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ON SET SYSTEMS DETERMINED BY INTERSECTIONS

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The set systems determined by intersections are studied and a sufficient condition for this property is given. For case of graphs a necessary and sufficient condition is established. Some connections to other results are discussed.

In this paper we study the question when a given set system is determined by its intersection graph. We say that a system $A_1, A_2, \ldots, A_n$ of subsets of a given set X is *determined by intersections* if for every other system $A'_1, A'_2, \ldots, A'_n$ of subsets of X such that

$$A_i \cap A_j \neq \emptyset \text{ iff } A'_i \cap A'_j \neq \emptyset \quad \text{for every } i \text{ and } j$$

there exists a permutation π of elements of X such that

$$\pi[A_i] = A'_i \quad \text{for all } i = 1, 2, \ldots, n.$$

A set system $\mathcal{S} = (X; A_1, A_2, \ldots, A_n)$ is determined by intersections if and only if

(a) $\mathcal{S}$ is the only minimal set representation of the intersection graph $\Omega(\mathcal{S})$, and

(b) every automorphism of the graph $\Omega(\mathcal{S})$ is induced by an automorphism of the set system $\mathcal{S}$.

These two properties has been studied separately. In [1] the uniquely intersectable graphs were introduced as those which have (up to isomorphism) the unique minimal set representation. Thus, a set system $\mathcal{S}$ has the property (a) iff $\Omega(\mathcal{S})$ is uniquely intersectable. An example of property (b) is given by the Theorem of Whitney [10] which states that for arbitrary graph G with more than 5 vertices every automorphism of the line graph $L(G)$ is induced by an automorphism of the graph G.

Our paper is divided into three sections. In Section 1, which has an introductory character, some propositions concerning the set systems determined by intersections are established. In Section 2 we give a sufficient condition for a set system to be determined by intersections. This strengthens a result of Wang [9] who proved that the intersection graph of a set system formed by all k-subsets of a given

n-set ($k \leq \frac{1}{2}n$) is uniquely intersectable. From our result follows e.g. that almost all systems of k-subsets of n-set are determined by intersections. In [5] and [2] those set systems $\mathcal{S} = (X;\ A_1, A_2, \ldots, A_n)$ are investigated that every mapping $\varphi : \{A_i;\ 1 \leq i \leq n\} \to \{A_i;\ 1 \leq i \leq n\}$ preserving cardinality of intersections is induced by an automorphism of $\mathcal{S}$. In 2.5 we mention some connections to our results. In Section 3 a characterization of graphs determined by intersections is given. This extends the result of Alter and Wang [1] who proved that line graphs of complete graphs are uniquely intersectable.

1. Preliminaries

1.1. Let X be a finite set and let $\alpha = (A_i, i \in I)$ be a family of subsets of X. By a *set system* we understand a couple (X, α). A set system is called *simple* if α is a family of distinct sets. (We will use the notation $A \in \alpha$ also in the case that α is not simple.)

1.2. Let (X, α) be a set system, $\alpha = (A_i, i \in I)$. We define the graph $G = \Omega(X, \alpha)$ as follows: $V(G) = I$ and for $i \neq j$ $(i, j) \in E(G)$ iff $A_i \cap A_j \neq \phi$. The graph $G = \Omega(X, \alpha)$ is called the *intersection graph* of (X, α) and, conversely, (X, α) is called a *set representation* of G. It is well known that every graph has a set representation [6].

If (X, α) is a simple set system then we identify the vertices of $\Omega(X, \alpha)$ with subsets of X. For an arbitrary family α of subsets of X denote by α_S the simple family which consists of all sets appearing in α.

1.3. A set representation (X, α) of a graph G is called *minimal* if there is no representation (X', α') of G with $|X'| < |X|$. The cardinality of a minimal set representation of G is usually denoted by $\omega(G)$.

1.4. A set system $H = (X, \alpha)$ is a graph if it is simple and $\alpha = (A_i, i \in I)$ is a family of pairs. In this case $\Omega(H)$ is called the *line graph* of H and usually is denoted by $L(H)$.

1.5. To illustrate the definition of the set representation consider the following examples. If K_n is the complete graph with n vertices then the minimal set representation (X, α) is a couple where $|X| = 1$ and α is a family of n one-points sets. Notice also that isolated vertices may be always represented by empty subsets.

1.6. We say that two vertices x, y of a graph G are equivalent $(x \sim y)$ if $(x, y) \in E(G)$ and $(u, x) \in E(G)$ iff $(u, y) \in E(G)$ for every vertex $u \neq x, y$. Obviously, the relation $\sim$ is an equivalence on the set $V(G)$. If we identify all equivalent vertices of G we obtain a graph which will be denoted by $G/\sim$.

Clearly $G/\sim$ is an induced subgraph of G. Let (X, α) be a set system. If $A_i, A_j \in$., $i \neq j$ and $A_i = A_j$, then $i \sim j$ in the intersection graph $\Omega(X, \alpha)$. Thus if $G = G/\sim$, then every set representation of G is simple.

1.7. Let (X, α) and (Y, β) be two set systems with $\alpha = (A_i, i \in I)$ $\beta = (B_j, j \in J)$. A couple (f, φ) where f is a bijection between X and Y and φ is a bijection between I and J is called *isomorphism if* $f[A_i] = B_{\varphi(i)}$ for every $i \in I$. Clearly if (X, α) is a simple set system then every set system isomorphic to (X, α) is simple, too. In this case every isomorphism $\rho = (f, \varphi)$ is uniquely determined by the vertex mapping f and therefore we may identify ρ with f.

If $\rho = (f, \varphi)$ is an isomorphism between (X, α) and (Y, β), then φ is an isomorphism between corresponding intersection graphs $\Omega(X, \alpha)$ and $\Omega(Y, \beta)$. In this case we say that φ is *induced* by ρ and we put $\varphi = \Omega(\rho)$. An isomorphism $(X, \alpha) \to (X, \alpha)$ is called *automorphism*. The group of automorphisms of (X, α) is denoted by $\mathrm{Aut}(X, \alpha)$.

1.8. Definition. We say that a set system (X, α) is *determined by intersections* if every isomorphism φ between $\Omega(X, \alpha)$ and $\Omega(X, \beta)$ is induced by an isomorphism ρ between (X, α) and (X, β) for an arbitrary set system (X, β).

1.9. In [1] the uniquely intersectable graphs were introduced as those which have (up to isomorphism) only one minimal representation where the minimum is taken over all simple representations. For example, the complete graphs are not uniquely intersectable in this sense (see [1, Theorem 2.1]). It will be convenient for our purpose to change slightly this definition.

1.10. Definition. We say that a graph G is *uniquely intersectable* (u.i. graph) if G has (up to isomorphism) only one minimal representation where the minimum is taken over all set representations of G.

1.11. The both above definitions clearly coincide in case of graphs without equivalent vertices (cf. 1.6). They differ e.g. for the complete graphs as those are in our sense uniquely intersectable (cf. 1.5). In the following the uniquely intersectable graphs are considered only in our sence of Definition 1.10.

1.12. Clearly if (X, α) is a set system determined by intersections then $\Omega(X, \alpha)$ is uniquely intersectable.

1.13. Proposition. *Let G be a graph. Then G is u.i. iff $G/\sim$ is u.i.*

Proof. Let $G/\sim$ be not u.i. Let (X, α_S) and (X, β_S) be two nonisomorphic minimal representation of $G/\sim$. Then (X, α), (X, β) where families α resp. β are constructed by adding multiple sets are not isomorphic. Thus G is not u.i.

Suppose that G is not u.i. Let (X, α), (X, β) be two nonisomorphic minimal representations of G. Consider two cases:

(a) Equivalent vertices are represented by the same sets in both (X, α) and (X, β). Then the simple set system (X, α_S) and (X, β_S) are not isomorphic.

(b) There exists a pair x, y of equivalent vertices of G such that x and y are represented by distinct subsets of α i.e. $A_x \neq A_y$. Suppose that $|A_x| \leq |A_y|$. Consider two graphs $G_1 \simeq G_2 \simeq G/\sim$ such that G_1 and G_2 are induced subgraphs of G, $x \in V(G_1)$, $y \in V(G_2)$ and $V(G_1) - \{x\} = V(G_2) - \{y\}$. Put

$$\alpha_1 = \{A_u \in \alpha;\ u \in V(G_1)\},$$

$$\alpha_2 = \{A_u \in \alpha:\ u \in V(G_2) - \{y\}\} \cup \{A'_y\}, \quad \text{where} \quad A'_y = A_x \cup A_y.$$

Clearly (X, α_1), resp. (X, α_2) are representations of G_1, resp. G_2 coincident in all vertices with exception of x and y. As $|A_x| < |A'_y|$ the set systems (X, α_1) and (X, α_2) are not isomorphic. Thus $G/\sim$ is not u.i.

1.14. Proposition. *A set system (X, α) is determined by intersections iff $\Omega(X, \alpha)$ is u.i. and* $\mathrm{Aut}(X, \alpha) = \mathrm{Aut}\, \Omega(X, \alpha)$.

For the proof it is sufficient to realize that if (X, α) is a minimal set representation of G, then $\Omega: \mathrm{Aut}(X, \alpha) \to \mathrm{Aut}\, G$ is an injective mapping (cf. 1.7).

1.15. Combining 1.13 and 1.14 we get the following

Proposition. *For an arbitrary set system (X, α) holds: (X, α) is determined by intersections iff (X, α_S) is determined by intersections.*

1.16. If convenient we shall denote by n the set $\{1, 2, \ldots, n\}$. Denote by $\exp_k n$ (resp. $\exp_{<k} n$) the set system formed by all k-element resp. $<k$ element subsets of n.

Proposition. *The only minimal representation of the graph $G = \Omega(n, \exp_{[n/2]} n)$ is the set system $(n, \exp_{[n/2]} n)$.*

Proof. Let (X, α) be a minimal representation of G. The system α must form an antichain (i.e. α does not contain two sets A, B with $A \subset B$). By Sperner Theorem [8] it follows that $|X| = n$, and the only antichains on n with cardinality $|V(G)|$ are $\exp_{[n/2]} n$ and $\exp_{[(n+1)/2]} n$. If n is even then $[n/2] = [(n+1)/2]$, if n is odd then $\Omega(n, \exp_{[(n+1)/2]} n)$ is complete. Thus

$$(X, \alpha) = (n, \exp_{[n/2]} n).$$

1.17. Notice that the statement converse to 1.12 does not hold. An example is given by set system $(2n, \exp_n 2n)$. By 1.16 $G = \Omega(2n, \exp_n 2n)$ is u.i. but

$(2n, \exp_n 2n)$ is not determined by intersections as

$$|\mathrm{Aut}(2n, \exp_n 2n)| = (2n)!,$$
$$|\mathrm{Aut}\, G| = \left(\frac{1}{2}\binom{2n}{n}\right)!\, 2^{\frac{1}{2}\binom{2n}{n}}.$$

1.18. Another example of a set system which is not determined by intersections but the intersection graph of which is u.i. was suggested by Dr. Jarik Nešetřil.

Let $\mathscr{G} = (\mathscr{P}, \mathscr{L})$ where $\mathscr{P}$ is the set of points and $\mathscr{L} \subset \exp \mathscr{P}$ is the system of lines be projective plane. Denote by $\mathscr{G} + x$ a set system (X, α) defined by $X = \mathscr{P} \cup \{x\}$, $\alpha = \mathscr{L} \cup \{\{x, p\};\ p \in \mathscr{P}\}$. Clearly the graph $\Omega(X, \alpha)$ is isomorphic to the graph $G = (\mathscr{P} \cup \mathscr{L}, E)$ where

$$(p, p') \in E \quad \text{for all } p, p' \in \mathscr{P}, p \neq p';$$
$$(l, l') \in E \quad \text{for all } l, l' \in \mathscr{L}, l \neq l';$$
$$(p, l) \in E \quad \text{iff } p \in l, \text{ for } p \in \mathscr{P}, l \in \mathscr{L}.$$

The following sets of vertices are obviously all cliques of the graph G:

$$\mathscr{P}, \quad \mathscr{L}, \quad \{K(p);\ p \in \mathscr{P}\}, \quad \{K(l);\ l \in \mathscr{L}\}$$

where

$$K(p) = \{p\} \cup \{l \in \mathscr{L};\ p \in l\} \quad \text{for } p \in \mathscr{P},$$
$$K(l) = \{l\} \cup \{p \in \mathscr{P};\ p \in l\} \quad \text{for } l \in \mathscr{L}.$$

Both $\{\mathscr{P}\} \cup \{K(p);\ p \in \mathscr{P}\}$ and $\{\mathscr{L}\} \cup \{K(l);\ l \in \mathscr{L}\}$ are systems of cliques covering all edges of the graph G and there is no other such a system with cardinality $\leq |\mathscr{P}| + 1$. As the minimal representations of the graph are in 1–1 correspondence with minimal systems of cliques covering edges (see 17.4, Proposition 1 in [3]) the only minimal representation of G are $\mathscr{G} + x$ and $\mathscr{G}^* + x$ (where $\mathscr{G}^*$ is the projective plane dual to $\mathscr{G}$).

If $\mathscr{G} \simeq \mathscr{G}^*$ (this holds for Galois planes), then the graph G is uniquely intersectable because $\mathscr{G} + x \simeq \mathscr{G}^* + x$, and $\mathscr{G} + x$ is not determined by intersections as

$$\mathrm{Aut}\, G = \mathbf{Z}_2 \times \mathrm{Aut}(\mathscr{G} + x)$$

(where $\mathbf{Z}_2$ is the cyclic group of order two).

If $\mathscr{G} \not\simeq \mathscr{G}^*$, then G is not u.i. but $\mathrm{Aut}\, G = \mathrm{Aut}(\mathscr{G} + x)$.

2. Set systems determined by intersections

2.1. Definition. Let (X, α), (X, β) be two set systems. We say that α is a *refinement* of β ($\alpha < \beta$) if for every $B \in \beta$ and $x \in B$ there exists an $A \in \alpha$ such that $x \in A \subset B$.

2.2. Lemma. *Let (X, α), (X, α'), (X, β) be set systems such that $\Omega(X, \alpha) \simeq \Omega(X, \alpha')$ and $\alpha < \beta$. Then for every isomorphism*

$$\varphi : \Omega(X, \alpha) \to \Omega(X, \alpha')$$

there exists a set system (X, β') and isomorphism

$$\bar{\varphi} : \Omega(X, \alpha \cup \beta) \to \Omega(X, \alpha' \cup \beta')$$

such that $\bar{\varphi} \mid \alpha = \varphi$.

Proof. Without loss of generality we may suppose that (X, α) and (X, β) are simple. Let us define the mapping $\bar{\varphi}$ by

$$\bar{\varphi}(B) = \bigcup \{\varphi(A); A \in \alpha, A \subset B\} \quad \text{for } B \in \beta,$$
$$\bar{\varphi}(A) = \varphi(A) \quad \text{for } A \in \alpha$$

and put

$$\beta' = \{\bar{\varphi}(B);\ B \in \beta\}.$$

It is easy to verify that $\bar{\varphi}$ has the required properties.

2.3. The following theorem is the main result of [9]. Using Sperner Theorem [8] we give an easy proof different to that given in [9].

Theorem. *The graph $\Omega(n, \exp_k n)$ is uniquely intersectable for $k \leq n/2$.*

Proof. Let (n, α') be a set system such that

$$\Omega(n, \exp_k n) \simeq \Omega(n, \alpha')$$

and let φ be an isomorphism between these graphs. Clearly $\exp_k n < \exp_{[n/2]} n$ holds. If we apply Lemma 2.2 to $\alpha = \exp_k n$ and $\beta = \exp_{[n/2]} n$ we obtain an isomorphism

$$(+) \qquad \varphi : \Omega(n, \exp_k n \cup \exp_{[n/2]} n) \to \Omega(n, \alpha' \cup \beta')$$

which extends φ and thus

$$\Omega(n, \exp_{[n/2]} n) \simeq \Omega(n, \beta').$$

It follows by 1.16 that $\beta' = \exp_{[n/2]} n$ and hence we may rewrite (+) as

$$\bar{\varphi} : \Omega(n, \exp_k n \cup \exp_{[n/2]} n) \to \Omega(n, \alpha' \cup \exp_{[n/2]} n).$$

Take an arbitrary $A \in \exp_k n$. The number of $[n/2]$-sets disjoint with A obviously equals to the number of those disjoint with $\bar{\varphi}(A)$. Thus $|\bar{\varphi}(A)| = k = |A|$ and as the mapping $\bar{\varphi}$ is 1–1 we obtain that $(n, \exp_k n) = (n, \alpha)$.

2.4. Theorem 2.3 may be strengthened to the following

Theorem. *Let $k < n/2$ then the set system $(n, \exp_k n)$ is determined by intersections.*

Proof. Let $\varphi : \Omega(n, \exp_k n) \to \Omega(n, \alpha)$ be an isomorphism. By 2.3 $(n, \alpha) \simeq (n, \exp_k n)$ and thus φ is an automorphism of $\Omega(n, \exp_k n)$. Choose an arbitrary point $x \in n$ and consider a family

$$\mathcal{S}_x = \{\varphi(A); x \in A \in \exp_k n\},$$

which obviously satisfies:

(i) all elements of $\mathcal{S}_x$ are k-sets, $k < n/2$,

(ii) every two elements of $\mathcal{S}_x$ have a nonempty intersection,

(iii) $\mathcal{S}_x$ is an antichain (in respect of $\subset$) with cardinality $\binom{n-1}{k-1}$.

From the Erdös–Ko–Rado Theorem [4] it follows that $|\bigcap \mathcal{S}_x| = 1$. Put $f(x) = \bigcap \mathcal{S}_x$. It is not difficult to verify that φ is induced by f (i.e. $\varphi = \Omega(f)$).

2.5. In [5] the following was proved.

Theorem. *Every* 1–1 *mapping* $\varphi : \exp_k n \to \exp_k n$ *such that*

$$(*) \qquad |A \cap B| = k-1 \Rightarrow |\varphi(A) \cap \varphi(B)| = k-1$$

is induced by a permutation of n provided $2k \neq n$.

We give a short proof of this Theorem as a consequence of 2.4. We may suppose that $2k < n$ as in the opposite case we can consider $\exp_{n-k} n$ and the mapping ψ defined by $\psi(A) = \varphi(n - A)$.

Consider a graph G with vertex set $V(G) = \exp_k n$ and with edges (A, B) for $|A \cap B| = k-1$. It follows from $(*)$ that φ is an automorphism of G and thus φ preserves distances of vertices in G. As

$$|A \cap B| = j \quad \text{iff} \quad \text{dist}_G(A, B) = k - j,$$

it follows that φ preserves all intersections and thus according 2.4, φ is induced by a permutation of n.

2.6. Definition. Let (X, α), (X, β) be two set systems. We say that β *separates* α if for every $A \in \alpha$ and $Y \subset X$ there is a $B \in \beta$ such that

$$A \cap B = \emptyset \quad \text{iff} \quad Y \cap B \neq \emptyset.$$

2.7. Theorem. *Let $\alpha < \beta$ and β separates α. If (X, β) is determined by intersections, then (X, α) is determined by intersections also.*

Proof. Let (X, α), (X, β) with above properties be given. Suppose that there is a system (X, α') and an isomorphism

$$\varphi : \Omega(X, \alpha) \to \Omega(X, \alpha').$$

Consider β' and the isomorphism

$$\bar{\varphi}: \Omega(X, \alpha \cup \beta) \to \Omega(X, \alpha' \cup \beta')$$

the existence of which follows by Lemma 2.2. If we restrict $\bar{\varphi}$ to β we obtain an isomorphism

$$\bar{\varphi}: \Omega(X, \beta) \to (X, \beta').$$

As (X, β) is determined by intersections there is an isomorphism

$$f: (X, \beta) \to (X, \beta')$$

such that $\varphi = \Omega(f)$, i.e.

$$(*) \qquad \bar{\varphi}(Y) = \{f(x);\ x \in Y\}$$

for every $Y \in \beta$.

We show that $(*)$ holds also for all $Y \in \alpha$. In a way of contradiction suppose that there is an $A \in \alpha$ such that $\varphi(A) \neq f[A]$. Then $A \neq f^{-1}[\varphi(A)]$ which contradicts to the fact that β separates α. Thus $\varphi = \Omega(f)$.

2.8. Corollary. *Let α be a system of $<\frac{1}{2}n$-subsets of n. If $\alpha < \exp_{[(n-1)/2]} n$, then (n, α) is determined by intersections.*

Proof. The set system $\exp_{[(n-1)/2]} n$ is determined by intersections by Theorem 2.4 and obviously separates $\exp_{<n/2} n$. Hence by Theorem 2.6 (n, α) is determined by intersections.

2.9. Corollary. *Let n, k be such that $k \leq \frac{1}{2}(n-3)$ and let*

$$L > \frac{2^k}{k} \cdot \frac{n^2(n-1)\cdots(n-k+1)}{(n-4)(n-6)\cdots(n-2k)}.$$

Then the probability that a simple set system (n, α) with

$$\alpha \subset \exp_k n \quad \text{and} \quad |\alpha| = L$$

is determined by intersections is bigger than $1 - ne^{-n}$

Proof. Let $\mathcal{S} = (n, \alpha)$ be an above set system. Then

$$\begin{aligned}
&\operatorname{Prob}[\alpha < \exp_{[(n-1)/2]} n] \\
&\geq 1 - \left[\frac{n-1}{2}\right]\binom{n}{[(n-1)/2]}\binom{\binom{n}{k} - \binom{[(n-1)/2]-1}{k-1}}{L} \Big/ \binom{\binom{n}{k}}{L} \\
&\geq 1 - n2^n\left(1 - \frac{k}{2^{k-1}} \cdot \frac{(n-4)(n-6)\cdots(n-2k)}{n(n-1)\cdots(n-k+1)}\right)^L \\
&\geq 1 - ne^{-n}.
\end{aligned}$$

Thus by Corollary 2.8 also

$$\text{Prob}[\mathcal{S} \text{ is determined by intersections}] \geq 1 - ne^{-n}.$$

3. Line graphs

3.1. The k-star is a connected graph with exactly k edges adjacent to one vertex. This vertex is called *central*. If G is a graph and x its vertex the *complete star* of x is the graph formed by all edges adjacent to x. The degree of a vertex x is denoted by $d(x)$. An edge (x, y) we will denote by xy. If convenient we shall consider a graph as the set of edges only.

3.2. Theorem. *Let G be a graph. Then $L(G)$ is u.i. iff G does not contain an edge-amalgamed triangle (i.e. there are no three vertices x, y, z such that $d(x) = 2$, $d(y)$, $d(z) > 2$ and x, y, z form a triangle in G).*

3.3. The following easy proposition asserts that we may further consider connected graphs only.

Proposition. *G is u.i. iff each component of G is u.i.*

3.4. Before proving Theorem 3.2 we introduce the notion of cover.

Definition. Let G be a graph. A system $\mathcal{U} = \{H_1, H_2, \ldots, H_p\}$ of subgraphs of G each of them is isomorphic either to a triangle or to a star is called a *cover* of G iff each pair of incident edges is contained in at least one H_i. A cover $\mathcal{U} = \{H_1, H_2, \ldots, H_p\}$ is called *minimal* if $p = \omega(L(G))$.

From the Proposition 1, Chapter 17, §4 in [3] and from the fact that cliques of $L(G)$ corresponds either to stars or to triangles of G we obtain immediately:

3.5. Lemma. *Representations of $L(G)$ are in a 1–1 correspondence with covers of G.*

Proof. The assignment of the above correspondence is constructed as follows: If $\mathcal{U} = \{H_1, H_2, \ldots, H_p\}$ is a cover of G we define a representation $(\mathcal{U}, \alpha)$ where $\alpha = \{A_e; e \in E(G)\}$ and $A_e = \{H_i; e \in E(H_i)\}$

3.6. Proof of Theorem 3.2. First we show that the statement of Theorem 3.2 is valid for $G = K_1, K_2, K_3, K_4$. It is easy to see that $\omega(L(K_1)) = \omega(L(K_2)) = 0$ and $\omega(L(K_3)) = 1$ as K_3 can be covered by triangle. Although the graph K_4 has two different minimal covers – one is formed by four triangles and the other by four 3-stars – both corresponding representations are isomorphic to K_4. Thus, the graphs K_1, K_2, K_3, K_4 are u.i.

Consider the following two cases:

(A) G has one minimal cover only (thus G is u.i.).

(B) G has two minimal representations $\mathcal{U}=\{H_1,\dots,H_p\}$, $\mathcal{U}'=\{H'_1,\dots,H'_p\}$ such that H_1 is a 2-star, H'_1 is triangle containing H_1 and $H'_i=H_i$ for all $i\geq 2$. Then the corresponding representations are not isomorphic and thus G is not u.i.

We show that for G distinct from K_i $(i=1,2,3,4)$ and fullfiling the condition of Theorem 3.2 (A) holds. If G contains an amalgamed triangle (B) holds.

For an arbitrary vertex x of G we distinguish the following five cases:

(0) $d(x)=1$.

(1) $d(x)=2$ and there are vertices y, z such that $d(y), d(z)>2$ and x, y, z is a triangle (thus x is a vertex of amalgamed triangle).

(2) $d(x)=2$ and there are vertices y, z such that $d(y)=2$, $d(z)>2$ and x, y, z is a triangle.

(3) There are vertices $z, y_1, y_2, \dots, y_k$, $k\geq 2$ such that x, y_i, z is a triangle and $d(y_i)=2$ for all $i=1,2,\dots,k$.

(4) None of the possibilities (0), (1), (2), (3) hold.

We say that the type of the vertex x is i, $t(x)=i$, if (i) holds.

In the following definition the above notation is used.

Definition. Let G be a graph. We say that a cover $\mathcal{U}$ is admissible if it can be constructed by the following way:

(1) For every vertex x with $t(x)=1$ exactly one of the graphs $\{xy, xz\}$, $\{xy, xz, yz\}$ is contained in $\mathcal{U}$.

(2) For every vertex with $t(x)=2$ take y and z as above and let $\{xy, yz, xz\}\in\mathcal{U}$.

(3) Let $t(x)=3$. If for all $y_1, y_2, \dots, y_k$ (obviously with $t(y_i)=1$) we have chosen triangles xy_iz in step (1) let either $\{xy_1, xy_2, \dots, xy_k\}\in\mathcal{U}$ or $\{xy_1, xy_2, \dots, xy_k, xz\}\in\mathcal{U}$. If there is y_j, $j=1,2,\dots,k$, for which the 2-star xy_j, zy_j has been chosen in step (1) let $\{xy_1, xy_2, \dots, xy_k, xz\}\in\mathcal{U}$.

(4) For every x with $t(x)=4$ the star which is formed by x and all vertices adjacent to x (the complete star) is an element of $\mathcal{U}$.

It is easily seen that the assumptions of the Theorem 3.2 are fullfiled iff there exists exactly one admissible cover. Thus the satatement of this Theorem follows from

3.7. Lemma. *A cover is minimal iff it is admissible.*

Proof. First we show that every minimal cover is admissible.

(a) First we prove that if $\mathcal{U}$ is minimal then each vertex x with $t(x)=4$ is a central vertex of exactly one star. It suffices to prove that such a vertex is a central vertex of at least one star since a cover which contains two stars with the same central vertex is clearly not minimal.

Suppose the contrary – thus there is an x with $t(x)=4$ and with each pair of edges incident to x covered by triangle. Thus x together with its neighbourhood form a complete graph. Denote the corresponding vertex set $\{x_1, x_2, \ldots, x_a, x_{a+1}, \ldots, x_r\}$ obviously $r \geq 4$. Let $x_1, x_2, \ldots, x_a$ be the vertices of K_r which are central vertices of some star of $\mathcal{U}$. From the fact that each component of G is different from K_4 it follows that if $r=4$, then $a \geq 1$. The number of triangles of cover $\mathcal{U}$ which are contained in K_r is at least $\binom{r}{3}-\binom{a}{3}$. If we replace all such triangles by new $r-a$ stars with central vertices $x_{a+1}, \ldots, x_r$ and the stars of cover $\mathcal{U}$ with central vertices $x_1, \ldots, x_a$ by complete stars with the same central vertices we get a new cover with a smaller cardinality, viz.

$$\binom{r}{3}-\binom{a}{3}>r-a \quad \text{for } r>4,\ r>a \text{ and } r=4, 4>a>0.$$

(b) If $\mathcal{U}$ is minimal, then $\mathcal{U}$ does not contain a triangle with vertices x, y, z subject to $t(x)=t(y)=t(z)=4$ as removing triangle xyz and replacing the stars with central vertices x, y, z by complete stars we get a new cover with smaller cardinality.

(c) If $\mathcal{U}$ is minimal, then each star with the central vertex x subject to $t(x)=4$ is complete. Suppose the contrary let S be a star with a central vertex x with $t(x)=4$ which is not complete, i.e. there is a pair of edges which is covered by a triangle xyz. As (b) holds and as every triangle containing a vertex of type 3 contains also a vertex of type 1 we may suppose that $t(y)=1$ or $t(y)=2$. The edge xy together with each edge incident to x different from xz must be contained in a star S, i.e. S contains all edges incident to x but xz. The edges xy', $y' \neq z$ and xz are contained in a triangle. From (b) it follows that $t(y')=1$ for all y. Thus $t(x)=3-a$ contradiction. As at vertices x with $t(x)=1, 2, 3$ the minimal cover satisfies (1), (2), (3), of definition and hence $\mathcal{U}$ is admissible.

As all admissible covers have, by definition, the same cardinality and as minimal cover is admissible it follows that every admissible cover is minimal.

3.8. Remark. We have shown that the definition of admissible cover gives an algorithm for computation of $\omega(L(G))$. This may be interesting with regard to the fact that the computation of $\omega(G)$ is in general case an NP-complete problem (see [7]).

3.9. Theorem. *Let G be a connected graph with at least five vertices. Then G is determined by intersections if and only if the following holds:*

(i) *G contains neither edge nor vertex amalgamed triangle,*

(ii) *degree of every vertex of G is at least* 2.

Proof. Let G be a connected graph with at least 5 vertices, satisfying (i) and (ii). From this follows that all vertices of G are of type 4 only. Thus the only admissible cover of G is formed by the system of complete stars. By Lemma 3.7, G is the only minimal set representation of $L(G)$. By the Whitney Theorem [10] every automorphism of $L(G)$ is induced by an automorphism of G. Hence G is determined by intersections.

If G does not satisfy one of (i), (ii) then $\omega(L(G)) < |V(G)|$ and thus G is not determined by intersections.

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