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GEOMETRIC STRUCTURES ON COMPACT COMPLEX ANALYTIC SURFACES

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ALTHOUGH the techniques of high-dimensional manifold topology have been successfully extended by Freedman [14] to the topology of 4-manifolds, the results of Donaldson [11] show that one must seek quite a different pattern in studying smooth 4-manifolds, for which low-dimensional techniques may be more appropriate. Since our most coherent account of three-dimensional topology is given by Thurston's geometrization theorem [47], this motivates the study of geometrical structures (in the sense of Thurston) in dimension 4.

By "*geometry* in the sense of Thurston" I understand a pair (X, G_X) with X a 1-connected manifold, G_X a Lie group acting transitively on X, such that:

(1) The stabilizer subgroup K_X in G_X is a point in X is compact (equivalently, X has a G_X-invariant Riemannian metric).
(2) G_X has discrete subgroups Γ such that $\Gamma\backslash X$ (or equivalently, $\Gamma\backslash G_X$) has finite volume: i.e. Γ is a lattice in the sense of [40].

A manifold M has a *geometric structure* of type (X, G_X) (or "modelled on X") if M has an atlas of charts mapping to X, with coordinate changes defined by elements of G_X. Such a structure is *complete* if it defines a homeomorphism of the universal cover $\tilde{M}$ with X, so that $M \cong \Gamma\backslash X$ for Γ, a torsion-free discrete subgroup of G_X. Equivalently [42, p. 403] one may talk in terms of a locally homogeneous metric on M, and metric completeness. In this paper (except the final §11) we shall suppose M to be compact, so that completeness is automatic.

As the four-dimensional geometries have been recently classified by Filipkiewicz [13], we embark here on the next step: the understanding of geometrical structures on closed manifolds. One significant difference from the three-dimensional case emerges at once—in most cases, a geometric structure carries a preferred complex structure, so we have a complex surface. We are then able to appeal to the extensive existing classification theory available for such surfaces, due for the most part to Enriques [12] (in the algebraic case) and to Kodaira [29] (in general).

Indeed, the notion of geometric structures in this situation is no novelty. As Riemann's uniformization theorem showed that the universal cover of any compact complex analytic 1-manifold was biholomorphic to the projective line $\mathbb{P}^1$, the affine line $\mathbb{C}$ or the upper half-plane H, work in this area is commonly known as uniformization theory, and there is already an extensive literature. It is our contention that the approach here using Thurston's ideas gives a better model for understanding geometric behaviour than those employed earlier: for example, affine structures (see §8 for references). As a first instance, observe that the groups of complex-analytic automorphisms of $\mathbb{P}^1$, $\mathbb{C}$ and H form, in each case, Lie groups whereas this is no longer the case for $\mathbb{C}^2$, for example [consider the automorphisms $(x, y) \to (x, y + \phi(x))$, ϕ being any entire function]. For a compact surface with a complex analytic affine connection, it does not even follow that the universal cover is the whole of $\mathbb{C}^2$. These remarks suggest that a geometric structure in the sense defined below may impose so stringent a condition that it excludes large classes of surfaces. However, while this is certainly true, these same classes are excluded by many of the older theories: the rigidity of Thurston geometries is largely compensated by their variety. Moreover, we will show that the assignment of the appropriate geometry (when available) gives a detailed insight into the intrinsic structure of the complex surface.

The plan of the paper is as follows. The first three sections are devoted to an abstract study of the geometries in question: in §1 we give the list, and also the list of compatible complex structures as determined in our previous paper [49]: here we give explicit models. In §2 we apply general results about lattices to the cases in question; these results are used in §3 to give a more accurate listing of the geometries themselves (the original list in §1 gives only the maximal connected group G_X).

Next, §4 recalls the main features of the Enriques–Kodaira classification in a convenient form. This classification is not as well known as it should be, and the results, beyond Kodaira's original papers [28], [29], are rather scattered, so we have presented (partly in §4 and partly in the next five sections) a rapid survey of the results. The recently published book [2] is also valuable in this area.

In §4 also we state our first main theorem: that a geometric complex surface has its place in the classification determined by the geometry X. Then in the main body §5–§9 of the paper, where different cases are discussed in turn, we prove this theorem and also discuss in each case necessary and sufficient conditions for a compact complex analytic surface to carry a compatible geometric structure: our best results are obtained for elliptic surfaces. These results are mostly already known in some form or another.

In §10 we extend our conclusion by contemplating geometric structures that need not be compatible with the complex structure, and also show that if a closed manifold M^4 admits a geometric structure of type X, then X is already determined by the homotopy type of M. Finally in §11 we collect miscellaneous observations about possible further lines of investigation.

§1. DESCRIPTION OF THE GEOMETRIES

Throughout this paper, the terms "geometry" and "geometric structure" will be used in the sense of Thurston: see Scott [42]. To rephrase slightly the above definition, we have a pair (X, G) where:

(i) X is a complete, simply-connected Riemannian manifold;
(ii) G is a group of isometries of X;
(iii) G acts transitively on X; and
(iv) G contains a discrete subgroup Γ with $\Gamma\backslash X$ of finite volume.

The metric is of use in describing the conditions above, but is not regarded as part of the geometric structure: note, however, that it implies that G acts with compact stabilizers. Thus we regard (X, G) and (X', G') as defining the same geometry if there exist a diffeomorphism of X on X' and an isomorphism of G on G' taking the first action to the second. Observe also that if G is contained in a larger group G'' of isometries of X, the geometries (X, G) and (X, G'') are very closely related and we shall often identify them. These situations will be further explored in §3 below: until then we suppose (except when otherwise specified) that $G = G_X^0$ is the maximal connected group in its equivalance class acting on X.

We now describe the geometries in dimensions ≤ 4. In dimension 2, these are well known; the classification in dimension 3 is due to Thurston (see Scott [42]) and in dimension 4 to Filipkiewicz [13] (I believe that there is also an account in lecture notes of Kulkarni).

Dimension 1. There is a unique geometry: that of the (real) Euclidean line: we denote it by E^1.

Dimension 2. There are three geometries: those of the sphere S^2, the Euclidean plane E^2 and the hyperbolic plane H^2.

Dimension 3. Again we have the sphere S^3, Euclidean space E^3 and hyperbolic space H^3. Next we have the products $S^2 \times E^1$ and $H^2 \times E^1$. The universal (infinite cyclic) cover $\widetilde{SL}_2$ of the unit tangent bundle $PSL_2(\mathbb{R})$ of H^2 is a group, and left translations give isometries, as do right translations by the induced cover (isomorphic to $\mathbb{R}$) of PSO_2: thus the isometry group can be written $\widetilde{SL}_2 \times_{Z} \mathbb{R}$.

We also have the group Nil^3—the unique simply-connected nilpotent group of this dimension—this has a circle group of outer automorphisms which yield isometries. Finally there is a solvable group Sol^3, the split extension $\mathbb{R}^2 \rtimes_\alpha \mathbb{R}$ where the quotient $\mathbb{R}$ acts on the subgroup $\mathbb{R}^2$ by

$$\alpha(t)(x, y) = (e^t x, e^{-t} y).$$

Dimension 4. Here we have the irreducible Riemannian symmetric spaces S^4, H^4, $P^2(\mathbb{C})$, $H^2(\mathbb{C}) = SU_{2,1}/S(U_2 \times U_1)$, and a list of products of lower-dimensional geometries: $S^2 \times S^2$, $S^2 \times E^2$, $S^2 \times H^2$, E^4, $E^2 \times H^2$, $H^2 \times H^2$, $S^3 \times E^1$, $H^3 \times E^1$, $\widetilde{SL}_2 \times E^1$, $Nil^3 \times E^1$ and $Sol^3 \times E^1$. There are the nilpotent Lie group $Nil^4 = \mathbb{R}^3 \rtimes_\beta \mathbb{R}$ and the solvable Lie groups $Sol^4_{m,n} = \mathbb{R}^3 \rtimes_{\gamma_{m,n}} \mathbb{R}$ where the action of the quotient is given respectively by

$$\beta(t) = \exp\left\{ t\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\right\}, \quad \gamma_{m,n}(t) = \exp\left\{ t\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}\right\}$$

where $a > b > c$ are real, $a + b + c = 0$, and e^a, e^b, e^c are the roots of $\lambda^3 - m\lambda^2 + n\lambda - 1 = 0$ with m and n positive integers. If $m = n$, then $b = 0$ and we can identify $Sol^4_{m,m} = Sol^3 \times E^1$: in general, $Sol^4_{m,n}$ and $Sol^4_{m',n'}$ are isomorphic iff the corresponding matrices are proportional.

The case of equal roots is excluded here, but the semidirect product $Sol^4_0 = \mathbb{R}^3 \rtimes_\delta \mathbb{R}$ with

$$\delta(t)(x, y, z) = (e^t x, e^t y, e^{-2t} z)$$

gives the next geometry: observe that there is a further circle of isometries here rotating the first two coordinates. There is a further solvable group Sol^4_1, conveniently represented as a matrix group

$$\left\{ \begin{pmatrix} 1 & b & c \\ 0 & \alpha & a \\ 0 & 0 & 1 \end{pmatrix}; \quad \alpha, a, b, c \in \mathbb{R}, \quad \alpha > 0 \right\}.$$

Finally we have the geometry F^4 with isometry group $\mathbb{R}^2 \rtimes SL_2(\mathbb{R})$ [with the natural action of $SL_2(\mathbb{R})$ on $\mathbb{R}^2$] and stabilizer $SO_2(\mathbb{R})$. This is the only geometry in the list to admit no compact models.

The list of geometries arranged by isotropy subgroups is shown in Table 1.

In the previous paper [49], after describing this list, I proved the following:

THEOREM 1.1.

(a) *X carries a complex structure compatible with G_X^0 if and only if it is one of* $P^2(\mathbb{C})$, $H^2(\mathbb{C})$, $S^2 \times S^2$, $S^2 \times E^2$, $S^2 \times H^2$, $E^2 \times H^2$, $H^2 \times H^2$, F^4, $\widetilde{SL}_2 \times E^1$, $Nil^3 \times E^1$, Sol^4_0, Sol^4_1.

Table 1

Isotropy	Geometries
SO_4	S^4, E^4, H^4
U_2	$P^2\mathbb{C}$, $H^2\mathbb{C}$
$SO_2 \times SO_2$	$S^2 \times S^2$, $S^2 \times E^2$, $S^2 \times H^2$, $E^2 \times H^2$, $H^2 \times H^2$
SO_3	$S^3 \times E^1$, $H^3 \times E^1$
SO_2	$\widetilde{SL}_2 \times E^1$, $Nil^3 \times E^1$, Sol_0^4
$(S^1)_{1,2}$	F^4
$\{1\}$	Nil^4, $Sol_{m,n}^4$ (including $Sol^3 \times E^1$), Sol_1^4

Here $(S^1)_{m,n}$ denotes the image of S^1 in $U_2 \subset SO_4$ by $z \mapsto (z^m, z^n)$.

(b) *The geometry* E^4 *(resp.* $S^3 \times E^1$*) carries a complex structure compatible with the smaller group* $\mathbb{R}^4 \rtimes U_2$ *(resp.* $U_2 \times \mathbb{R}$*), which still defines a geometry.*

(c) *The remaining geometries* S^4, H^4, $H^3 \times E^1$, Nil^4 *and* $Sol_{m,n}^4$ *do not admit a complex structure compatible with the geometric structure.*

(d) *In each case except* Sol_1^4*, the complex structure on the maximal relevant geometry is unique up to isomorphism. For* Sol_1^4 *there are just two isomorphism classes.*

This result was proved in [49] using Lie algebra calculations. The statements about existence or non-existence will be given alternative (less direct) proofs below. The same comment applies to the next result.

THEOREM 1.2. *The geometries in the list*

$$P^2(\mathbb{C}),\ H^2(\mathbb{C}),\ S^2 \times S^2,\ S^2 \times E^2,\ S^2 \times H^2,\ E^4,\ E^2 \times H^2,\ H^2 \times H^2,\ F^4$$

admit a Kähler structure compatible with the maximal relevant group of isometries. In the remaining cases of Theorem 1.1 (b), (c)

$$S^3 \times E^1,\ Nil^3 \times E^1,\ \widetilde{SL}_2 \times E^1,\ Sol_0^4,\ Sol_1^4,\ Sol'^4_1,$$

there is no Kähler structure compatible with a geometric structure.

Rather than repeat the earlier proofs, we now exhibit explicitly the complex and Kähler structures in question in the cases when they do exist.

We begin in fact by noting that the two-dimensional geometries S^2, E^2 and H^2 can be naturally identified with the complex projective line $P^1(\mathbb{C})$, affine line $\mathbb{C}$ and upper half-space $H^1(\mathbb{C})$ (denoted hereafter by H) and so acquire (complete) invariant Kähler metrics; hence so do their products. The geometries $P^2(\mathbb{C})$ and $H^2(\mathbb{C})$ are well known to be Kählerian symmetric spaces (and we shall not need the formulae here).

For the case of F^4, first define an action of $\mathbb{R}^2 \rtimes SL_2$ on $\mathbb{R}^2 \times H^2$ as follows. Take (x, y) as coordinates in $\mathbb{R}^2$, and z (with $\operatorname{Im} z > 0$) as (complex) coordinate in H. Then define an action of $\mathbb{R}^2$ by

$$(u, v)(x, y, z) = (u + x, v + y, z)$$

and an action of $SL_2(\mathbb{R})$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(x, y, z) = \left(ax + by, cx + dy, \frac{az + b}{cz + d}\right).$$

It is at once verified that these combine to give an action of $\mathbb{R}^2 \rtimes SL_2$ such that the stabilizer of $(0, 0, i)$ is $SO_2 \subset SL_2$. The action does *not* preserve the complex structure given by taking

$x+iy$ and z as complex coordinates; instead we set $w = x - yz$. Then the action becomes

$$(u, v)(w, z) = (u - vz + w, z)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(w, z) = \left(\frac{w}{cz+d}, \frac{az+b}{cz+d}\right)$$

and so does preserve the complex structure defined by taking $w \in \mathbb{C}$ and $z \in H$ as coordinates. It is not so easy to write down a Kähler structure, but following through the construction given in [49] leads to the formula

$$(\operatorname{Im} z)^{-2}(\mathrm{d}z \otimes \mathrm{d}\bar{z}) + (\operatorname{Im} z)^{-1}\left(\mathrm{d}w - \frac{\operatorname{Im} w}{\operatorname{Im} z}\mathrm{d}z\right) \otimes \left(\mathrm{d}\bar{w} - \frac{\operatorname{Im} w}{\operatorname{Im} z}\mathrm{d}\bar{z}\right).^{\dagger}$$

$S^3 \times E^1$. We identify (using polar coordinates)

$$S^3 \times E^1 = \mathbb{R}^4 - \{0\} \cong \mathbb{C}^2 - \{0\}$$

and give this the obvious complex structure. It is invariant under $U_2 \times \mathbb{R}$, which U_2 acts linearly on $\mathbb{C}^2$ and $\mathbb{R}$ by homotheties.

$Nil^3 \times E^1$. Define a multiplication on $\mathbb{C}^2$ by

$$(w, z)(w', z') = (w + w' - i\bar{z}z', z + z').$$

It is easy to verify that this satisfies the group axioms. The commutator of (w, z) and (w', z') is $(i(\bar{z}'z - \bar{z}z'), 0)$; this lies in the one-dimensional central subgroup $\mathbb{R} \times O$: the central subgroup $i\mathbb{R} \times O$ splits off as a direct factor. The group is thus isomorphic to $Nil^3 \times E^1$. The group SO_2 of outer automorphisms acts by

$$t(w, z) = (w, tz) \quad (t \in \mathbb{C}, |t| = 1)$$

so this also preserves the complex structure on $\mathbb{C}^2$.

$\widetilde{SL}_2 \times E^1$. We have natural actions of $SL_2(\mathbb{R})$ on H and on its bundle of non-zero tangent vectors: this lifts to the universal covers. Indeed, the action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$$

yields, on differentiation,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\mathrm{d}z) = \frac{(\mathrm{d}z)}{(cz+d)^2}.$$

To obtain the universal cover, write w to represent $\log(\mathrm{d}z)$: then we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(w, z) = \left(w - 2\log(cz+d), \frac{az+b}{cz+d}\right).$$

The isometry group of $\widetilde{SL}_2$ was $\widetilde{SL}_2 \times_{\mathbb{Z}} \mathbb{R}$. The central subgroup $\mathbb{Z}$ here corresponds to adding $2i\pi$ to the value of the logarithm: the group $\mathbb{R}$ acts by translating the w coordinate by an imaginary quantity. Translations of the other factor E^1 will translate w by a real amount.

Sol^4_0. Here we have a normal abelian subgroup of rank 3, which we identify with $\mathbb{C} \times \mathbb{R}$, and a quotient group $SO_2 \times \mathbb{R}$, which we identify with $\mathbb{C}^\times$. Define actions on $\mathbb{C}^2$ by letting

† Properties of this geometry are also discussed in R. BERNDT: Some differential operators in the theory of Jacobi forms. *IHES*, preprint, (March 1984). In particular, this metric appears there on p. 8.

$(a, b) \in \mathbb{C} \times \mathbb{R}$ and $\lambda \in \mathbb{C}^\times$ act on $(w, z) \in \mathbb{C} \times H$ by

$$(a, b)(w, z) = (w + a, z + b)$$

$$\lambda(w, z) = (\lambda w, |\lambda|^{-2} z).$$

This gives an action of the desired group which is transitive, preserves the complex structure, and has trivial stabilizer.

Sol_1^4. This group was already presented as a matrix group: again take (w, z) as coordinates in $\mathbb{C} \times H$, and define an action by

$$\begin{pmatrix} 1 & b & c \\ 0 & \alpha & a \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} w + bz + c \\ \alpha z + a \\ 1 \end{pmatrix}.$$

Again this is transitive and has trivial stabilizer. The other complex structure, which we denote by $Sol_1'^4$, is obtained by modifying this to give

$$(w + bz + c + i \log \alpha, \alpha z + a)$$

(note that $\alpha > 0$: $\log \alpha$ denotes the real logarithm). The verification of compatibility is again trivial.

The above explicit models give us the global complex structures of the various model spaces X.

For the case with real dimension 2, we have already identified

$$S^2 = P^1(\mathbb{C}), \quad E^2 = \mathbb{C}, \quad H^2 = H^1(\mathbb{C}) = H.$$

Products of pairs of these are self-explanatory, as is $P^2(\mathbb{C})$; $H^2(\mathbb{C})$ has a standard realization as the unit ball $|z|^2 + |w|^2 < 1$ in $\mathbb{C}^2$. The rest were identified above (up to holomorphic equivalence) as

$$S^3 \times E^1 \cong \mathbb{C}^2 - \{0\}$$

$$Nil^3 \times E^1 \cong \mathbb{C}^2$$

$$\widetilde{SL}_2 \times E^1 \cong Sol_0^4 \cong Sol_1^4 \cong Sol_1'^4 \cong F^4 \cong \mathbb{C} \times H.$$

In particular, we have

COROLLARY 1.3. *The ten complex two-dimensional geometries other than* $P^2(\mathbb{C})$, $P^1(\mathbb{C}) \times P^1(\mathbb{C})$, $P^1(\mathbb{C}) \times \mathbb{C}$ and $P^1(\mathbb{C}) \times H$ *are biholomorphic to domains in* $\mathbb{C}^2$.

§2. PLACEMENT OF DISCRETE SUBGROUPS

We collect here some general results about lattices in connected Lie groups which give some geometrical insight into the corresponding quotients, and which will be of use in the sequel.

Our general reference is Raghunathan [40].

Recall that we have a 1-connected manifold X, and a connected Lie group, here denoted G_X, acting transitively on X such that the isotropy subgroup K_X is compact. Thus if Γ is a discrete subgroup of G_X, $\Gamma \backslash X$ is of finite volume (resp. compact) if and only if $\Gamma \backslash G_X$ is so: thus Γ is a lattice in G_X in the sense of [40]. Now we have:

2.1. [40, 3.1]. *If G is solvable, then any lattice Γ in G is cocompact.*

2.2. (Mostow) [40, 3.3]. *If G is solvable, with nil radical N, and Γ is a lattice in G, then $\Gamma \cap N$ is a lattice in N.*

2.3. *If G is nilpotent, with derived subgroup G', and Γ is a lattice in G, then $\Gamma \cap G'$ is a lattice in G'.*

Proof. We can consider G as an algebraic group (cf. [40, 2.2])—then Γ is Zariski-dense [40, 2.3]. It follows that $[\Gamma, \Gamma]$ is Zariski-dense in $[G, G] = G'$ and hence [40, 2.3] cocompact in G'. Hence so is $\Gamma \cap G'$, which is of course still discrete.

2.4. (Borel). *If G is semisimple and algebraic, and Γ a lattice in G, then Γ is Zariski-dense in G. Any algebraic subgroup normalized by Γ is normalized by G. If, moreover, G has no compact connected normal subgroup, the centralizer of Γ is the centre of G.*

Proof. [40, 5.4, 5.16, 5.18].

2.5. (Bieberbach). *If Γ is a discrete subgroup of the isometry group of E^n, with compact quotient, then Γ contains a lattice of translations of rank n (and finite index in Γ).*

Proof. [40, 8.26].

2.6. (Wang). *If G is a connected Lie group with radical R, such that G/R has no compact factor, and Γ a lattice in G, then $\Gamma \cap R$ is a lattice in R.*

Proof. [40, 8.27].

Most of the above results give us a normal subgroup H of G such that $\Gamma \cap H$ is a lattice in H. It follows (indeed, is equivalent) that the image of Γ is a lattice in G/H. When G acts on X as above, we have a fibration $X \to X/H$, and the fact that $H\Gamma$ is closed shows that we have an induced Seifert fibration

$$\Gamma \backslash X (= \Gamma \backslash G/K) \to \Gamma \backslash X/H (= \Gamma \backslash G/HK).$$

We apply these to all the geometries of dimension ≤ 4.

Euclidean cases

For E^2, E^3 and E^4, (2.5) applies: Γ has a translation subgroup of finite index.

Nilpotent cases

For Nil^3, $Nil^3 \times E^1$, Nil^4, (2.3) applies. Here, $\Gamma \cap G'$ is a group of translations (of rank 1, 1, 2, respectively; the quotient is a subgroup of a vector space, so again a lattice of translations (of rank 2, 3, 2).

Solvable cases

For Sol^3, $Sol^4_{m,n}$, Sol^4_0 and Sol^4_1, (2.2) applies. In each but the final case, the nilradical is Euclidean: thus $\Gamma \cap N$ is a group of translations (of rank 2, 3, 3) and the quotient is infinite cyclic. If $G = Sol^4_1$, then $N \cong Nil^3$: thus $\Gamma \cap N$ is a lattice in Nil^3 (as discussed above); the quotient is a lattice in $\mathbb{R} \times SO_2$.

Mixed cases

For $H^2 \times E^1$, $\widetilde{SL}_2$, $\widetilde{SL}_2 \times E^1$, $H^3 \times E^1$ and F^4, (2.6) applies. (Note that we have listed X and not G_X.) The radical is Euclidean, of respective dimensions 1, 1, 2, 1, 2; so $\Gamma \cap R$ is a group of translations of the corresponding rank. The quotient is a lattice in $PSL_2(\mathbb{R})$ in all of these cases except $H^3 \times E^1$.

An alternative, more geometrical proof of this conclusion for $H^2 \times E^1$ and $\widetilde{SL}_2$ has been given by Thurston ([47]; see also Scott [42, pp. 461, 466]). This method of proof was extended in [49, 6.3] to give corresponding results for $\widetilde{SL}_2 \times E^1$, $Nil^3 \times E^1$ (as above) and also $H^2 \times E^2$.

Semisimple cases

For H^2, H^3, H^4, $H^2 \times H^2$ and $H^2(\mathbb{C})$, (2.4) applies: note, moreover, that in each of these cases the group G has no compact factor. These are the cases giving the richest and most interesting variety of geometric structures. This is in contrast to compact cases.

Compact cases

For S^2, S^3, S^4, $S^2 \times S^2$ and $P^2(\mathbb{C})$ the group Γ must be finite, and all cases may be enumerated.

Remaining cases

These are $S^2 \times E^1$, $S^2 \times E^2$, $S^2 \times H^2$, $S^3 \times E^1$; in each, X is of the form $A \times B$ with A compact. Thus G_X and hence Γ act properly on B: Γ is obtained by lifting this action (arbitrarily) and combining with a finite isometry group of A, normalized by this lifted group.

Only for the final group of geometries is X non-compact and not contractible: indeed, in all the other non-compact cases, X is homeomorphic to Euclidean space of the same dimension.

The grouping into cases given above gives a better insight into the structure of the several geometries. Combining with the results of §1 yields Table 2.

Table 2. Summary list of geometries of dim ≤ 4

			Dim 4		
	Dim 2	Dim 3	Kähler	Complex non-Kähler	Non-complex
Compact	S^2	S^3	$P^2(\mathbb{C})$ $S^2 \times S^2$		S^4
Compact factor		$S^2 \times E^1$	$S^2 \times E^2$ $S^2 \times H^2$	$S^3 \times E^1$	
Euclidean	E^2	E^3	E^4		
Nilpotent		Nil^3		$Nil^3 \times E^1$	Nil^4
Solvable		Sol^3		Sol_0^4 Sol_1^4	$Sol_{m,n}^4$
Mixed		$H^2 \times E^1$ $\widetilde{SL}_2$	$H^2 \times E^2$ F^4	$\widetilde{SL}_2 \times E^1$	$H^3 \times E^1$
Semisimple	H^2	H^3	$H^2(\mathbb{C})$ $H^2 \times H^2$		H^4

§3. MAXIMAL AND MINIMAL GEOMETRIES

In our listing of geometries in §1 we described, for each model X, the maximal connected group G_X° of isometries of X. In the first part of this section we will list those connected subgroups H_X of G_X° such that (X, H_X) is also a geometry: this result was in fact used in [49] and we shall make further occasional reference below. In the second part we will determine the full maximal group G_X of isometries of X.

Such descriptions seem necessary for an adequate understanding of the geometry X. Our results in the second part are relevant to the questions of orientation-reversing homeomorphisms and of anti-holomorphic equivalences: particularly complex conjugations giving real forms. The non-maximal geometries of the first part, though they introduce an annoying complication into the classification, are important in applications as the following four paragraphs illustrate.

It is shown in [42, pp. 445, 455] that a closed 3-manifold with geometric structure of type S^3 resp. E^3 in fact admits one of type (S^3, U_2) resp. $(E^3, E^3 \rtimes SO_2)$: it is these latter which correspond to Seifert fibre space structures. Similar examples in dimension 4 may be found below.

We saw in §1 that two geometries ($S^3 \times E^1$ and E^4) only admit a compatible complex structure when the group of the geometry is restricted (to $U_2 \times \mathbb{R}$ or $E^4 \rtimes U_2$, respectively).

A weighted homogeneous normal surface singularity admits, according to Neumann [38], a natural geometric structure of one of the types $S^3 \times E^1$, $Nil^3 \times E^1$ or $\widetilde{SL}_2 \times E^1$. It was shown by Dolgachev [10] that the singularity is Gorenstein if and only if there is a structure of the corresponding type (X, H_X), where $H_X \cong X$ acts on itself by translations.

We shall see in §9 below that manifolds with geometric structure of type Sol_0^4 are Inoue surfaces. In each case, the geometric structure can be taken to be non-maximal.

THEOREM 3.1. *The non-maximal geometries with connected isometry group in dimension ≤ 4 are as follows:*

Dimension 2: (E^2, E^2)

Dimension 3: (S^3, U_2), (S^3, SU_2), $(E^3, E^3 \rtimes SO_2)$, (E^3, E^3), $(\widetilde{SL}_2, \widetilde{SL}_2)$, (Nil^3, Nil^3).

Dimension 4: $(E^4, E^4 \rtimes K)$; $K = U_2,\ SU_2,\ SO_3,\ \underline{SO_2 \times SO_2},\ \underset{\sim}{SO_2},\ (S^1)_{m,n},\ \{1\}$; $(S^2 \times E^2, SO_3 \times E^2)$, $(H^2 \times E^2,\ PSL_2 \times E^2)$, $(\widetilde{SL}_2 \times E^1, \widetilde{SL}_2 \times E^1)$, $(S^3 \times E^1, U_2 \times E^1)$, $(S^3 \times E^1, SU_2 \times E^1)$, $(Nil^3 \times E^1, Nil^3 \times E^1)$, (Sol_0^4, H_λ^4) *(as listed below)*.

Proof. We organize the cases following the discussion in §2 of the different sorts of geometries.

Euclidean cases

Since any Γ has a translation subgroup of finite index, H_X must contain E^n. It follows that H_X is a split extension of E^n by a maximal compact subgroup K (the splitting follows since such a K has a fixed point on X): K can be any (connected) subgroup of SO_n.

Nilpotent cases

As Nil^4 already has trivial stabilizer, there can be no non-maximal geometry here. In the cases of Nil^3 and $Nil^3 \times E^1$ the stabilizer of G_X^0 is SO_2, hence that of H_X must be trivial. Now by (2.2) any lattice Γ in G_X^0 meets Nil^3 resp. $Nil^3 \times E^1$ in a lattice. Thus H_X must contain, hence coincide with Nil^3 resp. $Nil^3 \times E^1$.

Solvable cases

For Sol^3, $Sol_{m,n}^4$ and Sol_1^4 the connected isotropy group is trivial and there is nothing to prove. For Sol_0^4 we know by (2.2) that any lattice Γ meets $\mathbb{R}^3$ in a lattice in $\mathbb{R}^3$: thus H_X must contain $\mathbb{R}^3$. Since the stabilizer of G_X° is a circle, that of H_X must be trivial. Thus H_X is an extension of $\mathbb{R}^3$ by a 1-parameter subgroup L of the quotient $\mathbb{R} \times SO_2 \cong \mathbb{C}^\times$. Conversely, for such an extension to be a geometry, there must be a lattice in $\mathbb{R}^3$ invariant under some element $\lambda \in \mathbb{C}^\times$. Now the eigenvalues of λ on $\mathbb{R}^3$ are λ, $\bar{\lambda}$ and $|\lambda|^{-2}$: for there to exist an element of $SL_3(\mathbb{Z})$ with such eigenvalues, λ must be a unit in a complex cubic field.

Semisimple cases

If G_X° is semisimple with no compact factor then by (2.4) if Γ is a lattice in $H_X \subset G_X^\circ$ then as Γ normalizes H_X, so does G_X°. Thus H_X is normal and Zariski-dense in G_X°, so $H_X = G_X$.

Mixed cases

If G_X° is an extension of a vector group V by a semisimple group $\bar{G}$ with no compact factor, and Γ a lattice in G_X°, then by (2.6) the image of Γ in $\bar{G}$ is a lattice in $\bar{G}$, so arguing as above, $\bar{H} = \bar{G}$. Hence H_X is an extension of $H_X \cap V$ by a Levi complement for H. In the cases $X = H^2 \times E^1$, $H^3 \times E^1$, F^4 this is isomorphic to $\bar{H}$. But $H_X \cap V$ contains the full group $\Gamma \cap V$ of translations hence (being connected) is the whole of V. Thus again $H_X = G_X$. In the case $X = \widetilde{SL}_2$, the subgroup $\widetilde{SL}_2$ covering $\bar{G}$ already contains a lattice in V, so defines a non-maximal geometry. Similarly for $X = \widetilde{SL}_2 \times E^1$ we have non-maximal geometries with $H_X = \widetilde{SL}_2 \times W$ for some line $W \subset V$. All these cases are isomorphic.

For the remaining case $X = H^2 \times E^2$ in this group, we know that $\Gamma \cap E^2$ gives a lattice in E^2 and that the projection of Γ on PSL_2 defines a lattice there (see [49, 6.3]). As above, it follows that H_X must contain $PSL_2 \times E^2$, and so equals this if it is not maximal.

Compact factor cases

If $X = S^2 \times B$, then the projection of H_X on Isom (S^2) must be transitive, hence the whole of SO_3. As SO_3 is simple, and does not occur in Isom (B), it follows that $H_X = SO_3 \times C$, where (B, C) is a geometry. A similar argument works for the $S^3 \times E^1$ case to show that our geometry is a product of non-maximal geometries.

Compact cases

The only condition on H_X here is that it acts transitively on X. If $X = S^2$, then clearly H_X must be SO_3. For $X = S^3$, $H_X/\{\pm 1\} \subset SO_3 \times SO_3$ must project onto at least one of the factors: this leads quickly to the conclusion. Again for $X = S^2 \times S^2$, $SO_3 \times SO_3$ has no transitive subgroup. In the remaining cases $X = S^4$, $P^2(\mathbb{C})$, G_X has rank 2: here a subgroup of rank 1 is not large enough to be transitive, nor is one of rank 2 (necessarily $S^1 \times S^1$, $S^3 \times S^1$ or similar).

Collecting the results obtained above yields the result summarized in Theorem 3.1. The results were summarized rather inaccurately in [49]: in particular, the case $(Nil^3 \times E^1)'$ mentioned there does not admit a lattice. It is worth noting that apart from the somewhat exceptional case of Sol_0^4, the non-maximal geometries all belong to the family of Seifert fibre geometries.

We turn to the listing of maximal geometries. Here we describe the methods first and summarize the results at the end.

First, if we denote by K_X° the isotropy group of G_X°, then a group G_X with identity component G_X° will have stabilizer $K_X \subset O_4$ with identity component K_X°, and hence normalizing K_X°. This already enables us to deal with a dozen cases.

If $K_X^\circ = SO_4$ (cases $X = S^4$, E^4 or H^4) then $K_X = O_4$ as in each case X admits a reflexion.

If $K_X^\circ = U_2$ [cases $X = P^2(\mathbb{C})$, $H^2(\mathbb{C})$] then K_X equals the normalizer, $\tilde{U}_2$, say, of U_2 since in each case X admits a complex conjugation.

If $K_X^\circ = SO_2 \times SO_2$ then $K_X = O_2 \times O_2$ (cases $X = S^2 \times E^2, S^2 \times H^2, E^2 \times H^2$) or, if there is an isometry interchanging the factors (cases $X = S^2 \times S^2$, $H^2 \times H^2$), $K_X = \mathbb{Z}_2 \wr O_2$.

If $K_X^\circ = SO_3$ (cases $X = S^3 \times E^1, H^3 \times E^1$) then $K_X = O_3 \times O_1$: again the existence of the desired reflexions is clear in each case.

In general, if we have a Riemannian product $A \times B$ where A and B are irreducible and not isometric to each other, then, since such (de Rham) decompositions are unique we have (see, for example, [27, Vol. 1, p. 240] $\operatorname{Isom}(A \times B) = \operatorname{Isom} A \times \operatorname{Isom} B$. This line of argument recovers some of the cases above and also deals with $Nil^3 \times E^1$, where $K_X = O_2 \times O_1$; with $\widetilde{SL}_2 \times E^1$, where again $K_X \cong O_2 \times O_1$; and with $Sol^3 \times E^1$, where $K_X \cong D_8 \times O_1$ (here D_8 denotes the dihedral group of order 8): beware that these isomorphisms do not make explicit the representation $K_X \to O_4$.

In the remaining cases, we calculate directly:

F^4. Any automorphism α of $G_X^\circ = \mathbb{R}^2 \rtimes SL_2(\mathbb{R})$ must leave $\mathbb{R}^2$ invariant. As complements to $\mathbb{R}^2$ in the Lie algebra are all conjugate by $\mathbb{R}^2$, we may suppose α also leaves $SL_2(\mathbb{R})$ invariant. Now $\alpha | \mathbb{R}^2$ defines an element of $GL_2(\mathbb{R})$, whose class modulo $SL_2(\mathbb{R})$ is unaltered on changing α by an inner automorphism. If this class is trivial, we may suppose α the identity on $\mathbb{R}^2$: but now $\alpha | SL_2(\mathbb{R})$ is also the identity, since an element of $SL_2(\mathbb{R})$ is determined by the way it acts on $\mathbb{R}^2$. In general, as we are only interested in α of finite order, $\det(\alpha | \mathbb{R}^2)$ has finite order, thus equals ± 1. There are thus essentially only two cases for α, and $G_X = \mathbb{R}^2 \rtimes SL_2{}^{\pm}(\mathbb{R})$, where the $\pm$ denotes that matrices may have determinant ± 1.

In the remaining cases we calculate in the Lie algebra. Since we are seeking a compact automorphism group, which will act semisimply, any invariant subspace will have an invariant complement and we suppose our bases chosen to fit these: this explains the simplifications employed below.

Sol_0^4. G_X° is an extension of $\mathbb{R}^3$ by $\mathbb{R} \times SO_2$. Since $\mathbb{R} \times SO_2$ acts faithfully, any automorphism α not only preserves $\mathbb{R}^3$ but also is determined by its effect on $\mathbb{R}^3$. Further, α preserves the decomposition as $\mathbb{R}^2 \times \mathbb{R}$ into irreducible SO_2-modules. Thus $\alpha | \mathbb{R}^3 \in O_2 \times O_1$, and since these automorphisms are realizable, we deduce $K_X \cong O_2 \times O_1$.

The last three cases all have K_X° trivial.

Nil^4. The Lie algebra g has a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ with Lie bracket defined by $[\mathbf{e}_4, \mathbf{e}_1] = \mathbf{e}_2$, $[\mathbf{e}_4, \mathbf{e}_2] = \mathbf{e}_3$, other $[\mathbf{e}_i, \mathbf{e}_j] = 0$. Thus α preserves $g' = \langle \mathbf{e}_2, \mathbf{e}_3 \rangle$, $Z(g) = \langle \mathbf{e}_3 \rangle$; complements to these $\langle \mathbf{e}_1, \mathbf{e}_4 \rangle$ and $\langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4 \rangle$ so reduces to

$$\alpha(\mathbf{e}_1) = a\mathbf{e}_1 + b\mathbf{e}_4 \quad \alpha(\mathbf{e}_2) = r\mathbf{e}_2$$

$$\alpha(\mathbf{e}_4) = c\mathbf{e}_1 + d\mathbf{e}_4 \quad \alpha(\mathbf{e}_3) = s\mathbf{e}_3.$$

This defines an automorphism only if $r = ad - bc \neq 0$, $s = dr \neq 0$. In partitular, $d \neq 0$: we may thus suppose $b = c = 0$ and then take $a, d = \pm 1$.

$Sol_{m,n}^4$. Here g has basis $\{\mathbf{e}_i\}$ with

$$[\mathbf{e}_4, \mathbf{e}_1] = a\mathbf{e}_1, \quad [\mathbf{e}_4, \mathbf{e}_2] = b\mathbf{e}_2, \quad [\mathbf{e}_4, \mathbf{e}_3] = c\mathbf{e}_3.$$

Again α preserves $\langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \rangle$, and indeed each of the eigenspaces $\langle \mathbf{e}_i \rangle$ of $ad\,\mathbf{e}_4$. We may suppose $\alpha(\mathbf{e}_4) = \mathbf{e}_4$, and then $\alpha(\mathbf{e}_i) = \pm \mathbf{e}_i$ for the rest. The case $Sol^3 \times E^1$ is different here since the eigenvalues of $-\mathbf{e}_4$ are (in this case only) a permutation of those of $\mathbf{e}_4$, so we have an extra automorphism.

Sol_1^4. The Lie algebra g is defined by

$$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_2, \quad [\mathbf{e}_1, \mathbf{e}_3] = -\mathbf{e}_3, \quad [\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_4.$$

Thus $g' = \langle \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4 \rangle$, $g'' = Z(g) = \langle \mathbf{e}_4 \rangle$: our automorphism α fixes these and their complements, hence $\langle \mathbf{e}_1 \rangle$ and $\langle \mathbf{e}_2, \mathbf{e}_3 \rangle$, and indeed preserves or interchanges the eigenspaces $\langle \mathbf{e}_2 \rangle$, $\langle \mathbf{e}_3 \rangle$ of $ad\, \mathbf{e}_1$ on the latter. If α preserves all coordinate axes, we find $\alpha(\varepsilon, \eta)$ with

$$\alpha(\mathbf{e}_1) = \mathbf{e}_1, \quad \alpha(\mathbf{e}_2) = \varepsilon \mathbf{e}_2, \quad \alpha(\mathbf{e}_3) = \eta \mathbf{e}_3, \quad \alpha(\mathbf{e}_4) = \varepsilon\eta \mathbf{e}_4 \qquad (\varepsilon, \eta = \pm 1);$$

we also have the interchange

$$\tau(\mathbf{e}_1) = -\mathbf{e}_1, \quad \tau(\mathbf{e}_2) = \mathbf{e}_3, \quad \tau(\mathbf{e}_3) = \mathbf{e}_2, \quad \tau(\mathbf{e}_4) = -\mathbf{e}_4.$$

Thus $K_X^\circ \cong D_8$.

We now summarize these results.

THEOREM 3.2. (a) *The stabilizers K_X of the maximal geometries are given (up to isomorphism) by Table* 3:

Table 3

K_X	X	K_X	X
O_4	S^4, E^4, H^4	$O_2 \times O_1$	$Nil^3 \times E^1$, $\widetilde{SL}_2 \times E^1$, Sol_0^4
$\tilde{U}_2$	$P^2(\mathbb{C})$, $H^2(\mathbb{C})$	O_2	F^4
$\mathbb{Z}_2 \wr O_2$	$S^2 \times S^2$, $H^2 \times H^2$	$D_8 \times \mathbb{Z}_2$	$Sol^3 \times E^1$
$O_2 \times O_2$	$S^2 \times E^2$, $S^2 \times H^2$, $E^2 \times H^2$	D_8	Sol_1^4
$O_3 \times O_1$	$S^3 \times E^1$, $H^3 \times E^1$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$Sol_{m,n}^4$
		$\mathbb{Z}_2 \times \mathbb{Z}_2$	Nil^4

(b) *There exists an orientation-reversing element of K_X in all cases except $P^2(\mathbb{C})$, $H^2(\mathbb{C})$ and F^4.*

Conclusion (b) is obtained by checking the above descriptions in each case: for example, with Sol_1^4 the automorphisms α preserve orientation: τ changes it.

For a second summary we consider the effect on our complex structures. First, we complete the above by determining the maximal K_X for the relevant geometries (E^4, $E^4 \rtimes U_2$) and ($S^3 \times E^1$, $U_2 \times \mathbb{R}$). In the first case, $K_X = \tilde{U}_2$ [argument as for $P^2(\mathbb{C})$]. For the second, it suffices to calculate the normalizer of $U_2 \times \mathbb{R}$ in $O_4 \times \text{Isom}\, \mathbb{R}$: this equals $\tilde{U}_2 \times \text{Isom}\, \mathbb{R}$, with just four components.

THEOREM 3.3. (a) *For the* 15 *complex geometries (X, G_X°) of Theorem* 1.1, *the group $G^{\mathbb{C}}$ of holomorphic automorphisms has two components if $X = P^1(\mathbb{C}) \times P^1(\mathbb{C})$, $H \times H$, or Sol_1^4, and only one connected component in the remaining cases.* (b) *In all cases there exists an anti-holomorphic automorphism.*

Proof. For $P^2(\mathbb{C})$, $\mathbb{C}^2$ *and* $H^2(\mathbb{C})$ the unique non-identity component (of $\tilde{U}_2$, for example) is anti-holomorphic. For the five other products of two two-dimensional geometries, the interchange of factors (when possible) is holomorphic; conjugation in both factors together is anti-holomorphic; the other components reverse orientation.

For the three Neumann geometries $S^3 \times E^1$, $Nil^3 \times E^1$ and $\widetilde{SL}_2 \times E^1$, K_X has four components. Of these, two reverse orientation: the other non-trivial one is anti-holomorphic. This is immediate for $S^3 \times E^1$; for the others it follows easily from the Lie algebra calculations of [49]. Indeed, for Nil^3 we had $E_1 = \mathbf{e}_1 + i\mathbf{e}_2$, $E_2 = \mathbf{e}_3 + i\mathbf{e}_4$; and we have the automorphism changing the signs of $\mathbf{e}_1$ and $\mathbf{e}_3$; similarly for $\widetilde{SL}_2 \times E^1$ we have an automorphism replacing each of E_1, E_2 as given by their complex conjugates.

In the cases of F^4, the unique non-trivial component is anti-holomorphic. Perhaps the easiest way to see this is to note directly that F^4 admits the conjugation $(w, z) \to (\bar{w}, -\bar{z})$, and that this induces the automorphism of G_X^0 under which $(u, v) \to (u, -v)$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$.

There remain the cases Sol_0^4, Sol_1^4 and $Sol_1'^4$. For Sol_0^4, the group of components is generated by the elements α and β which reverse the signs of $\mathbf{e}_2$ and $\mathbf{e}_3$, respectively. Thus the non-trivial orientation-preserving case is $\alpha\beta$, which indeed takes both $E_1 = \mathbf{e}_4 + i\mathbf{e}_3$, $E_2 = \mathbf{e}_1 + i\mathbf{e}_2$ to their complex conjugates. For Sol_1^4, τ changes orientation; $\alpha(1, -1)$ is holomorphic and $\alpha(-1, 1)$ anti-holomorphic: recall that here $E_1 = \mathbf{e}_1 + i\mathbf{e}_2$ and $E_2 = \mathbf{e}_3 + i\mathbf{e}_4$. In the case of $Sol_1'^4$, where E_1 is redefined as $\mathbf{e}_1 + i(\mathbf{e}_2 + \mathbf{e}_4)$, $\alpha(-1, 1)$ is still anti-holomorphic, but $\alpha(1, -1)$ is no longer holomorphic.

This concludes the proof of Theorem 3.3. We remark that in all except the final case, every orientation-preserving element of G_X is either holomorphic or anti-holomorphic.

§4. CLASSIFICATION OF COMPACT COMPLEX ANALYTIC SURFACES

As it is our objective to relate geometric structures to the Enriques–Kodaira classification ([12], [29]), we now recall the main features of the latter. The most convenient presentation of this uses several more recent results also: I endeavour to give precise references, since these are not always easy to find, though the recently published [2] is very useful: see also [48].

There are two independent principles of classification, which we discuss in turn, corresponding to two invariants: the Kodaira dimension κ and the first Betti number b_1 modulo 2.

Let V be any compact complex manifold, K its canonical bundle (of holomorphic n-forms, where $n = \dim V$). The *plurigenus* $P_m(V)$ is the dimension of the space of holomorphic sections of the mth tensor power mK of K. The *Kodaira dimension* $\kappa(V)$ is defined [22] by:

if $P_m = 0$ for all $m \geq 1$, then $\kappa(V) = -1$ (many authors write $-\infty$)
otherwise, $\kappa(V) = \limsup_{m \to \infty} (\log P_m/\log m)$.

The following equivalent forms of this definition can be extracted from [22] (though they are not explicitly stated there):

$\kappa(V) = \sup_m \{\dim \phi_m(V) : \phi_m$ the map to projective space induced by $mK\}$.

$1 + \kappa(V) =$ transcendence degree over $\mathbb{C}$ of the "canonical" ring $\bigoplus_{m \geq 0} H^0(V; mK)$.

Thus the only possible values of $\kappa(V)$ are the integers r with $-1 \leq r \leq n$. It also follows from [22, Theorems 1 and 2] that:

given V with $\kappa(V) \geq 0$, there exists a positive integer m_0 such that $P_{km_0}(V)/(km_0)^{\kappa(V)}$ tends to a positive non-zero limit as $k \to \infty$ through positive integers.

Iitaka also shows (*loc. cit.*) that $\kappa(V)$ is a birational invariant, and proves

PROPOSITION 4.1. [22, Theorem 6]. *If $\tilde{V}$ is a finite unramified covering of V, then*

$$\kappa(\tilde{V}) = \kappa(V).$$

For the special case of surfaces there is a further invariance property.

PROPOSTION 4.2. [21, II]. *For S a compact complex surface, the plurigenera $P_m(S)$, and hence $\kappa(S)$ are invariant under deformations of S.*

The Betti numbers $b_i(S)$ of a compact complex surface S are defined in the usual topological sense (with, say, real coefficients). The importance of $b_1(S)$ mod 2 was recognized by Kodaira. Clearly, it is invariant under deformations! More significantly, we have

THEOREM 4.3. *The following conditions on S are equivalent:*

(i) *$b_1(S)$ is even.*
(ii) *S deforms to a projective algebraic surface.*
(iii) *S has a Kähler metric.*

Proof. For the equivalence of (i) and (ii), see Kodaira [29, Theorem 25]: their equivalence with (iii) was also conjectured by him. That (iii) implies (i) follows from elementary Hodge theory.

For the converse, we refer to the classification in [29, Theorem 22]. (We could use instead Theorems 4 and 11). For class I, $P_g = 0$ so by [29, Theorem 10] these surfaces are all projective algebraic, hence Kähler. Class II consists of $K3$ surfaces, which have Kähler metrics by Siu [44] (see also [4]). Class III consists of complex tori, which are well known to possess such metrics. Class IV consists of elliptic surfaces: the existence of a Kähler metric here (assuming b_1 even) was established by Miyaoka [32]. Finally Class V consists of surfaces with $\kappa = 2$, which are projective algebraic (by [8]: or see [29, Theorem 9]).

The invariant $b_1(S)$ mod 2, also, is stable under finite coverings.

PROPOSITION 4.4. *If $\tilde{S}$ is a finite unramified covering of S, then $\tilde{S}$ has a Kähler metric if and only if S does.*

For the pullback to $\tilde{S}$ of a Kähler metric on S gives a Kähler metric on $\tilde{S}$. For the converse, choose a covering $\bar{S}$ of $\tilde{S}$ which is a regular covering of S. Then a Kähler metric on $\tilde{S}$ pulls back to one on $\bar{S}$. Take the average of the transforms of this by the group of covering transformations of $\bar{S}$ over S. All these are Kähler; so is their average, since (i) positive definiteness of the real part defines a convex cone; and (ii) closedness of the imaginary part is a linear condition. We thus have an invariant Kähler metric on $\bar{S}$: this projects down to give a Kähler metric on S.

Since (as above) $\kappa(S) = 2$ implies $b_1(S)$ even, we have defined seven classes of surfaces. These do *not* coincide with the seven classes defined by Kodaira (*loc. cit.*) which do not, for example, enjoy the property of stability by finite unramified coverings. Various further properties are known about these classes, which will be mentioned in the appropriate sections below.

Our interest in these classes was motivated by their connection with geometric structures in the sense defined above. Our main conclusions can be (in part) summarized as follows:

THEOREM 4.5. *Let S be a compact manifold with a geometric structure of type X. Then S has a compatible complex structure with invariants related to X as in Table* 4:

Table 4

	b_1 *even*	b_1 *odd*
$\kappa = -1$	$S^2 \times S^2$, $S^2 \times E^2$, $S^2 \times H^2$, $P^2(\mathbb{C})$	$S^3 \times E^1$, Sol_0^4, Sol_1^4
$\kappa = 0$	E^4	$Nil^3 \times E^1$
$\kappa = 1$	$E^2 \times H^2$	$\widetilde{SL}_2 \times E^1$
$\kappa = 2$	$H^2 \times H^2$, $H^2(\mathbb{C})$	—

COROLLARY 4.6. *If X is one of $S^3 \times E^1$, $Nil^3 \times E^1$, $\widetilde{SL}_2 \times E^1$, Sol_0^4 and Sol_1^4, then X does not possess a Kähler structure compatible with the geometry.*

For such a structure would be inherited by surfaces modelled on S, contradicting the theorem. This proof is perhaps simpler, and certainly more enlightening, than the direct computation in [49].

In addition to proving this theorem we will seek to characterize, within each class of compact complex surfaces, those that possess a geometric structure: the results are precise in most cases.

The various cases will be treated in sections below as indicated in Table 5:

Table 5

	κ			
	-1	0	1	2
b_1 even	§5	§8	§7	§6
b_1 odd	§9	§7	§7	—

The uniqueness of geometric structures will be discussed in §10.

§5. RULED SURFACES

The conditions $\kappa(S) = -1, b_1(S)$ even are known to characterize the class of ruled surfaces: surfaces admitting a family of embedded copies of $P_1(\mathbb{C})$ whose union is dense.

Given a copy of $P_1(\mathbb{C})$ in a surface S with geometric structure, having X as universal cover, since $P_1(\mathbb{C})$ is simply-connected, the embedding of it in S lifts to one in X. Now since all global holomorphic functions on $P_1(\mathbb{C})$ are constants, there can be no embedding of X in $\mathbb{C}^2$. By Corollary 1.3 the only possibilities for X are now $P^2(\mathbb{C})$, $P^1(\mathbb{C}) \times P^1(\mathbb{C})$, $P^1(\mathbb{C}) \times \mathbb{C}$ and $P^1(\mathbb{C}) \times \mathbb{H}$. Even in these cases, X contains no exceptional curve [copy of $P^1(\mathbb{C})$ with self-intersection -1], hence nor does S: S is always minimal.

PROPOSITION 5.1. *Any complex surface (even non-compact) with geometric structure is minimal.*

PROPOSITION 5.2. *If S has geometric structure of type $P^2(\mathbb{C})$, $P^1(\mathbb{C}) \times P^1(\mathbb{C})$, $P^1(\mathbb{C}) \times \mathbb{C}$ or $P^1(\mathbb{C}) \times H$, then $b_1(S)$ is even and $\kappa(S) = -1$.*

For in these cases X is covered by embedded copies of $P^1(\mathbb{C})$, which all project down to rational curves covering S: thus S is ruled.

We now investigate the converse question of which ruled surfaces have geometric structure: we may suppose S minimal. Such surfaces have been classified (see, for example, [3, p. 30]): either $S \cong P_2(\mathbb{C})$ or there is a map $S \to B$ onto some smooth curve B with each fibre isomorphic to $P_1(\mathbb{C})$.

In this case, S is a $P_1(\mathbb{C})$-bundle over B: since $\dim B = 1$, this projective bundle is associated [3, p. 29] to a vector bundle $\mathbb{C}^2 \to E \to B$. We now need to refer to the classification of plane bundles over curves. However, if L is a line bundle, $L \otimes E$ and E have isomorphic projective bundles.

If B has genus 0, $B \cong P_1(\mathbb{C})$, any plane bundle is the sum of two line bundles, and line bundles are determined up to isomorphism by the Chern class, or degree, which we may regard as an integer. Thus $E = mH \oplus nH$ [where $c_1(H) = 1$], and $P(E)$ is determined up to isomorphism by $|m-n|$. These are called Hirzeburch surfaces after [17]: only the case $m-n=0$, giving $P^1(\mathbb{C}) \times P^1(\mathbb{C})$, has geometric structure. This is also [2] the only case when the ruling is not unique.

For the case when B has genus 1, plane bundles were classified by Atiyah [1]. Either we have the direct sum of two line bundles or we have an irreducible bundle: but all of these only define two distinct projective bundles A_0 and A_1, according to the parity of c_1. As to the reducible cases, we may suppose that the line bundles have respective degrees 0 and $d \geq 0$. According to [45], all those with fixed $d > 0$ (and a given elliptic base curve B) give isomorphic complex surfaces S_d, while those with $d = 0$ give a family $\mathscr{S}_0$ of surfaces parametrized by B itself (modulo automorphisms).

If S has a geometric structure, our bundle must be a flat bundle, induced by a representation of $\pi_1(B)$ in PU_2. As $\pi_1(B)$ is free abelian of rank 2, either this representation lifts to U_2, where it is reducible, or each generator interchanges the fixed points of the other on $P^1\mathbb{C}$, so the subgroup of PU_2 is isomorphic to the four group. In fact it is well known [36], [45] that each bundle with $d = 0$ admits a unique expression as a bundle induced from $U_1 \times U_1$: we thus obtain all the surfaces in $\mathscr{S}_0$. The flat bundle induced by sending the generators to (the projective classes of) $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ can be identified (see, for example, [45]) with A_1. The surfaces $S_d (d \geq 1)$, A_0 do not admit geometric structures.

If B has genus $g \geq 2$, plane bundles over B do not admit a simple classification, though a great deal of work has been done on their moduli space. This has tended to focus on Mumford's [34] concept of stable bundle. The key result for us is the following, due to Narasimhan and Seshadri [36].

THEOREM 5.3. *A vector bundle of rank n and degree q, with $0 \leq q < n$, is stable if and only if it is associated to an "irreducible unitary representation of $\pi_1(B)$ of type τ".*

It remains to explain the final term. In fact, we have a projective representation: if $\langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \rangle$ denotes a standard system of generators of $\pi_1(B)$, we require

$$\rho(\alpha_1)\rho(\beta_1)\rho(\alpha_1^{-1})\rho(\beta_1^{-1}) \ldots \rho(\beta_g^{-1}) = \exp(-2\pi i q/n) I.$$

Since any vector bundle of rank n can be tensored with a line bundle to have degree q as above, we deduce that if our surface S corresponds to a stable plane bundle, it is geometric.

Again, if we have a sum of two line bundles, each of degree 0, then we have a representation in $U_1 \times U_1$, and hence a geometric surface. But a sum of two line bundles of different degrees is not geometric in this sense (nor are the unstable bundles, which are irreducible, but extensions of a line bundle of degree d_1 by one of degree $d_2 > d_1$, so deform to such sums).

§6. CHARACTERISTIC NUMBERS

For any closed oriented Riemannian 4-manifold there are (see, for example, [27, Vol. 2, Chap. 12]) differential 4-forms ω_1, ω_2, defined in terms of the local geometry, whose integrals give the characteristic numbers of M (signature and Euler characteristic):

$$\sigma(M) = \int_M \omega_1 \quad \chi(M) = \int_M \omega_2.$$

Now the bundle of 4-forms has one-dimensional fibre, so at any point, ω_1 and ω_2 are linearly dependent. Thus if M is homogeneous, or at least locally homogeneous, a relation

$a_1\omega_1 + a_2\omega_2 = 0$ which holds at some point holds everywhere, and thus implies

$$a_1\sigma(M) + a_2\chi(M) = 0.$$

In particular, if M has a geometric structure modelled on X (say), the above applies with the addendum that the linear relations (a_1, a_2) depend only on X, not on the particular M.

To determine these relations, first observe that whenever G_X is not semisimple we found in §2 a normal subgroup V of G_X isomorphic to a real vector space such that for any lattice Γ in G_X, $\Gamma \cap V$ is a lattice L in V. This induces a foliation of the manifold $M = \Gamma\backslash G_X/K_X$ by copies of the torus $L\backslash V$. It follows from the fact that the tangent bundle of M splits as a direct sum that $\chi(M) = 0$: if the leaves have odd dimension we also obtain $\sigma(M) = 0$. The only cases remaining are $X = H^2 \times E^2$, when it is clear that σ vanishes for some (hence for all) corresponding M, and $X = F^4$, which does not correspond to any compact M.

THEOREM 6.1. *Let M^4 be a closed oriented geometric 4-manifold, modelled on X. Then the characteristic numbers of M satisfy:*

If $X = P^2(\mathbb{C}), H^2(\mathbb{C})$:	$\sigma(M) > 0,\ \chi(M) = 3\sigma(M)$.
If $X = S^2 \times S^2, H^2 \times H^2, S^4$:	$\sigma(M) = 0,\ \chi(M) > 0$.
If $X = S^2 \times H^2, H^4$:	$\sigma(M) = 0,\ \chi(M) < 0$.
Otherwise:	$\sigma(M) = 0,\ \chi(M) = 0$.

Proof. If $X = P^2(\mathbb{C})$, $S^2 \times S^2$ or S^4, we can take $M = X$ and compute the characteristic numbers: by the above, any linear relation is inherited by all other oriented geometric manifolds with the same model. The same argument also shows that the inequalities are inherited.

If $X = S^2 \times H^2, H^2 \times H^2$ we argue similarly, but replacing H^2 by a closed surface of genus $g > 1$ (which has model H^2).

For $X = H^2(\mathbb{C}), H^4$ there are no very simple models. The result in these cases (which is in any case well known) can be obtained by comparing the differential forms ω_1, ω_2 for X with those for the compact dual of X as symmetric space, viz. $P^2(\mathbb{C})$, S^4, respectively.

We turn to the interpretation in the complex case. Let S be a compact complex surface, homeomorphic to M as above. We have a dictionary of real and complex characteristic numbers:

$$\chi(M) = c_2(S), \quad \sigma(M) = \tfrac{1}{3}[c_1^2(S) - 2c_2(S)]$$

to which we may add for completeness the Pontrjagin number

$$p_1(M) = 3\sigma(M)$$

and the arithmetic genus

$$\chi(\mathcal{O}_S) = \tfrac{1}{12}[c_1^2(S) + c_2(S)].$$

The following relations between these and the classification are known.

For a minimal complex surface S,

If $\kappa(S) = 2$, then $c_1^2(S) > 0$ (see, for example, [2, p. 207]).
If $\kappa(S) = 0$ or 1, then $c_1^2(S) = 0$ (see [2, p. 188] or §§7, 8 below).
If $S = P^2\mathbb{C}$, $c_1^2(S) = 9$.
If S is a P^1-bundle over a surface of genus g, $c_1^2(S) = 8(1-g)$.
If $\kappa(S) = -1$ and $b_1(S)$ is odd, then $b_1(S) = 1$ (see §9)

so $\qquad c_2(S) = \chi(S) = b_2(S)$;

also $\quad \chi(\mathscr{C}_S) = 1 - q + p_g = 1 - 1 + 0 = 0 \quad$ [29, Th. 26 + Th. 11 + Th. 3]
so $\quad c_1^2(S) = -b_2(S) \leq 0.$

To include non-minimal surfaces note that blowing up a point subtracts 1 from c_1^2.

In particular, if $c_1^2(S) > 0$ then either $\kappa(S) = 2$ or S is rational. If S has geometric structure of type $H^2(\mathbb{C})$ or $H \times H$, it is clearly not rational (being aspherical), so has $\kappa(S) = 2$. There is no general structure theory for surfaces with $\kappa(S) = 2$, but a famous theorem of Yau [50] (cf. also [33]) characterizes one of our classes.

THEOREM 6.2. [50]. *For all surfaces with* $\kappa = 2$, $c_1^2(S) \leq 3c_2(S)$. *The equality* $c_1^2(S) = 3c_2(S)$ *holds if and only if S has a geometric structure of type* $H^2(\mathbb{C})$.

In fact until quite recently relatively few examples of surfaces with $c_1^2(S) \geq 2c_2(S)$ [equivalently, $\sigma(S) \geq 0$] were known, but a spate of new ones have now been constructed (see, for example, [19]). At one stage it was conjectured that all such surfaces were aspherical: this has been disproved by an example of Mandelbaum [31]. The conjecture that all have infinite fundamental group remains open (*), as does the problem of characterizing surfaces with geometric structure of type $H \times H$. To the best of my knowledge, no examples are known (*) of surfaces S with $\kappa(S) = 2$, $\sigma(S) = 0$ which are not of this type. Certainly such surfaces need not be (finitely covered by) products of curves: we can take Γ to be the group of units in a quaternion algebra over a real quadratic field, split at both infinite primes. A partial characterization is due to Enoki (private communication).

(*) *Added in proof* Examples of simply-connected surfaces with $c_1^2 > 2c_2$, and one example with $c_1^2 = 2c_2$, have now been discovered. See "Simply-connected algebraic surfaces of positive index," by B Moishezon & M Teicher (preprint, Columbia University).

§7. ELLIPTIC SURFACES

S is called an elliptic surface if there is an analytic map $\psi: S \to B$ whose general fibre is an elliptic curve. We will (as is customary) suppose that S is non-singular and that no fibre contains an exceptional curve of the first kind: such S is called "relatively minimal" (in fact, except when S is rational, this is equivalent to minimality in the usual sense).

A classification of possible fibres of ψ was given by Kodaira [28, 6.2]: the cases are labelled $\mathrm{I}_k (k \geq 0)$, II, III, IV, $\mathrm{I}_k^* (k \geq 0)$, II*, III* and IV*. Case I_0 means that the fibre is a smooth elliptic curve: all other types of fibre we shall call *singular*. By considering a neighbourhood of a fibre, Kodaira also (*loc. cit.*) defines a multiplicity $m \geq 1 : m > 1$ only for finitely many fibres, and these must all be of the type I_k for some k. A fibre is called *multiple* if $m \geq 2$ and *exceptional* if it is either singular or multiple.

A non-exceptional fibre $F_b (b \in B)$ has an invariant $j(F_b) \in \mathbb{C}$. This extends [28, §7] to a regular map $j: B \to P^1(\mathbb{C})$. We have $j = 0$ for fibres of types II, IV, II* and IV*; $j = 1$ for those of types III and III*; $j = \infty$ for those of types I_k and I_k^* with $k \geq 1$: for types I_0 and I_0^*, j can take any finite value. Kodaira calls j the functional invariant of the surface S.

Crucial for the study of elliptic surfaces is the following formula for the canonical divisor [29, Theorem 12]:

$$K_S = \psi^*(K_B + D) + \Sigma (m_i - 1) F_i$$

where the F_i (with multiplicities m_i) are the multiple fibres and

$$\deg D = 1 - q + p_g = \chi(\mathscr{C}_S)$$

is the arithmetic genus of S. A first deduction from the formula is that $K_S^2 = 0$. Then by Noether's formula

$$\chi(\mathscr{C}_S) = \tfrac{1}{12}(K_S^2 + \chi) = \tfrac{1}{12}\chi$$

where χ is the Euler characteristic of S, hence equal to the sum of those of the singular fibres, which are given by Table 6.

Table 6

Type of F	I_k	II	III	IV	I_k^*	II*	III*	IV*
$\chi(F)$	k	2	3	4	$k+6$	10	9	8

In particular, $\chi \geq 0$ and vanishes only if there are no singular fibres.

We often regard B as an orbifold (terminology of Thurston [47]) alias V-manifold (terminology of Satake [41]), with a $2\pi/m_i$ cone point at each point P_i corresponding to a fibre of multiplicity m_i. Then the orbifold Euler characteristic of B is (by definition)

$$\chi^{\text{orb}}(B) = 2 - 2g(B) - \Sigma(1 - m_i^{-1}).$$

The structure of S at a smooth multiple fibre is homeomorphic to the product of a circle with a multiple fibre in a Seifert fibration of a 3-manifold.

If F_i is the fibre over P_i, then as divisor $\psi^* P_i = m_i F_i$. Thus if m is divisible by all the m_i,

$$mK_S = \psi^*\left[m(K_B + D) + \Sigma m\left(1 - \frac{1}{m_i}\right)P_i\right] = \psi^*(D_m)$$

is the pullback of a divisor D_m of degree

$$m[2g(B) - 2 + \chi(\mathcal{O}_S)] + \Sigma m(1 - m_i^{-1}) = m[\chi(\mathcal{O}_S) - \chi^{\text{orb}}(B)] = m\tau_S,$$

where $\tau_S = \chi(\mathcal{O}_S) - \chi^{\text{orb}}(B)$.

LEMMA 7.1. *The Kodaira dimension of S is given by*

$$\kappa = \operatorname{sgn}(\tau_S)$$

i.e. $\kappa = -1$, 0 *or* 1 *according as* $\tau_S < 0$, $\tau_S = 0$ *or* $\tau_S > 0$.

Remark. This result appears to be well known, but I am unable to find it stated explicitly in the literature (cf. [2, p. 162]).

Proof. If m is divisible by the m_i, then

$$P_m(S) = H^0[S; \psi^*(D_m)] = H^0(B; D_m),$$

and $\deg D_m = m\tau_S$. If $\tau_S < 0$, we have negative degrees and these all vanish, hence so do all P_m. Thus $\kappa = -1$. If $\tau_S > 0$, then for large enough m, by the Riemann–Roch theorem,

$$P_m(S) = \deg D_m + 1 - g(B)$$

increases as a linear function of m, hence $\kappa = 1$. (Note that by (4.0) to compute κ it is enough to know all the P_{km_0} for any particular m_0.)

If $\tau_S = 0$, then $\deg D_m = 0$, so $P_m = 0$ or 1. It remains to show that the value 1 can occur. Now if $g(B) \geq 2$, we have $\tau_S > 0$. If $g(B) = 1$, then $\tau_S = 0$ only if there are no exceptional fibres. We return to this case below. Otherwise $g(B) = 0$: but then $\deg D_m = 0$ is enough to ensure $h^0(B; D_m) = 1$.

For the final case, if $b_1(S)$ is odd, as $b_1(S) \geq b_1(B) = 2$, S is in Kodaira's class VI [29, Th. 26], so $P_1(S) = p_g(S) \geq 1$. If $b_1(S)$ is even, then if $\kappa(S) = -1$, S is ruled. But any rational curve on S admits only constant maps to B so lies in a fibre which, too, is elliptic: a contradiction.

The importance of elliptic surfaces for the classification is shown by the first part of

LEMMA 7.2. (a) *If* $\kappa(S) = 1$, *or if* $\kappa(S) = 0$ *and* $b_1(S)$ *is odd, then* S *is elliptic.* (b) *In these cases the elliptic structure is unique.* (c) *If* S *is elliptic and* $b_1(S)$ *is odd, then* S *has no singular fibres.*

Proof. (a) follows easily from the results of Kodaira. By [29, Theorem 57], if $\kappa(S) \geq 0$ then $P_{12}(S) > 0$. Thus S is in one of the classes 2–6 of [29, Theorem 55]. But classes 2 and 3 ($K3$ surfaces and tori) have $\kappa(S) = 0$ and $b_1(S)$ even; class 5 is the set of surfaces with $\kappa(S) = 2$. The remaining classes 4 and 6 consist of elliptic surfaces.

Part (a) can also be seen directly as follows. If $\kappa = 1$, the pluricanonical map ϕ_m has one-dimensional fibres E. These satisfy $E^2 = 0$ and $K \cdot E = 0$ (since they are contracted by ϕ_m), hence have genus 1. Similarly, if the function field of S has transcendence degree 1, a non-trivial function defines a map from S to a curve, and the same argument shows that the fibres have genus 1 ([29, Theorem 4]).

For (c) we consider the fundamental group of S. If A is a finite subset of B containing the images of all exceptional fibres, and F a general fibre, we have an exact sequence

$$\{1\} \to \pi_1(F) \xrightarrow{i'_*} \pi_1[S - \psi^{-1}(A)] \xrightarrow{\psi'_*} \pi_1(B - A) \to \{1\}.$$

An examination of the neighbourhoods of exceptional fibres shows that this leads to an exact sequence

$$\pi_1(F) \xrightarrow{i_*} \pi_1(S) \xrightarrow{\psi_*} \pi_1^{\mathrm{orb}}(B) \to \{1\},$$

where $\pi_1^{\mathrm{orb}}(B)$ denotes the fundamental group of B as orbifold. If there are no singular fibres, i_* is injective; otherwise it is clearly not. More precisely, it can be seen [16], [21, II] that the image of i_* is finite cyclic (in fact [51], it is trivial). Thus ψ induces an isomorphism $H_1(S; \mathbb{R}) \cong H_1(B; \mathbb{R})$, so in particular $b_1(S) = b_1(B)$ is even.

As to (b), if we have any elliptic surface S then (as we have seen above) any pluricanonical map must collapse each fibre to a point. Thus if $\kappa(S) = 1$, fibres of ψ can only (cf. [21, II]) be the connected components of the fibres of any pluricanonical map with one-dimensional image. If $b_1(S)$ is odd, then the field of meromorphic functions on S has transcendence degree 1 (not 2, since S is not algebraic; not 0, since S is elliptic) so if C is a model for this field, and $B \xrightarrow{\pi} C$ the corresponding map, the map ψ must factor through π: in fact, the two must be equivalent.

Observe that both (a) and (b) may fail for algebraic surfaces with $\kappa = 0$: for example, $K3$ surfaces need not be elliptic (though they deform to ones that are) and Enriques surfaces have in general (hence $K3$ surfaces often) two distinct elliptic structures [2, 17.7]; similarly for complex tori.

It is time to return to our geometric structures.

PROPOSITION 7.3. (a) *Any complex surface with compatible geometric strucutre of type* $\mathbb{C} \times H$, $Nil \times \mathbb{R}$ *or* $\widetilde{SL}_2 \times \mathbb{R}$ *is elliptic.* (b) *Any elliptic surface with geometric structure compatible with its complex structure has no singular fibres.*

Proof. (a) S is the quotient by a discrete group Γ (acting on the left) of one of:

$$\mathbb{R}^2 \times SL_2(\mathbb{R})/SO_2, \quad \mathbb{R} \times (Nil^3 \rtimes SO_2)/SO_2, \quad \mathbb{R} \times (\mathbb{R} \times {}_{Z}\widetilde{SL}_2)/SO_2.$$

In each case we know [by (2.3) or (2.6): see the discussion in §2] that Γ meets the centre of G_X

(isomorphic to $\mathbb{R}^2$ in each case) in a lattice L in $\mathbb{R}^2$. Moreover, this subgroup has Lie algebra invariant under J in each case. Thus the elliptic curve $L\backslash\mathbb{C}$ acts on $\Gamma\backslash G_X/SO_2$ defining a fibration, and almost all the fibres are isomorphic to $L\backslash\mathbb{C}$ itself.

(b) If S has a singular fibre, then this is a union of rational curves, each with negative self-intersection number. But we have already seen that no surface with geometric structure carries any such curves.

We insert parenthetically here a construction giving many such examples with b_1 odd, which the author found helpful in giving insight. Let $(X, 0)$ be a normal surface singularity with good $\mathbb{C}^\times$-action (cf. [39]): choose $\lambda \in \mathbb{C}^\times$ with $|\lambda| > 1$, and factor $X - \{0\}$ by the subgroup generated by λ. The result is a compact surface with b_1 odd: the orbits of the $\mathbb{C}^\times$-action project to elliptic curves which fibre it. The base of the fibration is the orbifold which is the quotient of $X - \{0\}$ by $\mathbb{C}^\times$. This generalizes the original construction of Hopf [20]: the present version is largely due to Neumann [38].

We now consider the structure of our elliptic surfaces in greater depth. From Proposition 7.3 (and Lemma 7.2), we see that we need only study those without singular fibres. It follows that j cannot take the value ∞, so is constant on B. We must now consider multiple fibres.

An orbifold covering map $B' \to B$ induces an unramified covering $S' \to S$, with S' elliptic over B', with multiple fibres corresponding to the cone points of B'. Now any orbifold B is either [42], [47]

good: i.e. admits a finite orbifold cover B' with no cone points; its universal cover is then isomorphic to $P^1(\mathbb{C})$, $\mathbb{C}$ or H; or

bad: when B has genus 0 and either one cone point or two, with different multiplicities.

For an elliptic surface S with no singular fibres corresponding to a bad orbifold B, $\chi^{\mathrm{orb}}(B) > 0$, so $\tau_s < 0$, $\kappa = -1$.

Our objective is to prove the following, the case b_1 odd of which is closely related to a result due to Inoue: see [30].

THEOREM 7.4. *An elliptic surface S without singular fibres has a geometric structure if and only if its base is a good orbifold. The type of this structure is determined as follows:*

κ	-1	0	1
b_1 *even*	$\mathbb{C} \times P^1(\mathbb{C})$	$\mathbb{C}^2$	$\mathbb{C} \times H$
b_1 *odd*	$S^3 \times E^1$	$Nil^3 \times E^1$	$\widetilde{SL}_2 \times E^1$

Our plan of proof is first to prove the existence of a geometric structure of the indicated type, and then show that no structure of any other type is possible.

By hypothesis, B is a good orbifold; let B' be a finite cover of B free of cone points. The normalization S' of the pullback is a bundle over B; its (topological) monodromy group acts by analytic automorphisms on the fibre E, hence is finite [in fact, if it is non-trivial, the image of $\pi_1 E \to \pi_1 S$ is finite so $b_1(S)$ is even]. Taking a further cover B'', we may suppose the pulled back surface S'' has trivial monodromy. We first consider this case.

As the holomorphic monodromy can now only consist of translations of the fibres, the associated Jacobian fibration [28, §8] is a product:

$$JS = E \times B.$$

According to Kodaira [28, §10], such elliptic surfaces S are classified by $H^1(B; \Omega(JS))$. As JS

is trivial, we have an exact coefficient sequence

$$O \to \mathscr{L} \to \mathscr{C}_B \to \Omega(JS) \to O,$$

where $\mathscr{L}$ is a trivial bundle with fibre the lattice L corresponding to E:

$$O \to L \to \mathbb{C} \to E \to O.$$

This gives an exact cohomology sequence

$$H^1(B;\, \mathscr{L}) \to H^1(B;\, \mathscr{C}_B) \to H^1(B;\, \Omega(JS)) \xrightarrow{\delta} H^2(B;\, \mathscr{L}) \to H^2(B;\, \mathscr{C}_B).$$

In this sequence, $H^2(B; \mathscr{C}_B) = 0$ and $H^1(B; \mathscr{C}_B) = H^{1,0}(B)$ is a vector space over $\mathbb{C}$ of dimension $g(B)$. Also $H^2(B; \mathscr{L}) \cong L$, and the image $c(\eta)$ in L of the class η corresponding to S gives the topological part of the classification.

As we have a fibre bundle with trivial monodromy, $\pi_1(S)$ has a presentation of the form

$$\langle a_1, b_1, \ldots, a_g, b_g, c, d \mid \quad c, d \text{ central}; \quad a_1^{-1} b_1^{-1} a_1 b_1 \ldots a_g^{-1} b_g^{-1} a_g b_g = c^r d^s \rangle.$$

Here $L \cong \pi_1(E)$ is the free abelian group on c and d; we can identify the element $c(\eta)$ above with $c^r d^s$. It is thus clear that

if $\quad c(\eta) = 0, \quad b_1(S) = 2g+2 \quad$ is even

if $\quad c(\eta) \neq 0, \quad b_1(S) = 2g+1 \quad$ is odd.

We may, and in future will, choose the basis of L such that $c(\eta) = c^r$ for some $r \geq 0$. Observe also that with the current hypothesis, $\kappa(S) = \kappa(B)$ is determined in the usual way by $g(B)$.

We now consider the various cases.

$\kappa = 1$, b_1 *even*. We consider the group acting on $\mathbb{C} \times H$ generated by translations of $\mathbb{C}$ by L, and the operators

$$A_i(w, z) = (w + a_i,\, \alpha_i(z))$$

where the $\alpha_i \in PSL_2(\mathbb{R})$ generate a discrete group acting on H with quotient B, and can be normalized to satisfy

$$\alpha_1^{-1} \alpha_2^{-1} \alpha_1 \alpha_2 \ldots \alpha_{2g-1}^{-1} \alpha_{2g}^{-1} \alpha_{2g-1} \alpha_{2g} = 1.$$

The quotient by such a group gives a fibration over B with fibre $\mathbb{C}/L = E$. We appear to have $2g$ parameters a_i here as opposed to the g coming from $H^{1,0}(B)$ above: this is explained as follows.

Each holomorphic 1-form $\omega \in H^{0,1}(B)$ induces a form $\tilde{\omega}$ on the universal cover $\tilde{B} \cong H$. Such an $\tilde{\omega}$ is exact: say $\tilde{\omega} = \mathrm{d}\phi$. We have

$$\phi(\alpha_i(z)) = \phi(z) + p_i$$

where the p_i are the periods of ω. The map

$$\Phi(w, z) = (w + \phi(z),\, z)$$

gives an automorphism of $\mathbb{C} \times H$ transforming the group with translations by the a_i to one with translations by $a_i + p_i$. Thus sets of parameters $\mathbf{a} \in H^1(B; \mathbb{C})$ differing by an element of $H^{0,1}(B)$ give rise to isomorphic surfaces, and the image of $\mathbf{a}$ in $H^{1,0}(B)$ is the effective invariant.

This already shows that all surfaces of the type considered have a geometric structure modelled on $\mathbb{C} \times H$. For the exact classification, consider the above map $H^1(B; \mathscr{L}) \to H^1(B; \mathscr{C}_B)$. This is the composite of the inclusion $H^1(B; \mathscr{L}) \to H^1(B; \mathbb{C})$ (isomorphic to $L^{2g} \subset \mathbb{C}^{2g}$) with the projection of $H^1(B; \mathbb{C})$ on $H^{1,0}(B)$. The image is not in general discrete, and the composite fails to be injective precisely when there exist non-constant maps $B \to E$. All this is reflected in the geometric structure: in particular, factoring out $H^1(B; \mathscr{L})$ corresponds to the remark that only the values modulo L of the a_i are significant.

$\kappa = 1$, b_1 *odd*. We recall from §1 that the group acting on $\mathbb{C} \times H$ here is generated by translations in $\mathbb{C}$ together with the action of $\widetilde{SL}_2(\mathbb{R})$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(w, z) = \left(w - 2\log(cz + d), \frac{az + b}{cz + d}\right).$$

Thus to uniformize our surfaces we take the α_i as above, choose some lifts $\tilde{\alpha}_i$ to $\widetilde{SL}_2(\mathbb{R})$, and set

$$A_i(w, z) = \tilde{\alpha}_i(w, z) + (a_i, 0)$$

for some complex numbers a_i. Since $\widetilde{SL}_2$ commutes with the translations, the commutator product of the A_i coincides with that of the $\tilde{\alpha}_i$, which (of course) lies in the kernel of $\widetilde{SL}_2 \to PSL_2$. It was calculated in [37]: the result is $(2 - 2g)$ times the generator, determining a translation of $\mathbb{C}$ by $(2 - 2g)(2\pi i)$. Now since this corresponds to the Chern class of the bundle, it must coincide with the element $c(\eta)$ defined above. Since we normalized $c(\eta) = c^r$ (or additively: rc), the generator $c \in L$ must equal $r^{-1}(2 - 2g)(2\pi i) \in \mathbb{C}$. This identification is no restriction: it simply determines a preferred embedding of L in $\mathbb{C}$. Now the rest of the discussion is the same as in the case with b_1 even.

$\kappa = 0$, b_1 *even*. In this case, the model space is $\mathbb{C}^2$: for the cases under discussion we only need the translations.

Let $\alpha_1, \alpha_2 \in \mathbb{C}$ generate a lattice corresponding to B: we generate a group acting on $\mathbb{C}^2$ by translating the first coordinate (w) by elements of L and by $A_i(w, z) = (w + a_i, z + \alpha_i)$. The quotient gives a fibration over B with fibre E and monodromy of translations only: as in the case $\kappa = 1$, we see that we obtain all such fibrations.

$\kappa = 0$, b_1 *odd*. Here again the model space is $\mathbb{C}^2$; there is a group bijective to $\mathbb{C}^2$ which acts by

$$T_{a,b}(w, z) = (a + w - i\bar{b}z, b + z).$$

If α_1 and α_2 are as above, we again generate a group by translating w by elements of L and by T_{a_1,α_1} and T_{a_2,α_2}: the quotient gives a fibration over B with fibre E and monodromy consisting of translations. The first Chern class corresponds to the commutator of T_{a_1,α_1} and T_{a_2,α_2}, which is translation of w by $i(\bar{\alpha}_1\alpha_2 - \bar{\alpha}_2\alpha_1)$: twice the co-area of the lattice spanned by α_1 and α_2. As in the case $\kappa = 1$, provided we identify this vector with $rc \in L$, our conclusions follow as before.

$\kappa = -1$, b_1 *even*. When B has genus 0, the map c is an isomorphism. Thus for $c = 0$ we have a unique surface $B \times E = P^1(\mathbb{C}) \times E$, which is the quotient of $P^1(\mathbb{C}) \times \mathbb{C}$ by the usual group L of translations.

$\kappa = -1$, b_1 *odd*. In this case our model space is $\mathbb{C}^2 - \{\mathbf{0}\}$: there is a standard projection to $P^1(\mathbb{C})$, with fibres the punctured lines through $\mathbf{0}$. Choose λ such that $\mathbb{C}^\times/\langle\lambda\rangle$ is isomorphic to E: additively, we can think of $\mathbb{C}$ modulo the lattice generated by $2\pi i$ and $\log\lambda$. For suitable λ we can identify this with L in such a way that $2\pi i$ corresponds to $c \in L$. Then the quotient of $\mathbb{C}^2 - \{\mathbf{0}\}$ by $(x, y) \to (\lambda x, \lambda y)$ gives a fibration over $P^1(\mathbb{C})$ with fibre E and Chern class c. To obtain Chern class rc, we factor out the group generated by

$$(x, y) \to (\lambda^{1/r}x, \lambda^{1/r}y) \quad \text{and} \quad (x, y) \to (e^{2\pi i/r}x, e^{2\pi i/r}y).$$

This concludes the list of cases.

We now return to the general case contemplated at the beginning of the proof.

We have constructed a diagram

$$\begin{array}{ccccc} X & \to & S'' & \to & S \\ \big\downarrow{\scriptstyle \pi_X} & & \big\downarrow{\scriptstyle \pi''} & & \big\downarrow{\scriptstyle \pi} \\ Y & \to & B'' & \to & B \end{array}$$

where π, π'' are elliptic fibrations with fibre E; π_X has fibre $\mathbb{C}$; π'' has no multiple fibres and monodromy consisting only of translations; S'' is the quotient of X by a (cocompact) lattice Γ'' in G_X, and is a finite covering of S, which we may clearly suppose regular. Lifting this, we obtain a group Γ acting on X with quotient S, and containing Γ'' as normal subgroup with finite index. We will show that $\Gamma \subset G_X$, so preserves the geometric structure.

By construction, each covering transformation of S'' over S preserves the elliptic structure. We first consider the base. Here Γ, a finite extension of Γ'', acts properly and holomorphically on Y with compact quotient B. If $Y = H$ ($\kappa = 1$), it follows that this action is by a discrete cocompact subgroup of PSL_2. If $Y = \mathbb{C}$ ($\kappa = 0$), Γ is contained in the group of automorphisms of $\mathbb{C}$, which is the affine group. As it has a subgroup Γ'' of finite index, of translations, the projection to $GL_1(\mathbb{C})$ is finite, hence lies in the circle group, SO_2. If $Y = P^1(\mathbb{C})$ ($\kappa = -1$), Γ'' acts trivially, so Γ is finite, hence conjugate to a subgroup of $PU_2 = SO_3$.

Next we consider the effect on a fibre. Each fibre of S'' is isomorphic to E; the effect of a covering transformation over S coincides (modulo such identifications) with an automorphism of E. Such an automorphism lifts to one of $\mathbb{C}$ of the form

$$w \to \varepsilon w + a$$

where in general $\varepsilon^2 = 1$. If E is harmonic ($j = 0$), $\varepsilon^4 = 1$; if E is equianharmonic ($j = 1$), $\varepsilon^6 = 1$; thus in any case $\varepsilon^{12} = 1$. Since the monodromy (topological) in S'' is trivial, or the analytic one consists only of translations, the element ε corresponding to a given covering transformation cannot vary from fibre to fibre, but is constant.

Suppose b_1 even: then $X = \mathbb{C} \times Y$. For each $g \in \Gamma$ we have seen that g defines a geometric automorphism α_g of Y: from this and the effect on fibres we see that

$$g(w, z) = (\varepsilon w + a(z), \alpha_g(z)).$$

It remains to show that $a(z)$ is constant. But g normalizes the group generated by

$$A_i(w, z) = (w + a_i, \alpha_i(z)),$$

where the α_i generate a discrete group $\bar{\Gamma}''$ acting on Y with compact quotient. Since

$$g^{-1} A_i g(w, z) = (w + \varepsilon^{-1} a_i + \varepsilon^{-1} a(z) - a(\alpha_g^{-1} \alpha_i \alpha_g(z)),\ \alpha_g^{-1} \alpha_i \alpha_g(z)),$$

$\varepsilon^{-1} a(z) - a(\alpha_g^{-1} \alpha_i \alpha_g(z))$ is constant. Now if $\varepsilon = 1$, we deduce that a is constant on the orbits of $\bar{\Gamma}'' = \alpha_g^{-1} \bar{\Gamma}'' \alpha_g$, and hence factors through the quotient $Y/F'' = B''$, which is compact. Thus in this case $a(z)$ is constant. If $\varepsilon \neq 1$, the invariance condition defines a line bundle of finite order (that of ε) over B'', for which $a(z)$ defines a section. But since this bundle has degree zero and is non-trivial, the only holomorphic section is 0, so in this case $a(z) = 0$. This concludes the proof when b_1 is even.

If b_1 is odd but $\kappa = 0$ or 1, we only need slight modifications to the above. Again $X \cong \mathbb{C} \times Y$ and the action on Y is as before. However, in these cases for each α acting on Y there is a function $l_\alpha(z)$ on Y such that the geometric automorphisms are the

$$(w, z) \to (w + a + l_\alpha(z), \quad \alpha(z)).$$

The invariance of the Chern class shows that the automorphisms of the fibres must satisfy $\varepsilon = 1$. We can thus write our chosen $g \in \Gamma$ as

$$(w, z) \to (w + a(z) + l_\alpha(z), \quad \alpha(z))$$

and it normalizes a group of elements of the form

$$B\colon (w, z) \to (w + b + l_\beta(z), \quad \beta(z)).$$

Since $\alpha \to l_\alpha$ satisfies a cocycle condition, we deduce on computing $g^{-1}Bg$ that if this belongs to Γ'', then $a(z) - a(\alpha^{-1}\beta\alpha(z))$ is constant. The proof now concludes as before.

Finally, in the case when b_1 is odd and $\kappa = -1$, Γ induces a finite group acting on $P^1(\mathbb{C})$, which we may take as geometric. I claim that *any* lift α of such an element to an automorphism of $\mathbb{C}^2 - \{\mathbf{0}\}$ is linear: it suffices to verify this for the identity element. On each fibre $\mathbb{C}^\times$ this induces an automorphism which must be multiplication by some non-zero constant a (interchange of the ends is ruled out since α preserves the Chern class of the fibration). Now $a(z)$ depends holomorphically on $z \in P^1(\mathbb{C})$, hence is constant. This concludes the proof of Theorem 7.4.

COROLLARY 7.5. *Suppose S is an elliptic surface without singular fibres. Then the universal cover of S is biholomorphic to*

$$\begin{array}{lll} \mathbb{C} \times H & \textit{if} & \kappa(S) = 1 \\ \mathbb{C}^2 & \textit{if} & \kappa(S) = 0 \\ \mathbb{C} \times P^1\mathbb{C} & \textit{if} & \kappa(S) = -1, b_1(S) \textit{ even} \\ \mathbb{C}^2 - \{\mathbf{0}\} & \textit{af} & \kappa(S) = -1, b_1(S) \textit{ odd.} \end{array}$$

If the base B is a good orbifold, this is immediate; otherwise, if $\kappa = -1$ and b_1 is odd, we can appeal to [29, Theorem 28]—where a significant part of this corollary is proved; if $\kappa = -1$ and b_1 is even, by the classification, S is ruled over a curve with genus 1.

§8. TRIVIAL PLURICANONICAL BUNDLES

The following is an easy consequence of the main Kodaira classification.

THEOREM 8.1. *The following conditions on the compact complex surface S are equivalent*:

(i) $\kappa(S) = 0$.
(ii) *For some n, nK is a trivial bundle.*
(iii) *For some finite unramified covering surface $\tilde{S}$ of S, $K(\tilde{S})$ is trivial.*
(iv) *For some finite covering $\tilde{S}$ of S, $\tilde{S}$ is either a complex torus, a $K3$ surface or a Kodaira surface.*

Proof. (i) $\Rightarrow$ (iv) by the classification. (iv) $\Rightarrow$ (iii) by [29, Theorem 19]: indeed, Kodaira shows there that these are the only three types of surface with trivial canonical bundle. (iii) $\Rightarrow$ (ii) for we may suppose $\tilde{S}$ a regular covering of S, and choose a non-zero section σ. Each covering transformation g then has $g\sigma = a\sigma$ where a is a holomorphic on $\tilde{S}$, nowhere zero, hence constant. This defines a homomorphism from the covering group G (which is finite) to $\mathbb{C}^\times$: the image is cyclic, of order n, say. Thus σ projects to an n-valued section of K: this determines a nowhere zero section of nK. (ii) $\Rightarrow$ (i) for we have $P_{kn} = 1$ for all $k \in \mathbb{Z}$: as $P_k \leq P_{kn}$, every P_k is 0 or 1.

A direct proof of (i) $\Rightarrow$ (iii) is non-trivial: indeed, it is a significant step in the one version of the proof of the classification theorem.

The three classes of surfaces in (iv) are quite distinct, since $c_2 > 0$ for $K3$ surfaces and $c_2 = 0$ for the others; b_1 is odd for Kodaira surfaces and even for the others: these differences are inherited by S from $\tilde{S}$. By Lemma 7.2, surfaces with $\kappa = 0$ and b_1 odd are elliptic, and so by Theorem 7.4 geometric: they were fully discussed in §7.

The $K3$ surfaces are simply connected, and are certainly not geometric in the sense of this paper. If S is covered by a $K3$ surface, S is an Enriques surface.

There remains the case when $\tilde{S}$ is a complex torus, i.e. the quotient of $\mathbb{C}^2$ by a lattice. Such $\tilde{S}$ are clearly geometric. Surfaces S (not themselves tori) finitely covered by a torus are called hyperelliptic. They were classified long ago (see, for instance, [2, p. 148]) and are all elliptic—hence (or, better, by direct construction) geometric.

This completes the discussion of geometric structures.

Remark. Triviality of nK (for some n) in Pic (S) should not be confused with triviality of the characteristic class c_1 in $H^2(S; \mathbb{R})$ (which it clearly implies).

If b_1 is even, the conditions are equivalent. For as $c_1 = 0, c_1^2 = 0$: thus $\kappa \leq 1$. If $\kappa = 0$, some nK are trivial; otherwise if S is elliptic, $\tau_S \neq 0$ and c_1 equals τ_S times the class of a fibre, which is non-zero in cohomology. If $\kappa = -1$, S is ruled, and c_1 is non-zero on each curve of the ruling.

If b_1 is odd, $c_1 = 0$ for any elliptic surface, in particular for all minimal surfaces with $\kappa = 0$ or $\kappa = 1$; while for $\kappa = -1$,

$$c_1 = 0 \Rightarrow c_1^2 = 0 \Rightarrow \chi(S) = 0 \Rightarrow b_2(S) = 0 \Rightarrow c_1 = 0.$$

The result on elliptic surfaces holds since here the spectral sequence of the fibration is non-trivial [as $b_1(S) \neq b_1(B)$]; the differential kills the cohomology class of the fibre, and c_1 is a multiple of this.

As affine structures play an important role in the literature on uniformization, we insert here some further references. The closest to our approach is the paper [25], where the following are proved (in our terminology).

PROPOSITION 8.2. *If M is a compact complex surface then:*

(i) *if M has a holomorphic affine connection then $c_1(M) = c_2(M) = 0$ and M is minimal;*
(ii) *if $b_1(M)$ is even, M as in (i), M is either a complex torus or hyperelliptic;*
(iii) *if $b_1(M)$ is odd, M as in (i), then either $\kappa(M) = 0$ or 1 or $\kappa(M) = -1$ and $b_2(M) = 0$;*
(iv) *the converse holds (except for certain Hopf surfaces) and indeed such M have a holomorphic affine structure.*

The various possibilities for Hopf surfaces are also discussed in detail in [25]. We recognize these affine structures as subordinate to our geometric structures $\mathbb{C}^2$, $\widetilde{SL}_2 \times E^1$, $Nil^3 \times E^1$, $S^2 \times E^1$, Sol_0^4 and Sol_1^4 (see also §9 below).

Affine connexions play a key role in the paper [5] to which we refer in §9. Further useful references are [15], [46]: the former considers also other structures, in particular a projective connection: a necessary condition in this case is $c_1^2 = 3c_2$, which we encountered also in §6.

§9. SURFACES OF CLASS VII

It is customary, following Kodaira, to call surfaces with $\kappa = -1$ and b_1 odd "of class VII". This is not precisely the terminology introduced in [29, I] (though Kodaira's later version in [29, IV] is equivalent to it), but is (cf. [2, p. 188]) a more convenient usage. The class contains the significant subclass of *Hopf surfaces*, which are those with universal cover $\mathbb{C}^2 - \{\mathbf{0}\}$ (all Hopf surfaces are of class VII by [29, Theorem 30]). It was shown by Kodaira that for S of class VII, S is elliptic if and only if there are non-constant meromorphic functions on S [29, Theorem 4], and that all these elliptic surfaces are Hopf [29, Theorem 28]: thus we first discuss Hopf surfaces.

Kodaira calls a Hopf surface *primary* if its fundamental group is (infinite) cyclic: any Hopf surface S has a finite unramified cover $\tilde{S}$ which is primary, so we next discuss these. According to [29, Theorem 30], we have the quotient of $\mathbb{C}^2 - \{\mathbf{0}\}$ by one of

(a) $T(z_1, z_2) = (\alpha_1 z_1, \alpha_2 z_2), \quad 0 < |\alpha_1| \leq |\alpha_2| < 1;$

(b) $T(z_1, z_2) = (\alpha_2^m z_1 + \lambda z_2^m, \alpha_2 z_2), \quad 0 < |\alpha_2| < 1, \quad \lambda \neq 0.$

By [29, Theorem 31], a primary Hopf surface is elliptic if and only if it is of type (a) with $\alpha_1^k = \alpha_2^l$ for some positive integers k and l.

We first discuss the non-elliptic cases. By [29, Theorem 32] we then have $\pi_1(S) \cong \mathbb{Z} \times \mathbb{Z}/l$ (for $l \geq 1$) and the covering group is generated by T [as in (a) or (b) above] and U, where

$$U(z_1, z_2) = (\varepsilon_1 z_1, \varepsilon_2 z_2), \quad \varepsilon_1^l = \varepsilon_2^l = 1,$$

in case (b) we also need $\varepsilon_1 = \varepsilon_2^m$. The surfaces of type (a) contain two elliptic curves C_i (the image of the z_i-axis); those of type (b) only have the curve C_1: there are no further curves on these surfaces (cf. [35]).

The surface S is geometric if and only if it has type (a) with $|\alpha_1| = |\alpha_2|$; thus not all geometric surfaces are elliptic.

In the elliptic case, the elliptic structure is unique by Lemma 7.2 and defines a quotient orbifold B. We have shown in Theorem 7.4 that B is good if and only if S (or $\tilde{S}$) is geometric: the condition for this is that we have type (a) with $|\alpha_1| = |\alpha_2|$, and hence $k = l$. Otherwise, z_1^l/z_2^k defines the projection of an elliptic fibration of $\tilde{S}$ with C_1, C_2 as the only multiple fibres. The surface S is obtained from $\tilde{S}$ by factoring out an appropriate finite group of transformations, cyclic unless $\alpha_1 = \alpha_2$. The classification of topological types of these elliptic cases is (cf. Kato [26]) the same as that of Seifert fibre structures on S^3.

For surfaces S of class VII which are not Hopf and hence admit no non-constant functions we have the general results [29, Theorem 11] that $q = 1$ and $b_1 = 1$: thus $0 = \chi(\mathcal{O}_S)$, so $0 = c_1^2 + c_2$ and $c_2 = \chi = b_2$. Since $c_1^2 + c_2 = 0, \chi = -\sigma$: thus the quadratic form on $H^2(S; \mathbb{R})$ is negative definite. Other results are rather incomplete. An excellent summary of known results on the classification of surfaces S containing curves is given by Nakamura [35]; there are four classes which can be explicitly described: for the rest, the curves form a single cycle C of rational curves, with $C^2 < 0$, which do not span $H_2(S; \mathbb{R})$.

Of more immediate interest to us are the surfaces with $b_2 = 0$. Non-trivial examples (denoted S_M, S_N) were described by Inoue [24], who proved a characterization theorem in [23]. A paper by Bogomolov [5], [6] extends this result to show that there are no further surfaces with $b_2 = 0$. (However, I understand that there are doubts whether the arguments in [6] are complete.) We shall need this result in §10, and will indicate the results which depend on it by "(mod B)".

We now describe these surfaces. First, let $M \in SL_3(\mathbb{Z})$ have eigenvalues $\alpha, \beta, \bar{\beta}$ with $\alpha > 1$, $\beta \neq \bar{\beta}$. Choose a real eigenvector (a_1, a_2, a_3) belonging to α and an eigenvector (b_1, b_2, b_3) belonging to β. Now let G_M be the group of automorphisms of $\mathbb{H} \times \mathbb{C}$ generated by

$$g_0(w, z) = (\alpha w, \beta z)$$

$$g_i(w, z) = (w + a_i, z + b_i) \quad i = 1, 2, 3$$

and define S_M to be the quotient of $H \times \mathbb{C}$ by G_M. It is easy to see that G_M acts properly discontinuously and that S_M is compact. We see by inspection that (with the notation of §3) $g_i \in Sol_0^4$ and that the g_i generate a discrete cocompact subgroup of Sol_0^4. Conversely, by Theorem 3.2, for any lattice Γ in Sol_0^4, Γ meets $\mathbb{R}^3$ in a lattice L in $\mathbb{R}^3$ so Γ/L is a discrete cocompact subgroup of the quotient, $\mathbb{R} \times SO_2$. Now if this intersects SO_2 non-trivially, L is invariant under an element of finite order. Thus L contains a fixed vector $\mathbf{v}$ by this element. However, such a $\mathbf{v}$ is an eigenvector of $\mathbb{R}$, so we have a contradiction (a one-dimensional lattice cannot have an automorphism of infinite order.)

Thus Γ/L is cyclic: also it cannot intersect the factor $\mathbb{R}$ non-trivially, else L would admit an automorphism with real eigenvalues, of which two are equal. Hence all three are rational, and integral (since L is a lattice), contradicting the fact that the automorphism has infinite order.

Thus the generator g of Γ/L has no real eigenvalues, and is represented on L by a matrix M as above. It now follows that the action of Γ on $H \times \mathbb{C}$ is as described above.

Inoue's second family of examples has two subfamilies as follows.

S_N^+. Let $N \in SL_2(\mathbb{Z})$ have real eigenvalues α, α^{-1} with corresponding real eigenvectors (a_1, a_2), (b_1, b_2). Choose a non-zero integer r, a complex number t, and further real numbers c_1, c_2 satisfying an integrality condition to be made precise. Define automorphisms of $H \times \mathbb{C}$ by

$$g_0(w, z) = (\alpha w, z + t)$$

$$g_i(w, z) = (w + a_i, z + b_i w + c_i) \quad (i = 1, 2)$$

$$g_3(w, z) = (w, z + r^{-1}(b_1 a_2 - b_2 a_1))$$

Then these generate a discrete group acting freely, whose quotient we denote by S_N^+.

S_N^- is defined by modifying the above as follows. $N \in GL_2(\mathbb{Z})$ has real eigenvalues $\alpha > 1$, $-\alpha^{-1}$. The rest are as above, except that we do not choose a t, but define instead $g_0(w, z) = (\alpha w, -z)$.

We interpret these in terms of geometric structures of types Sol_1^4, $Sol_1'^4$: the latter occurring in the case $t \notin \mathbb{R}$: we also recall that the full group of holomorphic automorphisms in the case of Sol_1^4 has two components. We defined Sol_1^4 as a group of matrices, and the action by matrix multiplication

$$\begin{pmatrix} \varepsilon & b & c \\ 0 & \alpha & a \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ w \\ 1 \end{pmatrix} = \begin{pmatrix} \varepsilon z + bw + c \\ \alpha w + a \\ 1 \end{pmatrix}$$

for $t \in \mathbb{R}$, each of the above g_i belongs to this group.

To show that all discrete cocompact subgroups Γ are of the form above, let us write G_0 for the connected component ($\varepsilon = 1$) of Sol_1^4, G_1 for its derived group ($\varepsilon = \alpha = 1$) and G_2 for the derived group of G_1 ($\varepsilon = \alpha = 1$, $a = b = 0$), and set $\Gamma_i = \Gamma \cap G_i$. By (2.2), Γ_1 is a lattice in G_1, and by (2.3), Γ_2 is a lattice in G_2. Thus Γ_1/Γ_2 is a lattice in $G_1/G_2 \cong \mathbb{R}^2$, and is invariant under the generator $\bar{g}_0$ of Γ_0/Γ_1. If $\bar{g}_0 \to \alpha \in \mathbb{R}^\times$; Γ_1/Γ_2 has generators $\bar{g}_1 = (a_1, b_1)$ and $\bar{g}_2 = (a_2, b_2)$, and N is the matrix (with respect to this basis) of the automorphism of Γ_1/Γ_2 induced by $\bar{g}_0$ then we recover the above structure of Γ_0/Γ_2 (conjugating in G_0/G_2, we may easily arrange $a_0 = b_0 = 0$).

Next, $g_1^{-1} g_2^{-1} g_1 g_2$ is a non-zero element of Γ_2, so must be some power g_3^r of the generator g_3 of Γ_2: hence g_3 is as above. And $g_0^{-1} g_i^{-1} g_0 g_1^{n_{i1}} g_2^{n_{i2}}$ also belongs to Γ_2 ($i = 1, 2$), so must be a power of g_3: this yields the integrality conditions on c_1 and c_2 referred to above. We have thus recovered the above normal form for S_N^+.

If Γ meets the non-identity component of Sol_1^4, we first assert that Γ/Γ_1 is still cyclic. For suppose not: then Γ contains an element with $\varepsilon = -1, \alpha = +1$. Now if $a = 0$ this element has order 2, so does not act freely; if $a \neq 0$, its square defines a non-zero element of Γ_1/Γ_2 which is an eigenvalue of the action of Γ_0/Γ_1 and (compare the Sol_0^4 case above) this yields a contradiction. Now we can conjugate any element (with $\varepsilon = -1$, $\alpha^2 \neq 1$) into the normal form with $a = b = c = 0$. The rest of the argument is as before.

For the remaining cases recall that Sol_1' was interpreted as acting on $\mathbb{C} \times H$ by

$$\begin{pmatrix} 1 & b & c \\ 0 & \alpha & a \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} w + bz + c + i \log \alpha \\ \alpha z + a \\ 1 \end{pmatrix};$$

replacing the term $i \log \alpha$ by $im \log \alpha$ for any $m \neq 0$ gives an equivalent action. Since we have the same group, the discrete subgroups are as before; g_1, g_2 and g_3 are unchanged, but g_0 now acts by

$$g_0(w, z) = (\alpha w, z + t + im \log \alpha) \qquad t, m \in \mathbb{R};$$

but then $t + im \log \alpha$ gives an arbitrary complex number.

To summarize the above discussion, we have shown

PROPOSITION 9.1. *The Inoue surfaces are precisely the complex analytic surfaces with geometric structure of type*

S_M	Sol_0^4
S_N^+ (t real), S_N^-	Sol_1^4
S_N^+ ($t \notin \mathbb{R}$)	$Sol_1'^4$.

§10. TOPOLOGY AND GEOMETRIC STRUCTURE

In the previous sections of this paper, although topology has played a role, we have concentrated on complex analytic structure. In this section we will show that both the geometric and the analytic structures described earlier are determined (up to equivalence) by the topology. More precisely, we have:

THEOREM 10.1. *If M_i^4 ($i = 1, 2$) is a closed 4-manifold with geometric structure of type X_i ($i = 1, 2$), and M_1, M_2 are homotopy equivalent, then $X_1 = X_2$.*

THEOREM 10.2. (mod B). *If M^4 is a closed 4-manifold with geometric structure of type X; and S is a compact complex surface, homotopy equivalent to M, then the triple (X; $\kappa(S)$, $b_1(S)$ (mod 2)) appears in Table 4 (Theorem 4.5).*

We begin the proof of these theorems with some general remarks. First, we may replace the manifolds by appropriate finite covers as and when necessary: the homotopy equivalences lift, and we do not lose the information needed for the conclusion. It is thus enough to consider a manifold M (or manifolds M_1, M_2) given as $\Gamma \backslash X$ where $\Gamma \subset G_X^0$: in particular, we may suppose M orientable.

Next, the homotopy-theoretic information we will use is of three types: the homotopy type of the universal cover $\tilde{M}$ (namely X); the characteristic numbers $\sigma(M)$ and $\chi(M)$; and the fundamental group $\pi_1(M)$: our discussion accordingly in divided into several stages. We will treat both theorems together.

First case: $\tilde{M}$ compact

Then X is one of the geometries $P^2\mathbb{C}$, $S^2 \times S^2$ and S^4. The condition defining this case (as likewise with the other cases below) is invariant under homotopy equivalence of M, so for Theorem 10.1 we have both M_1 and M_2 as above. Since the three cases for $X = \tilde{M}$ are distinguished by their second Betti numbers, the conclusion is here immediate.

As to Theorem 10.2, we first observe that since no manifold homotopy equivalent to S^4 has an almost complex structure, the case $X = S^4$ cannot arise. Otherwise we have S homotopy equivalent of $P^2\mathbb{C}$ or to $P^1\mathbb{C} \times P^1\mathbb{C}$. The result in these cases follows from [2, pp. 135, 202].

Before turning to the remaining cases, we first observe that in all of them (cf. end of §2) X is homeomorphic to an open set in $\mathbb{R}^4$, with finitely generated homology. Now if M^4 contains a smoothly embedded 2-sphere with self-intersection number (= normal Euler class) ± 1, this sphere has a neighbourhood N with boundary a 3-sphere and M is a connected sum of $P^2\mathbb{C}$ with another manifold M_1. If also M has infinite fundamental group, its universal cover contains infinitely many copies of N, so $H_2(\tilde{M})$ is not finitely generated. If M' is homotopy equivalent to M, the same follows for M'.

Thus if M is geometric, and is homotopy equivalent to a complex surface S which is *not* minimal (so contains such a 2-sphere) we obtain a contradiction, since in all remaining cases, as $\tilde{M}$ is non-compact, $\pi_1(M)$ is infinite. Thus for the rest of the proof of Theorem 10.2, we may suppose S minimal.

Second case: $\tilde{M}$ not compact, $\sigma(M) \neq 0$

The only geometry arising here $H^2\mathbb{C}$, so again Theorem 10.1 is immediate. For Theorem 10.2 recall that the characteristic numbers satisfy $\sigma(M) > 0$, $\chi(M) = 3\sigma(M)$, so if the homotopy equivalence of S on M preserves orientation, $c_1^2(S) = 3c_2(S) > 0$ and as $c_1^2 > 0$ [and S is not rational, as $\pi_1(S)$ is infinite] we have $\kappa(S) = 2$ as required. If—as seems unlikely to be possible—the homotopy equivalence reverses the orientation, then $c_1^2(S) = c_2(S) > 0$ and we still conclude $\kappa(S) = 2$.

Third case: $\tilde{M}$ non-compact, $\sigma(M) = 0$, $\chi(M) \neq 0$

We have three relevant geometries here: $S^2 \times H^2$, $H^2 \times H^2$ and H^4. For Theorem 10.1, observe that in the first of these cases, $\tilde{M}$ is not contractible (as it is in the other two), while $\chi(M) > 0$ in the $H^2 \times H^2$ case and $\chi(M) < 0$ in the H^4 case.

As to Theorem 10.2, we find on checking the table giving the classification of minimal compact complex surfaces that $c_1^2 = 2c_2 \neq 0$ occurs only if $\kappa = 2$ (and here $c_1^2 = 2c_2 > 0$) or $\kappa = -1$ and b_1 is even, when we have a ruled surface over a curve of genus 0 (in which case $\tilde{S}$ is compact) or genus $g \geq 2$, in which case $c_1^2 = 2c_2 < 0$. Thus if M is geometric of type $H^2 \times H^2$ and is homotopy equivalent to S, $c_2(S) > 0$, so $\kappa(S) = 2$: if M is of type $S^2 \times H^2$ or H^4 then $c_2(S) < 0$, so $\kappa(S) = 0$, $b_1(S)$ is even, and S is ruled: in particular, the universal cover $\tilde{S}$ is not contractible. This gives a contradiction in the H^4 case, which concludes the proof.

Fourth case: $\tilde{M}$ non-compact and non-contractible; $\sigma(M) = \chi(M) = 0$

Here we have just two geometries: $S^2 \times E^2$ and $S^3 \times E^1$. As the two spaces X have different Betti numbers, Theorem 10.1 is again immediate.

Thus suppose (for Theorem 10.2) S a complex surface with $c_1^2(S) = c_2(S) = 0$ and $\tilde{S}$ non-contractible. Then $\kappa(S) = 2$ contradicts $c_1^2(S) = 0$. If $\kappa(S) = 0$ or 1, S is (or at least deforms to) an elliptic surface: since $c_2(S) = 0$ this has no singular fibres. But then by Theorem 7.4 it is geometric, with universal cover isomorphic to $\mathbb{C}^2$ or $\mathbb{C} \times H$, and hence contractible: a contradiction. Thus $\kappa(S) = -1$. Now if $b_1(S)$ is even, we conclude that S is a ruled surface over an elliptic curve, whence the universal cover $\tilde{S}$ is homeomorphic to $S^2 \times E^2$.

If $b_1(S)$ is odd, and hence equal to 1, we have $b_2(S) = \chi(S) = 0$. By Bogomolov's result [6] since $\tilde{S}$ is not contractible, S is a Hopf surface, so its universal cover is homeomorphic to $S^3 \times E^1$. (We can avoid this argument: see below.)

Fifth case: M aspherical; $\sigma(M) = \chi(M) = 0$

There still remain nine geometries, viz. E^4, $Nil^3 \times E^1$, Nil^4, $Sol^4_{m,n}$, Sol^4_0, Sol^4_1, $H^2 \times E^2$, $\widetilde{SL}_2 \times E^1$ and $H^3 \times E^1$. The argument in this case follows a different pattern from the preceding ones. First we have

LEMMA 10.3. (*mod B*). *If S is a compact complex surface which is aspherical and has $c_1^2(S) = c_2(S) = 0$, then S is geometric.*

Proof. We have seen above that S must be minimal. As $c_1^2(S) = 0$, $\kappa(S) \leq 1$. If $\kappa(S) = 1$ or if $\kappa(S) = 0$ and $b_1(S)$ is odd, then S is elliptic and [since $c_2(S) = 0$] has no singular fibres: by Theorem 7.4, S is geometric. If $\kappa(S) = 0$ and $b_1(S)$ is even, then since [as $c_2(S) = 0$] S is not a $K3$ or Enriques surface, it must be a complex torus or hyperelliptic: again S is geometric. If $\kappa(S) = -1$ and $b_1(S)$ is even, S is ruled and we deduce that $\tilde{S}$ is not contractible, a contradiction. Finally if $\kappa(S) = -1$ and $b_1(S)$ is odd, then S is not a Hopf surface (since $\tilde{S}$ is contractible), and $b_2(S) = c_2(S) = 0$. Thus by Bogomolov's theorem, S is an Inoue surface and hence geometric.

We deduce from the lemma that Theorem 10.2 in the remaining cases will follow from Theorem 10.1. It thus suffices to prove the following.

PROPOSITION 10.4. *Let X_1, X_2 be geometries from the list E^4, $Nil^3 \times E^1$, Nil^4, $Sol^4_{m,n}$, Sol^4_0, Sol^4_1, $H^2 \times E^2$, $\widetilde{SL}_2 \times E^1$, $H^3 \times E^1$. Let Γ_i be a discrete cocompact subgroup of G_{X_i} $(i = 1, 2)$. If Γ_1 is isomorphic to Γ_2, then $X_1 = X_2$.*

Proof. As before, we may pass to a convenient subgroup of finite index. We will, in particular, suppose that $\Gamma_i \subset G^o_{X_i}$. Setting $M_i = \Gamma_i \backslash X_i$ we have $\Gamma_i = \pi_1(M_i)$, so $b_1(M_i)$ is the torsion-free rank of $\Gamma_i/[\Gamma_i, \Gamma_i] = \Gamma_i^{ab}$: we define this to be $b_1(\Gamma_i)$.

The last three cases are distinguished from the others on the list by the fact that only in these cases is Γ not solvable: we consider these first. By (2.6), in the cases $H^2 \times E^2$, $\widetilde{SL}_2 \times E^1$, Γ has a normal subgroup which is free abelian of rank 2: the quotient is a Fuchsian group. For $H^3 \times E^1$ there is a normal subgroup $\mathbb{Z}$ of translations: the quotient $\bar{\Gamma}$ is a lattice in $SO^+_{3,1}$. This cannot have a normal abelian subgroup, since any such subgroup fixes a (unique) point in H^3 or on its ideal boundary, which would then have to be fixed by $\bar{\Gamma}$, contradicting cocompactness. Thus in the $H^3 \times E^1$ case, Γ does *not* have a normal subgroup which is free abelian of rank 2. We can distinguish the other two cases since we know that if Γ acts freely on $H^2 \times E^2$ resp. $\widetilde{SL}_2 \times E^1$, $b_1(\Gamma)$ is even resp. odd (and we achieve free actions by passing to a suitable subgroup of finite index).

In all the remaining cases except Sol^4_1 we saw in §2 that Γ contains a normal subgroup Γ_0 which is free abelian of rank 3: and indeed in each of these cases we may suppose (passing to a subgroup of finite index) that the quotient is a lattice in $\mathbb{R}$, hence infinite cyclic. For the case of Sol^4_1 we gave a detailed description of the possible groups Γ_i in §9: if γ does not belong to the derived group G_1, its centralizer is only two-dimensional; G_1 itself is three-dimensional but not abelian. Thus there is no normal subgroup here which is free abelian of rank 3.

To distinguish the remaining cases it will suffice to consider the monodromy of the corresponding extension of $\mathbb{Z}^3$ by $\mathbb{Z}$, provided we allow for (i) any non-uniqueness of the normal subgroup isomorphic to $\mathbb{Z}^3$, and (ii) use only properties which are not lost on passing to subgroups of finite index. In fact, by the descriptions of the various geometries we already know that the eigenvalues of the monodromy are as follows:

Sol_0^4: all distinct: one real and two complex

$Sol_{m,n}^4$: all distinct and real

E^4; $Nil^3 \times E^1$; Nil^4: all equal to 1: these cases are distinguished by the sizes of the Jordan blocks which are (1, 1, 1), (1, 2) and (3) in the three respective cases.

These distinctions are indeed clearly preserved on passing to subgroups of finite index—in fact, we have already so passed to reduce from quasi-unipotent to unipotent elements in the three final cases. In the Sol_0^4 and $Sol_{m,n}^4$ cases, the centralizer of an element not in $\mathbb{Z}^3$ has rank < 3; thus this subgroup is uniquely determined. Although this is not so in the other cases, it is easy to see that a different choice of $\mathbb{Z}^3$ leads to an equivalent situation. More directly, these cases are distinguished by the length of the lower central series.

This concludes the proof of Proposition 10.4, and hence of Theorems 10.1 and 10.2. The least satisfactory feature of the proof is the dependence on Bogomolov's result: in fact, we can avoid the use of this in some cases. In the proof of Theorem 10.2, this was used only to show the non-existence of surfaces of class VII_0 homotopy equivalent to a geometric M^4 with $\sigma(M) = \chi(M) = 0$. We can conclude more simply if we already know that for M^4—or indeed for some finite covering M_1^4—$b_1\ (M_1) \neq 1$. This argument excludes geometries of all the Kähler types X; it also excludes Nil^4 and $Sol^3 \times E^1$ (one of the $Sol_{m,n}^4$), where $b_1 = 2$; $Nil^3 \times E^1$ where (for a subgroup of finite index), $b_1 = 3$; and $\widetilde{SL}_2 \times E^1$ where b_1 can be made large by passing to a subgroup of finite index. There remain the cases $Sol_{m,n}^4$ and $H^3 \times E^1$. In the case of $H^3 \times E^1$, the general conjecture that any closed hyperbolic 3-manifold has a finite cover which is Haken would imply here that by passing to a finite cover we can achieve $b_1 \geq 2$.

We close this section by observing that although Theorems 10.1 and 10.2 are formulated in general terms, the fact that the conclusions are invariant under taking finite covers makes them rather selective in application. Indeed, we saw in §3 that though G_X/G_X^0 is non-trivial in many cases, hardly any of these yield holomorphic automorphisms. There are thus many geometric 4-manifolds of these types which (even if orientable) admit no complex structure. The most striking example is E^4: although there are only [2, p. 148] 8 possible isomorphism classes of fundamental group in the complex case, there are [7] 75 possible in the real case.

§11. GEOMETRIC STRUCTURES ON NON-COMPACT MANIFOLDS

It is not my intention in this section to be in any way systematic: merely to describe some interesting examples and perhaps signal the fact that there may be more to be discovered in this area.

Up to now, we have discussed almost exclusively discrete cocompact subgroups Γ of G_X which act freely on X, thus giving rise to compact quotients. A first generalization is to allow Γ to have fixed points on X. In the quotient $S = \Gamma/X$ we can either leave the resulting quotient singularities or resolve them to obtain a smooth surface $\tilde{S}$. An interesting case is the familiar one with $X = \mathbb{C}^2$ where we factor out by a lattice to obtain a complex torus and also by the map $x \to -x$, to obtain a Kummer surface, which resolves to a $K3$ surface. Again, some (perhaps all) elliptic surfaces with singular fibres but with j constant arise by resolving singularities in a singular geometric such surface. If fixed points are allowed, there are plenty of finite group actions on $P^2\mathbb{C}$. The quotients here are all unirational (hence rational), and include the weighted projective spaces (cf. [9]).

A second generalization (which may of course be combined with the above) is to allow Γ to be a non-cocompact lattice. The quotient then has finite covolume: in the cases with complex structure it seems to follow that there is a natural compactification as a complete surface (where again we may resolve the singularities). A famous class of examples here is that of Hilbert modular surfaces, arising as arithmetic quotients of $H^2 \times H^2$: see [18] and other papers of Hirzebruch. Such arithmetic quotients can also be obtained for $H^2(\mathbb{C})$: see [19] and papers of Hofzapfel.

In this connection the geometry F^4 (omitted in most of the above) comes again into its own. Recall that here $G_X^\circ = \mathbb{R}^2 \rtimes SL_2(\mathbb{R})$; $X = G_X^\circ/SO_2$ is thus fibred over H with fibre $\mathbb{R}^2 = \mathbb{C}$. If we first factor out $\mathbb{Z}^2$, we have an elliptic fibration over H, and this is such that the fibre F_τ over $\tau \in H$ has j-invariant precisely $j(\tau)$. Now if Γ is a lattice meeting $\mathbb{R}^2$ in $\mathbb{Z}^2$, $\Gamma\backslash X$ is an elliptic surface over $\bar{\Gamma}\backslash H$ [where $\bar{\Gamma} = \Gamma/\mathbb{Z}^2$ is the image of Γ in $SL_2(\mathbb{R})$], but there will be exceptional fibres corresponding to the fixed points (if any) of $\bar{\Gamma}$ on H. More precisely, let us regard j as taking values in an orbifold $\mathbb{C}_J$ with invariants $(2, 3, \infty)$: $P^1\mathbb{C}$ is punctured at ∞, and has cone points with angles π at $j = 1$ and $2\pi/3$ at $j = 0$. Then we have a commutative diagram

$$\begin{tikzcd}
H \arrow[rr, "J"] \arrow[dr] & & \mathbb{C}_J \\
& H/\bar{\Gamma} \arrow[ur, "\bar{j}"'] &
\end{tikzcd}$$

Here H is the universal orbifold cover of $\mathbb{C}_J$: thus $\bar{j}$ is also an orbifold covering.

Conversely, if we have a (compact) elliptic surface S such that when the singular fibres with $j = \infty$ are removed the resulting open surface S° has no singular fibres, and j/S° is an orbifold covering of $\mathbb{C}_J$, then S° has a geometric structure of type F^4. For if B, B° are the base curves of S, S°, the universal orbifold cover of B° must be H, and the elliptic surface over B° pulls back to one over H which we can identify with $\mathbb{Z}^2\backslash F^4$. Now we can argue as in the proof of Theorem 7.4 to show that the covering transformations over S° must lift to self-maps of F^4 induced by G_X°.

There is a close connection of the surfaces just discussed with the "elliptic modular surfaces" studied by Shioda [43].

The constructions discussed above seem in practice to give rise to projective algebraic surfaces. Our final examples, which do not fit so easily into the scheme of things, are non-Kählerian. These are the Inoue–Hirzebruch surfaces (cf. [18], [24, II]) where we take a group $M \rtimes V$ (M a module in a real quadratic field K; V a group of units)—which acts geometrically on $H \times H$, but *not* with finite covolume—let it act on $H \times \mathbb{C}$ instead, and compactify the quotient by a cycle of rational curves at each end. Other examples discussed in the survey [35] (half Inoue surfaces, parabolic Inoue surfaces, exceptional compactifications) seem to be of a similar nature, but using in some cases the geometry $Nil^3 \times E^1$.

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