \( \tau F[G] = \tau F[H] \). It remains to prove that \( \dim_k \tau F[H] < \infty \).

The group \( H \) has the so-called bounded generation, that is, there exist elements \( h_1, \ldots, h_n \in H \) such that \( H \) is the product of cyclic subgroups:

\[
H = \langle h_1 \rangle \cdots \langle h_n \rangle.
\]

Suppose that the vector space \( Fv \langle h_1 \rangle \cdots \langle h_i \rangle \) is finite dimensional. Let \( h_{i+1} = \text{Id}_V + a, a^n = 0 \). Then \( Fv \langle h_1 \rangle \cdots \langle h_{i+1} \rangle = v \langle h_1 \rangle \cdots \langle h_i \rangle F[a] \) is also finite dimensional. This finishes the proof that \( \dim_k \tau F[H] < \infty \). Theorem 2 is proved.

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