Whitney's 2-Switching Theorem, Cycle Spaces, and Arc Mappings of Directed Graphs

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We establish a directed analogue of Whitney's 2-switching theorem for graphs and apply it to settle the problem [J. London Math. Soc. (2) 3 (1971), 378–384] of Goldberg and Moon by showing that a strong tournament is uniquely determined, up to isomorphism or anti-isomorphism, by its arc set together with those arc sets that form directed 4-cycles. We obtain the corresponding result for directed Hamiltonian cycles in 10\(^{15}\)-connected tournaments. The proofs are based on investigations of the cycle space of a tournament.

1. Introduction

A fundamental result of Whitney [15] asserts that a 2-connected graph is completely characterized, modulo a series of 2-switchings, by a specification of its edge set together with those edge sets that form cycles. A 2-switching of a 2-connected graph \(G\) can be described as a graph \(G'\) which is obtained from \(G\) as follows: Let \(x_1, x_2\) be two vertices of \(G\) and let \(H\) be a connected component of \(G - \{x_1, x_2\}\). Now \(G'\) is obtained from \(G\) by removing each edge from a vertex \(z\) of \(H\) to \(x_i\) (\(i = 1\) or \(2\)) and putting it back as an edge from \(z\) to \(x_{3-i}\). Now \(G\) and \(G'\) have the same cycles in the sense that an edge set in \(G\) forms a cycle if and only if the corresponding edge set in \(G'\) forms a cycle. (We may think of \(G\) and \(G'\) as graphs whose edges are labelled with the same labels.)

Suppose conversely that \(G\) and \(H\) are 2-connected graphs and that there exists a bijection \(\pi: E(G) \rightarrow E(H)\) such that \(\pi\) and \(\pi^{-1}\) preserve cycles. Then Whitney's result asserts that there exists a series of 2-switchings of \(G\) resulting in \(G'\) such that \(\pi\) (regarded as a bijection of \(E(G')\) onto \(E(H)\)) is induced by an isomorphism of \(G'\) onto \(H\). In other words, \(G'\) and \(H\) are not only isomorphic but even isomorphic regarded as edge-labelled graphs.

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It is well known that the subsets of a given set $E$ may be regarded as a vector space over $GF(2)$ where the sum (which we call the modulo 2 sum) of two subsets is the symmetric difference of the sets. If $E$ is the edge set of a graph, then the subgraph generated by the cycles in $G$ is called the cycle space of $G$. It is easy to see that the cycle space of $G$ consists of all subgraphs of $G$ (or more precisely: the edge sets of those subgraphs) in which all vertices have even degree. Now suppose that $G$ and $H$ are graphs and that there is a bijection $\pi: E(G) \rightarrow E(H)$ and collections $S_G$ and $S_H$ of cycles in $G$ and $H$, respectively, generating the cycle spaces in $G$ and $H$, respectively, such that $\pi(S_G) = S_H$. Then $\pi$ and $\pi^{-1}$ preserve cycles (see Lemma 2.3 below). In particular, if $G$ and $H$ are 3-connected, then they are isomorphic, by Whitney's theorem. So results on generating sets of the cycle space of a graph may be convenient for extending Whitney's theorem. For example, Tutte [14] ([10] contains an alternative proof) showed that the cycle space of a 3-connected graph is generated by its induced non-separating cycles, i.e., those cycles which have no chords and whose vertex-deletion leaves a connected graph, and hence, by the remark above, these cycles determine a 3-connected graph completely up to isomorphism. (This was rediscovered by Kelmans [6].) Results on special types of cycles generating the cycle space in graphs with further constraints on the connectivity or minimum degree are given in [3, 5].

Since the dicycles (directed cycles) in a strong digraph (directed graph) generate the cycle space of the underlying undirected graph (we also refer to this as the cycle space of the digraph; see Lemma 3.2 below) we may obtain results of Whitney type for digraphs. In this paper we shall concentrate on tournaments. Goldberg and Moon [4] proved that, if $T$ and $T'$ are strong tournaments and there exists a bijection $\pi$ of the arc set $E(T)$ onto the arc set $E(T')$ such that $\pi$ and $\pi^{-1}$ preserve 3-dicycles and 4-dicycles, then $T$ and $T'$ are isomorphic or anti-isomorphic. (Here, anti-isomorphic means that $T$ is isomorphic to the converse of $T'$.) They showed that this need not be true if $\pi$ and $\pi^{-1}$ preserve 3-dicycles only, but they asked if it is sufficient that $\pi$ and $\pi'$ preserve 4-dicycles. This problem is also mentioned in the surveys [1, 2]. In this paper we settle that problem by investigating the cycle space of the subdigraph of a tournament which is the union of all 4-dicycles. We show that this cycle space is generated by all 3-dicycles and 4-dicycles. (In general, it will not be generated by the 4-dicycles only.) We also show that the space generated by the $(n-1)$-dicycles in a $10^6$-connected tournament of order $n$ has codimension at most 1, and we apply the proof of this to show that a $10^{15}$-connected tournament is uniquely determined up to isomorphism or anti-isomorphism by its arc set together with those arc sets that form Hamiltonian dicycles.
2. Terminology and Preliminaries

The terminology is the same as in [1] except that a directed path or cycle is called a dipath or dicycle, respectively. When we speak of path and cycle we always refer to an undirected graph or the underlying undirected graph of a digraph. An arc is a directed edge. The edge set of a graph $G$ and the arc set of a digraph $D$ are denoted $E(G)$ and $E(D)$, respectively. A cycle or dicycle with vertices $x_1, x_2, \ldots, x_k$ and edges (or arcs) $x_1x_2, x_2x_3, \ldots, x_kx_1$ is denoted $x_1x_2\cdots x_kx_1$ and is called a $k$-cycle or $k$-dicycle. A $k$-path and a $k$-dipath are defined similarly. When no confusion is possible we shall also refer to the edge set (or arc set) $\{x_1x_2, x_2x_3, \ldots, x_kx_1\}$ as a $k$-cycle or $k$-dicycle and similarly for a path or dipath. If a graph or digraph contains the edge (or arc) $xy$ we say that $xy$ joins $x$ and $y$ and that $x$ and $y$ are the ends of $xy$. In the digraph case we also say that $x$ dominates $y$ and refer to $x$ (respectively $y$) as the tail (respectively head) of the arc $xy$.

Two edges or arcs are independent if they have no end in common. If $D$ is a digraph, then the converse of $D$ is the digraph obtained by replacing each arc $xy$ by $yx$. An isomorphism (respectively anti-isomorphism) of a digraph $D$ onto a digraph $D'$ is a bijection $f$ of the vertex set $V(D)$ onto the vertex set $V(D')$ such that, for any vertices $x$ and $y$ in $D$, the number of arcs from $x$ to $y$ equals the number of arcs from $f(x)$ to $f(y)$ (respectively from $f(y)$ to $f(x)$). In either case, $f$ induces an arc bijection $\pi: E(D) \rightarrow E(D')$ such that $\pi$ and $\pi^{-1}$ preserve dicycles (i.e., each of $\pi$ and $\pi^{-1}$ maps each arc set of a dicycle onto the arc set of a dicycle).

A digraph $D$ is strong if, for any vertices $x$ and $y$, $D$ has a dipath from $x$ to $y$ and we say that $D$ is $k$-connected if the deletion of any set of fewer than $k$ vertices leaves a strong digraph. Similarly, an undirected graph is $k$-connected if the deletion of any set of fewer than $k$ vertices leaves a connected graph. A 2-connected undirected graph is called a block. A digraph is $k$-connected in the undirected sense if its underlying undirected graph is $k$-connected. A block of a digraph is a block of the underlying undirected graph.

In the Introduction we defined a 2-switching of an undirected graph. We define a directed 2-switching of a digraph $D$ in a similar way: Let $x_1, x_2$ be two vertices of $D$ and let $H$ be a connected component (in the undirected sense) of $D = \{x_1, x_2\}$. Then we remove each arc $zx_i$ (respectively $x_iz$) where $z \in V(H)$ and put it back as $x_3-iz$ (respectively $zx_3-i$) and then we reverse the direction of all arcs of $H$. The resulting digraph $D'$ is called a directed 2-switching of $D$. Clearly $D$ and $D'$ have the same cycles and the same dicycles in the sense that an arc set in $D$ forms a dicycle if and only if the corresponding arc set in $D'$ does. Also, $D'$ is strong iff $D$ is strong and $D'$ is 2-connected in the undirected sense iff $D$ is 2-connected in the undirected sense.
A tournament is a digraph such that any two distinct vertices are joined by precisely one arc. There are precisely two non-isomorphic tournaments on three vertices: the 3-dicycle and the transitive triple. If \( C : x_1 x_2 \cdots x_k x_1 \) is a dicycle in a tournament and \( z \) is a vertex not in \( C \) such that \( z \) is not dominated by all vertices of \( C \) and \( z \) does not dominate all vertices of \( C \), then there exists a dicycle of the form \( C' : x_1 x_2 \cdots x_j z x_{i+1} \cdots x_k x_1 \). We say that \( C' \) is an augmentation of \( C \) at the arc \( x_ix_{i+1} \). It is well known (and easy to see using augmentations if possible) that a strong tournament on \( n \) vertices has dicycles of all lengths 3, 4, ..., \( n \). In particular, if \( n \geq 4 \) then it has a vertex whose deletion leaves a strong tournament. We define \( Q_n \) as the tournament consisting of a dipath \( x_1 x_2 \cdots x_n \) and all arcs \( x_ix_j \) where \( i > j + 1 \) and get the following characterization of \( Q_n \):

**Proposition 2.1.** A strong tournament \( T \) on \( n \) vertices has three vertices \( y_1, y_2, y_3 \) such that \( T - y_i \) is strong for \( i = 1, 2, 3 \) unless \( T \) is isomorphic to \( Q_n \).

**Proof (by induction on \( n \)).** The statement is easily verified for \( n \leq 4 \) so assume \( n \geq 5 \) and let \( y_1 \) be a vertex such that \( T - y_1 \) is strong. If \( T - y_1 \) is not isomorphic to \( Q_{n-1} \), then it has three vertices \( y_2, y_3, y_4 \) such that \( T - \{ y_1, y_i \} \) is strong for each \( i = 2, 3, 4 \). Then for at least two \( i \) in \( \{ 2, 3, 4 \} \), say \( i = 2, 3 \), \( T - y_i \) is strong and the result follows. On the other hand, if \( T - y_1 \) is isomorphic to \( Q_{n-1} \), then \( T - \{ y_1, x_1 \} \) and \( T - \{ y_1, x_{n-1} \} \) are both strong and now one of \( T - x_1, T - x_2 \) and one of \( T - x_{n-2}, T - x_{n-1} \) are strong unless \( T \) is isomorphic to \( Q_n \).  

The same method of proof easily gives the following result which is a special case of a result of Las Vergnas [7]:

**Proposition 2.2 (Las Vergnas [7]).** A strong tournament on \( n \) vertices has at least \( n - 2 \) 4-dicycles unless it is isomorphic to \( Q_n \).

If \( T \) is a tournament on \( n \) vertices and \( k \) is a natural number, \( 3 \leq k \leq n \), then we denote by \( T(k) \) the subdigraph of \( T \) which is the union of all \( k \)-dicycles of \( T \). The above method shows that, if \( T \) is strong, then \( T(k) \) is strong and 2-connected in the undirected sense for \( k = 3, 4, \ldots, n \). If \( G \) is a graph, then the even cycle space of \( G \) consists of those edge sets in the cycle space having an even number of edges. It is easy to see that, if \( G \) is 2-connected, then the even cycle space is generated by the even cycles. Moreover, if \( G \) is connected and has \( n \) vertices and \( m \) edges, then the cycle space has dimension \( m - n + 1 \). If, in addition, \( G \) is not bipartite, then the even cycle space has dimension \( m - n \). We shall also use the following observation:
LEMMA 2.3. Suppose $G$ and $H$ are graphs and $S_G$ and $S_H$ are sets of cycles generating the cycle spaces of $G$ and $H$, respectively. Suppose further that $\pi : E(G) \to E(H)$ is a bijection such that $\pi(S_G) = S_H$. Then $\pi$ and $\pi^{-1}$ preserve cycles.

Proof. If $C$ is a cycle in $G$, then $C = C_1 + C_2 + \cdots + C_k$ (modulo 2) where $C_i \in S_G$ for $i = 1, 2, \ldots, k$. Now

$$\pi(C) = \pi(C_1) + \cdots + \pi(C_k) \pmod{2}$$

and so $\pi(C)$ belongs to the cycle space of $H$. Then $\pi(C)$ is the union of pairwise edge-disjoint cycles $C'_1, C'_2, \ldots, C'_m$ of $H$. Similarly, $\pi^{-1}(C'_1)$ belongs to the cycle space of $G$ and hence $m = 1$. So $\pi$ preserves cycles and, by a similar argument, $\pi^{-1}$ preserves cycles.

We shall also make use of the following observation:

LEMMA 2.4. Let $S_G$ be a set of cycles of a 2-connected graph $G$. If $S_G$ generates the cycle space of $G$ and the edges of $G$ are coloured in precisely two colours, say 1 and 2, then $S_G$ contains a cycle which is not monochromatic.

Proof. Since $G$ is 2-connected it has a cycle $C$ whose edges have both colours. Then

$$C = C_1 + C_2 + \cdots + C_k \pmod{2},$$

where each $C_i$ belongs to $S_G$. If each $C_i$ is monochromatic we can assume that $C_i$ has colour 1 for $i = 1, 2, \ldots, m$ and colour 2 for $i = m + 1, \ldots, k$. We must have $1 \leq m \leq k - 1$. Now $C_1 + C_2 + \cdots + C_m$ is a proper non-empty subgraph of $C$ and it is the edge-disjoint union of cycles, a contradiction.

A hypergraph is a set (called the vertex set) together with a collection of subsets (called hyperedges). A hypergraph is connected if, for any partition of its vertex set into two non-empty parts, there is a hyperedge intersecting each part. If $G$ is a graph we may regard its cycles as a hypergraph whose vertices are the edges of $G$ and whose hyperedges are the edge-sets of the cycles of $G$. We call this the cycle-hypergraph of $G$. If $G$ is 2-connected, then Lemma 2.4 asserts that any generating set of cycles forms a connected subhypergraph of the cycle-hypergraph. Our graphs and digraphs have no loops but may contain multiple edges and parallel arcs. All graphs or digraphs are finite unless otherwise stated.
3. The Directed Analogue of Whitney's 2-Switching Theorem

C. Berge (private communication) has observed that the arcs of any digraph can be coloured in two colours such that no dicycle is monochromatic (just arrange the vertices in a linear order and colour the arcs from right to left by one of the colours). As a counterpart to that we have:

**Lemma 3.1.** Let $D$ be a strong digraph whose arcs are coloured in two colours, say 1 and 2. Then $D$ contains a dicycle which is not monochromatic unless each block of $D$ is monochromatic.

Lemma 3.1 is an immediate consequence of Lemma 2.4 combined with the following lemma in [8]:

**Lemma 3.2.** The dicycles of a strong digraph generate the cycle space.

We can now prove the directed analogue of Whitney's 2-switching theorem. If $D'$ is a directed 2-switching of $D$, then clearly $D$ and $D'$ have the same dicycles. Conversely, we have:

**Theorem 3.3.** Let $D$ and $F$ be two strong digraphs with only one block. Let $S_D$ and $S_F$ be sets of dicycles generating the cycle spaces of $D$ and $F$, respectively, and let $\pi: E(D) \rightarrow E(F)$ be a bijection such that $\pi(S_D) = S_F$. Then there exists a series of directed 2-switchings of $D$ resulting in a digraph $D'$ such that $\pi$ is induced by an isomorphism or anti-isomorphism of $D'$ onto $F$.

**Proof.** Let $G$ and $H$ denote the underlying undirected graphs of $D$ and $F$, respectively. By Lemma 2.3, $\pi$ and $\pi^{-1}$ preserve cycles and hence, by Whitney's result, there exists a series of 2-switchings of $G$ resulting in a graph $G'$ such that $\pi$ is induced by an isomorphism $f$ of $G'$ onto $H$. For each of these 2-switchings we consider the corresponding directed 2-switching and thereby transform $D$ into a strong digraph $D'$ with only one block. We claim that the isomorphism of $G'$ onto $H$ corresponds to an isomorphism or an anti-isomorphism of $D'$ onto $F$. We assign colour 1 to those arcs $xy$ of $D'$ which are reversed by $\pi$ (i.e., $\pi(xy) = f(y)f(x)$) and colour 2 to those which are preserved by $\pi$. We claim that $D'$ is monochromatic. For otherwise, $D'$ has by Lemma 2.4, a dicycle $C$ in $S_D$ which is not monochromatic. This means that the corresponding cycle in $F$ (i.e., the cycle with arc set $\pi(E(C))$) is not a dicycle. But $\pi(S_D) = S_F$ consists of dicycles. This contradiction proves that $D'$ is monochromatic and hence that $\pi$ is induced by an isomorphism or an anti-isomorphism of $D'$ onto $F$.  $\blacksquare$
If $S_D$ and $S_F$ consist of all dicycles of $D$ and $F$, respectively, then Theorem 3.3 becomes the directed analogue of Whitney’s 2-switching theorem. As a consequence of Theorem 3.3 we also have:

**Corollary 3.4.** Let $D$ and $F$ be strong digraphs and let $S_D, S_F$ be sets of dicycles generating the cycle space of $D$ and $F$, respectively. If $\pi : E(D) \to E(F)$ is a bijection such that $\pi(S_D) = S_F$, then $\pi$ and $\pi^{-1}$ preserve dicycles.

**Proof.** By Lemma 2.3, $\pi$ and $\pi^{-1}$ preserve cycles and hence they preserve blocks and so we can apply Theorem 3.3 to each block of $D$ and $F$. \[\blacksquare\]

Let us now consider a digraph $D$ and assume that our only information about $D$ is its arc set and the collection of those arc sets that form dicycles. We consider the question to what extent $D$ is determined by this information. We shall assume that all arcs are contained in dicycles. The blocks of $D$ are the connected components of the cycle-hypergraph of $D$ and by Lemma 3.1, these components are determined by those hyperedges that are the dicycles of $D$. By Theorem 3.3, the blocks are uniquely determined modulo directed 2-switchings and reversal of all arcs in a block. Of course we have no information of how the blocks are pasted together. Christoph Reutenauer (private communication) asked the following question: Let $D$ be a digraph with vertices $v_1, v_2, \ldots, v_n$. Regard each arc of $D$ as a free variable and let $M$ denote the $n \times n$ matrix whose $ij$th entry is the sum of variables corresponding to the arcs from $v_i$ to $v_j$. To what extent is $D$ determined by $\det(I - M)$? A typical term in the expansion of this determinant is the product of arcs that form a collection of pairwise vertex-disjoint dicycles in $D$. Thus the determinant gives the list of dicycles and pairs of vertex-disjoint dicycles. We assume that all arcs are in dicycles and by the remarks of the preceding paragraph, $\det(I - M)$ tells us which arc sets constitute blocks. Furthermore, these blocks are uniquely determined by $\det(I - M)$ up to directed 2-switchings and arc reversals of all arcs in a block. Moreover, since $\det(I - M)$ tells us which arc sets form a pair of disjoint dicycles, $\det(I - M)$ gives some information of how the blocks of $D$ are pasted together. To illustrate this, let us consider the case where $D$ is obtained from two disjoint blocks $D_1, D_2$ by identifying a vertex $v_1$ in $D_1$ with a vertex $v_2$ in $D_2$. Now select another vertex $v'_1$ in $D_1$ and let $D'$ be obtained from $D_1 \cup D_2$ by identifying $v'_1$ and $v_2$. Then $\det(I - M)$ is the same for $D$ and $D'$ if and only if $D_1$ has no dicycle which contains one of $v_1, v'_1$ but not the other. It is easy to generalize this to the case where $D$ has more than two blocks and so the discussion of Reutenauer’s problem is completed by a characterization of the pairs of vertices that are in the same dicycles of a digraph. If $D_1$ and $D_2$ are two disjoint digraphs we can form
the union $D_1 \cup D_2$ and add two new vertices $x, y$ and some arcs from $x$ to $D_1$, some arcs from $D_1$ to $y$, some arcs from $y$ to $D_2$, and some from $D_2$ to $x$ and possibly arcs between $x$ and $y$. We say that $x$ and $y$ form a cyclic pair in the resulting digraph $D$. Clearly, any dicycle in $D$ which contains one of $x, y$ contains both $x$ and $y$. Conversely, we have:

**Proposition 3.5.** If $D$ is a strong digraph containing two vertices $x$ and $y$ such that any dicycle containing one of $x, y$ contains both of $x, y$, then $x$ and $y$ form a cyclic pair in $D$.

*Proof.* Let $D_1$ be the digraph induced by those vertices which in $D - y$ can be reached from $x$ by a dipath and let $D_2$ be the digraph induced by those vertices from which $x$ can be reached by a dipath in $D - y$. Since $D - y$ has no dicycle containing $x$, $D_1$ and $D_2$ are disjoint and there is no arc from $D_1$ to $D_2$. If $a$ is an arc from a vertex $z_1$ not in $D_1$ to a vertex $z_2$ in $D_1$, then $z_1 = x$, for otherwise any dicycle in $D$ containing $a$ would contain $y$ but not $x$. Similarly, a vertex in $D_2$ dominates no vertex outside $D_2$ except possibly $x$. This implies easily that $V(D) = V(D_1) \cup V(D_2) \cup \{x, y\}$ and that $x$ and $y$ are a cyclic pair in $D$. (For, if $z$ is a vertex not in $V(D_1) \cup V(D_2) \cup \{x, y\}$, then $D$ has dipaths $P_1, P_2$ from $y$ to $z$ and from $z$ to $y$, respectively, and $(P_1 \cup P_2) \cap (D_1 \cup D_2 \cup \{x\}) = \emptyset$ so $P_1 \cup P_2$ contains a dicycle through $y$ but not $x$, a contradiction).  

4. Cycle Spaces Generated by Small Dicycles in Tournaments

We first show that a strong tournament is uniquely determined by its subdigraph $T(4)$ consisting of all 4-dicycles.

**Theorem 4.1.** Let $T$ and $T'$ be strong tournaments of order $n$ defined on the same vertex set. If $T(4) = T'(4)$ and $n \geq 5$, then $T = T'$ unless $T$ and $T'$ are isomorphic to $Q_5$, and $T'$ is obtained from $T$ by reversing the direction of the arcs $x_3x_1, x_5x_3, x_5x_1$ (using the same notation as in Sect. 2).

*Proof (by induction on $n$).* Assume first that $n = 5$ and let $C: y_1, y_2, y_3, y_4, y_1$ be a 4-dicycle of $T$ and $T'$ and let $y_5$ be the vertex not in $C$. As $T$ is strong, $y_4$ is on some 4-dicycle. If $T$ has a 4-dicycle with only one arc in common with $C$, we can assume that it is of the form $C': y_1, y_2, y_4, y_5, y_1$. Now any pair of vertices which is not joined by an arc in $C \cup C'$ is connected by a dipath of length 3 in $C \cup C'$ and hence $T = T'$. On the other hand, if $T$ has no 4-dicycle $C'$ as above, then $T$ has a 4-dicycle of the form $C'': y_1, y_2, y_3, y_5, y_1$. If the arc between $y_4$ and $y_5$ has the same direction in $T$ and $T'$, then $T = T'$ as above so assume that $T$ contains $y_5y_4$ and $T'$ contains $y_4y_5$. If $T$ contains one of $y_4y_2, y_1y_3, y_2y_5$, then $T$ has a
4-dicycle which is not in T' so T contains y_2 y_4, y_3 y_1, and y_5 y_2. A similar argument shows that T' contains y_4 y_2, y_3 y_1, and y_2 y_5 and now T and T' are as described in Theorem 4.1 (where y_1 = x_2, y_2 = x_3, y_3 = x_4, y_4 = x_1, y_5 = x_5). Assume next that n ≥ 6 and let u, v be vertices of T such that T - u and T - v are strong. If T - u = T' - u and T - v = T' - v, then all arcs of T are in T' except possibly the arc between u and v. By [9, Theorem 3.2] and its succeeding remark, T has dipaths of all lengths 3, 4, ..., ⌊(n+1)/2⌋ connecting u and v. In particular, T has a 3-dipath from u to v or from v to u and since this dipath is also in T', we have T = T'. So we can assume that T - u ≠ T' - u. By the induction hypothesis, T - u consists of a dipath x_1 x_2 x_3 x_4 x_5 and all arcs x_ix_j, i ≥ j + 2, and T' - u is obtained from T by reversing the direction of x_3 x_1, x_5 x_3, and x_5 x_1. If T has a dipath of the type x_i u x_{i+1}, 1 ≤ i ≤ 4, then the 4-dicycle x_i u x_{i+1} x_{i+2} x_i or x_{i-1} x_i u x_{i+1} x_{i+1} is present in T but not T', a contradiction. So there is a k ∈ {1, 2, 3, 4} such that u dominates x_1, ..., x_k and is dominated by x_{k+1}, ..., x_5. Assume without loss of generality that k ≤ 2. Then T and T' contain the 4-dicycle u x_1 x_2 x_3 u. Since this is the only 4-dicycle in T containing u x_1, u must dominate x_4 in T' and a similar argument shows that x_5 dominates u in T'. Now T'(4) contains the 4-dicycle u x_4 x_1 x_5 u which is not in T(4). This contradiction proves Theorem 4.1.

Since a strong tournament T may contain many arcs whose reversal does not change T(3), Theorem 4.1 becomes false if we replace T(4) by T(3), even for n large. It is clear that the dicycles of length 3 or 4 in a strong tournament need not generate the cycle space of the tournament T since there may be many arcs not contained in small dicycles (as is the case for Q_n). Also, the 4-dicycles need not generate the cycle space of T(4) since T(4) may contain odd cycles. However, we have the following:

**Theorem 4.2.** If T is a strong tournament of order at least 4, then the dicycles of length 3 and 4 in T(4) generate the cycle space of T(4).

**Proof.** Since T(4) is strong it is sufficient, by Lemma 3.2, to show that any dicycle C of T(4) is the modulo 2 sum of 3-dicycles and 4-dicycles of T(4). We prove this by contradiction assuming that C: x_1 x_2 ··· x_k x_1 is a smallest counterexample (the indices are expressed modulo k). Clearly, k ≥ 5. Let T' be the subtournament of T with vertex set V(C). We shall derive some properties of T' and obtain a contradiction.

1. T' has no 4-dicycle with two consecutive arcs not in C.

**Proof of (1).** Suppose first that T' has a 4-dicycle containing two arcs x_i x_j, x_i x_j with 3 ≤ i ≤ j ≤ k - 1. Then C ∪ {x_i x_j} contains three dicycles whose modulo 2 sum equals C and whose lengths are smaller than
By the minimality of \( k \), these three dicycles are the modulo 2 sum of 3-dicycles and 4-dicycles in \( T(4) \) and so is \( C \), a contradiction.

Suppose next that \( T' \) has a 4-dicycle of the form \( x_1 x_j x_r x_1 \) where \( r < j < i \). For each arc \( a \) of this 4-dicycle, \( C \cup \{a\} \) contains a dicycle of length smaller than \( k \) and the modulo 2 sum of those four dicycles and \( x_1 x_j x_r x_1 \) equals \( C \). As above we obtain a contradiction.

Now if (1) fails and none of the two above cases occur, then \( T' \) has a 4-dicycle of the form \( x_1 x_j x_r x_{j+1} x_1 \) where \( 1 < j < i - 1 \). (Since the second case does not occur, \( T' \) has a 4-dicycle of the form \( x_j x_r x_1 x_i x_j \) where \( 1 < j < r \leq k \). Since the first case does not occur \( C \) contains one of \( x_j x_r, x_r x_1 \), say the former. Since the first case does not occur, \( j + 1 < i \leq k \).) For each arc \( a \in \{x_1 x_i, x_i x_j, x_{j+1} x_1\} \), \( C \cup \{a\} \) has a dicycle of length smaller than \( k \). The modulo 2 sum of those three dicycles and \( x_1 x_j x_r x_{j+1} x_1 \) equals \( C \). As above we obtain a contradiction which proves (1).

(2) \( \text{Any 4-dicycle of } T' \text{ contains three arcs of } C. \)

Proof of (2). If (2) were false, then by (1), \( T' \) contains a dicycle of the form \( x_1 x_2 x_j x_{i+1} x_1 \) where \( 4 \leq i \leq k - 2 \). Now \( C \cup \{x_2 x_i\} \) and \( C \cup \{x_{i+1} x_1\} \) each contain a dicycle of length smaller than \( k \) and the modulo 2 sum of these two dicycles and \( x_1 x_2 x_j x_{i+1} x_1 \) equals \( C \). As in the proof of (1) we obtain a contradiction which proves (2).

(3) \( T' \) has at most \( k - 3 \) distinct 4-dicycles.

Proof of (3). All 4-dicycles of \( T' \) are of the form \( x_i x_{i+1} x_{i+2} x_{i+3} x_i \) and hence \( T' \) has at most \( k \) 4-dicycles. If \( k - 2 \) arcs of the form \( x_{i+3} x_i \) are present, then \( k \neq 6 \) and \( T' \) contains a dicycle \( C' \) which has length smaller than \( k \) and which contains only arcs from \( C \) and arcs of the form \( x_{i+3} x_i \). (To see this we consider the three paths or cycles \( x_1 x_{k-2} x_{k-5} \cdots x_r, x_k x_{k-3} \cdots x_{r-1}, x_{k-1} x_{k-4} \cdots x_{r-2} \) where \( r \in \{1, 2, 3\} \). Since these are arc-disjoint one of them is a dipath or dicycle and can be extended into the desired dicycle \( C' \) by adding a dipath of length at most 3.) For each arc \( a \) in \( E(C') \setminus E(C) \) we let \( C_a \) denote the unique cycle of \( C \cup \{a\} \) containing \( a \). Now the modulo 2 sum of \( C' \) and all \( C_a \) is non-empty and contained in \( C \). Hence it equals \( C \) and, as in the proof of (1), we obtain a contradiction. This proves (3).

By (3) and Proposition 2.2, \( T' \) is isomorphic to \( Q_k \). In other words, we can assume without loss of generality that

(4) \( x_i \) dominates \( x_j \) whenever \( 1 \leq j \leq i - 2 \leq k - 2 \).

(5) \( T \) has no dipath of length 2 from \( x_1 \) to \( x_{k-1} \).
Proof of (5). If $T$ has a dipath $x_1z_kx_{k-1}$, and $k=5$, then $C$ is the modulo 2 sum of the dicycles $x_1z_kx_{k-1}x_kx_1$, $x_1x_2x_3x_4x_1$, and $x_1z_kx_{k-1}x_1$ which are all in $T(4)$ (since the third is in the union of the two first 4-dicycles). On the other hand, if $k \geq 6$, then $C$ is the modulo 2 sum of $x_1z_kx_{k-1}x_kx_1$, $x_1z_kx_{k-1}x_3x_1$, $x_1x_2x_3x_1$, and $x_3x_4 \cdots x_{k-1}x_3$. Since these are all in $T(4)$ and have length smaller than $k$ we have obtained a contradiction.

By a similar argument we prove

(6) $T$ has no dipath of length 2 from $x_2$ to $x_k$.

Since $C$ is in $T(4)$, the arc $x_kx_1$ is in a 4-dicycle $x_1z_kx_{k-1}x_1$ and by (5), (6) none of $z_1, z_2$ are in $T'$.

(7) $k \leq 8$, $k \neq 7$.

Proof of (7). If $k \geq 9$ or $k=7$, then $T(4)$ contains the dicycle $C': x_4x_1z_1z_2x_kx_{k-3}x_{k-6} \cdots$ (where $C'$ may contain one or two of the arcs $x_2x_3, x_3x_4$). Then $C$ is the modulo 2 sum of $C'$, the 4-dicycle $x_1z_1z_2x_kx_1$, and those 4-dicycles which contain one arc of $C'$ and three arcs of $C$. Since $C'$ has length smaller than $k$ we have obtained a contradiction which proves (7) and what remains is a finite problem.

(8) $k \neq 8$.

Proof of (8). Suppose $k=8$. We define $C': z_1z_2x_8x_3x_4x_1z_1$. By the proof of (7) we can assume that $C'$ is not the modulo 2 sum of 3-dicycles and 4-dicycles in $T(4)$ and hence, by (4), the subtournament $T''$ of $T$ with vertex set $V(C')$ is isomorphic to $Q_8$. But $Q_8$ does not contain a subdigraph isomorphic to $C' \cup \{x_8x_2, x_3x_1\}$. This contradiction proves that $k \neq 8$.

(9) $k \neq 6$.

Proof of (9). Suppose $k=6$. If $z_1$ dominates $x_2$, then $x_3x_1$ is in a 4-dicycle and now $C$ is the modulo 2 sum of the dicycles $x_1z_1z_2x_6x_3x_1$, $x_1z_1z_2x_6x_1$, $x_1x_2x_3x_1$, $x_3x_4x_5x_6x_3$ all of which are in $T(4)$ and have length smaller than 6, a contradiction. So we can assume that $x_2$ dominates $z_1$ and by a similar argument, $z_2$ dominates $x_5$. Now $C$ is the modulo 2 sum of the four 4-dicycles of the form $x_i z_1z_2x_jx_i$ (where $i \in \{1,2\}$ and $j \in \{5,6\}$) and the dicycles $x_1x_2x_3x_4x_2x_1, x_2x_3x_4x_5x_6x_2$, and $x_2x_3x_4x_5x_2$ each of which is in $T(4)$. This contradiction proves (9).

By (7), (8), (9), $k=5$. We claim that $x_5$ dominates $z_1$. For otherwise $C$ is the modulo 2 sum of $x_1z_1x_5x_3x_1, x_1z_1x_5x_1, x_1x_2x_3x_1, x_3x_4x_5x_3$. Also $x_3$
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dominates \( z_1 \). For otherwise, \( C \) is the modulo 2 sum of \( x_5 z_1 x_3 x_4 x_5, x_1 z_1 z_2 x_5 x_1, x_1 z_1 x_3 x_4 x_1, z_1 z_2 x_5 z_1, x_1 x_2 x_3 x_4 x_1 \). So we have shown that \( x_3 \) dominates \( z_1 \) and by similar arguments we can show that \( z_2 \) dominates \( x_1 \) and \( x_3 \). But now \( C \) is the modulo 2 sum of the dicycles \( z_1 z_2 x_5 x_3 z_1, z_1 z_2 x_3 x_1 z_1, z_1 z_2 x_3 z_1, x_1 z_1 z_2 x_5 x_1, x_1 x_2 x_3 x_1, x_3 x_4 x_5 x_3 \) all of which are in \( T(4) \). This contradiction proves the theorem.

The tournament \( Q_n \) shows that \( T(4) \) need not be 3-connected in the undirected sense. However, if we delete a separating set of two vertices from \( Q_n(4) \), then one of the components (in the undirected sense) of the resulting digraph has just one vertex. The next lemma shows that this holds in general.

**Lemma 4.3.** If \( T \) is a strong tournament and \( \{x, y\} \) is a separating vertex set of \( T(4) \) (in the undirected sense) then \( T(4) \setminus \{x, y\} \) has precisely two connected components (in the undirected sense) one of which has precisely one vertex unless \( T \) is isomorphic to \( Q_5 \).

**Proof (by induction on \( n = |V(T)| \).** The statement is easily verified for \( n = 4, 5 \) and for \( T \) isomorphic to \( Q_n \), so we can assume that \( n > 5 \). By Proposition 2.1, \( T \) has three vertices \( z_1, z_2, z_3 \) such that \( T - z_i \) is strong for \( i = 1, 2, 3 \). Since \( (T - z_i)(4) \) is 2-connected in the undirected sense for \( i = 1, 2, 3 \) we have \( \{x, y\} \cap \{z_1, z_2, z_3\} = \emptyset \). If \( T - z_1 \), say, is isomorphic to \( Q_5 \), then it is easy to verify the lemma. (With the same notation as in Proposition 2.1 we can assume that \( z_1 \) dominates \( x_3 \) since otherwise we replace \( T \) by its converse. If \( T \) contains one of the arcs \( x_1 z_1, x_5 z_1 \), then that arc and \( z_1 x_3 \) are in \( T(4) \) and the only possible separating sets of \( T(4) \) in the undirected sense are \( \{x_1, x_3\}, \{x_2, x_4\}, \{x_3, x_4\} \) and the lemma follows. On the other hand, if \( T \) contains both \( z_1 x_1 \) and \( z_1 x_5 \), then \( T \) contains one of \( x_2 z_1, x_4 z_1 \) since \( T \) is strong. That arc and \( z_1 x_3 \) are in \( T(4) \) and as above we complete the proof.) So we can assume by the induction hypothesis that \( (T - z_1)(4) \setminus \{x, y\} \) has precisely two connected components one of which consists of one vertex, say \( x_1 \).

If \( z_1 \) is adjacent in \( T(4) \) to some vertex distinct from \( x, y, x_1 \), then the proof is complete so assume that \( z_1 \) is adjacent to no such vertex. Assume without loss of generality that \( x_1 \neq z_2 \). As above we conclude that \( (T - z_2)(4) \setminus \{x, y\} \) has precisely two connected components (in the undirected sense) and that one of these must have precisely one vertex \( x_2 \). This implies that \( n = 6 \) and that \( z_1 \) and \( x_1 \) are adjacent in \( (T - z_2)(4) \) and that \( z_2 \) and \( x_2 \) are adjacent in \( (T - z_1)(4) \). Since \( T(4) \) contains the arc between \( z_1 \) and \( x_1 \) and the arc between \( z_2 \) and \( x_2 \), the notation can be chosen such that \( T(4) \) contains the 4-dicycles \( x v_1 u_2 x \) and \( x v_1 v_2 x \) where \( \{u_1, u_2\} = \{z_1, x_1\}, \{v_1, v_2\} = \{z_2, x_2\}, \) and \( V(T) = \{x, y, u_1, u_2, v_1, v_2\} \). Since \( \{x, y\} \) separates \( T(4) \) in the undirected sense, the dipaths \( v_2 xy u_1 \) and \( u_2 xy v_1 \) are not in
4-dicycles and hence $T$ contains the arcs $v_2u_1$ and $u_2v_1$. But then the 4-dicycle $v_2u_1u_2v_1v_2$ shows that $T(4)$ is 3-connected in the undirected sense. This contradiction proves Lemma 4.3.

Consider now a digraph $D$ with vertices $x_1, x_2$ such that $D - \{x_1, x_2\}$ has precisely two components (in the undirected sense) $D_1, D_2$. Suppose furthermore that $D_1 = \{z\}$ and that $z$ is incident with the arcs $x_1z$ and $zx_2$ only. The 2-switching corresponding to $D_1$ is simply an interchange between the arcs $x_1z$ and $zx_2$. The 2-switching corresponding to $D_2$ may be thought of as the 2-switching corresponding to $D_1$ followed by the reversal of all arcs when we think of $D$ as an arc-labelled but not vertex-labelled digraph. If we also ignore the arc labels, then one of the 2-switchings above leaves $D$ unchanged and the other replaces $D$ by its converse. By Lemma 4.3, this is also what happens if we perform a 2-switching on $T(4)$ when $T$ is a strong tournament. Combining Lemma 4.3, Theorem 4.2, Theorem 4.1, and Theorem 3.3 we therefore get the following extension of the result of Goldberg and Moon [4]. (As usual, a transposition is a permutation which permutes two elements only.)

**Theorem 4.4.** Let $T$ and $T'$ be strong tournaments and $\pi$ a bijection $E(T) \to E(T')$ such that $\pi$ and $\pi^{-1}$ preserve 4-dicycles and those 3-dicycles which are in $T(4)$ and $T'(4)$. Then $T$ and $T'$ are isomorphic or anti-isomorphic. Moreover, the restriction of $\pi$ to $T(4)$ is induced by an isomorphism or anti-isomorphism of $T(4)$ onto $T'(4)$ possibly followed by a series of transpositions of arcs incident with vertices of indegree and out-degree 1 in $T'(4)$.

In the result of Goldberg and Moon it was assumed that $\pi$ and $\pi^{-1}$ preserve all 3-dicycles and 4-dicycles. ($Q_n$ shows that this is a considerably stronger condition.) In that case the restriction of $\pi$ to $T(4) \cup T(3)$ is induced by an isomorphism or anti-isomorphism since it is easily proved by induction on the number of vertices in $T$ that $T(3) \cup T(4)$ is 3-connected in the undirected sense.

Goldberg and Moon [4] mentioned as unsolved the following extension of the first part of Theorem 4.4.

**Theorem 4.5.** If $T$ and $T'$ are strong tournaments and $\pi : E(T) \to E(T')$ is a bijection such that $\pi$ and $\pi^{-1}$ preserve 4-dicycles, then $T$ and $T'$ are isomorphic or anti-isomorphic.

Theorem 4.5 becomes false if we replace “4-dicycles” by “3-dicycles.” This can easily be seen as follows. Let $T_1, T_2, T_3$ be three strong tournaments of order $n$ such that $T_1, T_2, T_3$, and the three converse tournaments are all nonisomorphic. Now let $T$ denote the tournament obtained from the dis-
joint union $T_1 \cup T_2 \cup T_3$ by adding all arcs from $T_1$ to $T_2$, from $T_2$ to $T_3$, and from $T_3$ to $T_1$ and let $T'$ be obtained from $T$ by reversing all arcs of $T_1$ in $T$. Then there exists a bijection $\pi: E(T) \rightarrow E(T')$ such that $\pi$ and $\pi^{-1}$ preserve 3-dicycles but $T$ and $T'$ are not isomorphic or anti-isomorphic.

In the above tournament $T$ we can choose $T_1$, $T_2$, $T_3$ such that all arcs of $T$ are in 3-dicycles. Yet, the subhypergraph of the cycle-hypergraph which consists of the 3-dicycles of $T$ is disconnected. The next result shows that this cannot happen for 4-dicycles.

**Proposition 4.6.** If $T$ is a strong tournament and the arcs of $T(4)$ are coloured in precisely two colours, then $T$ has a 4-dicycle which is not monochromatic.

**Proof (by contradiction).** By Theorem 4.2 and Lemma 2.4, we can assume that $T(4)$ contains a 3-dicycle $C$ which is not monochromatic. If $C$ has no augmentation, then we define $A$ (respectively $B$) as the set of vertices dominated (in $T$) by (respectively dominating) each vertex of $C$ and since $T$ is strong, $T$ has an arc $e_1$ from $A$ to $B$. Now $C$ has an arc $e_2$ whose colour is different from that of $e_1$, and $T$ has a 4-dicycle through $e_1$ and $e_2$.

So assume that $C: xyzx$ has an augmentation $C_1: xyruz$. Since $C$ (and hence $zx$) is in $T(4)$, $T$ has a 4-dicycle $C_2: zxpqz$. Since $C$ is not monochromatic we can assume that all arcs of $C_i$ have colour $i$ for $i = 1, 2$. In particular, $y \neq p, q$. If $T$ contains one of the arcs $xq$, $py$ we get the desired 4-dicycle so assume that $T$ contains $qx$, $yp$. Then the 4-dicycle $ypqxy$ has two colours, a contradiction.

One can also give a direct (inductive) proof of Proposition 4.6 by using Proposition 2.1.

With the aid of Theorem 4.4 we shall prove Theorem 4.5 but first we establish analogues of Theorems 4.2 and 4.4 for long cycles in tournaments.

### 5. Cycle Spaces Generated by Long Cycles in Tournaments

**Proposition 5.1.** If $T$ is a tournament, then the transitive triples of $T$ together with the Hamiltonian dicycles of $T$ generate the cycle space of $T$.

**Proof.** The cycle space of $T$ is clearly generated by the 3-cycles, i.e., the transitive triples together with the 3-dicycles. If $T$ is not strong, then, for any 3-dicycle $xyzx$ in $T$, there is a vertex $u$ which either dominates or is dominated by each of $x$, $y$, $z$ and hence $xyzx$ is the modulo 2 sum of the three transitive triples containing $u$ and an arc of $xyzx$. So, each 3-cycle is
generated by the transitive triples, and consequently the cycle space is generated by the transitive triples. So, we can assume that $T$ is strong. By Lemma 3.2, it is sufficient to prove that every dicycle $C$ in $T$ is the modulo 2 sum of transitive triples and Hamiltonian dicycles in $T$. We do this by induction on $|V(T)| - |V(C)|$. Assume $|V(C)| < |V(T)|$ and let $v$ be a vertex not in $C$. If $v$ dominates (or is dominated by) all vertices of $C$ we argue as above so assume this is not the case. Then $T$ has a dicycle $C'$ which is an augmentation of $C$ and hence $E(C) = E(C') + (E(C') + E(C))$, where the latter term is a transitive triple. Proposition 5.1 now follows by induction.

The next result is a counterpart of Theorem 4.2.

**Theorem 5.2.** If $T$ is a 4-connected tournament of order $n$, then the cycle space of $T$ is generated by the dicycles of length $n$ and $n - 1$ of $T$.

**Proof.** Let $xy$, $yz$, $xz$ denote any transitive triple of $T$. Now $T - y$ is 3-connected and contains therefore, by [9, Corollary 5.3], a Hamiltonian dicycle $C$ through $xz$. Now $\{xy, yz, xz\} = E(C) + [E(C) + \{xy, yz, xz\}]$ and the term in the bracket is the arc set of a Hamiltonian dicycle of $T$. Hence each transitive triple is in the space generated by the dicycles of length $n$ and $n - 1$ and Theorem 5.2 now follows from Proposition 5.1.

The tournaments described after Theorem 4.5 show that the transitive triples need not generate the cycle space even if the tournament has large connectivity since a 3-dicycle with one vertex in each of $T_1$, $T_2$, $T_3$ is not a modulo 2 sum of transitive triples. (Any modulo 2 sum of transitive triples contains an even number of arcs from $T_1$ to $T_2$.) Also, it is easy to see that there is no tournament of order larger than 3 in which the Hamiltonian dicycles generate the cycle space (not even the even cycle space) because the modulo 2 sum of an even (respectively odd) number of Hamiltonian dicycles is a subdigraph whose vertices all have even (respectively odd) indegree and outdegree. So Proposition 5.1 is in a sense best possible. However, for tournaments of large connectivity we can strengthen Theorem 5.2 using the result in [12] that there exists a function $f(k)$ ($k$ being a natural number) such that any $k$-independent arcs in an $f(k)$-connected tournament are contained in a Hamiltonian dicycle. It was shown in [12] that $f(1) = 3$, $f(2) < 5 \cdot 10^7$, and $f(3) < 4 \cdot 10^{14}$. With this notation we have:

**Theorem 5.3.** If $T$ is an $(f(2) + 2)$-connected tournament of order $n$, then the dicycles of length $n - 1$ generate a subspace $S$ of codimension at most one in the cycle space of $T$. If $n - 1$ is even, then $S$ is the even cycle space of $T$. 
Proof. Let $S$ denote the subspace of the cycle space generated by the
dicycles of length $n - 1$. We first show that $S$ contains the modulo 2 sum of
any two transitive triples $\{x_1, y_1, z_1, x_1z_1\}, \{x_2, y_2, z_2, x_2z_2\}$ such that
$x_1, y_1, z_1, x_2, y_2, z_2$ are all distinct. Since $T - \{y_1, y_2\}$ is $f(2)$-connected it
has a Hamiltonian dicycle $C$ through $x_1z_1$ and $x_2z_2$. We can now augment
$C$ to two $(n - 1)$-dicycles whose modulo 2 sum is the modulo 2 sum of the
above transitive triples.

We next show that $S$ contains the modulo 2 sum of any two transitive
triples $R_1, R_2$. There exists a transitive triple $R_3$ which is (vertex) disjoint
from both $R_1$ and $R_2$ and now we have

$$R_1 + R_2 = (R_1 + R_3) + (R_2 + R_3)$$

which is contained in $S$.

If $C$ is any $(n - 3)$-dicycle of $T$, then $C$ can be augmented into an $(n - 2)$-
dicycle which can be augmented into an $(n - 1)$-dicycle $C'$. In other words,
there are two transitive triples $R_1, R_2$ such that

$$C = C' + R_1 + R_2 \quad \text{(modulo 2)}$$

and hence any $(n - 3)$-dicycle belongs to $S$. Consider now any transitive
triple $R = \{xy, yz, xz\}$ and the subtournament $T' = T - \{x, y, z\}$. We apply
Proposition 5.1 to $T'$ and conclude that, for every cycle $C$ in $T'$, either $C$ or
$C + R$ belongs to $S$. This holds for any $R$. Now, if $C$ is any 3-cycle of $T$,
then there exists a transitive triple $R'$ (vertex) disjoint from $C$ and so either
$C$ or $C + R'$ belongs to $S$. Since $R + R'$ belongs to $S$ we conclude that
$S \cup (R + S)$ equals the cycle space of $T$ and hence $S$ has codimension at
most 1. If $n - 1$ is even, then $S$ is contained in the even cycle space of $T$ and
since $S$ has codimension 1, $S$ equals the even cycle space. □

By induction on $k$ it follows that Theorem 5.3 remains valid for dicycles
of length $n - k$ ($k \geq 1$) provided that $T$ is $(f(2) + k + 1)$-connected.

For the sake of completeness we mention that Theorem 5.3 has an
analogue for transitive triples. These need not generate the cycle space of a
tournament since a tournament may contain an arc which is in no
transitive triple. Also, the transitive triples do not generate the cycle space
of the (unique) tournament on five vertices in which all vertices have
indegree and outdegree 2. Instead we have the following:

Proposition 5.4. The transitive triples in a tournament generate a
subspace of codimension at most 1 in the cycle space.

Proof. (By Induction on $n = |V(T)|$). If $n \leq 4$, the statement is trivial so
assume that $n > 4$ and let $R = \{xy, yz, xz\}$ be any transitive triple of $T$. By
the induction hypothesis, the transitive triples in $T - y$ generate a subspace of dimension at least $\binom{n - 1}{2} - n + 1$. For any vertex $v$ in $V(T) \setminus \{x, y, z\}$ there is a transitive triple $R_v$ containing the arc between $v$ and $y$ and one of $xy$, $yz$. Since the triples $R, R_v (v \in V(T) - \{x, y, z\})$ are linearly independent (and not contained in the cycle space of $T - y$), the transitive triples in $T$ generate a subspace of dimension at least

$$\binom{n - 1}{2} - n + 1 + n - 2 = \binom{n}{2} - n$$

and therefore have codimension at most 1. 

Theorem 5.2 combined with Theorem 3.3 implies that, if $\pi$ is an arc bijection from one 4-connected tournament onto another 4-connected tournament such that $\pi$ and $\pi^{-1}$ preserve all dicycles of length $n$ and $n - 1$, then $\pi$ is induced by an isomorphism or anti-isomorphism. With a little additional reasoning, Theorem 5.3 and the remark following it imply a similar result on arc bijections preserving dicycles of length $n - k$ in $(f(2) + k)$-connected tournaments. Instead of formulating this we prove a result on Hamiltonian dicycles.

**Theorem 5.5.** Let $T$ and $T'$ be 10-connected tournaments and suppose that $\pi : E(T) \rightarrow E(T')$ is an arc bijection such that $\pi$ and $\pi^{-1}$ preserve Hamiltonian dicycles. Then $\pi$ is induced by an isomorphism or anti-isomorphism of $T$ onto $T'$.

**Proof.** We first show that $\pi$ and $\pi^{-1}$ preserve 3-dicycles. Let $xyzx$ be any 3-dicycle in $T$. Since any two of $xy$, $yz$, $zx$ are in a Hamiltonian dicycle in $T$ the same holds for $\pi(xy)$, $\pi(yz)$, and $\pi(zx)$ in $T'$, and hence these three arcs form a 3-dicycle or a system of dipaths in $T'$. If the latter holds, then the three arcs are contained in a Hamiltonian dicycle in $T'$, by [12, Theorem 4.1] (because $f(3) < 4 \cdot 10^{14}$) but $xy$, $yz$, and $zx$ are not in a common Hamiltonian dicycle in $T$. So $\pi$ (and by symmetry also $\pi^{-1}$) preserves 3-dicycles. (Note that [12, Theorem 4.1] easily generalizes from independent arcs to arcs forming a dipath system since we can just delete all interior vertices in this dipath system and, whenever necessary, reverse the arc from the last vertex to the first vertex of each dipath.) We next show that $\pi$ and $\pi^{-1}$ preserve transitive triples. Let $R = \{x_0y_0, y_0z_0, x_0z_0\}$ be any transitive triple in $T$ and assume (reductio ad absurdum) that $\pi(R)$ is not a transitive triple. As in the previous paragraph, the connectivity of $T$ and $T'$ implies that two arcs of $T$ (respectively $T'$) are in a common Hamiltonian dicycle unless they have a common head or a common tail. Since $\pi$ and $\pi^{-1}$ preserve Hamiltonian dicycles, we conclude that $\pi(x_0z_0)$ and $\pi(x_0y_0)$ have a common tail or head, that $\pi(x_0z_0)$ and $\pi(y_0z_0)$ have a common
head or tail, and that \( \pi(x_0, y_0) \) and \( \pi(y_0, z_0) \) do not have a common tail or head. The assumption that \( \pi(R) \) is not a transitive triple then implies that \( \pi(R) \) is a path \( x'y'z'u' \) such that \( \pi(x_0, z_0) = z'y' \) and \( \{ \pi(x_0, y_0), \pi(y_0, z_0) \} = \{ x'y', z'u' \} \). Since \( T - \{ x_0, z_0 \} \) has connectivity at least 4 it contains four arc-disjoint 3-dicycles \( C_1, C_2, C_3, C_4 \) through \( y_0 \). Let \( C_1 : y_0wxy_0 \). As above, we conclude that \( \pi(uy_0) \) and \( \pi(x,y_0) \) have an end or tail in common, and that \( \pi(y,u) \) and \( \pi(y,z_0) \) have a head or tail in common. Hence \( \pi(C_1) \) (which is a 3-dicycle) contains two vertices of \( \{ x', y', z', u' \} \) and so \( \pi(C_1) \) contains an arc in the subtournament induced by \( \{ x', y', z', u' \} \). Since \( \pi(R), \pi(C_1), \pi(C_2), \pi(C_3), \pi(C_4) \) are pairwise arc-disjoint this gives a contradiction.

So \( \pi \) and \( \pi^{-1} \) preserve transitive triples and now Theorem 5.5 follows from Proposition 5.1 and Theorem 3.3.

6. Arc Bijections Preserving 4-Dicycles in Tournaments

In this section we prove Theorem 4.5. First we show that the bijection \( \pi \) in Theorem 4.5 may be very far from being induced by an isomorphism or anti-isomorphism of \( T(4) \) onto \( T'(4) \). Let \( M_n \) and \( N_n \) (n odd) be the tournaments obtained from \( Q_n \) as follows: \( Q_n \) may be regarded as a dipath \( p_1p_2 \cdots p_n \) together with the arcs \( p_jp_i, \ j \geq i + 2 \). In order to get \( M_n \) we add new vertices \( q_2, q_4, \ldots, q_{n-1} \) such that \( q_i \) dominates \( p_1, p_2, \ldots, p_i \) and no other vertex for \( i = 2, 4, \ldots, n - 1 \). In order to obtain \( N_n \) we add to \( Q_n \) vertices \( q_1, q_3, \ldots, q_{n-2} \) such that \( q_i \) dominates \( p_i, p_{i+1}, \ldots, p_1, q_1, q_3, \ldots, q_{i-2} \) and no other vertex for \( i = 1, 3, \ldots, n - 2 \). Then \( M_n \) and \( N_n \) are anti-isomorphic but not isomorphic and there exists a 4-dicycle preserving arc bijection \( E(M_n) \rightarrow E(N_n) \) which is far from being induced by an anti-isomorphism of \( M_n(4) \) onto \( N_n(4) \). Figure 1 shows \( M_7(4) \) and \( N_7(4) \) and indicates \( \pi \). All arcs not in the figure are directed from right to left.

We can define \( M_\infty \) and \( N_\infty \) in the obvious way and obtain

**Proposition 6.1.** There exist infinite strong tournaments \( M_\infty \) and \( N_\infty \) and an arc bijection \( \pi : E(M_\infty) \rightarrow E(N_\infty) \) such that \( \pi \) and \( \pi^{-1} \) preserve 4-dicycles and such that \( M_\infty \) and \( N_\infty \) are neither isomorphic nor anti-isomorphic.

The tournaments \( M_n \) and \( N_n \) will appear in the proof of Theorem 4.5 below. We now prove Theorem 4.5 by contradiction so let us assume that there exist strong tournaments \( T, T' \) and an arc bijection \( \pi : E(T) \rightarrow E(T') \) such that \( \pi \) and \( \pi^{-1} \) preserve 4-dicycles and such that \( T \) and \( T' \) are neither isomorphic nor anti-isomorphic. By Theorem 4.4 we can assume that \( T(4) \) contains a 3-dicycle \( C \) such that \( \pi(C) \) is not a 3-dicycle in \( T' \). Such a
3-dicycle will be called bad. We choose $T$, $T'$, and $\pi$ such that the total number of bad 3-dicycles in $T$ and $T'$ is minimum. We shall now prove a series of statements about $T$, $T'$, $\pi$ and finally obtain a contradiction. If $A$ and $A'$ are arc sets of $T$ and $T'$, respectively, we say that $A$ corresponds to $A'$ if $A' = \pi(A)$.

(1) $T$ has no bad 3-dicycle having augmentations at all arcs.

Proof of (1) (by contradiction). Suppose $xyzx$ is a bad 3-dicycle having augmentations at all arcs. Then $T$ has a 6-dicycle $xz_1yz_2zz_3x$. The arc between $z_3$ and $y$ is contained in a 4-dicycle which contains one of the dipaths $xz_1y$, $yz_2z$. Similar arguments for the arc between $z_1$ and $z$ and the arc between $z_2$ and $x$ show that at least two of the three dipaths $xz_1y$, $yz_2z$, $zz_3x$ are contained in at least two 4-dicycles, and hence at least two of these dipaths, say $xz_1y$ and $yz_2z$, are mapped onto dipaths of length 2. Then also the dipaths $yzx$ and $xzy$ are mapped onto dipaths of length 2. This implies, since $xyzx$ is a bad 3-dicycle, that $xyzx$ is mapped onto a dipath $y'z'x'u'$ where $\pi(zx) = z'x'$. Now $xy$ and $yz$ are contained in a 4-cycle not containing $zx$ but $\pi(xy)$ and $\pi(yz)$ are not contained in a 4-dicycle that avoids $\pi(zx)$. This contradiction proves (1).

(2) $T$ has a bad 3-dicycle which has augmentations at two distinct arcs.

Proof of (2). Suppose (reductio ad absurdum) that (2) is false and let $C: xyzx$ be any bad 3-dicycle of $T(4)$. We can assume that $C$ has no augmentation at any of the arcs $yz$, $zx$. Since $xy$ is in $T(4)$ there is a
4-dicycle $ywxy$, and the above assumption on augmentation implies that $v$ (respectively $u$) dominates (respectively is dominated by) all vertices of $C$. The arc $uv$ is contained in three 3-dicycles intersecting \{x, y, z\}. Since each of these has augmentations at two distinct arcs and since (2) is false we conclude that $\pi$ maps $C_x: uwxu$, $C_y: uwyu$, and $C_z: uwzu$ onto the 3-dicycles $u'v'x'u'$, $u'v'y'u'$, and $u'v'z'u'$, respectively. Each arc $a$ of $C$ is contained in a 4-dicycle $C_a$ which contains three arcs of $C_x \cup C_y \cup C_z$ and hence $\pi$ maps each arc of $C$ onto an arc connecting two of $x'$, $y'$, $z'$. Since the above three 4-dicycles $C_a$ have only $uv$ in common pair by pair we conclude that $\pi(C)$ cannot be a transitive triple. So $\pi(C)$ is a 3-dicycle, a contradiction which proves (2).

(3) If $C: xyzx$ is a bad 3-dicycle of $T$ with augmentations $C_1: xuyzx$ and $C_2: xyzvx$, then the arcs of $\pi(C)$ are not all incident with the same vertex.

Proof of (3). Suppose (3) is false. Then two of the arcs $\pi(xy)$, $\pi(yz)$, $\pi(zx)$ have a common head or tail (say common head). These arcs must be $\pi(xy)$ and $\pi(zx)$ because $\pi(C_1)$ and $\pi(C_2)$ are 4-dicycles. So $T'$ contains distinct vertices $x'$, $y'$, $z'$, $w'$ such that $\pi(yz) = y'x'$, $\pi(xy) = z'y'$, and $\pi(zx) = w'y'$. Furthermore, $\pi(C_1) = w'y'x'u'w'$ and $\pi(C_2) = z'y'x'v'z'$. We consider first the case \{v', u'\} \cap \{z', w'\} = \emptyset. Since $\pi^{-1}(w'y'x'u')$ cannot be extended to a 4-dicycle, $T'$ contains the arc $w'w'$, and, by a similar argument, $T'$ contains $z'u'$. Now $z'u'w'w'z'$ is a 4-dicycle of $T'$ which must correspond to a 4-dicycle of the form $vxuwv$ in $T$ where $w \notin \{x, y, z, v, u\}$ (since this 4-dicycle contains precisely one arc in $\{xu, uy\}$, precisely one arc in $\{zv, ux\}$, and no arc in $C$). In particular, $\pi(xu) = u'w'$ and $\pi(vx) = v'z'$ and $\pi(uv) = \{w'v', z'u'\}$. Now $w$ is dominated by $z$ since otherwise $xuwzx$ is a 4-dicycle which is not mapped onto a 4-dicycle in $T'$. Similarly, $w$ dominates $y$. Also, $z$ dominates $u$ since otherwise $uzxwu$ would be a 4-dicycle in $T$ (giving a contradiction in $T'$) and similarly, $v$ dominates $y$. So $yz$ is contained in a 4-dicycle $C_{uw}$ containing $uw$ and in a 4-dicycle $C_{wc}$ containing $ww$. The corresponding 4-dicycles in $T'$ must be $v'y'x'w'v'$ and $u'y'x'z'u'$. In $T$, $u$ must dominate $v$ since otherwise $uzyuw$ is a 4-dicycle (giving a contradiction in $T'$) and so $T$ contains the 4-dicycle $uzyuw$. This 4-dicycle contains $yz$ and precisely one additional arc of each of $C_{uw}$ and $C_{wc}$. In other words, its image under $\pi$ contains precisely one of $v'y'$, $x'w'$, and precisely one of $u'y'$, $x'z'$. But this leads to a contradiction in $T'$.

So we can assume that $u' = z'$ and $v' \notin \{w', z'\}$. Now $u$ must dominate $v$ since otherwise $uzyuw$ would be a 4-dicycle giving a contradiction in $T'$. If $y$ dominates $v$, then $yxuyy$ is a 4-dicycle, so $\pi(yxuyy)$ is a 4-dicycle. But $\pi(vx)$ is either $x'v'$ or $v'z'$, neither of which is in a 4-dicycle with $\pi(xu)$ and $\pi(yv)$ (which are $x'z'$ and $z'w'$) so $v$ dominates $y$ in $T$. In $T'$ $v'z'w'y'$ is not contained in a 4-dicycle (for otherwise, we would get a contradiction in $T$) so
contains $v'y'$. By a similar argument (considering the dipath $w'y'x'v'$ and the corresponding arcs in $T$), $w'$ dominates $v'$. Considering the dipath $x'v'z'w'$ and the corresponding arcs in $T$ we conclude that $w'$ dominates $x'$ iff $u$ dominates $z$ (and if that holds, then $\pi(xu) = z'w'$ and $\pi(uz) = w'x'$).

Now the subtournaments induced by $\{x, y, z, u, v\}$ and $\{z', y', x', w', v'\}$, respectively, are isomorphic. (An isomorphism is indicated by the order in which the vertices occur above.) As $T$ and $T'$ are assumed to be non-isomorphic, $T$ has order at least 6. We now define a partition of $V(T) \setminus \{x, y, z, u\}$ into sets $A, B, D$ as follows: $D$ are the vertices dominating $x, y$ but not $z$; $B$ are the vertices dominating $x, y, z$; and $A$ are the vertices dominated by $x, y, z$. (Note that $C$ has no augmentation at $yz$ and no augmentation at $xy$ except $C_1$ since the existence of another augmentation of $C$ at $xy$ would give the first case in the proof of (3). Also note that any vertex which is not in $A \cup B \cup \{x, y, z, u\}$ can play the role of $v$ and must therefore be in $D$.)

Suppose first that $A \neq \emptyset$. Then there is a vertex $a \in A$ such that $a$ dominates either $u$ or some vertex in $B \cup D$. Suppose first that $a$ dominates $u$. Then $auyza$ is a 4-dicycle in $T$ and it must correspond to a 4-dicycle $y'x'z't'y'$ in $T'$. In particular $\pi(uy) = x'z'$ and now $C' : x'z'y'x'$ is a bad 3-dicycle in $T'$ whose corresponding arcs in $T$ are $xy, uy,$ and $yz$ which are all incident with $y$. Considering the augmentations $y'x'z't'y'$ and $y'x'v'z'y'$ gives us the situation for $T'$ and $zP1$ treated in the first case of the proof of (3). So we can assume that $a$ does not dominate $u$.

If $a$ dominates some vertex $b$ in $B$, then the arc $ab$ is contained in three 4-dicycles each of which contains precisely one arc of $C : xyzx$ and such that they have only $ab$ in common pair by pair. But $\pi(ab)$ cannot have the same property in $T'$. (For, if $\pi(ab) = a'b'$, then clearly $a' \neq y'$. But then both arcs $y'a'$ and $a'b'$ are in $n(zxabz)$ and $n(xyabx)$, a contradiction.) So we conclude that a vertex in $A$ dominates no vertex outside $A \cup D$. Also note that each vertex in $D$ can replace $v$ in the preceding arguments and so each vertex in $D$ is dominated by $u$.

We now claim that no vertex $b$ in $B$ is dominated by $u$. For if that were the case, then $uhzxux$ is a 4-dicycle and would correspond to $\pi'w'y'h'z'$ in $T'$.

In particular, $\pi(xu) = z'w'$. The dipath $x'z'y'$ cannot be extended to a 4-dicycle and so $x'$ dominates $b'$. Then $x'b'z'y'x'$ is a 4-dicycle in $T'$ but the corresponding arcs in $T$ are not in a 4-dicycle. This proves the claim that $u$ is dominated by all vertices of $B$. Now it follows that each of the arcs $uy$ and $zx$ is contained in only one 4-dicycle, namely $C_1 : xuyzx$ and neither of them is contained in a non-bad 3-dicycle. If $\pi(uy) = x'z'$, then we denote by $\tau$ the permutation of $E(T)$ which permutes $uy$ and $zx$. Then $\pi \circ \tau$ and $(\pi \circ \tau)^{-1}$ preserve 4-dicycles and all those 3-dicycles which are preserved by $\pi$ and $\pi^{-1}$. Furthermore, $\pi \circ \tau$ maps $xyzx$ onto $x'z'y'x'$. This contradicts the minimality property of $\pi$ and $\pi^{-1}$. So we can assume that $\pi(uy) = z'w'$. Then $z$ dominates $u$ since otherwise the 4-dicycle $uzvxu$ gives a contra-
diction in \( T' \). (It was earlier seen that \( w' \) dominates \( x' \) iff \( u \) dominates \( z \), so \( x' \) dominates \( w' \).) Clearly, every 3-dicycle containing \( xu \) is bad. We claim that \( xu \) is not contained in any 4-dicycle \( C_3 \), where \( C_3 \neq C_1 \). For, if \( C_3 \) exists, then \( C_3 \) must contain a vertex of \( D \) and we can without loss of generality assume that \( C_3 \) contains \( v \). Since \( \pi(vx) \in \{x'v', v'z'\} \), \( C_3 \) does not contain \( vx \) and hence \( C_3 \) is of the form \( xuwx \) where \( b \in B \cup D \). Since \( uvyzv \) is a 4-dicycle, \( \pi(vb) \) has either \( v' \) as tail or \( y' \) as head. But then \( \pi(C_3) \) cannot exist, and this contradiction shows that \( xu \) is in only one 4-dicycle. Now we let \( \tau \) be the permutation of \( E(T) \) such that \( \pi(xz) = xu, \pi(xu) = uy \), and \( \tau(uy) = zx \) and each other arc is fixed. Then we consider \( \pi \circ \tau \) instead of \( \pi \) and obtain a contradiction to the minimality property of \( \pi, \pi^{-1} \). This proves (3).

(4) If \( C \) is a bad 3-dicycle with augmentations at two distinct arcs as in (3), then \( \pi(C) \) is a collection of one or two or three vertex-disjoint dipaths.

Proof of (4). Suppose (4) is false. Then two of the arcs of \( \pi(C) \) have a common head or tail (say tail) and these arcs must be \( \pi(xy) \) and \( \pi(xz) \). Since \( \pi(C_1) \) and \( \pi(C_2) \) are 4-dicycles having only \( \pi(yz) \) in common and since \( \pi(yz) \) is not incident with the tail of \( \pi(xy) \) by (3), \( \pi(yz) \) must be incident with the head of either \( \pi(xy) \) or \( \pi(xz) \). So we can assume that \( T' \) has four distinct vertices \( x', y', z', w' \) such that \( \pi(xy) = z'y', \pi(yz) = y'x' \), and \( \pi(xz) = z'w' \). Moreover, \( \pi \) maps \( \{xu, uy\} \) onto \( \{x'z', w'y'\} \) and \( C_2 \) onto a 4-dicycle \( z'y'x'v'z' \). Now \( x'z'y'x' \) is a bad 3-dicycle in \( T' \) and by (3) we have \( \pi(xu) = x'z' \) (and hence \( \pi(uy) = w'y' \)). The dipaths \( zvxu \) and \( uxuy \) cannot be extended to 4-dicycles and hence \( T \) contains the arcs \( zu \) and \( vy \). Similarly, \( T' \) contains the arcs \( v'y' \) and \( x'w' \). If \( v \) dominates \( u \) in \( T \), then \( uvyzv \) can be extended into a 4-dicycle which implies that \( v' \) dominates \( w' \) in \( T' \) (and \( \pi(zv) = x'v' \)). By a similar argument, the arc \( v'w' \) in \( T' \) implies \( vu \) in \( T \) so the tournaments induced by \( \{x, u, z, y, v\} \) and \( \{x', y', z', v', w'\} \) are isomorphic. Clearly \( C_1 \) is the only augmentation of \( C \) at \( xy \). Next define the partition of \( V(T) \backslash \{x, y, z, u\} \) into \( A \), \( B \) and \( D \) as in the proof of (3). As in (3) we conclude that there is no arc from \( A \) to \( B \). From this it follows easily that \( zy \) is in only one 3-dicycle of \( T \), namely \( xyzy \), and that any 4-dicycle of \( T \) containing \( zy \) also contains \( xu \). Since \( C': x'z'y'x' \) is a bad 3-dicycle of \( T' \) contradicting (4) with \( \pi^{-1} \) instead of \( \pi \), we can repeat all arguments above for \( C' \) and conclude that \( x'z' = \pi(xu) \) is only in one 3-dicycle of \( T' \) (namely \( C' \)) and that each 4-dicycle of \( T' \) containing \( x'z' \) also contains \( z'w' = \pi(xz) \). So \( zy \) and \( xu \) are in precisely the same 4-dicycles and none of them are in non-bad 3-dicycles. So if \( \tau \) denotes the transposition of \( zy \) and \( xu \), then \( \pi \circ \tau \) and \( (\pi \circ \tau)^{-1} \) preserve 4-dicycles and have fewer bad 3-dicycles than \( \pi \) and \( \pi^{-1} \). This contradiction proves (4).

(5) One of \( T, T' \) (say \( T \)) contains a bad 3-dicycle \( xyzx \) with augment-
tations \( C_1 : xuyzx \) and \( C_2 : xzyvx \) such that \( \pi(C_1) \cup \pi(C_2) \) includes six vertices.

Proof of (5). Let \( C : xyzx \) be a bad 3-dicycle of \( T \) with augmentations \( C_1 : xuyzx \) and \( C_2 : xzyvx \). Let \( C_1 = \pi(C_1) : y'z'x'u'y' \) (with \( \pi(yz) = y'z' \)). If (5) is false, then we can assume that \( \pi(C_2) = C_2 : x'y'z'v'x' \). By (4), \( \pi(xy) \neq x'y' \) and \( \pi(zx) \neq z'x' \). By (1) \( C \) has no augmentation at \( yz \) and if \( C \) has an augmentation at \( xy \) or \( zx \) other than \( C_1, C_2 \), then we consider that augmentation instead of \( C_1 \) or \( C_2 \) and verify (5). So we assume that \( C \) has no augmentation other than \( C_1 \) and \( C_2 \) and we partition \( V(T) \setminus \{x, y, z, u, v\} \) into sets \( A \) and \( B \) by letting \( A \) (respectively \( B \)) be those vertices of \( T \) which are dominated by (respectively which dominate) each of \( x, y, \) and \( z \).

We first claim that there is no arc from \( A \) to \( B \). For suppose there is such an arc, say \( ab \). Then, for any arc in \( C \), that arc and \( ab \) are in a unique 4-dicycle, and the three 4-dicycles \( C_1, C_2, C_3 \) obtained in this way have no arc but \( ab \) in common pair by pair and none of \( C_1, C_2, C_3 \) contain arcs in the dipaths \( xuy \) or \( zvx \). Using this it is easy to see that for some two of \( C_1, C_2, C_3 \) (say \( C_1, C_2 \)), \( \pi(C_1) \cup \pi(C_2) \) includes at least six vertices. (If \( \pi(ab) \) has no end in common with any arc of \( \pi(C) \), we prove (5) by selecting two independent arcs of \( \pi(C) \) and then consider the two corresponding 4-dicycles among \( C_1, C_2, C_3 \). It is easy to see that \( \pi(ab) \) has no end in common with \( \pi(yz) \) and if \( \pi(ab) \) has an end in common with one of \( \pi(xy), \pi(zx) \), then we consider the 4-dicycle \( C' \) containing that arc and also the 4-dicycle \( C' \) containing \( yz \).) Now \( C_1 \) and \( C_2 \) are augmentations of a 3-dicycle containing \( ab \) and one of \( x, y, z \) and since \( \pi(C_1) \cup \pi(C_2) \) has at least six vertices, that 3-dicycle must be bad and can play the role of \( C \) in (5). This contradiction proves that \( T \) has no arc from \( A \) to \( B \).

It is easy to see that \( zvxu \) cannot be extended to a 4-dicycle and so \( z \) dominates \( u \). Similarly \( v \) dominates \( y \). Since \( x'y'z'x' \) is a bad 3-dicycle in \( T' \) we conclude by similar arguments that \( T' \) contains the arcs \( v'y', z'u' \). Now it is easy to see that any 4-dicycle in \( T \) which contains \( zx \) also contains \( xu \) and that any 4-dicycle in \( T \) containing \( xy \) also contains \( vx \) and we get similar statements for \( z'x', x'u', x'y', \) and \( v'x' \) in \( T' \).

Now suppose \( v \) dominates \( u \), i.e., \( T \) contains the 4-dicycle \( uyzvu \). The corresponding 4-dicycle in \( T' \) must be \( u'y'z'v'u' \) and hence \( v' \) dominates \( u' \). Similarly, the arc \( v'u' \) implies \( v \) dominates \( u \) so the subtournaments induced by \( \{x, y, z, u, v\} \) and \( \{x', y', z', u', v'\} \) are isomorphic. Since \( T \) and \( T' \) are not isomorphic at least one of the sets \( A, B \) (say \( B \)) is non-empty and since \( T \) is strong, there is an arc from a vertex \( a \) in \( A \) to \( u \) or \( v \). If a dominates \( v \) but not \( u \), then the 4-dicycles \( axyza \) and \( axuva \) give a contradiction in \( T' \). So in any case \( a \) dominates \( u \). Now we consider the 4-dicycle \( auyza \) which must correspond to a 4-dicycle \( a'u'y'z'a' \). In particular, \( \pi(uy) = u'y' \) and
hence \( \pi(\alpha x) = \alpha'u' \) and \( \pi(\alpha xu) = \alpha'x' \) (because \( \pi(\alpha x) \neq \alpha'x' \) by (4)). Since any 4-dicycle which contains \( \alpha x \) also contains \( \alpha xu \) and the same holds for \( \alpha'x' \) and \( \alpha'u' \) we conclude that \( \alpha x \) and \( \alpha xu \) are in the same 4-dicycles. None of them are in non-bad 3-dicycles (since each of \( \alpha x \) and \( \alpha'x' \) is in only one 3-dicycle). If \( \tau \) denotes the transposition of \( \alpha x \) and \( \alpha xu \), then \( \pi \circ \tau \) and \( (\pi \circ \tau)^{-1} \) preserve 4-dicycles and have the same number of bad 3-dicycles as \( \pi \) and \( \pi^{-1} \) and now we have a contradiction to (4). This proves (5).

Let \( C \) be a bad 3-dicycle with augmentations \( C_1, C_2 \) such that (5) is satisfied. Then \( \pi(C_1) = u'y'z'x_1u' \) and \( \pi(C_2) = x_2' y'z'v'x_2' \) where \( x_1', x_2', u', y', z', v' \) are all distinct and \( \pi(yz) = y'z' \). Moreover, we can assume, without loss of generality, that \( \pi(\alpha x) \neq u'y' \) and \( \pi(\alpha y) \neq z'v' \). (In other words, in the system of dipaths \( \pi(C) \), \( \pi(\alpha x) \) is not the “predecessor” of \( \pi(yz) \) and \( \pi(\alpha y) \) is not the “successor” of \( \pi(yz) \). This can be achieved by considering the converse of \( T' \), if necessary. Note that if \( \pi(C) \) is a dipath, then \( \pi(yz) \) must be the mid-arc of this dipath.)

If \( \pi(C) \) form three independent arcs we can assume that \( \pi(zv) = z'v' \) (again by replacing \( T' \) by its converse if necessary). With this notation we have:

\[
(6) \ T' \text{ contains the arc } x'_2 x'_1 \text{ and } T \text{ contains the arcs } zu, vy, vu. \text{ Moreover, } \pi(yz) = y'z', \pi(zu) = z'u', \pi(uy) = u'y', \pi(zu) = z'v'. \]

**Proof of (6).** If \( x'_1 \) dominates \( x'_2 \), then the 4-dicycle \( x'_1 x'_2 y'z' x'_1 \) contains \( y'z' \) and precisely one more arc from each of \( \pi(C_1), \pi(C_2) \). The corresponding 4-dicycle in \( T' \) must be \( uy'z'u' \). But then \( \pi(C) \) consists of three independent arcs and now the assumption \( \pi(zv) = z'v' \) leads to a contradiction. Hence \( x'_2 \) dominates \( x'_1 \). It is easy to see that there is no 4-dicycle in \( T' \) corresponding to the 4-dicycle \( zuvxu \) so \( z \) dominates \( u \). Similarly, \( v \) dominates \( y \).

Suppose now (reductio ad absurdum) that \( u' \) dominates \( v' \). Then the 4-dicycle \( u'v'x'_2 x'_1 u' \) contains precisely one arc from each of \( C'_1, C'_2 \) and none of these arcs are \( y'z' \). Since \( C \) has no augmentation at \( yz \), the 4-dicycle \( u'v'x'_2 x'_1 u' \) must correspond to a 4-dicycle \( vxuqv \) in \( T \) and hence \( \pi(\alpha x) = x'_2 x'_1 \), \( \pi(\alpha xu) = x'_1 u' \), and \( \{\pi^{-1}(x'_2 x'_1), \pi^{-1}(u'v')\} = \{uv, qv\} \) and hence also \( \pi(\alpha x) = x'_2 x'_1, \pi(\alpha y) = x'_2 y', \pi(\alpha u) = u'y', \) and \( \pi(zv) = z'v' \). Now the dipath \( zvuq \) cannot be extended to a 4-dicycle and hence \( z \) dominates \( q \). Similarly, \( q \) dominates \( y \). Now the two 4-dicycles \( uqvyu \) and \( yqvuy \) must correspond to the dicycles \( x'_2 x'_1 y'z'x'_2 \) and \( u'y'z'u' \) in \( T' \). This means that \( \pi \) maps one of the arcs \( zu, vy \) into \( \{x'_1 y', z'_x'_2\} \) and the other into \( \{v'y', z'u'\} \). Since there is no 4-dicycle containing the dipath \( u'y'z'u' \), \( u \) must dominate \( v \) and now the above remark on \( zu \) and \( vy \) shows that there is no 4-dicycle in \( T' \) corresponding to the 4-dicycle \( uqvyu \). This contradiction
proves that $v'$ dominates $u'$ and so $u'v'y'z'u'$ is a 4-dicycle in $T'$. This 4-dicycle must correspond to $uyzuu$ in $T$. This completes the proof of (6).

(7) *The dipath $vxu$ is not contained in a 4-dicycle in $T$.*

**Proof of (7).** Suppose (reductio ad absurdum) that $vxyuv$ is a 4-dicycle in $T$. The corresponding 4-dicycle in $T'$ is either $x'_2'y'x'u'x'_2$ or $v'x'_2'z'x'_1v'$. Suppose the latter holds. (If the former holds the argument is similar.) Then $y'$ dominates $v'$ since otherwise $v'y'z'x'_1v'$ is a 4-dicycle in $T'$ giving a contradiction in $T$. Also $x'_2$ dominates $u'$ since otherwise $u'x'_2'y'v'u'$ is a 4-dicycle giving a contradiction in $T$ (since $T$ has no 4-dicycle containing $xy$ and $vu$). But now the 4-dicycle $x'_2'u'y'v'x'_2$ leads to a contradiction in $T$ (since $T$ has no 4-dicycle containing $vx$ and $uy$). This proves (7).

(8) *$C$ has no augmentation $C_3: yzxu_1y$ such that $p(C_3) = y'z'x'_1v'y'$.*

**Proof of (8).** Suppose (reductio ad absurdum) that such a $C_3$ exists. Then $T'$ contains the arcs $x'_1y'$ and $x'_2u'$ since otherwise $x'_1v'u'$ can be extended to a 4-dicycle, but $T$ has no 4-dicycle containing $vu$ and one of $xu_1$, $u_1y$. Also, $T'$ contains $z'u'$ since otherwise $T'$ contains a 4-dicycle containing $z'x'_1 = \pi(zx)$ and $v'u' = \pi(vu)$ leading to a contradiction in $T$. Finally, $z'$ dominates $x'_2$ since $z'x'_1 = \pi(zx)$ and $v'x'_2$ is not contained in a 4-dicycle (because $v'x'_2 \in \{\pi(vx), \pi(xy)\}$). In $T$, $z$ dominates $u_1$ since $zuuu_1$ cannot be extended into a 4-dicycle. Also, $u_1$ dominates $v$ since $u_1yz$ cannot be extended to a 4-dicycle, and then by (7), $u_1$ dominates $u$. Under the assumption that (8) is false we now prove

(8a) *The 4-dicycle $y'z'x'_2x'_1y'$ in $T'$ corresponds to $yzxu_1vy$ in $T$.*

**Proof of (8a).** Let $y'z'x'_2x'_1y'$ correspond to $yzp_1p_2y$. Assume first that $p_2 \neq v$ (but possibly $p_1 = u_1$ or $p_1 = u$). Since $u'y'z'x'_2u'$ is a 4-dicycle, we have $\pi(zp_1) = z'x'_2$ and $\pi^{-1}(x'_2u') = p_1u$ (in particular, $p_1 \neq u$). Now $x$ dominates $p_1$ because $xyzp_1$ cannot be extended to a 4-dicycle, and $p_2$ dominates $v$ because $p_2yz$ cannot be extended to a 4-dicycle. So $xp_1p_2vx$ is a 4-dicycle in $T$. This corresponds to a 4-dicycle $v'x'_2x'_1p'v'$ in $T$ (in particular, $\pi(vx) = v'x'_2$ and $\pi(p_1p_2) = x'_1x'_1$, and $\pi(p_2v)$ is one of $x'_1p'$, $p'v'$). Since $\pi(p_1p_2) = x'_2x'_1$ and $\pi(zx) = z'x'_1$ are not in a common 4-dicycle, $z$ dominates $p_2$ in $T$, and hence $T$ has a 4-dicycle containing both $yz$ and $p_2v$ and containing no arc of $(C_1 \cup C_2 \cup C_3) \setminus \{yz\}$. But $T'$ has no 4-dicycle containing $y'z'$ and an arc of $x'_1p'v'$ and avoiding $\pi(C_1) \cup \pi(C_2) \cup \pi(C_3) \setminus \{y'z'\}$. This contradiction proves that $p_2 = v$.

Suppose next that $p_1 \neq u_1$. As above we have $\pi^{-1}(z'x'_2) = zp_1$ and $\pi^{-1}(x'_2u') = p_1u$. Now $u_1$ dominates $p_1$ since otherwise $u_1yzp_1$ can be
extended to a 4-dicycle giving a contradiction in $T'$. But now $u_1p_1vxu_1$ is a 4-dicycle giving a contradiction in $T'$. This proves (8a).

Since $u'y'z'x'u'$ is a 4-dicycle in $T'$ we have $\pi^{-1}(z'x'_2) = zu_1$ and $\pi^{-1}(x'_2u') = u_1u$. Note that the tournaments induced by $\{x, y, z, u, u_1, v\}$ and $\{x'_1, x'_2, y', z', u', v'\}$ are isomorphic, and hence $T$ and $T'$ have more than six vertices.

(8b) $T$ has no dipath of the form $u_1qv$.

Proof of (8b). Suppose (reductio ad absurdum) that $u_1qv$ exists. The 4-dicycle $xu_1qv$ corresponds to a 4-dicycle $x'_1v'x'_2q'x'_1$ in $T'$ (in particular $\pi(xu_1) = x'_1v'$ and $\pi(vx) = v'x'_2$). Now $zxu_1q$ cannot be extended to a 4-dicycle, and hence $z$ dominates $q$. Also, $q$ dominates $y$ since $qvy$ cannot be extended into a 4-dicycle. The 4-dicycle $qyuv$ corresponds to $y'z'x'q'y'$, and hence $\pi(qy) = q'y'$, $\pi(uq) = x'_2q'$, and $\pi(qv) = q'x'_1$. Now $qyvz$ cannot be extended to a 4-dicycle and so $q$ dominates $x$. The 4-dicycle $qvyxz$ corresponds to $q'x'_1y'z'q'$, and hence $\pi(qz) \in \{z'q', x'_1y'\}$. But then the 4-dicycle $xyvq$ gives a contradiction in $T'$. This proves (8b).

(8c) $T$ has no dipath $vqx$.

Proof of (8c). Suppose (reductio ad absurdum) that $vqx$ exists. Then the 4-dicycle $vqyv$ corresponds to a 4-dicycle $x'_1v'x'_2q'x'_1$. (In particular, $\pi(u_1v) = x'_2x'_1$, $\pi(xu_1) = x'_1v'$, and $\pi(vy) = x'_1y'$.) Since none of $qxy$ and $zu_1vq$ can be extended into a 4-dicycle, $q$ dominates $y$, and $q$ is dominated by $z$. Now the 4-dicycle $qyvq$ must correspond to $y'z'v'q'y'$, in particular, $\pi(qy) = q'y'$. Since $qyzxq$ is a 4-dicycle, $\pi(zq) = z'q'$. Since $q'y'z'x'_1$ cannot be extended to a 4-dicycle, $q'$ dominates $x'_1$, but now the 4-dicycle $q'x'_1y'z'q'$ gives a contradiction in $T$, and hence (8c) is proved.

(8d) $T$ has no dipath $xqu_1$.

Proof of (8d). Suppose (reductio ad absurdum) that $xqu_1$ exists. Then the 4-dicycle $xqu_1v$ must correspond to $v'x'_2x'_1q'v'$. In particular, $\pi(vx) = v'x'_2$ and $\pi(u_1v) = x'_2x'_1$. Now none of $qyu$ and $zu_1vq$ can be extended to a 4-dicycle and so $q$ dominates $y$ and is dominated by $z$. The 4-dicycles $qyu, yzq, qyvq$ show that $\pi(zq) = z'q'$, $\pi(qy) = q'y'$. Since $vzq$ cannot be extended to a 4-dicycle, $v$ dominates $q$, but now the 4-dicycle $vqyv$ gives a contradiction in $T'$. This proves (8d).

We now define $A$ (respectively $B$) as the set of vertices that are dominated by (respectively dominate) each vertex of $x, u_1, v$. By (8b), (8c), (8d), and the preceding discussion, $A, B$ is a partition of $V(T) \setminus \{x, u_1, v\}$, and $u, y \in A, z \in B$. 
Now the 3-dicycle $y'z'x'_2y'$ and its augmentations $y'z'v'x'_2y'$, $x'_2u'y'z'x'_2$, $x'_2x'_1y'z'x'_2$ have the same properties as $C$ and its augmentations, $C_2$, $C_1$, and $C_3$: $xu_1yzx$. Hence we can apply all the arguments in the proof of (8) to $T'$ and conclude (by the analogues of (8b), (8c), (8d)) that all vertices in $T'\setminus \{x'_1, x'_2, v'\}$ either dominate or are dominated by each of $x'_1, x'_2, v'$. We define $A'$ (respectively $B'$) as those vertices which are dominated by (respectively dominate) all vertices in $\{x'_1, x'_2, v'\}$.

(8e) $y'z$ is the only arc from $A$ to $B$.

**Proof of (8e).** Assume first there is an arc $yz_1$ where $z_1 \in B \setminus \{z\}$. Then $z$ dominates $z_1$ since $C$ has no augmentation at $yz$, and hence $xyz_1z \in x$ is a 4-dicycle in $T$ which must correspond to a 4-dicycle $x'_2 y'z'z'_1 x'_2$ in $T'$. But now the 4-dicycle $z_1 xu_1 yz_1$ gives a contradiction in $T'$. By a similar argument, there is no arc $y_1 z$ where $y_1 \in A \setminus \{y, u\}$, and by (7), there is no arc from $u$ to $B$. By similar arguments in $T'$, we deduce that there is no arc $y'_1 z'$ or $y'_1 z'_1$, where $y'_1 \in A' \setminus \{y', u'\}$, $z'_1 \in B' \setminus \{z'\}$, and by (7), there is no arc from $u'$ to $B'$. Suppose now that there is an arc $y_1 z_1$ where $y_1 \in A \setminus \{u, y\}$ and $z_1 \in B \setminus \{z\}$. The arc $y_1 z_1$ has the following property: For each arc of $xu_1 v$ there is a 4-dicycle containing that arc and $y_1 z_1$, and the three 4-dicycles obtained in this way are pairwise arc-disjoint (except for $y_1 z_1$), and they contain only one arc joining two of $x, y, z, u, u_1, v$. Since each arc of $xu_1 v$ is mapped by $\pi$ onto an arc of $x'_1 x'_1 v'_1 x'_2$ or an arc from $\{x'_2, x'_1, v'\}$ to $y'$, it is easy to see that $\pi(y_1 z_1)$ must be an arc $y'_1 z'_1$ where $y'_1 \in A' \setminus \{y', u'\}$, $z'_1 \in B' \setminus \{z'\}$. (For otherwise, the three 4-dicycles above would correspond to the form $y' y'_1 z'_1 x'_1 y'$, $y'_1 z'_1 x'_2 y'$, $y'_1 z'_1 y'_1 y'$ where $y'_1 \in A'$, $z'_1, z'_1, z'_1 \in B'$ and $\pi(y'_1 z'_1) = y'_1 y'$. Repeating the above arguments with $y'_1 z'_1 (i = 1, 2, 3)$ instead of $y_1 z_1$ shows that each of $\pi^{-1}(y'_1 z'_1) (i = 1, 2, 3)$ must have $y$ as tail, a contradiction.) It is also easy to see that $\pi$ maps the arcs of $u_1 vu_1$ onto the arcs of $x'_1 v'_1 x'_2$. Now the 4-dicycle $y_1 z_1 xu_1 y_1$ corresponds to $y'_1 z'_1 x'_1 y'_1$, and hence $\{\pi(z_1 x), \pi(u_1 v_1)\} = \{z'_1 x'_1, v'_1 y'_1\}$. But then the 4-dicycle $z_1 zu_1 y_1 z_1$ gives a contradiction in $T'$ and we have proved (8e).

From (8e) it follows that every 4-dicycle which contains an arc between $A \cup B$ and $\{x, u_1, v\}$ is either contained in the subtournament induced by $\{x, y, z, u, u_1\}$ or is of the form $yzpy$ where $p \in \{x, u_1, v\}$, $q \in B \setminus \{z\}$ (or of the form $yzpqy$ where $p \in \{x_1, u_1, v\}$, $q \in A \setminus \{y, u\}$). In particular, each arc of the form $qp$ (or $pq$) above is contained in only one 4-dicycle of $T$. The arc $py$ (or $zp$) in the 4-dicycle above is called the mate of $qp$ (or $pq$). Note that $\pi$ maps the arcs from $z$ to $\{x, u_1, v\}$ onto the arcs from $z'$ to $\{x'_1, x'_2, v'\}$, and if $z$ dominates some $q \in B \setminus \{z\}$, then we consider the three 4-dicycles $yzpqy$ where $p \in \{x, u_1, v\}$ and conclude easily that $\pi$ maps the three arcs from $\{x, u_1, v\}$ to $y$ onto the arcs from $\{x'_1, x'_2, v'\}$ to $y'$.
particular, in this case \( \pi \) maps \( xu_1yv \) onto \( x'_2x'_1v'x'_2 \). On the other hand, if \( z \) dominates no such \( q \) in \( B \setminus \{ z \} \), then the two arcs of the dipath \( xu_1y \) are in precisely the same 4-dicycles. The same holds for the dipaths \( vxy \) and \( u_1vy \).

We now define a permutation \( \tau \) of \( E(T) \) as follows: Each arc joining two vertices of \( A \cup B \) and each arc not in \( T(4) \) are fixed by \( \tau \). Furthermore, \( \tau \) permutes the arcs between the vertices of \( \{ x, u_1, v, y \} \) such that \( \pi \circ \tau \) maps \( xu_1yv \) onto \( x'_2x'_1v'x'_2 \), and we let \( \tau \) permute the arcs from \( z \) to \( \{ x, u_1, v \} \) such that the restriction of \( \pi \circ \tau \) to the subtournament induced by \( \{ x, y, z, u, u_1, v \} \) is induced by an isomorphism. Finally, \( \tau \) permutes the arcs having mates such that \( \pi \circ \tau \) and \( (\pi \circ \tau)^{-1} \) preserve 4-dicycles. Then \( \pi \circ \tau \) and \( (\pi \circ \tau)^{-1} \) have fewer bad 3-dicycles than \( \pi \) and \( \pi^{-1} \). This contradiction proves (8).

By applying (8) to the converse of \( T \) and \( T' \) we obtain:

(9) \( C \) has no augmentation

\( C_3; xyzv_1x \text{ such that } \pi(C_3) = z'u'x'_2y'z' \).

We define \( H \) (respectively \( K \)) as the set of vertices dominated by \( x \) and dominating \( y \) (respectively dominating \( x \) and dominated by \( z \)). Any vertex of \( H \) (respectively \( K \)) can play the role of \( u \) (respectively \( v \)) in the preceding arguments. In particular we have

(10) Each vertex of \( H \) is dominated by \( z \); each vertex of \( K \) dominates \( y \).

If \( u_1 \) is a vertex of \( H \), then the dicycle \( xu_1yvz \) corresponds to a dicycle \( x'_1u'_1y'z'x'_1 \), and we define \( H' = \{ u'_1| u_1 \in H \} \). We define \( K' \) similarly. If there are vertices \( u_1 \in H, v_1 \in K \) such that \( u'_1 = v'_1 \), then we replace \( v \) by \( v_1 \) in the preceding arguments and obtain a contradiction to (8). Combining this with (6) we get

(11) \( H' \cap K' = \emptyset \), and each vertex of \( H' \) is dominated by each vertex of \( K' \); each vertex of \( H \) is dominated by each vertex of \( K \).

We now extend (8), (9) by showing

(12) \( H = \{ u \}, K = \{ v \} \).

Proof of (12). Suppose (12) is false. Then we can assume that \( u_1 \in H \setminus \{ u \} \). (For if \( H = \{ u \} \) we consider the converse of \( T \) and \( T' \) instead of \( T \) and \( T' \).) Since there is no 4-dicycle in \( T \) that contains both an arc of \( xuy \) and an arc of \( xu_1y \) we have

(12a) \( x'_1 \) dominates \( y' \).
Since $xu_1yz$ and $u_1yzvu_1$ are 4-dicycles we have

\[(12b) \quad \pi(xu_1) = x'_1u'_1, \quad \pi(u_1y) = u'_1y', \quad \pi(zx) = z'x'_1, \quad \pi(xu) = x'_1u', \quad \text{and} \quad \pi(vu_1) = v'u'_1.\]

\[(12c) \quad x'_2 \text{ dominates } z'.\]

Proof of (12c). Suppose (reductio ad absurdum) that $z'$ dominates $x'_2$. Then $z'x'_2x'_1y'z'$ is a 4-dicycle corresponding to a 4-dicycle $yzpqy$ in $T$ where $p \neq x$ and $q \neq x$. Since none of the dipaths $xyzp$ and $qyzz$ can be extended into a 4-dicycle, $x$ dominates $p$ and is dominated by $q$. Hence $p \notin K$ and $q \notin H$. If $p \notin H$ and $q \notin K$, then $T$ has three 4-dicycles containing $pq$ and an arc of $xyzx$ which are pairwise arc-disjoint (except for $pq$). This gives a contradiction in $T'$ (because $\pi(zx) = z'x'_1$ by (12b)). On the other hand, if $p \in H$ and $q \in K$, then we can assume that $p = u_1$ and $qxu_1zvx$ is a 4-dicycle. But $T'$ has no 4-dicycle containing $\pi(xu_1) = x'_1u'_1$, $\pi(zx) = z'x'_1$, and $\pi(u_1q) = \pi(pq) \in \{z'x'_1, x'_1x', x'_1y'\}$. Finally, if $p \in H$ and $q \in K$, then we can assume that $q = v$ and then the 4-dicycle $xyvux$ gives a contradiction in $T'$. This proves (12c).

If $K \neq \{v\}$, then we can prove that $z'$ dominates $x'_2$ (in the same way as we proved (12a)) and hence (12c) implies

\[(12d) \quad K = \{v\}.\]

In $T$ there is no 4-dicycle that contains $zx$ and one of $vx$, $xy$. Hence $T'$ has no 4-dicycle containing $\pi(zx) = z'x'_1$ and $v'x'_2$ and so

\[(12e) \quad v' \text{ dominates } x'_1.\]

Since there is no 4-dicycle in $T$ containing $zx$ and $xu$ we conclude that $z'$ dominates $u'$. If $u'$ dominates $x'_2$ then $T'$ contains the 4-dicycles $u'x'_2z'v'u'$ and $u'x'_2y'z'u'$, but it is easy to see that the corresponding 4-dicycles cannot be present in $T$; hence

\[(12f) \quad T' \text{ contains the arcs } z'u' \text{ and } x'_2u'.\]

Now we consider the 3-dicycle $y'z'x'_1y'$ in $T'$. It has augmentations $C'_1: \ y'z'x'_1u'y', \ y'z'x'_1u'_1y'$, and $y'z'v'x'_1y'$, and the latter 4-dicycle corresponds to a 4-dicycle $yzvqy$ where $q \notin \{x, y, z, u, u_1, v\}$. So we can apply all the preceding arguments starting with (6) to the 3-dicycle $y'z'x'_1y'$. We have already noticed that $z'$ dominates $u'$, and (6) also implies that

\[(12g) \quad T' \text{ contains the arc } v'y', \text{ and } T \text{ contains the arc } qx.\]
Furthermore, (12c) and (12f) imply that

(12h) $q$ dominates $z$ and $u$,

and (12d) implies that

(12i) $x_1' y' z' x_1'$ has only one augmentation at $z' x_1'$.

We now define $A$ (respectively $B$) as the set of vertices dominated by (respectively dominating) each of $x, y, z$. Then $A, B, H, \{x, y, z, v\}$ is a partition of $V(T)$. We now claim:

(12j) There is no arc from $A$ to $v$.

Proof of (12j). Suppose (reductio ad absurdum) that $T$ contains an arc $av$ where $a \in A$. Then the 4-dicycle $avqy$ corresponds to a 4-cycle $v'x_1' y'a'v'$. Furthermore, the 4-dicycle $avyz$ must correspond to $a'v'y'z'a'$, and hence $\pi(av) = a'v'$ and $\pi(za) \in \{z'a', v'y'\}$. But now the 4-dicycle $avqza$ gives a contradiction in $T'$. This proves (12j).

Furthermore, we show

(12k) $T$ has no arc from $A$ to $B$.

Proof of (12k). Suppose (reductio ad absurdum) that $T$ has an arc $ab$ where $a \in A, b \in B$. For each arc $e$ in $\{xy, yz, zx, vz\}$, there is a unique 4-dicycle $C_e$ in $T$ containing $ab$ and $e$ and no other arc in $\{xy, yz, zx, vz\}$. Using this it is easy to see that $\pi(ab) = a'b'$ has no end in common with any of $\pi(yz) = y'z'$, $\pi(zx) = z'x'_1$, $\pi(zv) = z'v'$, and $\pi(xy)$ (which is one of $v'x'_2, x'_2 y'$). If $\pi(xy) = x'_2 y'$, then $T'$ has a 4-dicycle $a'b'x'_2 x'_1 a'$ which contains $a'b'$ and an additional arc from each of $\pi(C_{xy}), \pi(C_{zx})$. But this gives a contradiction in $T$. On the other hand, if $\pi(xy) = v'x'_2$, then the 4-dicycle $a'b'v'x'_1a'$ in $T'$ gives a contradiction in $T$. This proves (12k).

Using (12a)–(12k) and (10) and (7) (which applies to any vertex of $H$) we conclude that each of $xy, vx$ is only contained in one 4-dicycle, namely $xyzuvx$. By the same reasoning in $T'$, each of the arcs $x'_1 y', v'x'_1$ is contained in only one 4-dicycle, namely $x'_1 y'z'v'x'_1$. Furthermore, $xy$ is contained in only one 3-dicycle, and $vx$ is contained in no 3-dicycle in $T$. Similarly, $x'_1 y'$ is in only one 3-dicycle, and $v'x'_1$ is in no 3-dicycle in $T'$. Hence none of $xy, vx, qy, vq$ are in a non-bad 3-dicycle of $T$. So there exists a permutation $\tau$ of $\{xy, vx, qy, vq\}$ such that $\pi \circ \tau(C)$ is the 3-dicycle $x'_1 y'z'x'_1$ (we regard $\tau$ as a permutation of $E(T)$), and such that $\pi \circ \tau$ and $(\pi \circ \tau)^{-1}$ preserve 4-dicyles. (Just let $\tau$ be any permutation such that $\{\tau(xy), \tau(xy)\} = \{vq, qy\}$ and $\pi \circ \tau(xy) = x'_1 y'$.) Now $\pi \circ \tau$ and $(\pi \circ \tau)^{-1}$ have fewer bad 3-dicyles than $\pi$ and $\pi^{-1}$. This contradiction finally proves (12).
We now define $A$ (respectively $B$) as the set of vertices dominated by (respectively dominating) each of $x$, $y$, $z$. By (12), $A, B, \{x, y, z, u, v\}$ is a partition of $V(T)$.

(13) There is no arc from $A$ to $v$ or from $u$ to $B$.

Proof of (13). Suppose (13) is false. Then we can assume that $T$ has an arc $av$ where $a \in A$. Then the 4-dicycle $avxya$ corresponds to a 4-dicycle $a'v'x'zy'a'$ in $T'$. The 4-dicycle $avyza$ corresponds to $a'u'y'z'a'$, and hence $\pi(ay) = a'y'$, $\pi(zy) = z'y'$, and $\{\pi(ay), \pi(zy)\} = \{v'y', z'a'\}$. By (7), $a$ dominates $u$, and now the 4-dicycle $avyza$ corresponds to $a'u'y'z'a'$, and hence $\pi(za) = z'a'$, $\pi(zy) = v'y'$, $\pi(au) = a'u'$. Now none of the dipaths $v'y'z'x'$ and $x_1u'y'a'$ can be extended into a 4-dicycle, and hence $T'$ contains the arcs $v'x'_1$ and $x'_1a'$. Then the 4-dicycle $a'v'x'_1x'a'$ shows that $\pi^{-1}(v'x'_2) = vx$ (and hence $\pi(xy) = x'_2y'$), and hence $\pi(xz) = z'x'_1$ and $\pi(yp) = y'x'_1$.

If $x'_2$ dominates $z'$, then the 4-dicycle $x'_2z'a'v'x'_1$ gives a contradiction in $T$. If there is an arc from $u$ to $B$, then the above reasoning applied to the converse of $T$ and $T'$ implies that $x'_2$ dominates $z'$ which, as we have just seen, gives a contradiction in $T$. This shows that

(13a) $z'$ dominates $x'_2$ and there is no arc from $u$ to $B$.

Since the dipath $x'_2x'_1a'u'$ cannot be extended to a 4-dicycle we have

(13b) $x'_2$ dominates $u'$.

We next prove:

(13c) There is no arc from $A$ to $B$.

Proof of (13c). Suppose (reductio ad absurdum) that $T$ contains such an arc $a_1b_1$. Then we consider the three 4-dicycles containing $a_1b_1$ and an arc of $xyza$. It is easy to see that $\pi(a_1b_1)$ has no end in common with any of $\pi(xy) = y'z'$, $\pi(xy) = x'_2y'$, and $\pi(zx)$ (which is one of $z'x'_1$, $x'_1u'$). (The reason being that, for each arc $e$ in $\{\pi(xy), \pi(yz), \pi(zx)\} = \pi(C)$, there is a 4-dicycle which contains $\pi(a_1b_1)$ and $e$ and no arc in $\pi(C) \setminus \{e\}$.) We now consider the 4-dicycle containing $\pi(a_1b_1)$ and $x'_2x'_1$ (if $\pi(zx) = z'x'_1$) or $x'_2u'$ (if $\pi(zx) = x'_1u'$). The corresponding 4-dicycle in $T$ would have to contain $a_1b_1$ and two additional arcs from the above three 4-dicycles. But this is impossible and we have proved (13c).
(13d) Each of the arcs $zx, xu$ is contained in only one 4-dicycle, namely $zxuyz$, and the only 3-dicycle which contains one of $zx, xu$ is $xyzx$.

The 3-dicycle $y'z'x'_2y'$ has augmentations $x'_2u'y'z'x'_2$ and $y'z'v'x'_2y'$, and $a'$ is dominated by $y'$ and dominates $v'$. So we can apply all arguments in the proof of (13) to $T'$, and we obtain the statement corresponding to (13d):

(13e) Each of the arcs $z'x'_2, x'_2u'$ is contained in only one 4-dicycle, and $y'z'x'_2y'$ is the only 3-dicycle containing one of $z'x'_2, x'_2u'$.

From (13d), (13e) it follows that we can define an appropriate permutation $\tau$ of $\{zx, xu, \pi^{-1}(z'x'_2), \pi^{-1}(x'_2u')\}$ such that $\pi \circ \tau$ and $(\pi \circ \tau)^{-1}$ preserve 4-cycles and all those 3-cycles that are preserved by $\pi$ and $\pi^{-1}$, and such that $(\pi \circ \tau)^{-1}$ maps the 3-dicycle $y'z'x'_2y'$ onto $xyzx$. (Just let $\tau$ be any permutation such that $\{\tau(zx), \tau(xu)\} = \{\pi^{-1}(z'x'_2), \pi^{-1}(x'_2u')\}$ and $\pi \circ \tau(zx) = z'x'_2$.) This contradiction proves (13).

(14) There is no arc from $A$ to $B$.

Proof of (14). Suppose (reductio ad absurdum) that $T$ has an arc $ab$ where $a \in A$, $b \in B$. By (13), $T$ has, for each arc $e \in \{xy, yz, zx, uy, zv\}$ a 4-dicycle $C_e$ containing $ab$ and $e$ and no other arc of the above arc set. This implies that $\pi(ab) = a'b'$ has no end in $\{x'_1, x'_2, u', y', z', v'\}$. (Clearly, $a'b'$ has no end in $\{y', z'\}$, and $a' \neq u', b' \neq v'$. If $b' = u'$, then any 4-dicycle containing $a'b' = \pi(ab)$ and $y'z' = \pi(yz)$ also contains $u'y' = \pi(uy)$, a contradiction. So $b' \neq u'$, and similarly, $a' \neq v'$. Since there is a 4-dicycle containing $u'y'$ and $a'b'$, we have $a' \neq x'_2$. If $x'_2 = b$, then any 4-dicycle containing $a'b'$ and $y'z'$ also contains $x'_2y'$ which is in $\{\pi(xy), \pi(xy)\}$. This contradiction shows that $x'_2 \notin \{a', b'\}$. Similarly, $x'_1 \notin \{a', b'\}$.) Since the above 4-cycles $C_{xy}$ and $C_{uv}$ have the dipath $yab$ in common, we have $\pi(xy) = x'_2y'$. By a similar argument, $\pi(zx) = z'x'_1$, and hence the arcs of $\pi(C)$ are the arcs of the dipath $x'_2y'z'x'_1$. But then the 4-dicycle $a'h'x'_2x'_1a'$ contains $\pi(ab)$ and an additional arc from each of $\pi(C_{xy}), \pi(C_{xy})$. This leads to a contradiction in $T$, and the proof of (14) is complete.

By (13), (14) we have

(15) Each arc joining $x$ with one of $u, y, z, v$ is contained in only one 4-dicycle and in no non-bad 3-dicycle, and each 4-dicycle that contains $yz$ contains one of $uy, vz$. 

Next we show:

(16) \( T' \) contains the arc \( x'_2 z' \).

Proof of (16). Suppose (reductio ad absurdum) that \( z' \) dominates \( x'_2 \). Since every 4-dicycle in \( T \) that contains \( yz \) also contains one of \( uy, \tau v \) we conclude that \( y' \) dominates \( x'_1 \) (since otherwise \( z'x'_2 x'_1 y'z' \) is a 4-dicycle giving a contradiction in \( T \)). Each of the arcs \( xu, xy, zx, vx \) is contained in only one 4-dicycle, and hence the dipath \( x'_2 y'x'_1 u' \) cannot be extended to a 4-dicycle. This shows that \( x'_2 \) dominates \( u' \), and now \( x'_2 y'z'x'_2 \) is a bad 3-dicycle with augmentations \( x'_2 u'y'z'x'_2 \) and \( x'_2 y'z'v'x'_2 \), and hence we can apply all the preceding arguments to this 3-dicycle. In particular, each of the arcs \( x'_2 u', x'_2 y', z'x'_2, v'x'_2 \) is contained in only one 4-dicycle. The 4-dicycle \( z'x'_2 u'y'z' \) must correspond to a 4-dicycle \( axyza \) in \( T \). Since each of the arcs \( xu, zx, au, za, vx, xy \) is contained in only one 4-dicycle and in no non-bad 3-dicycle, there is a permutation \( \tau \) of \( E(T) \) such that \( \tau \) preserves 4-dicycles and those 3-dicycles which are preserved by \( \tau \), and such that \( \tau \) maps \( yz \) onto \( x'_1 y'z'x'_2 \). (Just let \( \tau \) be any permutation of \( \{xz, xu, za, au, vx, xy\} \) such that \( \{\tau(zx), \tau(xu)\} = \{za, au\}, \{\tau(za), \tau(au)\} = \{zx, xu\}, \tau(xy) = x'_2 y', \tau(zx) = z'x'_2 \).) This contradiction proves (16).

If we apply (16) to the converse of \( T \) and \( T' \), we obtain

(17) \( T' \) contains the arc \( y'x'_1 \).

Since each arc joining \( x \) with \( \{u, y, z, v\} \) is contained in only one 4-dicycle, we have

(18) \( T' \) contains the arcs \( x'_2 u', v'x'_1 \) and hence also the arcs \( z'u', v'y' \).

(For, if \( T' \) contains \( u'z' \), say, then \( T' \) has two 4-dicycles containing \( x'_1 u' \).)

We can assume without loss of generality that \( A \neq \emptyset \) and since \( T \) is strong, \( T \) has, by (13), (14), an arc \( au \) where \( a \in A \). The 4-dicycle \( axyza \) corresponds to a 4-dicycle \( a'u'y'z'a' \).

(19) \( a' \) is dominated by \( x'_1 \).

Proof of (19). Suppose (reductio ad absurdum) that \( a' \) dominates \( x'_1 \). Since \( x'_1 u' \) is contained in only one 4-dicycle, \( a' \) dominates \( y' \). Since every 4-dicycle containing \( yz \) also contains one of \( uy, \tau v \) (by (15)), the dipath \( v'y'z'a' \) cannot be extended into a 4-dicycle, and hence \( v' \) dominates \( a' \). Now the 3-dicycle \( a'y'z'a' \) has two augmentations satisfying (5), and then (15) implies that each of \( a'u', a'y', z'a', v'a' \) is contained in only one
4-dicycle and in no non-bad 3-dicycle. The 4-dicycle \(a'y'z'v'a'\) corresponds to a 4-dicycle \(yzvby\) where \(b \in B\). Now there exists a permutation \(\tau\) of the arcs \(za, au, zx, xu, vx, xy, vb, by\) by such that \(\pi \circ \tau \) and \( (\pi \circ \tau)^{-1} \) preserve 4-dicyles and those 3-dicyles preserved by \(\pi\) and \(\pi^{-1}\) and such that \(\pi \circ \tau\) maps \(xyzx\) onto \(a'y'z'a'\). (The above arcs are paired such that a 4-dicycle contains one arc in a pair iff it contains the other. So we just choose \(\tau\) such that it permutes the pairs and such that \(\pi \circ \tau(zx) = z'a', \pi \circ \tau(xy) = a'y'\).) This contradiction proves (19).

By (19), the 3-dicycle \(C'' : y'x' u' y'\) has two augmentations \(C''_1 : x'_1 a'u'y'x'_1\) and \(C''_2 : x'_1 u'y'z'x'_1\). We claim that \(C''_1, C''_2\) satisfy (5). Since \(\pi^{-1}(u'y') = uy\) and \(\pi^{-1}(x'y') = zx, xu\) it follows that \(C''\) is bad. Since \(\pi^{-1}(a'u') = za, au\) and \(C''_1\) has precisely the arcs \(a'u', u'y'\) in common with \(a'u'y'z'a' = \pi(auyz)\), it follows that \(\pi(au) = a'u'\) and \(\pi(za) = z'a'\). Furthermore, \(\pi^{-1}(C''_1)\) is of the form \(auyx, a\) where \(x_1 \in A\). In particular, \(\pi^{-1}(C''_1) \cup \pi^{-1}(C''_2)\) has six vertices. Hence \(C''_1, C''_2\) satisfy (5), and so we can apply (6)–(19) to \(C''_1, C''_2\).

By (17), \(u\) dominates \(x_1\). If \(T''\) has a vertex dominated by each of \(u', y', x'_1\), then (19) (applied to \(u'y'x'_1 u'\) instead of \(xyzx\)) implies that the 3-dicycle \(ux_1 au\) has an augmentation at \(x_1 a\). As in the preceding paragraph we conclude that \(ux_1 au\) satisfies (5) and so we can apply (6)–(19) to \(ux_1 au\) instead of \(xyzx\). Note that the set of vertices dominated by each of \(u, x_1, a\) is a proper subset of \(A\). If \(T''\) has no vertex dominated by each of \(u', y', x'_1\), then it is easy to see that \(A = \{x_1, a\}\).

So far, we have considered the bad 3-dicycle \(xyzx\) in \(T\), then the bad 3-dicycle \(x'_1 u'y'z'x'_1\) in \(T'\), then the 3-dicycle \(x_1 au x_1\) which is bad if \(|A| > 2\). We continue like this until we get a bad 3-dicycle in \(T\) or \(T''\) such that there is no vertex which is dominated by all vertices of that bad 3-dicycle. In the course of this we determine completely the structure of the subtournament of \(T\) induced by \(A \cup \{x, y, z, u, v\}\). More precisely, this subtournament is isomorphic to a tournament of the form \(N_k\) or \(N_k - \{p_1, q_1\}\). By considering the converse of \(T\) and \(T''\) we conclude that \(B \cup \{x, y, z, u, v\}\) induces a subtournament of type \(M_k\) or \(M_k - \{q_{k-1}, p_k\}\). Similar arguments apply to \(T''\) and it now follows easily (since \(T\) and \(T''\) have the same number of vertices) that one of \(T, T''\) is of the form \(M_k\) and the other is of the form \(N_k\). Since \(M_k\) and \(N_k\) are anti-isomorphic the proof of Theorem 4.5 is complete. 

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WHITNEY'S 2-SWITCHING THEOREM

REFERENCES