JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 98, 435–457 (1984)

Inequalities for Absolute Moments of a Distribution: From Laplace to Von Mises

PAUL R. BEESACK

Carleton University, Ottawa K1S 5B6, Canada

Submitted by R. P. Boas

INTRODUCTION

Inequalities involving the absolute moments, ν_r, of a probability distribution have a long and honourable history dating (at least) back to Laplace [1] in 1810 and Gauss [2] in 1821. For example, Gauss stated the inequality $5\nu_4 \geqslant 9\nu_2^2$ in 1821; this is a special case of the so-called Gauss–Winckler inequality $[(n+1)\nu_n]^{1/n} \leqslant [(r+1)\nu_r]^{1/r}$ first stated by Winckler [3] in 1866 (with a false proof). Laplace dealt with a probability distribution function Q corresponding to a mass (1) on $[0, \alpha] = I$ with $Q(x)$ concave on I, and proved that $\nu_2 \leqslant (\frac{1}{3})\alpha^2$. Extensive generalizations of this inequality were given by Winckler and seem to have been ignored since, although some special cases appear in the 1950 monograph of Fréchet [10].

In this paper, which is of a semi-historical nature, we deal with a large number of such inequalities from a more modern point of view. We correct, resurrect, or extend these inequalities and present a few new ones. In this introductory section we shall give an outline of what appears in subsequent sections. To this end, let $F: \mathbb{R} \to [0, 1]$ be a proability distribution function, let $a \in \mathbb{R}$, and set

$$\nu_r = \int_{\mathbb{R}} |x-a|^r \, dF(x), \qquad P(x) = Pr(|X-a| \geqslant x), \qquad Q(x) = 1 - P(x).$$

In Sections 1–4 we give Čebyšev type inequalities of the form $P(x_1) \leqslant A(r, \nu_r, x_1)$ giving upper bounds for $P(x_1)$. In particular Section 2 gives a correct proof of an inequality due to Camp [6], Section 3 does the same for two inequalities of von Mises [8], and Section 4 gives a comparison of these results when both apply. (In most cases the upper bounds of von Mises are better than that of Camp.) Special cases of von Mises' bounds are given in Section 5 for which it is also noted that the case $r = 2$ was given by Gauss and the cases $r \geqslant 1$ by Winckler.

In Section 6, which constitutes almost a third of the paper, we present

0022-247X/84 $3.00

versions of more than twenty inequalities of Winckler involving $\chi_r(x) = \int_0^x t^r\, dQ(t)$ $(0 \leqslant x \leqslant \alpha \leqslant \infty)$ and $\nu_r = \chi_r(\alpha)$. As examples of such inequalities, *in case Q is concave on $[0, \alpha)$ we have*

$$\chi_r(x) \geqslant \left(1 + \frac{1}{r}\right)^r x_1^r [Q(x) - Q(x_1)] \qquad \text{if} \quad Q(x_1) > \frac{r}{r+1} Q(x), \quad (28)$$

$$[(n+1)\nu_n]^{1/n} \leqslant \left\{ \frac{r}{r+1} \left(\frac{r}{r-n}\right)^{1/n} \right\} [(r+1)\nu_r]^{1/r} \qquad (0 < n < r), \quad (30)$$

$$\chi_r(x) \leqslant \frac{x^r}{r+1} Q(x) \qquad (r > 0, 0 \leqslant x \leqslant \alpha), \quad (32)$$

$$(r+1)\, x^n \chi_r(x) - (n+1)\, x^r \chi_n(x) + \frac{n(n+1)}{r+n} \chi_{r+n}(x)$$
$$\leqslant \frac{n(n+1)}{(r+n)(r+n+1)} x^{r+n} Q(x) \qquad (r, n, x > 0), \quad (34)$$

$$m(n+1)\, x^m \chi_n(x) - n(m+1)\, x^n \chi_m(x)$$
$$\leqslant (m-n)\, x^{m+n} Q(x) \qquad (m \geqslant n > 0, x \geqslant 0), \quad (41)$$

$$\left(1 + \frac{1}{n}\right) \frac{\nu_n}{\alpha^n} - \left(1 + \frac{1}{m}\right) \frac{\nu_m}{\alpha^m} \leqslant \frac{1}{n} - \frac{1}{m} \qquad (m \geqslant n > 0, \alpha \text{ finite}). \quad (43)$$

Moreover, *if $Q'(x)$ is continuous and nonincreasing on $[0, \alpha)$ then*

$$(n+1)\, x^m \chi_n(x) \geqslant (m+1)\, x^n \chi_m(x) \qquad (m \geqslant n \geqslant 0), \quad (45)$$

$$(n+1)(r-m) \frac{\nu_n}{\alpha^n} + (r+1)(m-n) \frac{\nu_r}{\alpha^r}$$
$$\geqslant (m+1)(r-n) \frac{\nu_m}{\alpha^m} \qquad (m \geqslant n \geqslant 0, r \geqslant m+n, \alpha \text{ finite}). \quad (48)$$

The special case $r = 2$, $x = \alpha$ (so $\chi_r(x) = \nu_r$) of (28) was stated in 1821 by Gauss, as well as a corresponding inequality for the case $Q(x_1) \leqslant (r/(r+1))\, Q(x)$, also for $r = 2$, $x = \alpha$. Inequality (30) is what was proved (correctly) by Winckler. By an incorrect modification of the proof he also "obtained" the improved Gauss–Winckler inequality, which is (30) with the factor $\{...\}$ replaced by 1.

The case $x=\alpha$, $r=2$ of (32) gives the Laplace inequality mentioned above. In case $0<\alpha<\infty$, inequalities of the form (34), (41), (45) yield corresponding inequalities involving the absolute moments ν_r, ν_n, etc., on setting $x=\alpha$; in fact (43) follows from (41) in this way.

The first valid proof of the Gauss–Winckler inequality appears to be due to Faber [7] in 1926. In Section 7 we give a modification of a proof of it by von Mises [8] which makes use of *Liapounov's inequality*; a short proof of this latter inequality is also given in Section 7. The paper concludes by using this same method of proof to give a powerful extension of the Gauss–Winckler inequality, and an application of it to obtain the (probably) new inequality

$$\{\Gamma(n+k+1)\}^{1/n} \leqslant \{\Gamma(r+k+1)\}^{1/r} \qquad (0 \leqslant n \leqslant r, k=1, 2,...), \quad (50)$$

involving the gamma function of analysis.

1

Let $F\colon \mathbb{R} \to [0, 1]$ be a probability distribution function and let $a \in \mathbb{R}$. Then the rth *absolute moment of F about a* is defined by

$$\nu_r = \nu_{r,a} = \int_{-\infty}^{\infty} |x-a|^r \, dF(x), \qquad r \geqslant 0. \quad (1)$$

It will be more convenient to deal with the probabilities

$$P(x) = Pr(|X-a| \geqslant x), \qquad Q(x) = 1-P(x) = Pr(|X-a|<x), \quad (2)$$

than with F in what follows. In particular we note that

$$Q(x) = F(a+x) - F(a-x), \qquad x \geqslant 0,$$

from which it follows that

$$\int_0^{\infty} x^r \, dQ(x) = \nu_r. \quad (3)$$

Note that $Q(0)=0$, $Q(\infty)=1$ and Q is *nondecreasing* on $[0, \infty)$. We shall give several inequalities relating ν_r and $P(x_1)$ under appropriate hypotheses on Q. These results are essentially due to Gauss [2], Winckler [3], Camp [6], and von Mises [9]. We shall first deal with the inequalities of Camp and von Mises since those of Winckler and Gauss then follow as special cases.

2. Camp's Inequality

Suppose that $x_0 > 0$, $r > 0$, $p > 1$, $k = p(1 + (1/r))$, *and* $x_1 = px_0$. *If* $Q'(x)$ *exists and is nonincreasing on the interval* $[x_0, kx_0]$, *then*

$$P(x_1) \leqslant \frac{(p-1)(r+1)}{1 + (p-1)(r+1)\, p^r(1 + 1/r)^r} \left\{ \frac{v_r}{x_0^r} + \frac{1}{(p-1)(r+1)} \right\}. \tag{4}$$

To prove (4), we shall use the following lemma, also due to Camp [6].

Lemma. *Let* P *be nonincreasing on* $[x_0, kx_0]$ *with* $P'(x)$ *nondecreasing on this interval, where* $x_0 > 0$, $k = p(1 + (1/r))$ *with* $p > 1$, $r > 0$. *Let* $y(x) = P(x_0) + m(x - x_0)$ *be the equation of the secant line to the graph of* P *through the points* $(x_0, P(x_0))$ *and* $(x_1, P(x_1))$, *where* $x_1 = px_0$. *Then*

$$\int_{x_0}^{kx_0} x^{r-1} P(x)\, dx \geqslant \int_{x_0}^{kx_0} x^{r-1} y(x)\, dx. \tag{5}$$

We first use (5) to prove (4) where P, Q are related by (2), noting that the hypotheses on P are implied by those on Q. To this end, we have

$$\begin{aligned} v_r &= \int_0^{\infty} x^r\, d(1-P) \geqslant \int_{x_0}^{\infty} x^r\, d(1-P) = -\int_{x_0}^{\infty} x^r\, dP \\ &= x_0^r P(x_0) + r \int_{x_0}^{\infty} x^{r-1} P(x)\, dx, \end{aligned}$$

since $X^r P(X) = X^r \int_X^{\infty} d(1-P) \leqslant \int_X^{\infty} x^r\, d(1-P) = \int_X^{\infty} x^r\, dQ \to 0$, as $X \to \infty$. Hence,

$$v_r \geqslant x_0^r P(x_0) + r \int_{x_0}^{kx_0} x^{r-1} P(x)\, dx \geqslant x_0^r P(x_0) + r \int_{x_0}^{kx_0} x^{r-1} y(x)\, dx,$$

using the lemma. Using $m = [P(x_1) - P(x_0)]/(p-1)x_0$, this eventually reduces to

$$\begin{aligned} v_r &\geqslant x_0^r \left\{ -\frac{P(x_0)}{(p-1)(r+1)} + P(x_1) \frac{1 + (p-1)(r+1)\, p^r(1 + (1/r))^r}{(p-1)(r+1)} \right\}, \\ &\geqslant x_0^r \left\{ -\frac{1}{(p-1)(r+1)} + P(x_1) \frac{1 + (p-1)(r+1)\, p^r(1 + (1/r))^r}{(p-1)(r+1)} \right\}, \end{aligned}$$

from which (4) follows.

Proof of Lemma. Let $z(x) = P(x_1) + P'(x_1)(x - x_1)$ be the equation of the tangent line to the graph of P at the point $(x_1, P(x_1))$. Since P' is nondecreasing we have $P(x) \geqslant z(x)$ on $[x_0, kx_0]$, so that (5) will follow if we can show that

$$\int_{x_0}^{kx_0} x^{r-1} z(x)\,dx \geqslant \int_{x_0}^{kx_0} x^{r-1} y(x)\,dx. \tag{6}$$

Using the preceding computation,

$$r\int_{x_0}^{kx_0} x^{r-1} y(x)\,dx = x_0^r \left\{ -\left[1 + \frac{1}{(p-1)(r+1)}\right] P(x_0) + \frac{1 + (p-1)(r+1)\,p^r(1+(1/r))^r}{(p-1)(r+1)} P(x_1) \right\}.$$

A similar computation gives

$$r\int_{x_0}^{kx_0} x^{r-1} z(x)\,dx = x_0^r \left\{ [p^r(1+(1/r))^r - 1]\,P(x_1) + \left(p - \frac{r}{r+1}\right) x_0 P'(x_1) \right\}.$$

From this we see after further reduction that (6) is equivalent to

$$\frac{P(x_1) - P(x_0)}{x_0(p-1)} \frac{r(p-1)+p}{r+1} \leqslant P'(x_1) \frac{r(p-1)+p}{r+1},$$

or by the mean value theorem, to

$$P'(X) \leqslant P'(x_1), \qquad \textit{where} \quad x_0 < X < x_1 = px_0.$$

Since P' is nondecreasing, the result follows.

Remark 1. It is easy to see that equality is attained in (4) for the case that $Q(x) = 0$ for $0 \leqslant x \leqslant x_0$, $Q(x) = 1$ for $x \geqslant kx_0$, and $Q(x) = (x - x_0)/(k-1)x_0$ for $x_0 \leqslant x \leqslant kx_0$. In this case of course, $P(x) \equiv y(x) \equiv z(x)$ in the lemma, and $P(x_0) = 1$.

Remark 2. In [6], a different notation was used from that used here. The proof of (6) given in [6] does not appear to be valid. Moreover in [6] only the special case that $F(x) = \int_{-\infty}^{x} f(t)\,dt$ was considered, where F had mean value 0, and it was assumed that $f(x)$ was a monotonic decreasing function of $|x|$ for $|x| \geqslant x_0$. Taking $a = 0$ in the preceding, $Q'(x) = f(x) - f(-x)$ is thus nonincreasing for $x \geqslant x_0$ as required.

Remark 3. In case $Q'(x)$ exists and is nonincreasing for $x \geqslant 0$, we may set $x_0 = c\sigma$, $x_1 = \lambda\sigma$, $p = \lambda/c$ in (4) and let $c \to 0$ to obtain

$$P(\lambda\sigma) \leqslant v_r/[\lambda\sigma(1 + (1/r))]^r, \tag{7}$$

precisely as in Camp [6]. As noted there, this is an improvement of Pearson's inequality [5].

Remark 4. Note that the case $x_0 > 0$ allows one to handle the case of a *bimodal distribution* F corresponding to a continuous probability density function f having maxima at $x = a \pm x_0$.

3. Von Mises' Inequality [9]

Under the assumptions of Section 1, we now weaken somewhat the hypotheses on Q. Here *we only assume that there are three values* x_0, x_1, z, *with* $0 \leqslant x_0 < x_1 < z$, *such that the graph of* $Q(x)$ *for* $x \geqslant x_0$ *always lies under the line* D *passing through the two points* (x_1, Q_1) *and* $(z, 1)$, *where* $Q_1 = Q(x_1)$. *That is, we assume*

$$Q(x) \leqslant Q_1 + \frac{1 - Q_1}{z - x_1}(x - x_1) \qquad \textit{for} \quad x \geqslant x_0. \tag{8}$$

Then

$$P(x_1) = 1 - Q(x_1) \leqslant \frac{v_r}{c^r}, \tag{9}$$

where c *is the unique solution* $(c > x_1)$ *of the equation*

$$\frac{x_0^{r+1}}{c^r} + rc = (r + 1)x_1. \tag{10}$$

Moreover, if

$$(r + 1)x_1 < \frac{x_0^{r+1}}{\alpha^r} + r\alpha, \tag{11}$$

where α *is the unique solution* $(\alpha > x_0)$ *of the equation*

$$\alpha^{r+1} - x_0^{r+1} = (r + 1)\, v_r(\alpha - x_0), \tag{12}$$

then

$$P(x_1) \leqslant \frac{\alpha - x_1}{\alpha - x_0}. \tag{13}$$

Moreover, when (11) *holds, the estimate* (13) *is always better than* (9), *while if* (12) *has a solution* $\alpha > x_1$, *which does not satisfy* (11), *then* (9) *is better than* (13).

To prove (9), let Q_0 be the distribution function with $Q_0(x) \equiv 0$ for $x < 0$, $Q_0(x) \equiv Q(x_0)$ for $0 \leqslant x \leqslant x_0$, $Q_0(x) \equiv 1$ for $x \geqslant z$, and $Q_0(x) = Q_1 + (1 - Q_1)(x - x_1)(z - x_1)^{-1}$ for $x_0 \leqslant x \leqslant z$. Then

$$\nu_r = \nu_r(Q) \geqslant \int_0^{x_0} x^r \, dQ + \nu_r(Q_0), \tag{14}$$

where $\nu_r(Q_0)$ denotes the rth moment about 0 of the distribution Q_0. For,

$$\begin{aligned}
\nu_r &= \int_0^{x_0} x^r \, dQ + \int_{x_0}^{z} x^r \, dQ + \int_z^{\infty} x^r \, dQ, \\
&\geqslant \int_0^{x_0} x^r \, dQ + z^r Q(z) - \left\{ x_0^r Q(x_0) + r \int_{x_0}^{z} x^{r-1} Q(x) \, dx \right\} + z^r [1 - Q(z)], \\
&\geqslant \int_0^{x_0} x^r \, dQ - \left\{ x_0^r \, Q_0(x_0) + r \int_{x_0}^{z} x^{r-1} Q_0(x) \, dx \right\} + z^r Q_0(z), \\
&= \int_0^{x_0} x^r \, dQ + \nu_r(Q_0),
\end{aligned}$$

where we have only used the fact that Q is nondecreasing on $[z, \infty)$, and $-Q(x) \geqslant -Q_0(x)$ for $x_0 \leqslant x \leqslant z$. It is easy to see that equality holds in (14) if and only if $Q(x) = Q_0(x)$ for $x \geqslant x_0$, and that $\nu_r(Q) = \nu_r(Q_0)$ if and only if $Q = Q_0$.

From (14), it follows that

$$\nu_r \geqslant \int_{x_0}^{z} x^r \frac{1 - Q_1}{z - x_1} \, dx = \frac{1 - Q_1}{z - x_1} \frac{z^{r+1} - x_0^{r+1}}{r + 1},$$

or

$$P(x_1) = 1 - Q_1 \leqslant (r + 1)\nu_r \frac{z - x_1}{z^{r+1} - x_0^{r+1}}. \tag{15}$$

Now (15) is satisfied for some $z > x_1$, by hypothesis. On the other hand, the right side of (15) assumes its maximum value for $z > x_1$ when its derivative with respect to z has the value 0, that is, when $z = c$, where

$$c^{r+1} - x_0^{r+1} = (r+1)\,c^r(c - x_1), \qquad c > x_1, \tag{10$'$}$$

or where c satisfies (10). Moreover this maximum value is found to be equal to ν_r/c^r, proving (9).

We note that the function g defined by

$$g(x) = x_0^{r+1}x^{-r} + rx, \qquad x \geqslant x_0,$$

has $g(x_1) = rx_1 + x_0(x_0/x_1)^r < (r+1)x_1$, and $g'(x) = r[1 - (x_0/x)^{r+1}] > 0$ for $x > x_0$, so that since $g(x) \to \infty$, as $x \to \infty$, Eq. (10) *has* a unique solution $c > x_1$.

For the second part, we observe that since $Q(x_0) \geqslant 0$, the slope of the line D cannot exceed that of the line D_0 passing through the points $(x_0, 0)$, (x_1, Q_1), so that z necessarily satisfies

$$\frac{1 - Q_1}{z - x_1} \leqslant \frac{Q_1}{x_1 - x_0},$$

or

$$z \geqslant x_0 + \frac{x_1 - x_0}{Q_1} \equiv z_0. \tag{16}$$

If the value $z = c$ given by (10) is $\geqslant z_0$, then the distribution $Q_0(x)$ corresponding to *this* value of z is admissible, and for it the equality sign holds in (14)—with $Q = Q_0$—and in (15)—with $z = c$—hence also in (9). Thus (9) is "best possible" in this case. On the other hand, suppose the solution c of (10) is less than z_0. Then the admissible values of the right side of (15), that is, values for $z \geqslant z_0$, are all less than or equal to its value for $z = z_0$, since $h(z) = (z - x_1)/(z^{r+1} - x_0^{r+1})$ is strictly decreasing for $z \geqslant c$. In this case we have

$$P(x_1) \leqslant (r+1)\nu_r \frac{z_0 - x_1}{z_0^{r+1} - x_0^{r+1}}. \tag{17}$$

From (16),

$$\frac{z_0 - x_1}{z_0 - x_0} = 1 - Q_1 = P(x_1),$$

whence (17) can be written as

$$\frac{z_0^{r+1} - x_0^{r+1}}{z_0 - x_0} \leqslant (r+1)\nu_r. \tag{18}$$

Since both $k_1(x) = (x^{r+1} - x_0^{r+1})/(x - x_0)$ and $k_2(x) = (x - x_1)/(x - x_0)$ are increasing functions for $x > x_0$ and $x > x_1$, respectively, it follows from (12) and (18) that $z_0 \leqslant \alpha$, and hence that

$$P(x_1) = \frac{z_0 - x_1}{z_0 - x_0} \leqslant \frac{\alpha - x_1}{\alpha - x_0},$$

proving (13) under the assumption $c < z_0$. Inequality (13) is also "best possible" in this case because equality holds in (17) for the distribution $Q = Q_0$, corresponding to the line D_0, for which $z = z_0$, and $\alpha = z_0$.

However, as noted by Fréchet [10, p. 161], the condition

$$c < z_0 = x_0 + \frac{x_1 - x_0}{Q_1}, \tag{19}$$

involves the very quantity Q_1 which is being estimated. Suppose now that α satisfies the condition (11). Since $g(x) = x_0^{r+1} x^{-r} + rx$ is increasing for $x > x_0$, it follows from (10) that $c < \alpha$. Now if $P(x_1) > (c - x_1)/(c - x_0)$ then it follows from $P(x_1) = (z_0 - x_1)/(z_0 - x_0)$ that $z_0 > c$, and so (13) holds as proved above. If $P(x_1) \leqslant (c - x_1)/(c - x_0)$ then since $c < \alpha$, it follows that

$$P(x_1) \leqslant \frac{c - x_1}{c - x_0} < \frac{\alpha - x_1}{\alpha - x_0},$$

so (13) again holds. Thus (13) holds whenever α satisfies (11). Moreover, in this case ($c < \alpha$) the estimate (13) is always *better* than the estimate (9). For, $c < \alpha$ together with (12) and (10′) implies

$$\frac{\alpha - x_1}{\alpha - x_0} = (r+1)\nu_r \frac{\alpha - x_1}{\alpha^{r+1} - x_0^{r+1}} < (r+1)\nu_r \frac{c - x_1}{c^{r+1} - x_0^{r+1}} = \frac{\nu_r}{c^r}. \tag{20}$$

We observe that it is impossible to have both $c < \alpha$ and $z_0 \leqslant c$ for the distribution $Q = Q_0$ corresponding to the line D with $z = c$, since if this were the case we would then have

$$P(x_1) = \frac{\nu_r}{c^r} > \frac{\alpha - x_1}{\alpha - x_0},$$

the equality sign holding since $z_0 \leqslant c$ (as noted after (16)), and the inequality sign holding by (20) since $c < \alpha$. We thus have a contradiction since (13) holds when $c < \alpha$.

Similarly, if Eq. (12) has a solution $\alpha > x_1$ such that (11) does not hold, then

$$g(\alpha) = \frac{x_0^{r+1}}{\alpha^r} + r\alpha \leqslant (r+1)x_1 = g(c)$$

implies that $\alpha \leqslant c$, and so (20) holds with $<$ replaced by $\geqslant$. Thus (9) is better than (13) in this case.

Remark 1. Condition (11) is equivalent to the condition $c < \alpha$, as noted above, and can be applied without computing c. In general, the situation with regard to the solutions α of (12) is as follows: either (12) has *no* solution $\alpha > x_0$ (this holds if $x_0^r \geqslant \nu_r$ clearly), or has a solution α, with $x_0 < \alpha \leqslant x_1$ (this holds if $x_0^r < \nu_r \leqslant (x_1^r + \cdots + x_0^r)/(r+1)$), and in both cases, (9) is our *only* estimate; or (12) has a solution α, with $x_1 < \alpha$. In this latter case, either (11) holds, in which case $c < \alpha$ and (13) is a better estimate than (9), or $\alpha \leqslant c$ and (9) is better than (13).

Remark 2. Von Mises [9] gave a geometric "proof" of the basic inequality (14). In his book, Fréchet [10, p. 159]—who must have remained unconvinced by von Mises' proof—altered the argument somewhat, and claimed that

$$\int_{x_0}^{z} x^r \, dQ \geqslant \int_{x_0}^{z} x^r \, dQ_0$$

which would be enough to prove (14). Fréchet's "proof" was also geometric. Unfortunately, the last inequality is false in general as can be seen by taking $x_0 = 0$, $x_1 = \frac{1}{2}$, $Q(x) = x^{1/2}$ for $0 \leqslant x \leqslant 1$, $Q_0(x) = Q_1 + Q_1'(x - \frac{1}{2}) = (1/(2\sqrt{2})) + (x/\sqrt{2})$, so $z = 3/(2\sqrt{2})$, and $r > \frac{1}{2}$.

4. Comparison of Bounds of Camp and von Mises

First we note that if Camp's hypothesis, that $Q'(x)$ is nonincreasing on $[x_0, kx_0]$, is satisfied then taking $Q_0(x) = Q_1 + Q_1'(x - x_1)$ we see that $z = x_1 + P(x_1)(Q_1')^{-1}$. Hence von Mises' hypothesis will be satisfied if and only if $x_1 + P_1(Q_1')^{-1} \leqslant kx_0 = px_0(1 + (1/r))$, which reduces to $P_1 \leqslant px_0 Q_1' r^{-1}$. This not only involves the quantity $P_1 = P(x_1)$ to be estimated, but in general will not be satisfied for large r. Hence in order that both estimates apply *we shall assume that $Q'(x)$ is nonincreasing on $[x_0, \infty)$, and that $x_0 > 0$.* (The useful case $x_0 = 0$ is excluded in Camp's estimate.)

We now show that when Eq. (12) *has a solution $\alpha > c$, then the von Mises estimate* (13) *is better than the estimate* (4) *of Camp*, that is

$$V_1 \equiv \frac{\alpha - x_1}{\alpha - x_0} = \frac{\alpha - px_0}{\alpha - x_0}$$
$$\leqslant \frac{(p-1)(r+1)}{1 + (p-1)(r+1)\,p^r(1 + (1/r))^r} \left\{ \frac{\nu_r}{x_0^r} + \frac{1}{(p-1)(r+1)} \right\} \equiv C_1.$$

Moreover, equality can hold only if $\alpha = px_0(1 + (1/r))$.

Writing $(\alpha - px_0)/(\alpha - x_0) = 1 - [(p-1)x_0/(\alpha - x_0)]$, and using the fact that

$$\frac{\alpha^{r+1} - x_0^{r+1}}{(r+1)(\alpha - x_0)} = v_r$$

by (12), it is easy to verify that $V_1 < C_1$ if and only if

$$(r+1)p^r\left(1+\frac{1}{r}\right)^r \alpha x_0^r < \alpha^{r+1} + x_0^{r+1}p(r+1)p^r\left(1+\frac{1}{r}\right)^r,$$

or

$$G(x) = x^{r+1} - (x - px_0)\,x_0^r(r+1)p^r\left(1+\frac{1}{r}\right)^r > 0 \qquad \text{for} \quad x = \alpha.$$

Now $G(px_0) = (px_0)^{r+1} > 0$, while

$$G'(x) = (r+1)\left[x^r - x_0^r\,p^r\left(1+\frac{1}{r}\right)^r\right], \qquad G'(px_0) < 0,$$

and $G''(x) > 0$ for all $x > 0$. Moreover $G'(x) = 0$ for $x = px_0(1 + (1/r))$, while $G(px_0(1 + (1/r))) = 0$. It follows that $G(x) \geqslant 0$ for all $x > px_0$, with strict inequality unless $x = px_0(1 + (1/r))$. Since $\alpha > c > x_1 = px_0$, this proves the assertion.

For example, for the standard normal distribution $N(x; 0, 1)$, we take $a = 0$ in Section 1, and $r = 1$, $x_0 = 0.5$, $p = 1.2$, so $x_1 = 0.6$. One finds $\alpha = 1.09577$, that (11) is satisfied, and $V_1 = 0.83215$, $C_1 = 0.83587$. Since $(1 + (1/r))\,px_0 = 1.2$, is close to α, one would expect V_1, C_1 to be close, from the above analysis. In this case, $v_1 = \sqrt{2/\pi} = 0.7978846$, $c = 0.9316625$, and the larger estimate, $v_1/c = 0.85641$; from tables, the actual value of $P(x_1)$ is 0.5485.

In case $\alpha \leqslant c$, or even more generally, the estimate (4) of Camp is only sometimes better than the estimate (9) of von Mises. Consider the simple case $r = 1$, when the bounds (4), (9), respectively, are

$$C_2 = \frac{2(p-1)}{1+4p(p-1)}\left\{\frac{v_1}{x_0} + \frac{1}{2(p-1)}\right\}, \qquad V_2 = \frac{v_1}{c},$$

where $c^2 - 2px_0 + x_0^2 = 0$, so $c = (p + \sqrt{p^2-1})x_0$. The condition $C_2 \leqslant V_2$ becomes

$$\frac{2(p-1)}{1+4p(p-1)}\frac{v_1}{x_0} + \frac{1}{1+4p(p-1)} \leqslant \frac{1}{p+\sqrt{p^2-1}}\frac{v_1}{x_0}.$$

Since

$$\frac{1}{p+\sqrt{p^2-1}} > \frac{1}{2p} = \frac{2(p-1)}{4p(p-1)} > \frac{2(p-1)}{1+4p(p-1)},$$

for all $p > 1$, we see that $C_2 < V_2$ holds for large p. However, if $v_1 < x_0$, then $V_2 < C_2$ holds for p near 1. We note that in this case, $\alpha = 2v_1 - x_0$, whence $\alpha \leqslant c$ holds precisely when $v_1 \leqslant \frac{1}{2}(p+1+\sqrt{p^2-1})x_0$. The latter condition also holds for large p so that Camp's estimate (4) is better than the better of von Mises' estimates for large p, when $r = 1$.

For general r, we again denote the estimates (4) and (9) by C_2 and V_2. Computation shows that $C_2 \leqslant V_2$ if and only if

$$v_r \left\{ x_0^r + (p-1)(r+1) \left[x_0^r p^r \left(1+\frac{1}{r}\right)^r - c^r \right] \right\} > x_0^r c^r, \tag{21}$$

where c is the unique solution ($c > x_1 = px_0$) of Eq. (10), that is, of

$$g(c) \equiv x_0^{r+1} x^{-r} + rc = (r+1)\, px_0.$$

Since $g(px_0(1+(1/r))) > (r+1)\, px_0$, it follows that $c < px_0(1+(1/r))$ so the term in square brackets in (21) is positive. Unfortunately, c is a function of both r and p, and I have found no simple condition that will assure (21). One can show that $(1+(\beta/r))\, px_0 < c(<(1+(1/r))\, px_0)$ if $\beta \in [0, 1)$ is such that

$$\frac{1}{(1+(\beta/r))^r} < (1-\beta)\, p^{r+1},$$

which is clearly satisfied for any $p > 1$ and all large r, or any $r > 0$ and all large p. Replacing v_r/c^r by $v_r/[(1+(\beta/r))\, px_0]^r$ we see that $C_2 < V_2$ if

$$v_r \left\{ 1 + (p-1)(r+1)\, p^r \left[\left(1+\frac{1}{r}\right)^r - \left(1+\frac{\beta}{r}\right)^r \right] \right\} > \left(1+\frac{\beta}{r}\right)^r p^r x_0^r,$$

and this is clearly true for fixed $r > 0$ and all large p.

As a first example with $\alpha \leqslant c$, we use the standard normal distribution and take $r = 1$, $x_0 = 1$, $p = 2$, so $x_1 = 2$. One finds $\alpha < 0.6 < c = 2 + \sqrt{3}$, and $C_2 = 0.15269$, $V_2 = 0.21379$, so (4) is better than (9) in this case. The actual value of $P(x_1)$ is 0.0445.

On the other hand, also using the standard normal distribution, with $r = 2$, $x_0 = 1$, $p = 2$, so $x_1 = 2$, $v_2 = 1$, we find $\alpha = 1 < c = 2.9422$. Now, $C_2 = \frac{1}{7} = 0.142857$, $V_2 = 0.115519$, so (9) is better than (4). The actual value of $P(x_1)$ is still 0.0455. Note that both estimates with $r = 2$ are better than those with $r = 1$.

5. The Case $x_0 = 0$: Inequalities of Gauss and Winckler

Von Mises' results are much simpler in the case $x_0 = 0$. Here the hypothesis is that for numbers x_1, z $(0 < x_1 < z)$, the graph of $Q(x)$ for $x \geqslant 0$ lies under the line D, passing through the two points (x_1, Q_1) and $(z, 1)$, where $Q_1 = Q(x_1)$. Stating only the better bound where applicable, (9) and (13) reduce to

$$P(x_1) \leqslant \frac{\nu_r}{(1+(1/r))^r x_1^r} \qquad \textit{if} \quad x_1 \geqslant \frac{r}{r+1}\,[(r+1)\nu_r]^{1/r}, \tag{22}$$

$$P(x_1) \leqslant 1 - \frac{x_1}{[(r+1)\nu_r]^{1/r}} \qquad \textit{if} \quad x_1 < \frac{r}{r+1}\,[(r+1)\nu_r]^{1/r}. \tag{23}$$

It is easy to verify that both the upper bounds in (22), (23) remain less than unity for all $r > 0$. In case $x_1 = \{r/(r+1)\}[(r+1)\nu_r]^{1/r}$, both upper bounds reduce to $1/(1+r)$.

The bounds (22), (23) for the case $r = 2$ were given without proof by Gauss [2, Art. 10]. For $r \geqslant 1$ these bounds were also given by Winckler [3, Sect. 8], but under more restrictive hypotheses. See also Section 6. In addition, the conditions given by Winckler for (22), (23) were that $P(x_1) \leqslant 1/(r+1)$, $P(x_1) > 1/(r+1)$, respectively. By (19) these are equivalent to the conditions $c \geqslant z_0$, $c < z_0$, when $x_0 = 0$, and so also follow from our analysis. Additional special cases are given in Fréchet [10, 164–167].

6. Inequalities of A. Winckler

In a paper published in 1866, Winckler [3] obtained a large number of inequalities dealing with a probability distribution F derived from a continuous probability density function φ which is unimodal and symmetric on a bounded interval $[-\alpha, \alpha]$, so that $\varphi(-x) \equiv \varphi(x)$, φ is decreasing on $[0, \alpha]$, $\varphi(-\alpha) = \varphi(\alpha) = 0$, and $\int_{-\alpha}^{\alpha} \varphi\, dx = 1$. We shall restate some of these results under the less restrictive hypotheses for F listed in Section 1, adding additional hypotheses as required. In particular, we use the notation of Section 1, except that now we assume F is defined on $[a - \alpha, a + \alpha]$, and Q is defined and nondecreasing on $[0, \alpha]$, where $0 < \alpha \leqslant +\infty$, and

$$\nu_r = \int_0^{\alpha} t^r\, dQ(t), \qquad Q(\alpha) = 1. \tag{24}$$

We also define

$$\chi_r(x) = \int_0^{x} t^r\, dQ(t), \qquad 0 \leqslant x \leqslant \alpha. \tag{25}$$

6.1

First we give Winckler's proof of (22), (23) under the hypotheses given in Section 5. That is, we suppose that the graph of $Q(t)$ for $0 \leqslant t \leqslant x$ lies under a straight line D, through the point $(x_1, Q(x_1))$, which we may assume has an equation $y = u + [Q(x_1) - u](t/x_1)$, where $0 \leqslant u < Q_1 = Q(x_1)$. That is,

$$Q(t) \leqslant u + \frac{Q(x_1) - u}{x_1} t, \qquad or \qquad t \geqslant \frac{Q(t) - u}{Q(x_1) - u} x_1$$

for $0 \leqslant t \leqslant x$. It follows that

$$\begin{aligned} \chi_r(x) &= \int_0^r t^r \, dQ(t) \geqslant \frac{x_1^r}{[Q(x_1) - u]^r} \int_0^x [Q(t) - u]^r \, dQ(t) \\ &= \frac{x_1^r}{[Q(x_1) - u]^r} \int_0^{Q(x)} (y - u)^r \, dy, \end{aligned}$$

or

$$\chi_r(x) \geqslant \frac{x_1^r}{r+1} \frac{[Q(x) - u]^{r+1}}{[Q(x_1) - u]^r} \qquad (0 < x_1 \leqslant x). \tag{26}$$

Now let $g(u) = (Q - u)^{r+1}(Q_1 - u)^{-r}$, and compute

$$g'(u) = (Q - u)^r (Q_1 - u)^{-(r+1)} \{u - [(r+1)Q_1 - rQ]\}.$$

If $Q_1 \leqslant (r/(r+1))Q$, $g'(u) > 0$ for $0 < u < Q_1$, so $g(u) \geqslant g(0)$, and thus

$$\chi_r(x) \geqslant \frac{x_1^r}{r+1} \frac{(Q(x))^{r+1}}{(Q(x_1))^r} \qquad if \quad Q(x_1) \leqslant \frac{r}{r+1} Q(x). \tag{27}$$

If, however, $(r/(r+1))Q < Q_1$ $(\leqslant Q)$, then g has an absolute minimum for $u = (r+1)Q_1 - rQ > 0$. In this case $Q - u = (r+1)(Q - Q_1)$, and $Q_1 - u = r(Q - Q_1)$, whence

$$\chi_r(x) \geqslant \left(1 + \frac{1}{r}\right)^r x_1^r [Q(x) - Q(x_1)] \qquad if \quad Q(x_1) > \frac{r}{r+1} Q(x). \tag{28}$$

When $x = \alpha$, $Q(x) = 1$, $\chi_r(x) = v_r$, and (27), (28) reduce to (23), (22), respectively.

6.2

We may rewrite (27), (28) in the form

$$x_1^r \leqslant (r+1)\, \chi_r(x) \frac{Q^r(x_1)}{Q^{r+1}(x)}, \qquad x_1^r \leqslant \frac{1}{(1 + (1/r))^r} \chi_r(x) [Q(x) - Q(x_1)]^{-1},$$

whence if $n > 0$ also,

$$x_1^n \leqslant [(r+1)\chi_r(x)]^{n/r} \frac{Q^n(x_1)}{[Q(x)]^{n+(n/r)}} \qquad \textit{if} \quad Q(x_1) \leqslant \frac{r}{r+1} Q(x),$$

$$x_1^n \leqslant \frac{\chi^{n/r}(x)}{(1+(1/r))^n} \frac{1}{[Q(x)-Q(x_1)]^{n/r}} \qquad \textit{if} \quad \frac{r}{r+1} Q(x) \leqslant Q(x_1) < Q(x).$$

If now *we assume Q is concave on* $[0, x]$, then we may take D to be a tangent line at $(x_1, Q(x_1))$ for each $x_1 \in [0, x]$. Then the last inequalities hold for $0 \leqslant x_1 < x$, $r > 0$, $n > 0$. Now integrate these inequalities with respect to $dQ(x_1)$ over $[0, x_1(x)]$, $[x_1(x), x]$, respectively, where $Q(x_1(x)) = (r/(r+1))Q(x)$, and add the corresponding integrals. A change of variables $y = Q(x_1)$ in the integrals on the right side leads to the inequality

$$\left\{\frac{(n+1)\chi_n(x)}{Q(x)}\right\}^{1/n} \leqslant \frac{r}{r+1}\left(\frac{r}{r-n}\right)^{1/n} \left\{\frac{(r+1)\chi_r(x)}{Q(x)}\right\}^{1/r}, \tag{29}$$

provided $0 < n < r$, *and Q is concave on* $[0, x]$. For $x = \alpha$ this reduces to

$$[(n+1)\nu_n]^{1/n} \leqslant \left\{\frac{r}{r+1}\left(\frac{r}{r-n}\right)^{1/n}\right\} [(r+1)\nu_r]^{1/r}, \qquad 0 < n < r. \tag{30}$$

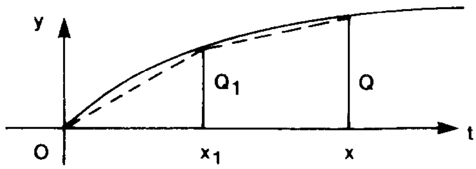
Observe that the factor

$$\frac{r}{r+1}\left(\frac{r}{r-n}\right)^{1/n} > 1 \Leftrightarrow \left(\frac{r}{r+1}\right)^n = \left(1 - \frac{1}{r+1}\right)^n > 1 - \frac{n}{r}.$$

If $r > n \geqslant 1$ the latter inequality follows from Bernoulli's inequality, so that the first factor on the right side of (30) is larger than 1 in this case. By calculus one can show it exceeds 1 for $0 < n < r$. In [3] Winckler used (26) and a false argument to obtain (29) and (30) with this factor replaced by 1 when $1 \leqslant n < r$. The results *are* true and we shall later (Section 7) obtain them using a different proof. (The argument in [3] was as follows: for each $x_1 \in (0, x]$ we have an inequality of the form (26) when Q is concave on $[0, x]$, just as above; solve as above for x_1^r, then x_1^n, and integrate the resulting inequality with respect to $dQ(x_1)$ over the interval $[Q^{-1}(u), x]$ and finally set $u = 0$ to obtain the result. The falsity of the argument lies in the fact that in (26), u is a function of x_1—in fact $u = Q(x_1)$—$x_1 Q'(x_1)$.)

6.3

In this subsection we give some generalizations of Winckler of earlier inequalities of Laplace and Gauss. *From now on we assume throughout that Q is concave on* $[0, \alpha]$, *and nondecreasing with* $Q(0) = 0$, $Q(\alpha) = 1$, *where* $0 < \alpha \leqslant \infty$.



In this case, if $0 < x_1 < x$, with $Q_1 = Q(x_1)$, $Q = Q(x)$, we have

$$Q(t) \geqslant \frac{Q_1}{x_1} t, \qquad 0 \leqslant t \leqslant x_1,$$

$$Q(t) \geqslant Q_1 + \frac{Q - Q_1}{x - x_1}(t - x_1), \qquad x_1 \leqslant t \leqslant x.$$

Hence

$$\chi_r(x) = \int_0^x t^r \, dQ(t) = \int_0^{x_1} t^r \, dQ(t) + \int_{x_1}^x t^r \, dQ(t)$$
$$\leqslant \int_0^{x_1} \left(\frac{x_1}{Q_1}\right)^r Q^r(t) \, dQ(t) + \int_{x_1}^x \left\{ x_1 + \frac{x - x_1}{Q - Q_1} [Q(t) - Q_1] \right\}^r dQ(t),$$

and this reduces to

$$\chi_r(x) \leqslant \frac{Q(x)(x^{r+1} - x_1^{r+1}) - xQ(x_1)(x^r - x_1^r)}{(r+1)(x - x_1)} \qquad (0 < x_1 < x), \qquad (31)$$

valid for any $r > 0$. Taking limits as $x_1 \to x$, or as $x_1 \to 0$ we obtain

$$\chi_r(x) \leqslant \frac{x^r}{r+1} Q(x), \qquad x \geqslant 0, r > 0. \qquad (32)$$

Moreover, if $Q(\alpha) = 1$, *with* α *finite, then*

$$\nu_r \leqslant \frac{\alpha^{r+1} - x^{r+1} - \alpha Q(x)(\alpha^r - x^r)}{(r+1)(\alpha - x)}, \qquad 0 \leqslant x < \alpha, r > 0. \qquad (33)$$

The inequality (33), for $x = 0$, $r = 2$, gives the result $\nu_2 \leqslant \alpha^2/3$, first proved by Laplace [1] in 1810.

For additional results, rewrite (29) in the form

$$xQ(x_1)(x^r - x_1^r) \leqslant Q(x)(x^{r+1} - x_1^{r+1}) - (r+1)\chi_r(x)(x - x_1),$$

multiply by $x^{-1}x_1^{n-1}$, and integrate with respect to x_1 over $[0, x]$. An integration-by-parts of the integral on the left, and algebraic simplification leads to the inequality

$$(r+1)\,x^n\chi_r(x) - (n+1)\,x^r\chi_n(x) + \frac{n(n+1)}{r+n}\chi_{r+n}(x)$$
$$\leqslant \frac{n(n+1)}{(r+n)(r+n+1)}\,x^{r+n}Q(x), \tag{34}$$

valid for $x > 0$, $r > 0$, $n > 0$. In particular, if $Q(\alpha) = 1$ with α finite, we may take $x = \alpha$ above, so $Q(x) = 1$. On division by α^{r+n} we obtain

$$(r+1)\frac{\nu_r}{\alpha^r} - (n+1)\frac{\nu_n}{\alpha^n} \leqslant \frac{n(n+1)}{(r+n)(r+n+1)}\left\{1 - (r+n+1)\frac{\nu_{r+n}}{\alpha^{r+n}}\right\}, \tag{35}$$

for $r > 0$, $n > 0$.

We may also rewrite (31) in the form

$$Q(x_1) \leqslant \frac{Q(x)}{x}x_1 + \frac{x^rQ(x) - (r+1)\,\chi_r(x)}{x}\,\frac{x - x_1}{x^r - x_1^r}.$$

Now multiply by x_1^{n-1} $(n > 0)$ and again integrate over $[0, x]$ to obtain after some simplification,

$$x^{r+n}Q(x) \leqslant (n+1)\,x^r\chi_n(x)$$
$$+\, n(n+1)[x^{r+n}Q(x) - (r+1)\,x^n\chi_r(x)]\,\lambda(n, r), \tag{36}$$

where

$$\lambda(n, r) = x^{r-n-1}\int_0^x \frac{(x - x_1)x_1^{n-1}}{x^r - x_1^r}\,dx_1 = \int_0^1 \frac{1-t}{1-t^r}\,t^{n-1}\,dt. \tag{37}$$

One could obtain additional inequalities from (36) by multiplying by x^{m-1} and integrating over $[0, X]$, and so on. Instead, we observe that on expanding $(1 - t^r)^{-1}$ and integrating, we obtain

$$\lambda(n, r) = \sum_{k=0}^{\infty} \frac{1}{(kr+n)(kr+n+1)}$$
$$= \frac{1}{n(n+1)} + \frac{1}{(r+n)(r+n+1)} + \cdots > \frac{1}{n(n+1)},$$

for all $n > 0$, $r > 0$. Moreover, if $r \geqslant 1$, $\lambda(n, r) \geqslant 1/(rn)$ for all $n > 0$, and $\lambda(n, r) \leqslant 1/[r(n - r + 1)]$ for all $n > r$. In particular, since

$n(n+1)\lambda(n,r) > 1$ for $n > 0$, $r > 0$, it follows that we may rewrite (36) in the form

$$Q(x) \geqslant \frac{n+1}{n(n+1)\lambda - 1} \left\{ n(r+1)\lambda \frac{\chi_r(x)}{x^r} - \frac{\chi_n(x)}{x^n} \right\}, \qquad n > 0, r > 0. \quad (38)$$

If $Q(\alpha) = 1$, we obtain

$$n(r+1)\lambda \frac{v_r}{\alpha^r} - \frac{v_n}{\alpha^n} \leqslant n\lambda - \frac{1}{n+1} \qquad \textit{for} \quad n > 0, r > 0. \tag{39}$$

where $\lambda = \lambda(n,r)$ *is defined by* (37). If $r = 1$, $\lambda = n^{-1}$, whence

$$\frac{2v_1}{\alpha} \leqslant \frac{v_n}{\alpha^n} + \frac{n}{n+1} \qquad \textit{for} \quad n > 0. \tag{40}$$

For $n = 1$, (40) gives the same result as (32) for $r = 1$, $x = \alpha$.

Note that (32) can be written in the form

$$\int_0^x t^r \varphi(t)\,dt \leqslant \frac{1}{x} \left(\int_0^x t^r\,dt\right)\left(\int_0^x \varphi(t)\,dt\right),$$

with φ nonincreasing on $[0, x]$, and thus is an early special form of Čebyšev's inequality. This approach was pushed somewhat further in [3], but with only partial success. Instead of pursuing this, we return to (32), and rewrite it in the form

$$(n+1)\int_0^{x_1} t^n\,dQ(t) \leqslant x_1^n Q(x_1), \qquad n > 0, x_1 \geqslant 0.$$

Multiply by x_1^{m-n-1} $(m > n)$, and integrate over $[0, x]$ to obtain

$$(n+1)\frac{x^{m-n}}{m-n}\chi_n(x) - \frac{(n+1)}{m-n}\chi_m(x) \leqslant \frac{x^m}{m} Q(x) - \frac{1}{m}\chi_m(x).$$

or after multiplication by $m(m-n)x^n$,

$$m(n+1)\,x^m \chi_n(x) - n(m+1)\,x^n \chi_m(x) \leqslant (m-n)\,x^{m+n} Q(x), \tag{41}$$

valid for $m \geqslant n > 0$, $x \geqslant 0$. Similarly we may multiply (41) by $x^{r-m-n-1}$ (where $r > m+n$, $m > n$), integrate by parts, and then multiply by $r(r-m)(r-n)x^{m+n}$ to obtain

$$\begin{aligned} &mr(r-m)(n+1)\,x^{m+r}\chi_n(x) + rn(n-r)(m+1)\,x^{r+n}\chi_m(x) \\ &\qquad + nm(m-n)(r+1)\,x^{n+m}\chi_r(x) \leqslant (m-r)(r-n)(n-m)\,x^{m+n+r}Q(x), \end{aligned}$$

provided $0 < n \leqslant m$, $m + n \leqslant r$. (42)

If $Q(\alpha)=1$ with α finite, we may take $x=\alpha$ in (41), (42) and, dividing by mn, and by $r^2m^2n^2$, respectively, obtain

$$\left(1+\frac{1}{n}\right)\frac{\nu_n}{\alpha^n}-\left(1+\frac{1}{m}\right)\frac{\nu_m}{\alpha^m}\leqslant\frac{1}{n}-\frac{1}{m} \qquad \textit{if} \quad m\geqslant n>0, \tag{43}$$

$$\frac{1}{n}\left(1+\frac{1}{n}\right)\left(\frac{1}{m}-\frac{1}{r}\right)\frac{\nu_n}{\alpha^n}+\frac{1}{r}\left(1+\frac{1}{r}\right)\left(\frac{1}{n}-\frac{1}{m}\right)\frac{\nu_r}{\alpha^r}$$
$$\leqslant\frac{1}{m}\left(1+\frac{1}{m}\right)\left(\frac{1}{n}-\frac{1}{r}\right)\frac{\nu_m}{\alpha^m}+\left(\frac{1}{m}-\frac{1}{r}\right)\left(\frac{1}{n}-\frac{1}{r}\right)\left(\frac{1}{n}-\frac{1}{m}\right), \tag{44}$$

provided $0<n\leqslant m$, $m+n\leqslant r$.

6.4

In this final subsection we shall obtain some reverse inequalities to those in 6.3 by assuming somewhat more, namely, that Q is not only nondecreasing and concave, but, in fact, *that* Q' *is nonincreasing and continuous on* $[0,\alpha]$ $(0<\alpha\leqslant\infty)$. In place of (32) we may then write

$$\chi_n(x)=\int_0^x t^nQ'(t)\,dt\geqslant\int_0^x t^nQ'(x)\,dt=\frac{x^{n+1}}{n+1}Q'(x), \qquad n\geqslant 0.$$

We now proceed as in the analysis leading to (41)–(44). Multiply the last inequality by x^{m-n-1} $(m>n)$, and integrate to obtain

$$\frac{x^{m-n}}{m-n}\chi_n(x)-\frac{1}{m-n}\chi_m(x)\geqslant\frac{1}{n+1}\int_0^x t^mQ'(t)\,dt=\frac{1}{n+1}\chi_m(x),$$

or after some reduction

$$(n+1)x^m\chi_n(x)\geqslant(m+1)\,x^n\chi_m(x) \qquad \textit{if} \quad m\geqslant n\geqslant 0. \tag{45}$$

Similarly, multiply through (45) by $x^{r-m-n-1}$ $(r>m+n)$, and integrate over $[0,x]$. After multiplication by $(r-n)(r-m)x^{m+n}$, we have

$$(n+1)(r-m)\,x^{r+m}\chi_n(x)+(r+1)(m-n)\,x^{m+n}\chi_r(x)$$
$$\geqslant(m+1)(r-n)\,x^{r+n}\chi_m(x), \tag{46}$$

valid for $m\geqslant n\geqslant 0$, $r\geqslant m+n$, $x>0$.

If $Q(\alpha) = 1$ where α is finite, we may set $x = \alpha$ in (45), (46) to obtain

$$(n+1)\frac{v_n}{\alpha^n} \geqslant (m+1)\frac{v_m}{\alpha^m} \qquad \text{if} \quad m \geqslant n \geqslant 0. \tag{47}$$

$$\begin{aligned}(n+1)(r-m)\frac{v_n}{\alpha^n} &+ (r+1)(m-n)\frac{v_r}{\alpha^r} \\ &\geqslant (m+1)(r-n)\frac{v_m}{\alpha^m} \qquad \text{if} \quad m \geqslant n \geqslant 0, r \geqslant m+n.\end{aligned} \tag{48}$$

7. Inequalities of Liapounov and Gauss–Winckler

Again we return to the rth absolute moment of Section 1,

$$v_r = v_{r,a} = \int_{-\infty}^{\infty} |x-a|^r \, dF(x) = \int_0^{\infty} t^r \, dQ(x) \qquad (r \geqslant 0).$$

We observe that $v_0 = 1$, and *if v_n exists for some $n > 0$, then v_r exists for $0 \leqslant r < n$, and*

$$v_r^{1/r} \leqslant v_n^{1/n}. \tag{49}$$

This follows from Hölder's inequality with conjugate exponents $p = n/r > 1$, and $q = n/(n-r)$, which gives

$$v_r = \int_0^{\infty} x^r \cdot 1 \, dQ(x) \leqslant \left(\int_0^{\infty} x^n \, dQ\right)^{r/n} \left(\int_0^{\infty} 1 \, dQ\right)^{(n-r)/n},$$

and hence (49).

We next prove that *the function $g(r) \equiv \log v_r$ is convex* on $[0, \infty)$, that is,

$$g(\lambda s + \mu t) \leqslant \lambda g(s) + \mu g(t) \qquad \text{for} \quad 0 \leqslant s \leqslant t, 0 < \lambda < 1, \lambda + \mu = 1. \tag{50}$$

Indeed, using Hölder's inequality with $p = \lambda^{-1}$, $q = \mu^{-1}$, we have

$$\begin{aligned} g(\lambda s + \mu t) &= \log \int_0^{\infty} (x^s)^{\lambda} (x^t)^{\mu} \, dQ \leqslant \log \left(\int_0^{\infty} x^s \, dQ\right)^{\lambda} \left(\int_0^{\infty} x^t \, dQ\right)^{\mu} \\ &= \lambda g(s) + \mu g(t). \end{aligned}$$

Inequality (50) may be written in a different form, when it is called *Liapounov's inequality* [4]:

$$v_b^{c-a} \leqslant v_a^{c-b} v_c^{b-a} \qquad \text{if} \quad 0 \leqslant a \leqslant b \leqslant c. \tag{50'}$$

This is just (50) with $a = s$, $c = t$, $\lambda = (c-b)/(c-a)$, $\mu = (b-a)/(c-a)$.

We shall use (50) to prove the so-called *Gauss–Winckler inequality*, namely,

$$[(n+1)\nu_n]^{1/n} \leqslant [(r+1)\nu_r]^{1/r} \qquad \textit{if} \quad 0 \leqslant n \leqslant r, \tag{51}$$

provided Q' is continuous and nonincreasing on $(0, \infty)$. Note that this is an improvement of (30); as noted following (30), Winckler obtained (51) by an invalid argument in 1866. According to Fréchet [10, p. 68], the first proof of (51) was due to Faber [7] in 1926. The following proof is a modification of von Mises' proof [8] given in 1931. The special case $n=2$, $r=4$, of (51), namely,

$$\nu_4 \geqslant \tfrac{9}{5}\nu_2^2 \tag{52}$$

was stated without proof by Gauss [2, Art. 10].

To prove (51), observe that the existence of $\nu_r = \int_0^\infty t^r Q'\,dt$ under the hypotheses on Q' implies that

$$\int_{a/2}^{\infty} t^r Q'(t)\,dt \geqslant \int_{a/2}^{a} t^r Q'(t)\,dt \geqslant \left(\frac{a}{2}\right)^r Q'(a)\cdot\frac{a}{2} \to 0 \qquad as \quad a \to \infty,$$

and

$$\int_0^{\varepsilon} t^r Q'(t)\,dt \geqslant Q'(\varepsilon)\frac{\varepsilon^{r+1}}{r+1} \to 0 \qquad as \quad \varepsilon \to 0.$$

Hence,

$$\nu_r = \frac{Q'(t)t^{r+1}}{r+1}\bigg|_0^{\infty} + \frac{1}{r+1}\int_0^{\infty} t^{r+1}\,d[-Q'(t)],$$

or

$$(r+1)\nu_r = \int_0^{\infty} t^{r+1}\,d[-Q'(t)], \tag{53}$$

and, in particular, the latter integral exists. Similarly, since ν_0 exists, $J_1 = \int_0^\infty t\,d[-Q'(t)] < \infty$ follows in the same way, and the function Q_1 defined by $Q_1(t) = 0$, $t \leqslant 0$,

$$Q_1(t) = J_1^{-1}\int_0^t u\,d[-Q'(u)], \qquad t > 0,$$

is a probability distribution function. We may write the preceding equality in the form

$$(r+1)\nu_r = J_1\int_0^{\infty} t^r\,dQ_1(t) = J_1\tilde{\nu}_r,$$

where $\tilde{\nu}_r$ denotes the rth moment about 0 of Q_1 (and thus exists). Now apply (50) to the moments $\tilde{\nu}_r = J_1^{-1}(r+1)\nu_r$, with $s = 0$, $t = r$, $\mu = (n/r) < 1$, to obtain

$$\log[J_1^{-1}(n+1)\nu_n] \leqslant \left(1 - \frac{n}{r}\right)\log[J_1^{-1} \cdot 1 \cdot \nu_0] + \frac{n}{r}\log[J_1^{-1} \cdot (r+1)\nu_r].$$

Since $\nu_0 = 1$, this reduces to (51). We observe that this proof is valid even in the case that $Q'(0+) = -\infty$. In case $0 < Q'(0) < \infty$, a slightly simpler proof can be given, using $\tilde{Q}_1(x) = 1 - \{Q'(x)/Q'(0)\}$ in place of $Q_1(x)$.

If, in addition to the preceding hypotheses on Q, we assume that $-Q''$ is continuous, positive and decreasing, then from (53) we may write

$$(r+1)\nu_r = -\int_0^\infty Q''(t)\, t^{r+1}\, dt = \frac{t^{r+2}}{r+2}[-Q''(t)]\Big|_0^\infty + \frac{1}{r+2}\int_0^\infty t^{r+2}\, dQ''(t).$$

This reduces as before to

$$(r+1)(r+2)\nu_r = \int_0^\infty t^{r+2}\, dQ''(t) = J_2\int_0^\infty t^r\, dQ_2(t),$$

where $J_2 = \int_0^\infty t^2\, dQ''(t)$ exists (since J_1 does), and

$$Q_2(t) = J_2^{-1}\int_0^t u^2\, dQ''(u).$$

From this, the same analysis as before shows that

$$[(n+1)(n+2)\nu_n]^{1/n} \leqslant [(r+1)(r+2)\nu_r]^{1/r} \qquad \textit{if} \quad 0 \leqslant n \leqslant r.$$

More generally, *if Q is a probability distribution function with $Q(x) = 0$ for $x \leqslant 0$, $Q(0) = 0$, $Q(\infty) = 1$, and if $(-1)^{k-1}Q^{(k)}$ is positive, continuous, and decreasing on $(0, \infty)$ for $k = 1, 2, \ldots, N$, then*

$$\begin{aligned}&\{(n+1)(n+2)\cdots(n+k)\nu_n\}^{1/n} \\ &\qquad \leqslant \{(r+1)(r+2)\cdots(r+k)\nu_r\}^{1/r} \qquad \textit{if} \quad 0 \leqslant n \leqslant r, \end{aligned} \tag{54}$$

holds for $1 \leqslant k \leqslant N$. As an application of (54), consider the case $Q(x) = 1 - e^{-x}$, $x \geqslant 0$, for which the hypotheses of (54) are satisfied for arbitrary $N > 0$. Since $\nu_r = \Gamma(r+1)$ in this case, (54) reduces to

$$\{\Gamma(n+k+1)\}^{1/n} \leqslant \{\Gamma(r+k+1)\}^{1/r}, \qquad 0 \leqslant n \leqslant r, k = 1, 2, 3, \ldots. \tag{55}$$

ACKNOWLEDGMENTS

The author's attention was first drawn to these matters some years ago by Professor D. S. Mitrinovič, who kindly supplied a copy of the long-forgotten paper of Winckler, and to whom I acknowledge my indebtedness.

REFERENCES

1. P. S. LAPLACE, "Mémoires de l'Institut. Année," I Partie, p. 327, 1810.
2. C. F. GAUSS, "Theoria combinationis observationum," 1821; translated into German *in* "Abhandlungen zur Methode der kleinsten Quadrate, *1887*," pp. 9, 12, Neudruck, Würzburg, 1964.
3. A. WINCKLER, Allgemeine Sätze zur Theorie der unregelmässigen Beobachtungsfehler, *Sitzungsber. Math.-Natur. Kl. K. Akad. Wiss. Wien, Zweite Abt.* **53** (1866), 6–41.
4. A. LIAPOUNOV, Nouvelle forme du théorème sur la limite de probabilité, *Mém. Acad. Sci. St. Pétersburg* **12** (5) (1901).
5. K. PEARSON, On generalised Tchebycheff theorems in the mathematical theory of statistics, *Biometrika* **12** (1919), 284–296.
6. B. H. CAMP, A new generalization of Tchebycheff's statistical inequality, *Bull. Amer. Math. Soc.* **28** (1922), 427–432.
7. G. FABER, Bermerkungen zu Sätzen der Gausschen theoria combinationis observationum, Heft 1, *Sitzungsber. Bayer Akad. Wiss.* (1922), 7–21.
8. R. VON MISES, Ueber einige Abschätzungen von Erwartungswerten, *J. reine angewand Math.* **165** (1931), 184–193.
9. R. VON MISES, Sur une inégalité pour les moments d'une distribution quasi-convexe, *Bull. Sci. Math.* (2) **62** (1938), 68–71.
10. M. FRÉCHET, "Recherches Théorique Modernes sur le Calcul des Probabilites, Vol. 1, Genéralités sur les Probabilites. Éléments aléatoires," 2nd ed. Gauthier–Villars, Paris, 1950.