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$\mathcal{N} = 2$ superconformal Newton–Hooke algebra and many-body mechanics

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ABSTRACT

A representation of the conformal Newton–Hooke algebra on a phase space of n particles in arbitrary dimension which interact with one another via a generic conformal potential and experience a universal cosmological repulsion or attraction is constructed. The minimal $\mathcal{N} = 2$ superconformal extension of the Newton–Hooke algebra and its dynamical realization in many-body mechanics are studied.

1. Introduction

Recent proposals for a non-relativistic version of the AdS/CFT correspondence [1,2][1] stimulated renewed interest in non-relativistic conformal (super)algebras [3–13] (for related earlier studies see [14–18]). Matching of symmetries in the bulk to those on the boundary is one of the principal ingredients of the correspondence.

There are two competing approaches to constructing non-relativistic conformal algebras. The first option is to minimally extend the Galilei algebra by the generators of dilatations and special conformal transformations which form the $so(1,2)$ subalgebra together with the generator of time translations. The resulting algebra is known as the Schrödinger algebra [19,20]. An alternative possibility is to consider non-relativistic contractions of the conformal algebra $so(d+1,2)$ (see e.g. the discussion in [3,5,7,8,11,12]). This yields a larger algebra which goes under the name of conformal Galilei algebra. Because the conformal Galilei algebra requires vanishing mass, the Schrödinger algebra is likely to have a better prospect for quantum mechanical applications.

An analogue of the Galilei algebra in the presence of a universal cosmological repulsion or attraction is the Newton–Hooke algebra [21–23]. It can be derived from the (anti) de Sitter algebra by a non-relativistic contraction in much the same way as the Galilei algebra is obtained from the Poincaré algebra [21]. In contrast to the Galilei transformations, however, the bracket relation involving the generators of time and space translations is modified to yield the boost: $[H, P^\alpha] = \pm\frac{1}{R^2}K^\alpha$. Here R is the radius of the parent (anti) de Sitter space. The positive sign on the right-hand side is realized in a non-relativistic spacetime with a negative cosmological constant. The corresponding algebra is conventionally denoted as nh^-. The negative sign is realized in a spacetime with a positive cosmological constant, the shorthand for the algebra being nh^+.

While in arbitrary dimension the Newton–Hooke algebra admits only one central charge, in $(2+1)$-dimensions the second central charge is allowed which leads one to the so-called exotic Newton–Hooke symmetry [22,24–26]. Generalizations of the Newton–Hooke algebra associated with non-relativistic strings and branes were studied in [10,27–29]. Various extensions of the Newton–Hooke algebra by extra vector generators and their dynamical realizations were discussed in [30–32].

That the Newton–Hooke algebra can be extended by conformal generators was known for a long time [33]. However, its dynamical realization in many-body mechanics as well as supersymmetric extensions have not yet been studied. It is natural to expect that many-body models with superconformal Newton–Hooke symmetry may provide new insight into the non-relativistic version of the AdS/CFT correspondence. Another motivation stems from the desire to construct new exactly solvable many-body models in a non-relativistic spacetime with a cosmological constant and to explore novel correlations.

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[1] By now there is an extensive literature on the subject. For a more complete list of references see e.g. a recent work [3].

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The purpose of this work is to construct a representation of the conformal Newton–Hooke algebra on a phase space of n particles in arbitrary dimension which interact with one another via a generic conformal potential and experience a universal cosmological repulsion or attraction. We also study the minimal $\mathcal{N}=2$ superconformal extension[2] of the Newton–Hooke algebra and provide its dynamical realization.

In Section 2 we discuss the minimal conformal Newton–Hooke algebra and give its dynamical realization in many-body mechanics. Section 3 is devoted to the minimal $\mathcal{N}=2$ superconformal Newton–Hooke algebra and its representation on a phase space of n superparticles in arbitrary dimension. In each section, the negative and positive values of a cosmological constant are treated separately.

2. Conformal extension of the Newton–Hooke algebra

The Newton–Hooke algebra describes symmetries of a nonrelativistic spacetime with a cosmological constant [21–23]. The generators of time translations H, space translations P^α, space rotations $M^{\alpha\beta}$, and boosts K^α obey the following Lie brackets

$$
\begin{aligned}
&\left[H, K^\alpha\right]=-P^\alpha, \qquad \left[P^\alpha, K^\beta\right]=-M\delta^{\alpha\beta},\\
&\left[H, P^\alpha\right]=\pm\frac{1}{R^2}K^\alpha, \qquad \left[M^{\alpha\beta}, P^\gamma\right]=\delta^{\alpha\gamma}P^\beta-\delta^{\beta\gamma}P^\alpha,\\
&\left[M^{\alpha\beta}, K^\gamma\right]=\delta^{\alpha\gamma}K^\beta-\delta^{\beta\gamma}K^\alpha,\\
&\left[M^{\alpha\beta}, M^{\gamma\delta}\right]=\delta^{\alpha\gamma}M^{\beta\delta}+\delta^{\beta\delta}M^{\alpha\gamma}-\delta^{\beta\gamma}M^{\alpha\delta}-\delta^{\alpha\delta}M^{\beta\gamma},
\end{aligned}
\tag{1}
$$

where M is the central charge and $\alpha=1,\ldots,d$. The algebra coincides with the (centrally extended) Galilei algebra but for the leftmost bracket entering the second line in (1).

Conformal extensions of the Newton–Hooke algebra were discussed in [33]. Below we construct a representation of the minimal conformal Newton–Hooke algebra on a phase space of n identical particles in arbitrary dimension.

2.1. Negative cosmological constant

We first consider nh^-. Apart form the generators displayed above, the minimal conformal extension of nh^- involves the generators of dilatations t_1 and special conformal transformations t_2. Along with the generator of time translations $t_0=H$ they form the $so(1,2)$ subalgebra[3]

$$
[t_0, t_1]=-\frac{2}{R}t_2, \qquad [t_0, t_2]=\frac{2}{R}t_1, \qquad [t_1, t_2]=\frac{1}{2R}t_0. \tag{2}
$$

Other non-vanishing Lie brackets include

$$
\begin{aligned}
&\left[t_1, P^\alpha\right]=-\frac{1}{2R}P^\alpha, \qquad \left[t_1, K^\alpha\right]=\frac{1}{2R}K^\alpha,\\
&\left[t_2, P^\alpha\right]=\frac{1}{2R^2}K^\alpha, \qquad \left[t_2, K^\alpha\right]=\frac{1}{2}P^\alpha.
\end{aligned}
\tag{3}
$$

It is straightforward to verify that the Jacobi identities hold for the algebra determined by the structure relations (1), (2), (3) where $[H, P^\alpha]=\frac{1}{R^2}K^\alpha$.

In order to construct a representation of the conformal nh^- algebra, let us consider a set of n identical particles (of unit mass) which are parameterized by the coordinates x_i^α and momenta p_i^α, $i=1,\ldots,n$, obeying the standard Poisson bracket

$$
\left\{x_i^\alpha, p_j^\beta\right\}=\delta^{\alpha\beta}\delta_{ij}. \tag{4}
$$

The Hamiltonian which governs the dynamics of the system is chosen in the form

$$
t_0=H=\frac{1}{2}p_i^\alpha p_i^\alpha+V(x)+\frac{1}{2R^2}x_i^\alpha x_i^\alpha. \tag{5}
$$

The last term in (5) is designed to describe a universal cosmological attraction [23], while $V(x_1,\ldots,x_n)$ is supposed to be a generic conformal potential compatible with the translation and rotation invariance

$$
\begin{aligned}
&x_i^\alpha\partial_{\alpha i}V(x)+2V(x)=0, \qquad \sum_{i=1}^n\partial_{\alpha i}V(x)=0,\\
&\left(x_i^\alpha\partial_{\beta i}-x_i^\beta\partial_{\alpha i}\right)V(x)=0,
\end{aligned}
\tag{6}
$$

where we denoted $\partial_{\alpha i}=\frac{\partial}{\partial x_i^\alpha}$.

Guided by the previous study of the conformal algebra in the context of the oscillator potential [35], we introduce the notation

$$
C=\frac{1}{2}x_i^\alpha x_i^\alpha, \qquad D=-\frac{1}{2}x_i^\alpha p_i^\alpha, \tag{7}
$$

and construct the following quantities

$$
\begin{aligned}
&t_1=\frac{1}{R}D\cos(2t/R)+\frac{1}{2}H\sin(2t/R)-\frac{1}{R^2}C\sin(2t/R),\\
&t_2=\frac{1}{R^2}C\cos(2t/R)-\frac{1}{2}H\cos(2t/R)+\frac{1}{R}D\sin(2t/R),\\
&P^\alpha=\left(\sum_{i=1}^n p_i^\alpha\right)\cos(t/R)+\frac{1}{R}\left(\sum_{i=1}^n x_i^\alpha\right)\sin(t/R),\\
&K^\alpha=\left(\sum_{i=1}^n x_i^\alpha\right)\cos(t/R)-R\left(\sum_{i=1}^n p_i^\alpha\right)\sin(t/R),\\
&M^{\alpha\beta}=x_i^\alpha p_i^\beta-x_i^\beta p_i^\alpha.
\end{aligned}
\tag{8}
$$

Note that they explicitly depend on time. It is a matter of straightforward calculation to verify that Eqs. (5), (8) contain the set of functions which are conserved in time and form a representation of the conformal nh^- algebra under the Poisson bracket. The value of the central charge for this particular representation is $M=n$. For particles of mass m one would have $M=nm$. So M is interpreted as the mass of a system.

In order to implement the flat space limit, one first redefines the generators t_1 and t_2 so as to restore the conventional dimensions $t_1\to Rt_1$, $t_2\to R^2(t_2+\frac{1}{2}t_0)$ and then sends R to infinity which corresponds to the vanishing cosmological constant. Taking into account the relation $\lim_{x\to 0}\frac{\sin x}{x}=1$ one then precisely reproduces the representation of the Schrödinger algebra on a phase space of n identical particles constructed in [4].

Concluding this section we note that in [16,35,36] (see also a recent work [37]) it was demonstrated that the whole motion of a free particle can be associated to a half-period of the harmonic oscillator via a specific local diffeomorphism. With the use of this transformation one can construct the wave function of the oscillator starting from that of the free particle. A possibility to generalize the transformation in [16,35–37] to the case of many-body conformal mechanics in the harmonic trap is an interesting open problem which we hope to address in the near future.

2 Recent studies of $\mathcal{N}=4$ superconformal many-body models in one-dimension (see e.g. [34] and references therein) indicate that $\mathcal{N}=2$ is likely to be the maximal superextension compatible with the translation invariance in higher-dimensions.

3 Redefining the generators as $\tilde{t}_0=Rt_0$, $\tilde{t}_1=2Rt_1$, $\tilde{t}_2=2Rt_2$, one gets the standard structure relations of the $so(1,2)$ algebra: $[\tilde{t}_0,\tilde{t}_1]=-2\tilde{t}_2$, $[\tilde{t}_0,\tilde{t}_2]=2\tilde{t}_1$, $[\tilde{t}_1,\tilde{t}_2]=2\tilde{t}_0$.

2.2. Positive cosmological constant

We now turn to discuss the minimal conformal extension of nh^+. As compared to the previous case the structure relations are slightly modified so as to take into account the negative sign in $[H, P^\alpha] = -\frac{1}{R^2} K^\alpha$ in a way compatible with the Jacobi identities[4]

$$[t_0, t_1] = \frac{2}{R} t_2, \qquad [t_0, t_2] = \frac{2}{R} t_1, \qquad [t_1, t_2] = -\frac{1}{2R} t_0,$$
$$\big[t_1, P^\alpha\big] = -\frac{1}{2R} P^\alpha, \qquad \big[t_1, K^\alpha\big] = \frac{1}{2R} K^\alpha,$$
$$\big[t_2, P^\alpha\big] = \frac{1}{2R^2} K^\alpha, \qquad \big[t_2, K^\alpha\big] = -\frac{1}{2} P^\alpha. \tag{9}$$

A representation of the algebra on a phase space of n identical particles is constructed by analogy with the previous case. One starts with the Hamiltonian

$$t_0 = H = \frac{1}{2} p_i^\alpha p_i^\alpha + V(x) - \frac{1}{2R^2} x_i^\alpha x_i^\alpha, \tag{10}$$

where $V(x)$ is a generic conformal potential obeying the constraints (6). The last term describes a universal cosmological repulsion [22,23]. It proves sufficient to replace the trigonometric functions entering (8) by the hyperbolic functions and adjust the number coefficients so as to get the conserved charges

$$t_1 = \frac{1}{R} D \cosh(2t/R) + \frac{1}{2} H \sinh(2t/R) + \frac{1}{R^2} C \sinh(2t/R),$$
$$t_2 = \frac{1}{R^2} C \cosh(2t/R) + \frac{1}{2} H \cosh(2t/R) + \frac{1}{R} D \sinh(2t/R),$$
$$P^\alpha = \left(\sum_{i=1}^n p_i^\alpha\right) \cosh(t/R) - \frac{1}{R} \left(\sum_{i=1}^n x_i^\alpha\right) \sinh(t/R),$$
$$K^\alpha = \left(\sum_{i=1}^n x_i^\alpha\right) \cosh(t/R) - R \left(\sum_{i=1}^n p_i^\alpha\right) \sinh(t/R),$$
$$M^{\alpha\beta} = x_i^\alpha p_i^\beta - x_i^\beta p_i^\alpha. \tag{11}$$

Explicit calculation then shows that they do reproduce the structure relations of the conformal nh^+ algebra under the Poisson bracket, the corresponding central charge being $M = n$.

3. $\mathcal{N} = 2$ superconformal extension of the Newton–Hooke algebra

To the best of our knowledge, the minimal $\mathcal{N} = 2$ superconformal extension of the Newton–Hooke algebra has not yet been studied in literature. Below we establish the structure relations of the algebra and construct a representation on a phase space of n superparticles in d-dimensions. Depending on the sign of a cosmological constant, the analysis proceeds along different lines. Given a spacetime with a negative cosmological constant, our strategy is to start with reasonable supercharges. On the one hand, they should yield a Hamiltonian which reduces to the one from the preceding section in the bosonic limit. On the other hand, the supercharges ought to be compatible with the conformal generators. Other generators prove to follow from the requirement of the closure of the full superalgebra.

In a spacetime with a positive cosmological constant the construction of a conventional supersymmetric extension is problematic [22]. The problem connects to the difficulty to define conserved positive energy in the parent de Sitter space. In this case we construct a modified superalgebra in which the bracket of two supersymmetry charges yields the conformal generator t_2 which is treated as a kind of a regularized Hamiltonian.

In order to accommodate $\mathcal{N} = 2$ supersymmetry in many-body mechanics, one introduces the fermionic variables ψ_i^α and $\bar\psi_i^\alpha$ which are complex conjugates of each other and obey the brackets

$$\big\{\psi_i^\alpha, \psi_j^\beta\big\} = 0, \qquad \big\{\bar\psi_i^\alpha, \bar\psi_j^\beta\big\} = 0,$$
$$\big\{\psi_i^\alpha, \bar\psi_j^\beta\big\} = -i\delta^{\alpha\beta}\delta_{ij}. \tag{12}$$

Brackets among the bosonic and fermionic variables vanish.

3.1. Negative cosmological constant

We first consider the minimal $\mathcal{N} = 2$ superconformal extension of nh^-. Our ansatz for the supersymmetry charges is

$$Q = \psi_i^\alpha \left(p_i^\alpha + i\partial_{\alpha i} U(x) + \frac{i}{R} x_i^\alpha\right),$$
$$\bar Q = \bar\psi_i^\alpha \left(p_i^\alpha - i\partial_{\alpha i} U(x) - \frac{i}{R} x_i^\alpha\right). \tag{13}$$

Here $U(x)$ is a prepotential which obeys the system of partial differential equations

$$\big(x_i^\alpha \partial_{\beta i} - x_i^\beta \partial_{\alpha i}\big) U(x) = 0, \qquad \sum_{i=1}^n \partial_{\alpha i} U(x) = 0,$$
$$x_i^\alpha \partial_{\alpha i} U(x) = RZ, \tag{14}$$

where Z is a constant. The first two constraints in (14) are needed to ensure the rotation and translation invariance. The last restriction guarantees the conformal invariance of a resulting supersymmetric mechanics. Note that these restrictions coincide with those underling many-body quantum mechanics with $\mathcal{N} = 2$ Schrödinger supersymmetry [6].

Computing brackets among Q and $\bar Q$

$$\{Q, Q\} = 0, \qquad \{\bar Q, \bar Q\} = 0, \qquad \{Q, \bar Q\} = -2iH, \tag{15}$$

one finds the Hamiltonian which governs the dynamics of the system

$$t_0 = H = \frac{1}{2} p_i^\alpha p_i^\alpha + \frac{1}{2} \partial_{\alpha i} U(x) \partial_{\alpha i} U(x) + \frac{1}{2R^2} x_i^\alpha x_i^\alpha$$
$$- \partial_{\alpha i} \partial_{\beta j} U(x) \psi_i^\alpha \bar\psi_j^\beta - \frac{1}{R} \psi_i^\alpha \bar\psi_i^\alpha + Z. \tag{16}$$

Because the Hamiltonian is defined up to a constant, if desirable, one can disregard the constant contribution $Z = x_i^\alpha \partial_{\alpha i} U(x)/R$.

An immediate corollary of the structure relations (15) is that Q and $\bar Q$ are conserved in time. Note also that comparing the bosonic limit of (16) with (5) one gets

$$V(x) = \frac{1}{2} \partial_{\alpha i} U(x) \partial_{\alpha i} U(x). \tag{17}$$

That the latter is conformal invariant is assured by the last constraint in (14).

As the next step we modify the conformal generators in (8) so as to take into account the fermionic degrees of freedom

$$t_1 = \frac{1}{R} D \cos(2t/R) + \frac{1}{2} H \sin(2t/R) - \frac{1}{R^2} C \sin(2t/R)$$
$$+ \frac{1}{2R} J \sin(2t/R),$$
$$t_2 = \frac{1}{R^2} C \cos(2t/R) - \frac{1}{2} H \cos(2t/R) + \frac{1}{R} D \sin(2t/R)$$
$$- \frac{1}{2R} J \cos(2t/R). \tag{18}$$

[4] The redefinition $\tilde t_0 = 2Rt_2$, $\tilde t_1 = Rt_0$, $\tilde t_2 = 2Rt_1$ yields the conventional structure relations of $so(1,2)$.

Here C, D are defined as in (7), H is the Hamiltonian (16) and J is the generator of $U(1)$ R-symmetry transformations which affect only the fermionic variables

$$J = \psi_i^\alpha \bar{\psi}_i^\alpha - RZ. \tag{19}$$

The constant contribution to J might seem odd. It was included so as to avoid the appearance of a fictitious central charge in structure relations of the superalgebra. It is straightforward to verify that t_1, t_2 and J are conserved in time. Along with $t_0 = H$ they form a closed subalgebra (vanishing brackets are omitted)

$$\{t_0, t_1\} = -\frac{2}{R} t_2, \qquad \{t_0, t_2\} = \frac{2}{R} t_1,$$
$$\{t_1, t_2\} = \frac{1}{2R} t_0 + \frac{1}{2R^2} J. \tag{20}$$

Thus the $so(1,2)$ subalgebra is modified and includes the R-symmetry generator.

Computing brackets among the supersymmetry charges and the conformal ones (see Appendix A for explicit relations) one gets two new generators

$$S = \left(x_i^\alpha \psi_i^\alpha + \frac{i}{2} R Q\right) e^{\frac{2it}{R}}, \qquad \bar{S} = \left(x_i^\alpha \bar{\psi}_i^\alpha - \frac{i}{2} R \bar{Q}\right) e^{-\frac{2it}{R}}, \tag{21}$$

which correspond to superconformal transformations. It is readily verified that these functions are conserved in time. However, because they exhibit explicit dependence on time, they yield non-vanishing brackets with the Hamiltonian (see Appendix A).

As for the tensor generators, the conserved charges corresponding to translations and Galilei boosts prove to maintain their form (see Eq. (8) above). The generator of rotations is to be modified so as to take into account the fermionic degrees of freedom

$$M^{\alpha\beta} = x_i^\alpha p_i^\beta - x_i^\beta p_i^\alpha - i\left(\psi_i^\alpha \bar{\psi}_i^\beta - \psi_i^\beta \bar{\psi}_i^\alpha\right). \tag{22}$$

Computing brackets among the generators of translations and supertranslations (see Appendix A for explicit relations) one finds two more fermionic conserved charges

$$L^\alpha = \left(\sum_{i=1}^n \psi_i^\alpha\right) e^{\frac{it}{R}}, \qquad \bar{L}^\alpha = \left(\sum_{i=1}^n \bar{\psi}_i^\alpha\right) e^{-\frac{it}{R}}. \tag{23}$$

At this stage it is straightforward to verify that the full algebra closes and no further generators appear. Thus one ultimately arrives at a representation of the minimal $\mathcal{N} = 2$ superconformal Newton–Hooke algebra on a phase space of n superparticles in d-dimensions. Structure relations of the superalgebra are gathered in Appendix A. In arbitrary dimension two central charges M and Z_1 may enter the algebra (see Appendix A). The representation constructed above corresponds to $M = Z_1 = n$.

3.2. Positive cosmological constant

For a conformal mechanics in a space with a positive cosmological constant the energy spectrum is not bounded from below. This is reminiscent of the fact that in the (parent) de Sitter space there does not exist conserved positive energy. By this reason, a conventional supersymmetric extension $\{Q, \bar{Q}\} \sim H$ which implies $H \geqslant 0$ is problematic [22].

As was mentioned above, the conformal generator t_2 can be viewed as a kind of a regularized Hamiltonian. Below we demonstrate that discarding $\{Q, \bar{Q}\} \sim H$ in favour of $\{Q, \bar{Q}\} \sim t_2$ allows one to construct a reasonable $\mathcal{N} = 2$ supersymmetric extension of the conformal nh^+ algebra.

Our starting point is the following representation of the conformal generators

$$t_0 = H = \frac{1}{2} p_i^\alpha p_i^\alpha + \frac{1}{2} \partial_{\alpha i} U(x) \partial_{\alpha i} U(x) - \frac{1}{2R^2} x_i^\alpha x_i^\alpha - \partial_{\alpha i} \partial_{\beta j} U(x) \psi_i^\alpha \bar{\psi}_j^\beta,$$
$$t_1 = \frac{1}{R} D \cosh(2t/R) + \frac{1}{2} H \sinh(2t/R) + \frac{1}{R^2} C \sinh(2t/R),$$
$$t_2 = \frac{1}{R^2} C \cosh(2t/R) + \frac{1}{2} H \cosh(2t/R) + \frac{1}{R} D \sinh(2t/R) + \frac{1}{2R} J, \tag{24}$$

where C, D are defined as in (7) and J is given in (19). It is assumed that the prepotential $U(x)$ obeys the constraints (14). It is readily verified that t_1 and t_2 are conserved in time. Together with t_0 they form a closed subalgebra (see Appendix B). Note that J was included in t_2 so as to reconcile the conformal generators with supercharges to be introduced below.

In order to fix the form of supersymmetry generators, we impose the relations

$$\{Q, \bar{Q}\} = -4it_2, \qquad \{Q, Q\} = 0, \qquad \{\bar{Q}, \bar{Q}\} = 0 \tag{25}$$

and demand Q and $\bar{Q}$ to be conserved with respect to the Hamiltonian t_0. These requirements yield

$$Q = \psi_i^\alpha \left(\left(p_i^\alpha + i\partial_{\alpha i} U(x)\right)\left(\cosh(t/R) + i \sinh(t/R)\right) - \frac{i}{R} x_i^\alpha \left(\cosh(t/R) - i \sinh(t/R)\right) \right),$$
$$\bar{Q} = \bar{\psi}_i^\alpha \left(\left(p_i^\alpha - i\partial_{\alpha i} U(x)\right)\left(\cosh(t/R) - i \sinh(t/R)\right) + \frac{i}{R} x_i^\alpha \left(\cosh(t/R) + i \sinh(t/R)\right) \right). \tag{26}$$

Superconformal generators S and $\bar{S}$ are then determined by computing brackets among Q, $\bar{Q}$ and the conformal generators (see Appendix B for explicit relations)

$$S = \psi_i^\alpha \left(\left(p_i^\alpha + i\partial_{\alpha i} U(x)\right)\left(\cosh(t/R) - i \sinh(t/R)\right) + \frac{i}{R} x_i^\alpha \left(\cosh(t/R) + i \sinh(t/R)\right) \right),$$
$$\bar{S} = \bar{\psi}_i^\alpha \left(\left(p_i^\alpha - i\partial_{\alpha i} U(x)\right)\left(\cosh(t/R) + i \sinh(t/R)\right) - \frac{i}{R} x_i^\alpha \left(\cosh(t/R) - i \sinh(t/R)\right) \right). \tag{27}$$

These are conserved in time.

It turns out that the generators of translations P^α, Galilei boosts K^α, and rotations $M^{\alpha\beta}$ can be chosen as in Eqs. (11) and (22), respectively. They are conserved in time and form a closed algebra together with other generators provided the fermionic partners of K^α are taken in the form

$$L^\alpha = \sum_{i=1}^n \psi_i^\alpha, \qquad \bar{L}^\alpha = \sum_{i=1}^n \bar{\psi}_i^\alpha. \tag{28}$$

Because $U(x)$ is translation invariant, L^α and $\bar{L}^\alpha$ are automatically conserved in time. The full list of structure relations of the minimal $\mathcal{N} = 2$ superconformal extension of nh^+ algebra is given in Appendix B. Like in the preceding case, two central charges M

and Z_1 enter the algebra. The representation above corresponds to $M = Z_1 = n$.

4. Concluding remarks

To summarize, in this work we have constructed a representation of the minimal conformal Newton–Hooke algebra on a phase space of n particles in arbitrary dimension. The minimal $\mathcal{N}=2$ superconformal extension of the algebra and its dynamical realization in many-body mechanics were proposed.

Turning to possible further developments, it is interesting to construct a Lagrangian formulation and to uncover global symmetries which correspond to the conserved charges considered above. Then it is tempting to explore if the decoupling similarity transformation of [4,6] can be applied in the context of many-body quantum mechanics in a spacetime with a cosmological constant. Finally, it is interesting to study if larger (super)conformal Newton–Hooke algebras can be derived by non-relativistic contractions.

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Appendix A

In this appendix we display structure relations of the minimal $\mathcal{N}=2$ superconformal Newton–Hooke algebra for the case of a negative cosmological constant. The non-vanishing graded Lie brackets read

$$\{Q,\bar{Q}\}=-2it_0,\qquad \{Q,\bar{S}\}=2R(t_2+it_1),$$
$$\{\bar{Q},S\}=-2R(t_2-it_1),\qquad \{S,\bar{S}\}=-\frac{i}{2}R^2t_0-iRJ,$$
$$[t_1,Q]=\frac{i}{R^2}S,\qquad [t_2,Q]=\frac{1}{R^2}S,\qquad [t_1,\bar{Q}]=-\frac{i}{R^2}\bar{S},$$
$$[t_2,\bar{Q}]=\frac{1}{R^2}\bar{S},\qquad [t_0,S]=\frac{2i}{R}S,\qquad [t_0,\bar{S}]=-\frac{2i}{R}\bar{S},$$
$$[t_1,S]=-\frac{i}{4}Q,\qquad [t_2,S]=\frac{1}{4}Q,\qquad [t_1,\bar{S}]=\frac{i}{4}\bar{Q},$$
$$[t_2,\bar{S}]=\frac{1}{4}\bar{Q},\qquad [Q,J]=iQ,\qquad [\bar{Q},J]=-i\bar{Q},$$
$$[S,J]=iS,\qquad [\bar{S},J]=-i\bar{S},\qquad \left[Q,P^\alpha\right]=\frac{i}{R}L^\alpha,$$
$$\left[Q,K^\alpha\right]=-L^\alpha,\qquad \left[\bar{Q},P^\alpha\right]=-\frac{i}{R}\bar{L}^\alpha,\qquad \left[\bar{Q},K^\alpha\right]=-\bar{L}^\alpha,$$
$$\left[S,P^\alpha\right]=\frac{1}{2}L^\alpha,\qquad \left[S,K^\alpha\right]=-\frac{i}{2}RL^\alpha,\qquad \left[\bar{S},P^\alpha\right]=\frac{1}{2}\bar{L}^\alpha,$$
$$\left[\bar{S},K^\alpha\right]=\frac{i}{2}R\bar{L}^\alpha,\qquad \left[t_0,L^\alpha\right]=\frac{i}{R}L^\alpha,\qquad \left[t_0,\bar{L}^\alpha\right]=-\frac{i}{R}\bar{L}^\alpha,$$
$$\left\{Q,\bar{L}^\alpha\right\}=\frac{1}{R}K^\alpha-iP^\alpha,\qquad \left\{\bar{Q},L^\alpha\right\}=-\frac{1}{R}K^\alpha-iP^\alpha,$$
$$\left\{L^\alpha,\bar{L}^\beta\right\}=-iZ_1\delta^{\alpha\beta},\qquad \left\{S,\bar{L}^\alpha\right\}=\frac{1}{2}RP^\alpha-\frac{i}{2}K^\alpha,$$
$$\left\{\bar{S},L^\alpha\right\}=-\frac{1}{2}RP^\alpha-\frac{i}{2}K^\alpha,\qquad \left[L^\alpha,J\right]=iL^\alpha,$$
$$\left[\bar{L}^\alpha,J\right]=-i\bar{L}^\alpha,\qquad \left[t_1,P^\alpha\right]=-\frac{1}{2R}P^\alpha,$$
$$\left[t_1,K^\alpha\right]=\frac{1}{2R}K^\alpha,\qquad \left[t_2,P^\alpha\right]=\frac{1}{2R^2}K^\alpha,\qquad \left[t_2,K^\alpha\right]=\frac{1}{2}P^\alpha,$$
$$[t_0,t_1]=-\frac{2}{R}t_2,\qquad [t_0,t_2]=\frac{2}{R}t_1,\qquad [t_1,t_2]=\frac{1}{2R}t_0+\frac{1}{2R^2}J,$$
$$\left[t_0,K^\alpha\right]=-P^\alpha,\qquad \left[P^\alpha,K^\beta\right]=-M\delta^{\alpha\beta},\qquad \left[t_0,P^\alpha\right]=\frac{1}{R^2}K^\alpha,$$
$$\left[M^{\alpha\beta},P^\gamma\right]=\delta^{\alpha\gamma}P^\beta-\delta^{\beta\gamma}P^\alpha,$$
$$\left[M^{\alpha\beta},K^\gamma\right]=\delta^{\alpha\gamma}K^\beta-\delta^{\beta\gamma}K^\alpha,$$
$$\left[M^{\alpha\beta},L^\gamma\right]=\delta^{\alpha\gamma}L^\beta-\delta^{\beta\gamma}L^\alpha,\qquad \left[M^{\alpha\beta},\bar{L}^\gamma\right]=\delta^{\alpha\gamma}\bar{L}^\beta-\delta^{\beta\gamma}\bar{L}^\alpha,$$
$$\left[M^{\alpha\beta},M^{\gamma\delta}\right]=\delta^{\alpha\gamma}M^{\beta\delta}+\delta^{\beta\delta}M^{\alpha\gamma}-\delta^{\beta\gamma}M^{\alpha\delta}-\delta^{\alpha\delta}M^{\beta\gamma}.$$

Appendix B

In this appendix we give structure relations of the minimal $\mathcal{N}=2$ superconformal Newton–Hooke algebra for the case of a positive cosmological constant. The non-vanishing graded Lie brackets read

$$\{Q,\bar{Q}\}=-4it_2,\qquad \{Q,\bar{S}\}=-4i\left(\frac{1}{2}t_0+it_1\right),$$
$$\{\bar{Q},S\}=-4i\left(\frac{1}{2}t_0-it_1\right),\qquad \{S,\bar{S}\}=-4i\left(t_2-\frac{1}{R}J\right),$$
$$[t_1,Q]=-\frac{1}{2R}S,\qquad [t_0,Q]=\frac{i}{R}S,$$
$$[t_1,\bar{Q}]=-\frac{1}{2R}\bar{S},\qquad [t_0,\bar{Q}]=-\frac{i}{R}\bar{S},\qquad [t_0,S]=-\frac{i}{R}Q,$$
$$[t_0,\bar{S}]=\frac{i}{R}\bar{Q},\qquad [t_1,S]=-\frac{1}{2R}Q,\qquad [t_2,S]=-\frac{i}{R}S,$$
$$[t_1,\bar{S}]=-\frac{1}{2R}\bar{Q},\qquad [t_2,\bar{S}]=\frac{i}{R}\bar{S},\qquad [Q,J]=iQ,$$
$$[\bar{Q},J]=-i\bar{Q},\qquad [S,J]=iS,\qquad [\bar{S},J]=-i\bar{S},$$
$$\left[Q,P^\alpha\right]=-\frac{i}{R}L^\alpha,\qquad \left[Q,K^\alpha\right]=-L^\alpha,\qquad \left[\bar{Q},P^\alpha\right]=\frac{i}{R}\bar{L}^\alpha,$$
$$\left[\bar{Q},K^\alpha\right]=-\bar{L}^\alpha,\qquad \left[S,P^\alpha\right]=\frac{i}{R}L^\alpha,\qquad \left[S,K^\alpha\right]=-L^\alpha,$$
$$\left[\bar{S},P^\alpha\right]=-\frac{i}{R}\bar{L}^\alpha,\qquad \left[\bar{S},K^\alpha\right]=-\bar{L}^\alpha,\qquad \left[t_2,L^\alpha\right]=-\frac{i}{2R}L^\alpha,$$
$$\left[t_2,\bar{L}^\alpha\right]=\frac{i}{2R}\bar{L}^\alpha,\qquad \left\{Q,\bar{L}^\alpha\right\}=-\frac{1}{R}K^\alpha-iP^\alpha,$$
$$\left\{\bar{Q},L^\alpha\right\}=\frac{1}{R}K^\alpha-iP^\alpha,$$
$$\left\{L^\alpha,\bar{L}^\beta\right\}=-iZ_1\delta^{\alpha\beta},\qquad \left\{S,\bar{L}^\alpha\right\}=\frac{1}{R}K^\alpha-iP^\alpha,$$
$$\left\{\bar{S},L^\alpha\right\}=-\frac{1}{R}K^\alpha-iP^\alpha,$$
$$\left[L^\alpha,J\right]=iL^\alpha,\qquad \left[\bar{L}^\alpha,J\right]=-i\bar{L}^\alpha,\qquad \left[t_1,P^\alpha\right]=-\frac{1}{2R}P^\alpha,$$
$$\left[t_1,K^\alpha\right]=\frac{1}{2R}K^\alpha,\qquad \left[t_2,P^\alpha\right]=\frac{1}{2R^2}K^\alpha,$$
$$\left[t_2,K^\alpha\right]=-\frac{1}{2}P^\alpha,$$
$$[t_0,t_1]=\frac{2}{R}t_2-\frac{1}{R^2}J,\qquad [t_0,t_2]=\frac{2}{R}t_1,$$
$$[t_1,t_2]=-\frac{1}{2R}t_0,$$
$$\left[t_0,K^\alpha\right]=-P^\alpha,\qquad \left[P^\alpha,K^\beta\right]=-M\delta^{\alpha\beta},$$
$$\left[t_0,P^\alpha\right]=-\frac{1}{R^2}K^\alpha,$$
$$\left[M^{\alpha\beta},P^\gamma\right]=\delta^{\alpha\gamma}P^\beta-\delta^{\beta\gamma}P^\alpha,$$

$$
\begin{aligned}
&\left[M^{\alpha\beta}, K^{\gamma}\right] = \delta^{\alpha\gamma} K^{\beta} - \delta^{\beta\gamma} K^{\alpha}, \\
&\left[M^{\alpha\beta}, L^{\gamma}\right] = \delta^{\alpha\gamma} L^{\beta} - \delta^{\beta\gamma} L^{\alpha}, \qquad \left[M^{\alpha\beta}, \bar{L}^{\gamma}\right] = \delta^{\alpha\gamma} \bar{L}^{\beta} - \delta^{\beta\gamma} \bar{L}^{\alpha}, \\
&\left[M^{\alpha\beta}, M^{\gamma\delta}\right] = \delta^{\alpha\gamma} M^{\beta\delta} + \delta^{\beta\delta} M^{\alpha\gamma} - \delta^{\beta\gamma} M^{\alpha\delta} - \delta^{\alpha\delta} M^{\beta\gamma}.
\end{aligned}
$$

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