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Semipositone Higher-Order Differential Equations

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Abstract—Krasnoselskii's fixed-point theorem in a cone is used to discuss the existence of positive solutions to semipositone conjugate and (n, p) problems.

Keywords—Existence, Positive solution, Semipositone, Conjugate and (n, p) problems.

1. INTRODUCTION

This paper presents existence results for semipositone higher-order boundary value problems. In particular, we discuss the conjugate boundary value problem

$$\begin{aligned} (-1)^{n-p} y^{(n)}(t) &= \mu f(t, y(t)), && 0 < t < 1, \\ y^{(i)}(0) &= 0, && 0 \le i \le p-1, \\ y^{(i)}(1) &= 0, && 0 \le i \le n-p-1, \end{aligned} \tag{1.1}$$

where $n \ge 2$, $1 \le p \le n-1$, and $\mu > 0$ are constants. We note that our nonlinearity f may take *negative* values. Problems of this type are referred to as semipositone problems in the literature and they arise naturally in chemical reactor theory [1]. The constant μ is usually called the Thiele modulus, and in applications one is interested in showing the existence of positive solutions for $\mu > 0$ small (see [2] for a nonexistence result). Most papers [3–7] in the literature discuss (1.1) when f takes nonnegative values (i.e., positone problems) and only a handful of papers (see [2,8,9] and the references therein) have appeared discussing the semipositone problem. This paper attempts to fill part of this gap in the literature. Moreover, our technique should enable

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the reader to see that other types of boundary data could also be considered. To illustrate this, we will also briefly discuss the (n,p) problem

$$\begin{aligned} y^{(n)}(t) + \mu f(t, y(t)) &= 0, \qquad 0 < t < 1, \\ y^{(i)}(0) &= 0, \qquad 0 \le i \le n-2, \\ y^{(p)}(1) &= 0, \end{aligned}$$

where $n \ge 2$ and $1 \le p \le n-1$ is fixed. Existence in this paper will be established using Krasnoselskii's fixed-point theorem in a cone, which we state here for the convenience of the reader.

THEOREM 1.1. *Let $E = (E, \|\,.\,\|)$ be a Banach space and let $K \subset E$ be a cone in E. Assume Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\bar{\Omega}_1 \subset \Omega_2$ and let $A : K \cap (\bar{\Omega}_2 \backslash \Omega_1) \to K$ be continuous and completely continuous. In addition, suppose either*

$$\|Au\| \le \|u\|, \quad \text{for } u \in K \cap \partial\Omega_1, \qquad \text{and} \qquad \|Au\| \ge \|u\|, \quad \text{for } u \in K \cap \partial\Omega_2,$$

or

$$\|Au\| \ge \|u\|, \quad \text{for } u \in K \cap \partial\Omega_1, \qquad \text{and} \qquad \|Au\| \le \|u\|, \quad \text{for } u \in K \cap \partial\Omega_2,$$

hold. Then A has a fixed point in $K \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

2. SEMIPOSITONE PROBLEMS

In this section, we first discuss the conjugate boundary value problem

$$\begin{aligned} (-1)^{n-p} y^{(n)}(t) &= \mu f(t, y(t)), \qquad 0 < t < 1, \\ y^{(i)}(0) &= 0, \qquad 0 \le i \le p-1, \\ y^{(i)}(1) &= 0, \qquad 0 \le i \le n-p-1, \end{aligned} \tag{2.1}$$

where $n \ge 2$, $1 \le p \le n-1$, and $\mu > 0$ are constants. Of physical interest is the existence of solutions which are positive on $(0,1)$.

Before we prove our main result, we first recall two well-known results from the literature which will be used in our proof. The first lemma can be found in [3,4] and the second in [10, p. 18].

LEMMA 2.1. *Suppose $y \in C^{n-1}[0,1] \cap C^n(0,1)$ satisfies*

$$\begin{aligned} (-1)^{n-p} y^{(n)}(t) &\ge 0, \qquad \text{for } t \in (0,1), \\ y^{(i)}(0) &= 0, \qquad 0 \le i \le p-1, \\ y^{(i)}(1) &= 0, \qquad 0 \le i \le n-p-1. \end{aligned}$$

Then

$$y(t) \ge t^p (1-t)^{n-p} |y|_0, \qquad \text{for } t \in [0,1];$$

here $|y|_0 = \sup_{t \in [0,1]} |y(t)|$.

LEMMA 2.2. *The boundary value problem*

$$\begin{aligned} (-1)^{n-p} y^{(n)}(t) &= 1, \qquad \text{for } t \in (0,1), \\ y^{(i)}(0) &= 0, \qquad 0 \le i \le p-1, \\ y^{(i)}(1) &= 0, \qquad 0 \le i \le n-p-1, \end{aligned}$$

has a solution w with

$$w(t) \le \frac{1}{n!} t^p (1-t)^{n-p}, \qquad \text{for } t \in [0,1].$$

We now use Lemma 2.1, Lemma 2.2, and Krasnoselskii's fixed-point theorem to establish our main result.

THEOREM 2.3. *Suppose the following conditions are satisfied:*

$$f : [0,1] \times [0,\infty) \to \mathbf{R} \text{ is continuous and there exists a constant } M > 0, \text{ with } f(t,u) + M \geq 0 \text{ for } (t,u) \in [0,1] \times [0,\infty), \tag{2.2}$$

$$f(t,u) + M \leq \psi(u) \text{ on } [0,1] \times [0,\infty), \text{ with } \psi : [0,\infty) \to [0,\infty) \text{ continuous and nondecreasing and } \psi(u) > 0 \text{ for } u > 0, \tag{2.3}$$

$$\exists\, r \geq \frac{\mu M}{n!}, \text{ with } \frac{r}{\psi(r)} \geq \mu \sup_{t\in[0,1]} \int_0^1 (-1)^{n-p} G(t,s)\, ds, \tag{2.4}$$

$$\text{there exists } a \in (0,1/2) \text{ (choose and fix it) and a continuous, nondecreasing function } g : (0,\infty) \to (0,\infty), \text{ with } f(t,u) + M \geq g(u) \text{ for } (t,u) \in [a, 1-a] \times (0,\infty), \tag{2.5}$$

and

$$\exists\, R > r, \text{ with } \frac{R}{g(\epsilon R A_0)} \leq \mu \int_a^{1-a} (-1)^{n-p} G(\sigma,s)\, ds; \tag{2.6}$$

here $\epsilon > 0$ is any constant (choose and fix it) so that $1 - \mu M/Rn! \geq \epsilon$ (note ϵ exists since $R > r \geq \mu M/n!$),

$$A_0 = \begin{cases} a^p(1-a)^{n-p}, & \text{if } n \leq 2p, \\ (1-a)^p a^{n-p}, & \text{if } n > 2p, \end{cases}$$

$G(t,s)$ is the Green's function (see [4,10] for an explicit representation) for

$$\begin{aligned} y^{(n)} &= 0, && \text{on } (0,1), \\ y^{(i)}(0) &= 0, && 0 \leq i \leq p-1, \\ y^{(i)}(1) &= 0, && 0 \leq i \leq n-p-1, \end{aligned}$$

and $0 \leq \sigma \leq 1$ is such that

$$\int_a^{1-a} (-1)^{n-p} G(\sigma,s)\, ds = \sup_{t\in[0,1]} \int_a^{1-a} (-1)^{n-p} G(t,s)\, ds.$$

Then (2.1) has a solution $y \in C^{n-1}[0,1] \cap C^n(0,1)$ with $y(t) > 0$ for $t \in (0,1)$.

PROOF. To show (2.1) has a nonnegative solution, we will look at the boundary value problem

$$\begin{aligned} (-1)^{n-p} y^{(n)}(t) &= \mu f^\star(t, y(t) - \phi(t)), && 0 < t < 1, \\ y^{(i)}(0) &= 0, && 0 \leq i \leq p-1, \\ y^{(i)}(1) &= 0, && 0 \leq i \leq n-p-1, \end{aligned} \tag{2.7}$$

where $\phi(t) = \mu M w(t)$ (w is as in Lemma 2.2) and

$$f^\star(t,v) = \begin{cases} f(t,v) + M, & v \geq 0, \\ f(t,0) + M, & v \leq 0. \end{cases}$$

We will show, using Theorem 1.1, that there exists a solution y_1 to (2.7) with $y_1(t) \geq \phi(t)$ for $t \in [0,1]$ (note $\phi(t) > 0$ for $t \in (0,1)$). If this is true, then $u(t) = y_1(t) - \phi(t)$ is a nonnegative solution (positive on $(0,1)$) of (2.1) since for $t \in (0,1)$ we have

$$\begin{aligned} (-1)^{n-p} u^{(n)}(t) &= (-1)^{n-p} y_1^{(n)}(t) - \mu M = \mu f^\star(t, y(t) - \phi(t)) - \mu M \\ &= \mu\,[f(t, y(t) - \phi(t)) + M] - \mu M = \mu f(t, u(t)). \end{aligned}$$

As a result, we will concentrate our study on (2.7). Let $E = (C[0,1], |\,.\,|_0)$ and

$$K = \left\{u \in C[0,1] : u(t) \geq t^p(1-t)^{n-p}|u|_0, \text{ for } t \in [0,1]\right\}.$$

Clearly, K is a cone of E. Let

$$\Omega_1 = \{u \in C[0,1] : |u|_0 < r\} \qquad \text{and} \qquad \Omega_2 = \{u \in C[0,1] : |u|_0 < R\}.$$

Next let $A : K \cap (\bar{\Omega}_2 \backslash \Omega_1) \to C[0,1]$ be defined by

$$Ay(t) = \mu \int_0^1 (-1)^{n-p} G(t,s) f^\star(s, y(s) - \phi(s))\, ds.$$

First we show $A : K \cap (\bar{\Omega}_2 \backslash \Omega_1) \to K$. If $u \in K \cap (\bar{\Omega}_2 \backslash \Omega_1)$, then $(-1)^{n-p} G(t,s) \geq 0$ for $(t,s) \in [0,1] \times [0,1]$ (see [4]) and (2.2) guarantee that

$$\begin{aligned}
(-1)^{n-p}(Au)^{(n)}(t) &\geq 0, && \text{on } (0,1),\\
(Au)^{(i)}(0) &= 0, && 0 \leq i \leq p-1,\\
(Au)^{(i)}(1) &= 0, && 0 \leq i \leq n-p-1,
\end{aligned}$$

and so Lemma 2.1 implies $Au(t) \geq t^p(1-t)^{n-p}|Au|_0$ for $t \in [0,1]$. Consequently, $Au \in K$ so $A : K \cap (\bar{\Omega}_2 \backslash \Omega_1) \to K$. It is well known [3] that $A : K \cap (\bar{\Omega}_2 \backslash \Omega_1) \to K$ is continuous and compact.

We now show

$$|Ay|_0 \leq |y|_0, \qquad \text{for } y \in K \cap \partial\Omega_1. \tag{2.8}$$

To see this, let $y \in K \cap \partial\Omega_1$. Then $|y|_0 = r$ and $y(t) \geq t^p(1-t)^{n-p} r$ for $t \in [0,1]$. Now, for $t \in [0,1]$, we have

$$\begin{aligned}
Ay(t) &= \mu \int_0^1 (-1)^{n-p} G(t,s) f^\star(s, y(s) - \phi(s))\, ds\\
&\leq \mu \int_0^1 (-1)^{n-p} G(t,s) \psi(y(s))\, ds\\
&\leq \mu \psi(|y|_0) \int_0^1 (-1)^{n-p} G(t,s)\, ds\\
&\leq \mu \psi(r) \sup_{t \in [0,1]} \int_0^1 (-1)^{n-p} G(t,s)\, ds,
\end{aligned}$$

since for $s \in (0,1)$ (note $y(s) \geq 0$),

$$f^\star(s, y(s) - \phi(s)) = \begin{cases} f(s, y(s) - \phi(s)) + M \leq \psi(y(s) - \phi(s)) \leq \psi(y(s)), & \text{if } y(s) - \phi(s) \geq 0,\\ f(s, 0) + M \leq \psi(0) \leq \psi(y(s)), & \text{if } y(s) - \phi(s) < 0; \end{cases}$$

in fact, one can show $y(s) - \phi(s) \geq 0$ for $s \in (0,1)$ (see the argument below). This together with (2.4) yields

$$|Ay|_0 \leq \mu \psi(r) \sup_{t \in [0,1]} \int_0^1 (-1)^{n-p} G(t,s)\, ds \leq r = |y|_0,$$

so (2.8) holds.

Next, we show

$$|Ay|_0 \geq |y|_0, \qquad \text{for } y \in K \cap \partial\Omega_2. \tag{2.9}$$

To see this, let $y \in K \cap \partial\Omega_2$ so $|y|_0 = R$ and $y(t) \geq t^p(1-t)^{n-p}R$ for $t \in [0,1]$. Let ϵ, a, and A be as in the statement of Theorem 2.3. For $t \in (0,1)$, we have from Lemmas 2.1 and 2.2 that

$$\begin{aligned} y(t) - \phi(t) = y(t) - \mu M w(t) &\geq y(t) - \frac{\mu M}{n!} t^p (1-t)^{n-p}, \\ &\geq y(t)\left[1 - \frac{\mu M}{n! R}\right] \geq \epsilon y(t), \\ &\geq \epsilon t^p (1-t)^{n-p} |y|_0 = \epsilon t^p (1-t)^{n-p} R. \end{aligned}$$

As a result,

$$y(t) - \phi(t) \geq \epsilon A_0 R, \qquad \text{for } t \in [a, 1-a]. \tag{2.10}$$

Now with σ as in the statement of Theorem 2.3, we have

$$\begin{aligned} Ay(\sigma) &= \mu \int_0^1 (-1)^{n-p} G(\sigma, s) f^\star(s, y(s) - \phi(s))\, ds \\ &\geq \mu \int_a^{1-a} (-1)^{n-p} G(\sigma, s) f^\star(s, y(s) - \phi(s))\, ds \\ &\geq \mu g(\epsilon A_0 R) \int_a^{1-a} (-1)^{n-p} G(\sigma, s)\, ds, \end{aligned}$$

since for $s \in [a, 1-a]$ we have from (2.10) that

$$f^\star(s, y(s) - \phi(s)) = f(s, y(s) - \phi(s)) + M \geq g(y(s) - \phi(s)) \geq g(\epsilon A_0 R).$$

This together with (2.6) yields

$$Ay(\sigma) \geq \mu g(\epsilon A_0 R) \int_a^{1-a} (-1)^{n-p} G(\sigma, s)\, ds \geq R = |y|_0.$$

Thus, $|Ay|_0 \geq |y|_0$, so (2.9) holds.

Now Theorem 1.1 implies A has a fixed point $y_1 \in K \cap (\bar{\Omega}_2 \backslash \Omega_1)$, i.e., $r \leq |y_1|_0 \leq R$ and $y_1(t) \geq t^p(1-t)^{n-p} r$ for $t \in [0,1]$. To finish the proof, we need to show $y_1(t) \geq \phi(t)$ for $t \in [0,1]$. This is immediate since Lemma 2.2 with the fact that $r \geq \mu M / n!$ implies for $t \in (0,1)$ that

$$y_1(t) \geq t^p (1-t)^{n-p} r \geq \frac{\mu M}{n!} t^p (1-t)^{n-p} = \mu M w(t) = \phi(t). \qquad \blacksquare$$

EXAMPLE. Consider (2.1) with $f(t,u) = u^m - 1$, $m > 1$, and $\mu \in (0, n!]$. Then (2.1) has a solution y with $y(t) > 0$ for $t \in (0,1)$.

To see this, we will apply Theorem 2.3 with (here $R > 1$ will be chosen later)

$$M = 1, \qquad \psi(u) = g(u) = u^m, \qquad \epsilon = \frac{1}{2}\left(1 - \frac{\mu}{R n!}\right), \qquad \text{and} \qquad a = \frac{1}{4}.$$

Clearly, (2.2), (2.3), and (2.5) hold. In addition, we have [10, p. 18] that

$$\sup_{t \in [0,1]} \int_0^1 (-1)^{n-p} G(t,s)\, ds = \frac{1}{n!} \sup_{t \in [0,1]} t^p (1-t)^{n-p},$$

so (2.4) is true with $r = 1$ since

$$\frac{\mu M}{n!} = \frac{\mu}{n!} \leq 1 = r \qquad \text{and} \qquad \mu \sup_{t \in [0,1]} \int_0^1 (-1)^{n-p} G(t,s)\, ds \leq \frac{\mu}{n!} \leq 1 = \frac{r}{\psi(r)}.$$

Finally, notice (2.6) is satisfied for R large since

$$\frac{R}{g(\epsilon A_0 R)} = \frac{1}{\epsilon^m A_0^m R^{m-1}} \to 0, \qquad \text{as } R \to \infty.$$

Thus, all the conditions of Theorem 2.3 are satisfied so existence is guaranteed.

Next we consider the (n,p) boundary value problem

$$\begin{aligned} y^{(n)}(t) + \mu f(t, y(t)) &= 0, \qquad 0 < t < 1, \\ y^{(i)}(0) &= 0, \qquad 0 \le i \le n-2, \\ y^{(p)}(1) &= 0, \end{aligned} \tag{2.11}$$

where $n \ge 2$, $1 \le p \le n-1$ is fixed and $\mu > 0$ is a constant. The following two results are well known, see [3] for the first and [10, pp. 21, 85] for the second.

LEMMA 2.4. *Suppose $y \in C^{n-1}[0,1] \cap C^n(0,1)$ satisfies*

$$\begin{aligned} y^{(n)}(t) &\le 0, \qquad \text{for } t \in (0,1), \\ y^{(i)}(0) &= 0, \qquad 0 \le i \le n-2, \\ y^{(p)}(1) &= 0. \end{aligned}$$

Then

$$y(t) \ge t^{n-1} |y|_0, \qquad \text{for } t \in [0,1].$$

LEMMA 2.5. *The boundary value problem*

$$\begin{aligned} y^{(n)}(t) + 1 &= 0, \qquad \text{for } t \in (0,1), \\ y^{(i)}(0) &= 0, \qquad 0 \le i \le n-2, \\ y^{(p)}(1) &= 0, \end{aligned}$$

has a solution w with

$$w(t) \le \frac{1}{(n-1)!(n-p)} t^{n-1}, \qquad \text{for } t \in [0,1].$$

Essentially the same reasoning as in Theorem 2.3 (only obvious adjustments are needed) establishes the following result.

THEOREM 2.6. *Suppose (2.2) and (2.3) hold. In addition, assume the following conditions are satisfied:*

$$\exists\, r \ge \frac{\mu M}{(n-1)!(n-p)}, \qquad \text{with } \frac{r}{\psi(r)} \ge \mu \sup_{t \in [0,1]} \int_0^1 G_1(t,s)\, ds, \tag{2.12}$$

there exists $a \in (0, 1/2)$ (choose and fix it) and a continuous, nondecreasing function $g : (0,\infty) \to (0,\infty)$, with $f(t,u) + M \ge g(u)$ for $(t,u) \in [a,1] \times (0,\infty)$, (2.13)

and

$$\exists\, R > r, \qquad \text{with } \frac{R}{g\left(\epsilon R a^{n-1}\right)} \le \mu \int_a^1 G_1(\sigma, s)\, ds; \tag{2.14}$$

here $\epsilon > 0$ *is any constant (choose and fix it) so that* $1 - \mu M/R(n-1)!(n-p) \geq \epsilon$, G_1 *is the Green's function (see [4,10] for an explicit representation) for*

$$\begin{aligned} -y^{(n)} &= 0, && (0,1), \\ y^{(i)}(0) &= 0, && 0 \leq i \leq n-2, \\ y^{(p)}(1) &= 0, \end{aligned}$$

and $0 \leq \sigma \leq 1$ *is such that*

$$\int_a^1 G_1(\sigma, s)\, ds = \sup_{t \in [0,1]} \int_a^1 G_1(t,s)\, ds.$$

Then (2.11) has a solution $y \in C^{n-1}[0,1] \cap C^n(0,1)$ *with* $y(t) > 0$ *for* $t \in (0,1]$.

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