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An Existence Theorem for Evolution Inclusions Involving Opposite Monotonicities

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In this paper we examine evolution inclusions of the subdifferential type with the set-valued perturbation being nonconvex valued and dissipative. Under certain generally mild hypotheses on the data, we prove the existence of a strong global solution, extending earlier analogous results by M. Ôtani and A. Cellina-V. Staicu. An example of a distributed parameter system is also presented in detail. © 1998 Academic Press

1. INTRODUCTION

Evolution inclusions involving a difference term of subdifferentials were first considered by Y. Koi and J. Watanabe [10] and M. Ôtani [11]. More recently A. Cellina and V. Staicu [7] replaced the second subdifferential in the difference term by an upper semicontinuous, not necessarily convex-valued multifunction, which is cyclically monotone. The lack of convexity precludes the possibility for the multifunction to be maximal cyclically monotone and so we can only say that the multifunction (orientor field) $F(x)$ satisfies $F(x) \subseteq \partial\psi(x)$, with $\psi(\cdot)$ being a proper, convex, and lower semicontinuous function. A. Cellina and V. Staicu [7] studied such an

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evolution equation in $\Re^N$ and using techniques from the theory of maximal monotone differential inclusions and the theory of integrable multifunctions, they proved a local existence theorem.

In this paper we combine the approaches of M. Ôtani [11] and A. Cellina and V. Staicu [7] with ideas and techniques from nonsmooth and multivalued analysis to extend their results to a class of evolution inclusions in a separable Hilbert space. Compared to Ôtani's paper, our work replaces the second subdifferential by a nonconvex-valued, upper semicontinuous, and cyclically monotone orientor field. Compared to the paper of A. Cellina and V. Staicu, our work considers evolution inclusions in a Hilbert space and our existence result is global. We should also mention the works of M. Ôtani [12, 13] who considered time-varying subdifferential inclusions with nonmonotone, demiclosed, convex-valued perturbations and N. S. Papageorgiou [16] who had a nonconvex-valued, lower semicontinuous multivalued perturbation.

2. PRELIMINARIES

Let $T = [0, b]$ and let H be a separable Hilbert space. In what follows by $\mathbf{P}_f(H)$ we will denote the collection of all nonempty and closed subsets of H. On $\mathbf{P}_f(H)$ we can define a generalized metric, known in the literature as the "Hausdorff metric" by setting

$$h(A, B) = \max[h^*(A, B), h^*(B, A)],$$

where

$$h^*(A, B) = \sup_{a \in A}\Big[\inf_{b \in B} \|a - b\|\Big] \qquad (\text{the generalized separation of } A \text{ from } B)$$

and

$$h^*(B, A) = \sup_{b \in B}\Big[\inf_{a \in A} \|a - b\|\Big] \qquad (\text{the generalized separation of } B \text{ from } A).$$

A multifunction $F\colon D \subseteq H \to 2^H \setminus \{\emptyset\}$ is said to be *Hausdorff upper semicontinuous* (denoted henceforth by h-u.s.c.) if

(h-u.s.c.) for every $x_0 \in D$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $h^*(F(x), F(x_0)) < \varepsilon$, for all $x \in D$ with $\|x - x_0\| < \delta$ (i.e., $F(x) \subseteq F(x_0) + \varepsilon B$, with $B = \{h \in H\colon \|h\| < 1\}$).

For further details on h-upper semicontinuity and exhaustive comparisons with other continuity notions for multifunctions, we refer to the interesting paper of F. S. De Blasi and J. Myjak [9].

Let $\varphi: H \to \overline{\mathfrak{R}} = \mathfrak{R} \cup \{+\infty\}$ be a proper, convex, and lower semicontinuous (l.s.c.) function, with effective domain $\operatorname{dom} \varphi = \{x \in H: \varphi(x) < +\infty\}$. The class of all such functions will be denoted by $\Gamma_0(H)$. A function $\varphi \in \Gamma_0(H)$ is said to be of "compact type" if for each $\lambda > 0$ the level set $\{x \in H: \|x\|^2 + \varphi(x) \leq \lambda\}$ is compact in H.

The subdifferential of $\varphi(\cdot)$ at $x \in H$ is defined to be the set

$$\partial\varphi(x) = \{y \in H: (y, z - x) \leq \varphi(z) - \varphi(x), \forall z \in \operatorname{dom} \varphi\}$$

(here $(\cdot, \cdot)$ denotes the inner product of H). It is well known (see, for example, H. Brezis [5]) that $\partial\varphi(\cdot)$ is maximal monotone (i.e., $(y_1 - y_2, x_1 - x_2) \geq 0$ for all $y_i \in \partial\varphi(x_i)$, $i = 1, 2$ (monotonicity), and is not properly included in any other monotone subset of $H \times H$ (maximality)). In fact $\partial\varphi(\cdot)$ is maximal cyclically monotone (i.e., for any $n \geq 1$ and any $y_i \in \partial\varphi(x_i)$, $i = 0, 1, \ldots, N$ we have $(y_0, x_0 - x_1) + \cdots + (y_{n-1}, x_{n\text{-}1} - x_n) + (y_n, x_n - x_0) \geq 0$ (cyclical monotonicity) and is maximal with respect to graph inclusion (maximality)). Let $\operatorname{dom} \partial\varphi = \{x \in H: \partial\varphi(x) \neq \emptyset\}$. Because of the maximal monotonicity of $\partial\varphi(\cdot)$, for every $x \in \operatorname{dom} \partial\varphi$, $\partial\varphi(x)$ is a nonempty, closed, and convex subset of H. So for every $x \in \operatorname{dom} \partial\varphi$, the set $\partial\varphi(x)$ contains an element of minimum norm (the projection of the zero vector on $\partial\varphi(x)$). This unique element is denoted by $\partial^\circ\varphi(x)$. Thus we have $\partial^\circ\varphi(x) \in \partial\varphi(x)$ and $\|\partial^\circ\varphi(x)\| = \inf\{\|y\|: y \in \partial\varphi(x)\}$. Finally recall (see H. Brezis [5, p. 27]) that $\partial\varphi(\cdot)$ is demiclosed; i.e., if $x_n \xrightarrow{s} x$ and $y_n \xrightarrow{w} y$ in H, with $y_n \in \partial\varphi(x_n)$, then $y \in \partial\varphi(x)$.

We will be examining the following multivalued Cauchy problem

$$\begin{cases} -\dot{x}(t) \in \partial\varphi(x(t)) - F(x(t)) & \text{a.e.} \\ x(0) = x_0. \end{cases} \tag{1}$$

By a "strong solution" of (1) we mean a function $x \in C(T, H)$, which is absolutely continuous on every compact subinterval of T, $x(0) = x_0$, $x(t) \in \operatorname{dom} \partial\varphi(\cdot)$ a.e. and $-\dot{x}(t) \in \partial\varphi(x(t)) - f(t)$ a.e., with $f \in L^2(T, H)$, $f(t) \in F(x(t))$ a.e. Recall that an absolutely continuous, H-valued function is almost everywhere strongly differentiable and so the derivative $\dot{x}(\cdot)$ involved in the above definition is a strong derivative.

3. EXISTENCE THEOREM

In this section we prove a global existence theorem for (1), extending the corresponding results of M. Ôtani [11] and A. Cellina and V. Staicu [7].

We will need the following hypotheses on the data of (1).

H(φ) $\varphi \in \Gamma_0(H)$, $\varphi \geq 0$, and is of compact type.

H(ψ) $\psi \in \Gamma_0(H)$ and $\operatorname{dom} \partial\varphi \subseteq \operatorname{dom} \partial\psi$.

H(F) F: $\operatorname{dom} \partial\psi \to \mathbf{P}_f(H)$ is a multifunction such that

(1) $F(x) \subseteq \partial\psi(x)$ for all $x \in \operatorname{dom} \partial\psi$,

(2) $F(\cdot)$ is h-u.s.c.,

(3) $|F(x)|^2 = \sup\{\|y\|^2 \colon y \in F(x)\} \leq \beta \|\partial^\circ\varphi(x)\|^2 + \tau(\|x\|)\,(\varphi(x)^2 + c_1)$ with $\beta \in (0,1)$, $\tau\colon \Re_+ \to \Re_+$ nondecreasing, $c_1 > 0$, and for all $x \in \operatorname{dom} \partial\varphi$,

(4) $(-v + w, x) + \eta\varphi(x) \leq c_2(\|x\|^2 + 1)$, with $\eta > 0$ and $c_2 > 0$, for all $x \in \operatorname{dom} \partial\varphi$, $v \in \partial\varphi(x)$, and $w \in F(x)$.

(H_0) $x_0 \in (\operatorname{int} \operatorname{dom} \partial\psi) \cap \operatorname{dom} \partial\varphi$.

THEOREM 1. *If Hypotheses* H(φ), H(ψ), H(F), *and* (H_0) *hold then problem* (1) *admits a strong solution.*

Because of Hypothesis (H_0), we can find $r > 0$ such that if $\mathring{B}(x_0, r) = \{x \in H\colon \|x\| < r\}$, then $\psi/_{\mathring{B}(x_0,r)}$ is Lipschitz continuous, with Lipschitz constant $L > 0$. Choose $\hat{b} = r/2(L + \|(\partial^\circ\varphi(x_0)\|)$. Now we shall consider a sequence of functions defined in $[0, \hat{b}]$ and prove that a subsequence converges to a solution of the Cauchy problem. For every $n \in N$ we set $T_k^n = [0, t_k^n]$, where $t_k^n = k(\hat{b}/n)$, $k \in \{1, \dots, n\}$ and we are going to construct two functions $f_n, x_n\colon [0, \hat{b}] \to H$. Pick $y_0^n \in F(x_0)$ and define f_n on T_1^n by $f_n(t) = y_0^n$ for all $t \in T_1^n$. Then consider the following Cauchy problem

$$\begin{cases} -\dot{x}(t) \in \partial\varphi(x(t)) - f_n(t) & \text{a.e. on } T_1^n \\ x(0) = x_0. \end{cases}$$

From H. Brezis [5, Theorem 3.6, p. 72], we know that the above Cauchy problem has a unique solution $x_n \in C(T_1^n, H)$. Let $q(0)(\cdot) \in C(T_1^n, H)$ be the unique solution of $-\dot{x}(t) \in \partial\varphi(x(t))$ a.e. on T_1^n, $x(0) = x_0$ (i.e., there is no perturbation term). We have

$$\|x_n(t) - x_0\| \leq \|x_n(t) - q(0)(t)\| + \|q(0)(t) - x_0\|, \qquad t \in T_1^n.$$

From Benilan's inequality (see, for example, H. Brezis [5, Lemma 3.1, p. 64]), we have that

$$\|x_n(t) - q(0)(t)\| \leq \int_0^t \|f_n(s)\|\,ds, \qquad t \in T_1^n.$$

Also $\|q(0)(t) - x_0\| \leq \int_0^t \|\dot{q}(0)(s)\|\, ds$. Using Theorem 3.1 of H. Brezis [5, p. 54], we have that

$$\|q(0)(t) - x_0\| \leq \int_0^t \|\partial^\circ \varphi(q(0)(s))\|\, ds \leq \int_0^t \|\partial^\circ \varphi(x_0)\|\, ds$$
$$= t\|\partial^\circ \varphi(x_0)\|, \qquad t \in T_1^n$$

and $t \to \|\partial^\circ \varphi(q(0)(t))\|$ is nonincreasing on T_1^n. Therefore

$$\|x_n(t) - x_0\| \leq \int_0^t \|f_n(s)\|\, ds + t\|\partial^\circ \varphi(x_0)\|, \qquad t \in T_1^n.$$

Recall that $f_n(t) = y_0^n \in F(x_0)$ for all $t \in T_1^n$ and $|F(x_0)| \leq \sup\{\|v\|: v \in \partial\psi(x_0)\} \leq L$. So we get

$$\|x_n(t) - x_0\| \leq \frac{\hat{b}}{n}[L + \|\partial^\circ \varphi(x_0)\|] < \frac{r}{n}$$

therefore $x_n(t) \in \mathring{B}(x_0, r/n)$, for all $t \in T_1^n$.

Now assumed that f_n and x_n have been defined on the initial interval T_k^n, we shall extend these functions to the interval T_{k+1}^n, for all $k \in \{1, \dots, n-1\}$. Taking $y_k^n \in F(x_n(t_k^n))$, we define f_n on $(t_k^n, t_{k+1}^n]$ by $f_n(t) = y_k^n$ for all $t \in (t_k^n, t_{k+1}^n]$. Again let $x_n \in C(T_{k+1}^n, H)$ be the unique strong solution of the Cauchy problem

$$\begin{cases} -\dot{x}(t) \in \partial\varphi(x(t)) - f_n(t) & \text{a.e. on } T_{k+1}^n \\ x(0) = x_n(t_k^n). \end{cases}$$

Then as above $x_n(t) \in \mathring{B}(x_0, k(r/n))$ for all $t \in T_{k+1}^n$. So we have obtained two sequences of functions, $(f_n)_n$ and $(x_n)_n$, defined on $T = [0, \hat{b}]$ and with values in H. Now we set $\theta_n: [0, \hat{b}] \to [0, \hat{b}]$ defined by

$$\theta_n(t) = \sum_{k=1}^{n} t_{k-1}^n \chi_{T_k^n}(t) \qquad (\text{where } t_o^n = 0).$$

From the above construction, we have that

$$\begin{cases} -\dot{x}_n(t) \in \partial\varphi(x_n(t)) - f_n(t) & \text{a.e. on } \hat{T} = [0, \hat{b}] \\ x_n(0) = x_0 \end{cases}$$

with $f_n \in L^2(\hat{T}, H)$, $f_n(t) \in F(x_n(\theta_n(t))$ a.e. on $\hat{T} = [0, \hat{b}]$. Furthermore we have $x_n(t) \in \operatorname{dom} \partial\varphi$ a.e. on $\hat{T}$ and $x_n(t) \in \mathring{B}(x_0, r)$ for all $t \in \hat{T}$.

Using once again Theorem 3.6 of H. Brezis [5], we get that

$$\left(\int_0^{\hat{b}} \| \dot{x}_n(t) \|^2 \, dt\right)^{1/2} \leq \left(\int_0^{\hat{b}} \| f_n(t) \|^2 \, dt\right)^{1/2} + \sqrt{\varphi(x_0)},$$

then we have

$$\|\dot{x}_n\|_{L^2(\hat{T}, H)} \leq L\sqrt{\hat{b}} + \sqrt{\varphi(x_0)} = M_1$$

therefore we can say that $(\dot{x}_n)_{n \geq 1}$ is relatively sequentially weakly compact in $L^2(\hat{T}, H)$.

Next let q: $L^2(\hat{T}, H) \to C(\hat{T}, H)$ be the solution map for the Cauchy problem $-\dot{x}(t) \in \partial\varphi(x(t)) + h(t)$ a.e. on $\hat{T}$, $x(0) = x_0$; i.e., for every $h \in L^2(\hat{T}, H)$, $q(h)(\cdot) \in C(\hat{T}, H)$ is the unique strong solution of the mentioned Cauchy problem. Since by Hypothesis H(φ), $\varphi(\cdot)$ is of compact type, from the H. Brezis–Konishi theorem [6], we know that the nonlinear semigroup of contractions generated by $-\partial\varphi(\cdot)$ is compact and so Theorem 1 of P. Baras [3] tells us that $q(\cdot)$ is sequentially continuous from $L^2(\hat{T}, H)$ equipped with the weak topology, into $C(\hat{T}, H)$ equipped with the strong topology. Let $W = \{h \in L^2(\hat{T}, H)$: $\|h(t)\| \leq L$ a.e. on $\hat{T}\}$. Then clearly W is sequentially weakly compact in $L^2(\hat{T}, H)$ and so $q(W)$ is compact in $C(\hat{T}, H)$. Note that $(x_n)_{n \geq 1} \subseteq q(W)$. So by passing to a subsequence if necessary we may assume that

$$x_n \xrightarrow{s} x \quad \text{in } C(\hat{T}, H)$$

$$\dot{x}_n \xrightarrow{w} \dot{x} \quad \text{in } L^2(\hat{T}, H)$$

and

$$f_n \xrightarrow{w} f \quad \text{in } L^2(\hat{T}, H).$$

Since $-\dot{x}_n(t) + f_n(t) \in \partial\varphi(x_n(t))$ a.e. on $\hat{T}$, we can say that $-\dot{x}_n + f_n \in \partial\Phi(x_n)$, where Φ: $L^2(\hat{T}, H) \to \overline{\mathfrak{R}}$ is the integral functional defined by

$$\Phi(x) = \begin{cases} \int_0^{\hat{b}} \varphi(x(t)) \, dt & \text{if } \varphi(x(\cdot)) \in L^1(\hat{T}) \\ +\infty & \text{otherwise} \end{cases}$$

and $\Phi(\cdot) \in \Gamma_0(L^2(\hat{T}, H))$ (see H. Brezis [5, Proposition 2.16, p. 47]). Recall that the subdifferential operator is demiclosed and $x_n \xrightarrow{s} x$ in $C(\hat{T}, H)$, $-\dot{x}_n + f_n \xrightarrow{w} -\dot{x} + f$ in $L^2(\hat{T}, H)$. Thus in the limit, we get $-\dot{x} + f \in \partial\Phi(x)$. Therefore we have $-\dot{x}(t) \in \partial\varphi(x(t)) - f(t)$ a.e., $x(0) = x_0$. Note that $\theta_n(t) \to t$ uniformly in $\hat{T}$ and so since $x_n \xrightarrow{s} x$ in $C(\hat{T}, H)$, we get that $x_n(\theta_n(t)) \to x(t)$ in $\hat{T}$ as $n \to +\infty$. Furthermore from Theorem 3.6 of H.

Brezis [5, p. 72] we have

$$\|\dot{x}_n(t)\|^2 + \frac{d}{dt}\varphi(x_n(t)) = (f_n(t), \dot{x}_n(t)) \qquad \text{a.e.}$$

and so

$$\|\dot{x}_n\|^2_{L^2(\hat{T}, H)} = \int_0^{\hat{b}} (f_n(t), \dot{x}_n(t))\, dt - \varphi\big(x_n(\hat{b})\big) + \varphi(x_0). \qquad (3.1)$$

Recall that $f_n(t) \in F(x_n(\theta_n(t))) \subseteq \partial\psi(x_n(\theta_n(t)))$ a.e. So we have

$$(f_n(t), \dot{x}_n(t)) \le \psi'\big(x_n(\theta_n(t))\big)(\dot{x}_n(t)) \qquad \text{a.e.}$$

with $\psi'(x)(h)$ being the directional derivative of $\psi(\cdot)$ at x in the direction h. Since $\psi/_{\mathring{B}(x_0, r)}$ is Lipschitz, we know that $\psi'(x)(h) = \psi^0(x)(h)$ for all $x \in \mathring{B}(x_0, r)$ and all $h \in H$, where $\psi^0(x)(h)$ is Clarke's generalized directional derivative at x in the direction h (see F. H. Clarke [8, Proposition 2.2.7, p. 36]). Also from Proposition 2.2.1 of F. H. Clarke [8, p. 25], we know that $(x, h) \to \psi'(x)(h) = \psi^0(x)(h)$ is u.s.c. from $\mathring{B}(x_0, r) \times H$ into $\Re$ and of course $h \to \psi'(x)(h) = \psi^0(x)(h)$ is sublinear. So by integrating first over $\hat{T} = [0, \hat{b}]$ and then applying Theorem 2.1 of G. Balder [2], we get that

$$\overline{\lim} \int_0^{\hat{b}} \psi'\big(x_n(\theta_n(t))\big)(\dot{x}_n(t))\, dt \le \int_0^{\hat{b}} \psi'(x(t))(\dot{x}(t))\, dt. \qquad (3.2)$$

From Theorem 2.3.10 of F. H. Clarke [8, p. 45] (see also J. P. Aubin [1, Proposition 5, p. 212]), we have that

$$\psi'(x(t))(\dot{x}(t)) = \psi^0(x(t))(\dot{x}(t)) = \frac{d}{dt}\psi(x(t)) \qquad \text{a.e.}$$

In addition, since $f_n(t) \in \partial\psi(x_n(\theta_n(t)))$ a.e. on $\hat{T}$ and $f_n \xrightarrow{w} f$ in $L^2(\hat{T}, H)$, from Theorem 3.1 of N. S. Papageorgiou [15] we have that $f(t) \in \overline{\text{conv}}\,$w-$\overline{\lim}\{f_n(t)\}_{n \ge 1} \subseteq \overline{\text{conv}}\,$w-$\overline{\lim}\, \partial\psi(x_n(\theta_n(t))) \subseteq \partial\psi(x(t))$ a.e. on $\hat{T}$, since $x_n(\theta_n(t)) \to x(t)$ in $\hat{T}$ and the subdifferential operator is demiclosed. Hence $f(t) \in \partial\psi(x(t))$ a.e. on $\hat{T}$ and we can apply Lemma 3.3 of H. Brezis [5, p. 73] and we get that

$$\frac{d}{dt}\psi(x(t)) = (f(t), \dot{x}(t)) \qquad \text{a.e. on } \hat{T}.$$

Then we can say that

$$\int_0^{\hat{b}} \psi'(x(t))(\dot{x}(t))\, dt = \int_0^{\hat{b}} (f(t), \dot{x}(t))\, dt$$

and therefore, using (3.2), we obtain

$$\overline{\lim} \int_0^{\hat{b}} (f_n(t), \dot{x}_n(t))\, dt \le \int_0^{\hat{b}} (f(t), \dot{x}(t))\, dt.$$

Going back to inequality (3.1) and taking $\overline{\lim}$ of both sides and recalling that $\varphi(\cdot)$ is l.s.c. (see hypothesis H(φ)), we get

$$\begin{aligned}\overline{\lim} \|\dot{x}_n\|^2_{L^2(\hat{T}, H)} &\le \int_0^{\hat{b}} (f(t), \dot{x}(t))\, dt - \varphi\big(x(\hat{b})\big) + \varphi(x_0) \\ &= \int_0^{\hat{b}} \left((f(t), \dot{x}(t)) - \frac{d}{dt} \varphi(x(t)) \right) dt\end{aligned}$$

while from Theorem 3.6 of H. Brezis [5, p. 72] we have

$$(f(t), \dot{x}(t)) - \frac{d}{dt} \varphi(x(t)) = \|\dot{x}(t)\|^2 \qquad \text{a.e. on } \hat{T}.$$

Thus finally we have

$$\overline{\lim} \|\dot{x}_n\|^2_{L^2(\hat{T}, H)} \le \|\dot{x}\|^2_{L^2(\hat{T}, H)}.$$

On the other hand, since $\dot{x}_n \xrightarrow{w} \dot{x}$ in $L^2(\hat{T}, H)$ and recalling that the norm is weakly l.s.c., we have

$$\|\dot{x}\|^2_{L^2(\hat{T}, H)} \le \underline{\lim} \|\dot{x}_n\|^2_{L^2(\hat{T}, H)}.$$

Therefore finally we get

$$\|\dot{x}_n\|_{L^2(\hat{T}, H)} \to \|\dot{x}\|_{L^2(\hat{T}, H)}.$$

This together with $\dot{x}_n \xrightarrow{w} \dot{x}$ in $L^2(\hat{T}, H)$ and since $L^2(\hat{T}, H)$ is a Hilbert space, imply that $\dot{x}_n \xrightarrow{s} \dot{x}$ in $L^2(\hat{T}, H)$ (see Proposition I.1.6 of D. Pascali and S. Sburlan [17]). Then by passing to a subsequence if necessary, we may assume that $\|\dot{x}_n(t)\| \le \delta_1(t)$ a.e., with $\delta_1(\cdot) \in L^2(\hat{T})$ and $\dot{x}_n(t) \to \dot{x}(t)$ a.e. on $\hat{T}$. Set $g_n(t) = -\dot{x}_n(t) + f_n(t)$ a.e., $g_n \in L^2(\hat{T}, H)$, $\|g_n(t)\| \le \delta_1(t) + L = \delta_2(t)$ a.e. on $\hat{T}$, and $\delta_2(\cdot) \in L^2(\hat{T})$. We have

$$\begin{aligned}&d\big(g_n(t), F(x(t)) - \dot{x}(t)\big) \\ &\quad = d\big(g_n(t) + \dot{x}(t), F(x(t))\big) \\ &\quad \le \|\dot{x}_n(t) - \dot{x}(t)\| + h^*\Big(F\big(x_n(\theta_n(t))\big), F(x(t))\Big) \to 0\end{aligned}$$

as $n \to +\infty$,

because of hypothesis H(F)(2) and since $x_n(\theta_n(t)) \to x(t)$ in $\hat{T}$ and $\dot{x}_n(t) \to \dot{x}(t)$ a.e. on $\hat{T}$. Set $\Gamma(t) = \overline{\lim}\{g_n(t)\}_{n \geq 1}$. From Theorem 3.5 of N. S. Papageorgiou [14], we know that $t \to \Gamma(t)$ is a $\mathbf{P}_f(H)$-valued Lebesgue measurable multifunction. So via Aumann's selection theorem (see, for example, D. Wagner [18, Theorem 5.10]), we can get $\gamma: \hat{T} \to H$ a measurable map such that $\gamma(t) \in \Gamma(t)$ a.e. Also let $t \in \hat{T} \setminus N$, $\lambda(N) = 0$ ($\lambda(\cdot)$ being the Lebesgue measure on T) be such that

$$d\big(g_n(t), F(x(t)) - \dot{x}(t)\big) \to 0 \qquad \text{as } n \to +\infty.$$

So given $\varepsilon > 0$, we can find $n_0(\varepsilon, t) \geq 1$ such that for all $n \geq n_0$, we have

$$g_n(t) + \dot{x}(t) \in F(x(t)) + \varepsilon \mathring{B}.$$

Since $\varepsilon > 0$ was arbitrary and $F(\cdot)$ is closed-value (see Hypothesis H(F)), we get

$$\Gamma(t) \subseteq F(x(t)) - \dot{x}(t) \qquad \text{a.e.}$$

Also from Theorem 3.1 of N. S. Papageorgiou [15] and since $\partial\varphi(\cdot)$ is demiclosed, we get

$$\Gamma(t) \subseteq \overline{\mathrm{conv}}\, \text{w-}\overline{\lim}\{g_n(t)\}_{n \geq 1} \subseteq \overline{\mathrm{conv}}\, \text{w-}\overline{\lim}\, \partial\varphi\big(x_n(t)\big) \subseteq \partial\varphi(x(t)) \\ \times \text{a.e. on } \hat{T}.$$

Therefore we have

$$\gamma(t) \in \partial\varphi(x(t)) \qquad \text{a.e. on } \hat{T}$$

hence

$$\gamma(t) + \dot{x}(t) = \tilde{f}(t) \in F(x(t)) \qquad \text{a.e. on } \hat{T}$$

and

$$\|\gamma(t)\| \leq \delta_2(t) \qquad \text{a.e. on } \hat{T}.$$

So $-\dot{x}(t) \in \partial\varphi(x(t)) - \tilde{f}(t)$ a.e. on $\hat{T}$, $x(0) = x_0$, and $\tilde{f}(t) \in F(x(t))$ a.e. on $\hat{T}$, with $\tilde{f} \in L^2(\hat{T}, H)$. Thus $x(\cdot) \in C(\hat{T}, H)$ is a local strong solution of (1) (a strong solution of (1) on $\hat{T} = [0, \hat{b}]$).

Next we will show that this local solution is in fact global (i.e., can be continued on all $T = [0, b]$). Until now, we have

$$-\dot{x}(t) \in \partial\varphi(x(t)) - \tilde{f}(t) \qquad \text{a.e. on } \hat{T}, x(0) = x_0,$$

with

$$\tilde{f} \in L^2(\hat{T}, H), \tilde{f}(t) \in F(x(t)) \qquad \text{a.e.}$$

Take the inner product of both sides with $x(t)$ and use Hypothesis H(F)(4) to get

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2 + \eta\varphi(x(t)) \le c_2\left(\|x(t)\|^2 + 1\right) \qquad \text{a.e. on } \hat{T}$$

and therefore

$$\|x(t)\|^2 + 2\eta\int_0^t \varphi(x(s))\,ds \le 2c_2\int_0^t\left(\|x(s)\|^2 + 1\right)ds + \|x_0\|^2,$$
$$\text{for all } t \in \hat{T}.$$

Since $\varphi \ge 0$ (see Hypothesis H(φ)) and $\eta \ge 0$, we get

$$\|x(t)\|^2 \le 2c_2\int_0^t\left(\|x(s)\|^2 + 1\right)ds + \|x_0\|^2, \qquad \text{for all } t \in \hat{T}.$$

Therefore by Gronwall's inequality, we deduce that there exists $M_1 > 0$ (independent of $\hat{b}$) such that $\|x(t)\| \le M_1$, for all $t \in \hat{T}$. Using this bound then, we see that there exists $M_2 > 0$ (also independent of $\hat{b}$) such that

$$\int_0^{\hat{b}} \varphi(x(t))\,dt \le M_2.$$

Recall that with $g(t) = -\dot{x}(t) + \tilde{f}(t) \in \partial\varphi(x(t))$ a.e. on $\hat{T}$, we have

$$\frac{d}{dt}\varphi(x(t)) + (g(t), \dot{x}(t)) = \left(g(t), \tilde{f}(t) - g(t)\right) \qquad \text{a.e. on } \hat{T}$$

hence we have

$$\frac{d}{dt}\varphi(x(t)) + \|g(t)\|^2 = \left(g(t), \tilde{f}(t)\right) \qquad \text{a.e. on } \hat{T},$$

and from Cauchy's inequality with $\varepsilon > 0$ we obtain

$$\frac{d}{dt}\varphi(x(t)) + \|g(t)\|^2 \le \frac{1}{2\varepsilon}\|g(t)\|^2 + \frac{\varepsilon}{2}\|\tilde{f}(t)\|^2 \qquad \text{a.e. on } \hat{T}.$$

So $(d/dt)\varphi(x(t)) + ((2\varepsilon - 1)/2\varepsilon)\|g(t)\|^2 \le (\varepsilon/2)\|\tilde{f}(t)\|^2$ a.e. on $\hat{T}$. Using Hypothesis H(F)(3), we get

$$\begin{aligned}
\frac{d}{dt}\varphi(x(t)) &+ \frac{2\varepsilon - 1}{2\varepsilon}\|g(t)\|^2 \\
&\le \frac{\varepsilon\beta}{2}\|\partial^\circ\varphi(x(t))\|^2 + \frac{\varepsilon}{2}\tau(\|x(t)\|)\left(\varphi(x(t))^2 + c_1\right) \\
&\le \frac{\varepsilon\beta}{2}\|g(t)\|^2 + \frac{\varepsilon}{2}\tau(M_1)\left(\varphi(x(t))^2 + c_1\right) \qquad \text{a.e. on } \hat{T}
\end{aligned}$$

(recall that $\tau(\cdot)$ is nondecreasing and for all $t \in \hat{T}$, $\|x(t)\| \leq M_1$). Hence

$$\frac{d}{dt}\varphi(x(t)) + \frac{2\varepsilon - 1 - \varepsilon^2\beta}{2\varepsilon}\|g(t)\|^2 \leq \frac{\varepsilon}{2}\tau(M_1)\left(\varphi(x(t))^2 + c_1\right)$$

$$\times ia.e.\ on\ \hat{\mathrm{T}}.$$

Choose $\varepsilon > 0$ so that $2\varepsilon - 1 - \varepsilon^2\beta > 0$ (this can be done since $\beta \in (0, 1)$). Then we have

$$\frac{d}{dt}\varphi(x(t)) \leq \frac{\varepsilon}{2}\tau(M_1)\left(\varphi(x(t))^2 + c_1\right) \qquad \text{a.e. on } \hat{T},$$

and so

$$\varphi(x(t)) \leq \varphi(x_0) + \frac{\varepsilon}{2}\tau(M_1)c_1 b + \frac{\varepsilon\tau(M_1)}{2}\int_0^t \varphi(x(s))^2\, ds,$$

for all $t \in \hat{T}$.

Set $\xi = \varphi(x_0) + (\varepsilon/2)\tau(M_1)c_1 b$. We have

$$\varphi(x(t)) \leq \xi + \frac{\varepsilon\tau(M_1)}{2}\int_0^t \varphi(x(s))\,\varphi(x(s))\, ds, \qquad \text{for all } t \in \hat{T}.$$

But recall that $\int_0^{\hat{b}}\varphi(x(s))\, ds \leq M_2$, with $M_2 > 0$ independent of $\hat{b}$. Since $\varphi \geq 0$, we have that $\varphi(x(\cdot)) \in L^1(\hat{T})$. So applying Gronwall's inequality, we get $M_3 > 0$ independent of $\hat{b}$, such that

$$\varphi(x(t)) \leq M_3, \qquad \text{for all } t \in \hat{T}.$$

Therefore $x(\hat{b}) \in \operatorname{dom}\varphi$ and $x(\hat{b}) \in \mathring{B}(x_0, r) \subseteq \operatorname{int}\operatorname{dom}\psi$. So we can apply the above local existence result on the initial datum $(\hat{b}, x(\hat{b}))$ and extend the solution on all $T = [0, b]$; i.e., $x(\cdot)$ is in fact a global solution.

Remarks. (1) If Hypotheses H(F)(3) and H(F)(4) are not present, the we can only conclude the existence of a local strong solution.

(2) It will be interesting to know whether we can have this result for nonautonomous systems (cf. M. Ôtani [12, 13] and N. S. Papageorgiou [16]).

4. AN EXAMPLE

As an application of our abstract existence theorem, we will consider a nonlinear parabolic feedback control system, with nonconvex control constraint set, and establish the existence of admissible "state-control" pairs.

So let $Z \subseteq \Re^N$ be a bounded domain with smooth boundary Γ and $T = [0, b]$. The system under consideration is

$$\begin{cases} \dfrac{\partial x}{\partial t} - \sum_{k=1}^{N} D_k\left(|D_k x|^{p-2} D_k x\right) + u(z) = 0 & \text{a.e. on } T \times Z \\ x/_{T\times\Gamma} = 0, \quad x(0,z) = x_0(z) \text{ a.e. on } Z, u(z) \in U(x(z)) \text{ a.e.} \\ 2 \le p < \infty \end{cases} \tag{2}$$

(here $D_k = \partial/\partial z_k$, $k = 1, \ldots, N$).

Let $H = L^2(Z)$ and let $\varphi\colon H \to \overline{\Re} = \Re \cup \{+\infty\}$ be defined by

$$\varphi(x) = \begin{cases} \dfrac{1}{p} \sum_{k=1}^{N} \int_z |D_k x|^p \, dz & \text{if } x \in W_0^{1,p}(Z) \\ +\infty & \text{otherwise.} \end{cases}$$

We know (see, for example, V. Barbu [4, Proposition 2.9]) that $\varphi \in \Gamma_0(H)$ and furthermore

$$\partial\varphi(x) = -\sum_{k=1}^{N} D_k\left(|D_k x|^{p-2} D_k x\right) = -\Delta_p x$$

(Δ_p being the pseudo-Laplace operator; for $p = 2$ we get the Laplacian) and dom $\partial\varphi = \{x \in W_0^{1,p}(Z)\colon \Delta_p x \in H\}$.

Note that for every $\lambda > 0$, the level set

$$L_\lambda = \left\{x \in H\colon \|x\|_2^2 + \varphi(x) \le \lambda\right\}$$

is bounded in $W_0^{1,p}(Z)$. Recall that $W_0^{1,p}(Z)$ is embedded compactly in $L^2(Z)$ (Sobolev embedding theorem). Hence L_λ is compact in H and so $\varphi(\cdot)$ is of compact type. Also note that $\varphi \ge 0$. So we have satisfied Hypothesis H(φ).

Concerning the control constraint multifunction, we will make the following hypothesis

H(U) $U(x) = \operatorname{sgn} x$, where recall (see, for example, A. Cellina and V. Staicu [7]) that $\operatorname{sgn} x = 1$ if $x > 0$, $\operatorname{sgn} x = -1$ if $x < 0$, and $\operatorname{sgn} 0 = \{-1, 1\}$. It is well known that $\partial|x| = \overline{\operatorname{conv}}\, U(x)$ for every $x \in \Re$. Then let $\psi\colon H \to \Re$ be defined by

$$\psi(x) = \int_Z |x(z)| \, dz.$$

Clearly this is a continuous, convex function. Furthermore "$v \in \partial\psi(x)$ if and only if $v(z) \in \overline{\operatorname{conv}}\, U(x(z))$ a.e. on z."

Also dom $\varphi = W_0^{1,p}(z) \subseteq$ dom $\psi = H = L^2(z)$. Thus we have satisfied Hypothesis H(ψ).

Then define $F\colon H \to \mathbf{P}_f(H)$ by

$$F(x) = \{v \in H = L^2(Z): v(z) = u(z) \text{ a.e. and } u(z) \in U(x(z)) \text{ a.e.}\}.$$

It is easy to see that $F(\cdot)$ is h-u.s.c. and $F(x) \subseteq \partial\psi(x)$. In addition if by $|Z|$ we denote the Lebesgue volume of Z, then we have $|F(x)|^2 = |Z|$. Therefore if we consider the nondecreasing function $\tau\colon \Re_+ \to \Re_+$ defined by $\tau(x) = x + 1$ and the constant $c_1 = |z|$ we get

$$|F(x)|^2 \leq \beta\|\partial^\circ\varphi(x)\|^2 + \tau(\|x\|)\left(\varphi(x)^2 + c_1\right) \qquad \text{for all } x \in \operatorname{dom} \partial\varphi$$

(being $\beta \in (0,1)$) and so we have satisfied Hypothesis H(F)(3). Next let $x \in \operatorname{dom} \partial\varphi$, $v \in \partial\varphi(x)$, $w \in F(x) \subseteq \partial\psi(x)$. We have

$$(-v + w, x)_{L^2(Z)} = (-v, x)_{L^2(Z)} + (w, x)_{L^2(Z)}.$$

Because $v \in \partial\varphi(x)$, if we consider the function $y(z) = 0$, for all $z \in Z$ (being $y \in \operatorname{dom} \varphi = W_0^{1,p}(z)$ and $\varphi(y) = 0$), then we have that

$$(v, -x)_{L^2(Z)} \leq -\varphi(x).$$

Moreover

$$(w, x)_{L^2(Z)} = \|x\|_{L^1(Z)} \leq |z|^{1/2}\|x\|^2_{L^2(Z)}.$$

Therefore we can say that

$$(-v + w, x)_{L^2(Z)} + \varphi(x) \leq |z|^{1/2}\|x\|^2_{L^2(Z)},$$

so we have satisfied H(F)(4).

Rewrite (2) in the following equivalent, control-free, evolution inclusion form

$$\begin{cases} -\dot{x}(t) \in \partial\varphi(x(t)) - F(x(t)) & \text{a.e.} \\ x(0) = x_0, \end{cases}$$

with $x_0 = x_0(\cdot) \in W^{2,p}(Z) \cap W_0^{1,p}(Z)$. Invoking Theorem 1 we get

THEOREM 2. *If Hypothesis* H(U) *holds,* $p \geq 2$, *and* $x_0(\cdot) \in W^{2,p}(Z) \cap W_0^{1,p}(Z)$ *then there exists a "state-control" pair* $[x, u] \in C(T, L^2(Z)) \times L^2(T \times Z)$ *satisfying* (2).

Remark. Our general framework incorporates systems of the form

$$\begin{cases} -\dot{x}(t) \in N_K(x(t)) - F(x(t)) & \text{a.e.} \\ x(0) = x_0 \in K, \end{cases} \tag{3}$$

where $K \subseteq H$ is nonempty, compact convex and $F: D \to \mathbf{P}_f(H)$ is h-u.s.c. and cyclically monotone with $K \subseteq \mathring{D}$. Here $N_K(x)$ is the normal cone to K at x and recall that $N_K(x) = \partial\delta_K(x)$ with $\delta_K(x) = 0$ if $x \in K$, $+\infty$ otherwise (the indicator function of the set K). Assume that H(F)(4) is satisfied and that $|F(x)|^2 \leq \tau(\|x\|)$ with $\tau(\cdot)$ nondecreasing (so that H(F)(3) is valid). Such evolution inclusions are known as "differential variational inequalities" and are important in mathematical economics (in the analysis of resource allocation problems) and in theoretical mechanics (in the study of unilateral problems). If $H = \mathfrak{R}^N$, then (3) is equivalent to $\dot{x}(t) \in$ proj$(F(x(t)); T_K(x))$ a.e., $x(0) = x_0$, with $T_K(x)$ being the tangent cone to K at x.

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