ELSEVIER
Topology and its Applications 129 (2003) 273–280
TOPOLOGY AND ITS APPLICATIONS
www.elsevier.com/locate/topol

André–Quillen spectral sequence for *THH*

Vahagn Minasian

University of Illinois at Urbana-Champaign, 1409 W. Green St, Urbana, IL 61801, USA

Received 1 November 2001

Abstract

We produce a topological analogue of André–Quillen's spectral sequence and prove that for connective spectra, $TAQ \simeq *$ if and only $\widetilde{THH} \simeq *$, where $\widetilde{THH}$ is the reduced topological Hochschild Homology.

MSC: 55P42

Keywords: Spectra; Homology theories; Spectra with additional structure

1. Introduction and notations

In [7] Quillen introduced a homology theory for commutative algebras (André–Quillen homology) and discussed its relationship with Hochschild homology. In particular, he observed that if $A \to B$ is a morphism of commutative rings, then the homology of B as just an associative A-algebra may be calculated as certain Tor groups in the homotopy category of simplicial modules over $B \overset{\mathbf{L}}{\otimes} B$, where $- \overset{\mathbf{L}}{\otimes} B$ is the left derived functor of $- \otimes B$. In other words, $B \overset{\mathbf{L}}{\otimes} B$ is the object $R \otimes B$, where R is a simplicial A-algebra resolution of B which in every dimension is flat as an A-module. In addition, Quillen produced a spectral sequence

$$E^2_{pq} = H_p\left(\bigwedge_q^B \mathbf{L}_{B/A}\right) \Longrightarrow \mathrm{Tor}_{p+q}^{B \overset{\mathbf{L}}{\otimes} B}(B, B)$$

which therefore may be regarded as a relation between the homologies as a commutative A-algebra and as just an associative A-algebra. This spectral sequence leads to a

E-mail address: minasian@math.uiuc.edu (V. Minasian).

0166-8641/02/$ – see front matter
doi:10.1016/S0166-8641(02)00184-0

decomposition of Hochschild homology in the rational case and is frequently referred to as the *fundamental spectral sequence for André–Quillen homology*.

The main objective of this paper is to develop an analogue of this spectral sequence for topological setting in the framework provided in [4]. We start by listing the categories in which our work takes place.

- $\mathcal{M}_A$ is the category of A-modules, where A is a cofibrant S-algebra.
- $\mathcal{C}_A$ is the category of commutative A-algebras.

Another description of $\mathcal{C}_A$ is obtained by observing that it coincides with $\mathcal{M}_A[\mathbb{P}]$ the category of $\mathbb{P}$-algebras where $\mathbb{P}: \mathcal{M}_A \to \mathcal{M}_A$ is the monad given by $\mathbb{P}M = \bigvee_{j \geqslant 0} M^j/\Sigma_j$. Here M^j denotes the j-fold smash power over A and $M^0 = A$. Clearly, $\mathbb{P}$ can viewed as a functor from $\mathcal{M}_A$ to the category of A-algebras augmented over A with the obvious augmentation map. A related construction is the monad $\mathbb{P}_1$ given by $\mathbb{P}_1 M = \bigvee_{j>0} M^j/\Sigma_j$. It leads to the next category of interest.

- $\mathcal{N}_A$ is the category of commutative non-unital A-algebras which coincides with $\mathcal{M}_A[\mathbb{P}_1]$. We adopt the terminology of [1] and call the objects of $\mathcal{N}_A$ $A - NUCA$'s.
- $\mathcal{C}_{A/B}$ is the category of commutative A-algebras over B, i.e., the objects of $\mathcal{C}_{A/B}$ are A-algebras C equipped with an algebra map $C \to B$ and the morphisms are maps of algebras over B.

Note that all of these categories are closed model categories [3] and for a discussion on their homotopy categories we refer to [4,1].

To obtain the desired spectral sequence we produce a tower of functors that approximates the forgetful (exponential) functor $\mathrm{Exp}: \mathcal{N}_A \to \mathcal{M}_A$. Using this tower we arrive to certain inverse limit systems, and it is the second quadrant spectral sequences associated to these systems that provide the topological analogue of Quillen's fundamental spectral sequence.

2. André–Quillen spectral sequence

First we construct a tower of functors (denoted by $\{I/I^{n+1}\}_{n \geqslant 1}$) which approximates Exp. However, before we begin we list some of the properties we would like our tower to have.

One frequently wants to work with $\mathcal{N}_A$ (or $\mathcal{C}_{A/A}$, or $\mathcal{M}_A$) not only up to isomorphism, but up to a weak equivalence as well, i.e., often it is essential that functors preserve weak equivalences. This can frequently be achieved by considering the derived version of functors. In other words, we evaluate our functors not at the objects themselves but rather at their cofibrant replacement, or equivalently, we precompose our functors with the functor Γ, which is the cofibrant replacement functor in $\mathcal{N}_A$ (see [4, Chapter 7]).

In addition, we want to build a tower which is computationally friendly. The following construction is a step in this direction. Recall that $(\mathbb{P}_1, \mu, \eta)$ is the free non-unital commutative algebra triple in the category $\mathcal{M}_A$ and for a cofibrant $A - NUCA\, N$ define

$B_*N = B_*(\mathbb{P}_1, \mathbb{P}_1, N)$ to be the simplicial object with n-simplices $B_n = \mathbb{P}_1^{n+1}$ and face and degeneracy maps operators given by

$$d_i = \mathbb{P}_1^i \mu_{\mathbb{P}_1^{n-i-1}} \quad 0 \leqslant i < n, \qquad d_n = \mathbb{P}_1^n \varepsilon,$$
$$s_i = \mathbb{P}_1^{i+1} \eta_{\mathbb{P}_1^{n-i}} \quad 0 \leqslant i \leqslant n.$$

This construction (along with a discussion of its properties) can be found in [4,1].

Of course, B_*N is a simplicial $A - NUCA$. For this construction to be useful calculationally, the simplicial spectrum B_*N must be proper in the following sense.

Definition 2.1. Let K_* be a simplicial spectrum and let $sK_q \subset K_q$ be the 'union' of the subspectra $s_j K_{q-1}$, $0 \leqslant j < q$. A simplicial A-module K_* is proper if the canonical map of A-modules $sK_q \to K_q$ is a c-cofibration for each $q \geqslant 0$.

In the above definition we use the term 'c-cofibration' (for classical cofibration) to distinguish it from cofibrations that are part of the model category structure. Thus, a c-cofibration of A-modules is simply a map $i : M \to \overline{M}$ of A-modules that satisfies the homotopy extension property in the category of A-modules. Of course, all cofibrations are c-cofibrations, but not conversely.

The main reason that proper simplicial A-modules K_* are computationally useful is that one can use the simplicial filtration to construct a well-behaved spectral sequence that converges to $\pi_*(E \wedge K)$ for any spectrum E [4, Theorem X.2.9]. In particular, if $f : K_* \to L_*$ is a map of proper simplicial A-modules which is a weak equivalence level-wise, then the geometric realization $|f|$ of f is also a weak equivalence.

Lemma 2.2. *For a cofibrant $A - NUCA\, N$, B_*N is a proper simplicial A-module.*

Proof. The condition of properness involves only the degeneracy operators (and not the face maps) of a simplicial A-module. The degeneracies are obtained from the unit map $\eta : N \to \mathbb{P}_1 N$. Of course, this map has a section $\mathbb{P}_1 N \to N$ which is a map of A-modules. Thus, it satisfies the homotopy extension property. The same argument proves that all degeneracies are c-cofibrations. $\square$

In addition, observe that we can consider $\varepsilon : B_*N \to N$ an augmented simplicial A-module and using the unit $\eta : N \to \mathbb{P}_1 N$ one can easily construct a contraction to it.

By [4, Chapter 7, 3.3] we have that the geometric realization in the category of $A - NUCA$'s is isomorphic to the geometric realization in $\mathcal{M}_A$, and thus $|B_*N|$ and N are homotopy equivalent in $\mathcal{M}_A$ and weakly equivalent in $\mathcal{N}_A$.

Note that even when N is cofibrant $|B_*N|$ is not necessarily cofibrant. Instead, if we form a simplicial object ΓB_*N, where the cofibrant replacement functor is applied level wise, then $|\Gamma B_*N|$ itself is the cofibrant approximation of $|B_*N|$. This is the case because both ΓB_*N and B_*N are proper and thus the fact that they have weakly equivalent simplices implies that their geometric realizations are weakly equivalent.

The strategy will be to precompose all our functors with the functor $\Gamma B_* \Gamma$. We make a definition to ease the notation.

Definition 2.3. For any functor $F : \mathcal{N}_A \to \mathcal{M}_A$ we define $F^w(N)$ to be $F(\Gamma B_* \Gamma N)$. Note that F^w is now a functor from the category of $A - NUCA$'s to the category $s\mathcal{M}_A$ of simplicial A-modules.

Now we are ready to produce the desired tower. First, for all $n > 1$ we define a functor $Q_n : \mathcal{N}_A \to \mathcal{M}_A$ by the following pushout in the category of A-modules

$$\begin{array}{ccc} N^{\wedge n} & \longrightarrow & * \\ \downarrow{\scriptstyle \mu} & & \downarrow \\ N & \longrightarrow & Q_n(N) \end{array}$$

and set $I/I^n(N) \stackrel{\text{def}}{=} Q_n(\Gamma B_* \Gamma N)$, where Q_n is applied degree-wise, and lastly,

$$I^n/I^{n+1}(N) \stackrel{\text{def}}{=} \text{hofiber}\big[I/I^{n+1}(N) \to I/I^n(N)\big].$$

Proposition 2.4. *For every $A - NUCA$ N there is a weak equivalence of simplicial objects*

$$\left[\bigwedge^n I/I^2(N)\right]_{h\Sigma_n} \simeq I^n/I^{n+1}(N).$$

Proof. We begin by showing that $I/I^n(N)$ is weakly equivalent to $Q_n(B_* \Gamma N)$. For $n = 2$ this is also proved in [1, Proposition 5.4]. We will need a key result from [1]:

If N is a cofibrant $A - NUCA$ and $\gamma : Y \to \mathbb{P}_1^n N$ is a cell A-module approximation then $\gamma^i/\Sigma_i : Y^i/\Sigma_i \to (\mathbb{P}_1^n N)^i/\Sigma_i$ is a weak equivalence for all $i > 0$.

Now let $\gamma : Y \to \mathbb{P}_1^k N$ be a cell A-module approximation of $\mathbb{P}_1^k N$, where N is a cofibrant $A - NUCA$. As an immediate corollary of the result above we have that $\mathbb{P}_1\gamma : \mathbb{P}_1 Y \to \mathbb{P}_1\mathbb{P}_1^k N$ is a weak equivalence. Hence so is $\Gamma\mathbb{P}_1 Y \to \Gamma\mathbb{P}_1\mathbb{P}_1^k N$, in fact this last map is a homotopy equivalence since the two algebras involved are cofibrant.

Consider the following commutative diagram:

$$\begin{array}{ccc} Q_n(\Gamma\mathbb{P}_1 Y) & \longrightarrow & Q_n(\Gamma\mathbb{P}_1\mathbb{P}_1^k N) \\ \downarrow & & \downarrow \\ Q_n(\mathbb{P}_1 Y) & \longrightarrow & Q_n(\mathbb{P}_1\mathbb{P}_1^k N) \end{array}$$

The top horizontal arrow is a homotopy equivalence by the above discussion, and so is the left vertical map. To see that the bottom arrow is a weak equivalence, observe that $Q_n(\mathbb{P}_1 Y) = Y \vee Y^2/\Sigma_2 \vee \cdots \vee Y^{n-1}/\Sigma_{n-1}$ and $Q_n(\mathbb{P}_1\mathbb{P}_1^k N) = (\mathbb{P}_1^k N) \vee (\mathbb{P}_1^k N)^2/\Sigma_2 \vee \cdots \vee (\mathbb{P}_1^k N)^{n-1}/\Sigma_{n-1}$, and hence the equivalence of $Q_n(\mathbb{P}_1 Y)$ and $Q_n(\mathbb{P}_1\mathbb{P}_1^k N)$ follows from the result from [1] stated above. Thus, since three of the maps in the diagram are weak equivalences, so is the fourth one, which proves that $I/I^n(N)$ and $Q_n(B_* \Gamma N)$ have equivalent simplices.

Hence we are entitled to conclude that the map $I/I^{n+1}(N) \to I/I^n(N)$ is weakly equivalent to a fibration which on the level of $(k+1)$-simplices is given by the evident map $(\mathbb{P}_1^k N) \vee (\mathbb{P}_1^k N)^2/\Sigma_2 \vee \cdots \vee (\mathbb{P}_1^k N)^n/\Sigma_n \to (\mathbb{P}_1^k N) \vee (\mathbb{P}_1^k N)^2/\Sigma_2 \vee \cdots \vee (\mathbb{P}_1^k N)^{n-1}/\Sigma_{n-1}$.

This implies that $I^n/I^{n+1}(N)$ is equivalent to a simplicial A-module whose $(k+1)$-simplices are $(\mathbb{P}_1^k \Gamma N)^{\wedge n}/\Sigma_n$. In particular, $I/I^2(N)$ is weakly equivalent to an object with $(k+1)$-simplices $\mathbb{P}_1^k \Gamma N$. Thus to complete the proof we need to show that the natural map $(\mathbb{P}_1^k \Gamma N)^{\wedge n}/h\Sigma_n \to (\mathbb{P}_1^k \Gamma N)^{\wedge n}/\Sigma_n$ is a weak equivalence.

Theorem 5.1 in Chapter 3 of [4] states that for a cell A-module Y, $Y^i/h\Sigma_i \to Y^i/\Sigma_i$ is a weak equivalence. Of course, this is not quite good enough, since as we indicated earlier, $\mathbb{P}_1^k \Gamma N$ is not cofibrant even when ΓN is. However, if we let $\gamma : Y \to \mathbb{P}_1^k \Gamma N$ be a cell A-module approximation, then we can form the following commutative diagram:

$$\begin{array}{ccc} (E\Sigma_n)_+ \wedge_{\Sigma_n} Y^{\wedge n} & \longrightarrow & (E\Sigma_n)_+ \wedge_{\Sigma_n} (\mathbb{P}_1^k \Gamma N)^{\wedge n} \\ \downarrow & & \downarrow \\ Y^{\wedge n}/\Sigma_n & \longrightarrow & (\mathbb{P}_1^k \Gamma N)^{\wedge n}/\Sigma_n \end{array}$$

where the left vertical arrow is a weak equivalence since Y is a cell A-module, and the bottom horizontal arrow is a weak equivalence by the above stated result from [1].

To see that the top horizontal arrow is also a weak equivalence, we simply observe that using the skeletal filtration of $E\Sigma_n$ one can set up a pair of natural spectral sequences

$$H_*\big(\Sigma_n, \pi_*\big(Y^{\wedge n}\big)\big) \Rightarrow \pi_*\big((E\Sigma_n)_+ \wedge_{\Sigma_n} Y^{\wedge n}\big),$$
$$H_*\big(\Sigma_n, \pi_*\big(\big(\mathbb{P}_1^k \Gamma N\big)^{\wedge n}\big)\big) \Rightarrow \pi_*\big((E\Sigma_n)_+ \wedge_{\Sigma_n} \big(\mathbb{P}_1^k \Gamma N\big)^{\wedge n}\big).$$

Now the desired weak equivalence follows from the isomorphism of $\pi_*(Y^{\wedge n})$ and $\pi_*((\mathbb{P}_1^k \Gamma N)^{\wedge n})$. One needs to be careful here, as even when we are given a weak equivalence $T \to S$, the induced map $T^{\wedge n} \to S^{\wedge n}$ may not be an equivalence if T or S is not cofibrant. However, this is not an issue in our case, since Y is cofibrant and $\mathbb{P}_1^k \Gamma N$ (being equal to a wedge of objects of the form $(\Gamma N)^{\wedge i}/H$ where H is a subgroup Σ_i) belongs to a class of objects (denoted by $\overline{\mathcal{F}}_A$ in [1]) whose smash powers are weakly equivalent to the smash powers of their cofibrant replacements (see Definition 3.3, Theorem 9.5 and Proposition 9.9 of [1]).

Thus, we get that the right vertical arrow is $(E\Sigma_n)_+ \wedge_{\Sigma_n} (\mathbb{P}_1^k \Gamma N)^{\wedge n} \to (\mathbb{P}_1^k \Gamma N)^{\wedge n}/\Sigma_n$ is also a weak equivalence which completes the proof. $\square$

Next we discuss the question of convergence of the inverse limit system $\{I/I^n\}$.

Proposition 2.5. *Assume A is a connective commutative S-algebra. Then for every* 0-*connected $A - NUCA$ N, the natural map*

$$\phi : \mathrm{Exp}^w(N) \to \operatorname{holim} I/I^n(N)$$

is a weak equivalence.

Proof. We start by observing that if N is 0-*connected*, then $\mathrm{Exp}^w(N) \to I/I^n(N)$ is at least $(n-1)$-*connected*. To prove this, it is enough to show that $(\Gamma B_* \Gamma N)^{\wedge n}$ is $(n-1)$-*connected*. Since A is connective, by Cellular Approximation Theorem [4, Chapter 3] $\Gamma B_* \Gamma N$ can be functorially replaced by a weakly equivalent CW A-module M. Using the argument presented at the end of the proof of the previous proposition we can claim that $(\Gamma B_*\Gamma N)^{\wedge n}$ is weakly equivalent to $M^{\wedge n}$. Since M is 0-*connected*, it has no cells in dimensions below 1, hence $M^{\wedge n}$ has no cells in dimensions less than n. Thus, $M^{\wedge n}$ (and hence also $(\Gamma B_*\Gamma N)^{\wedge n}$) is $(n-1)$-*connected.*

Thus, for each i we have an isomorphism from $\pi_i(\mathrm{Exp}^w(N))$ to $\lim_n \pi_i(I/I^n(N))$. To complete the proof of the proposition we observe that by Mittag–Leffler [2], $\pi_i(\mathrm{holim}_n I/I^n(N))$ is isomorphic to $\lim_n \pi_i(I/I^n(N))$, and hence ϕ induces an isomorphism on π_i, for each i. □

Remark 2.6. We would like to point out that the homotopy inverse limit system constructed above is nothing else but the Taylor tower of the functor Exp^w. More precisely, we can prove the following.

Claim. $P_n^w\mathrm{Exp}(N) \xrightarrow{\simeq} I/I^{n+1}(N)$, *where* $P_n^w\mathrm{Exp}(N)$ *is the* n*th Taylor polynomial of* Exp *evaluated at* $\Gamma B_*\Gamma N$.

The following easy observation is at the root of this statement:

$$cr_2\mathrm{Exp}(M, N) \cong \mathrm{Exp}(M) \wedge \mathrm{Exp}(N).$$

In general, functors satisfying the above identity along with a commutativity condition are called exponential, which explains our choice of notation for the forgetful functor $\mathcal{N}_A \to \mathcal{M}_A$. For a precise definition and a structure lemma for exponential functors into chain complexes we refer to [5].

Proof. By induction we need only to show that the induced maps

$$D_n\mathrm{Exp}(\Gamma B_*\Gamma N) \to \mathrm{hofiber}\big[I/I^{n+1}(N) \to I/I^n(N)\big]$$

are weak equivalences, where D_n are the layers of the Taylor tower $P_n^w\mathrm{Exp}$.

To do this, we use some equivalences from Goodwillie's Calculus to obtain a description of the layers D_n (for a detailed discussion on differential ∇ of a functor we refer to [5]):

$$\begin{aligned}
\nabla\mathrm{Exp}^w(N, M) &\stackrel{\text{def}}{=} D_1\big[cr_2\mathrm{Exp}^w(-, M) \vee \mathrm{Exp}^w(-)\big](N)\\
&\cong D_1\big[\mathrm{Exp}^w(-) \wedge \mathrm{Exp}^w(M) \vee \mathrm{Exp}^w(-)\big](N)\\
&\cong D_1\big[\mathrm{Exp}^w(-) \wedge \big(\mathrm{Exp}^w(M) \vee A\big)\big](N)\\
&\cong D_1 I^w(-)(N) \wedge \big[\mathrm{Exp}^w(M) \vee A\big],
\end{aligned}$$

where M and N are $A-NUCA$'s and the smash products as always are taken over A. Now from [5, Propositions 5.9, 5.11] it follows that

$$\nabla^n\mathrm{Exp}^w(N, M) \cong \bigwedge_{i=1}^{n} D_1\mathrm{Exp}^w(-)(N_i) \wedge \big[\mathrm{Exp}^w(M) \vee A\big]$$

and

$$D_n\mathrm{Exp}^w \cong \Big(\bigwedge^{n} D_1\mathrm{Exp}^w\Big)_{h\Sigma_n}.$$

Hence, the result follows immediately from the previous proposition by observing that $D_1\mathrm{Exp}^w$ is weakly equivalent to I/I^2. □

The promised spectral sequence for Topological Hochschild Homology follows as a corollary of the two propositions above.

Corollary 2.7. *There is a spectral sequence with*

$$E_1^{s,t} = \pi_{t-s}\big(\mathrm{Exp}^w\big)^{s-1} \big/ \big(\mathrm{Exp}^w\big)^s \quad \textit{for } t \geqslant s \geqslant 0,$$

where $d_1 : E_1^{s,t} \to E_1^{s+1,t}$ *is the composite*

$$\pi_{t-s}\big(\mathrm{Exp}^w\big)^{s-1} \big/ \big(\mathrm{Exp}^w\big)^s \to \pi_{t-s}\big(\mathrm{Exp}^w\big) \big/ \big(\mathrm{Exp}^w\big)^s$$
$$\to \pi_{t-s-1}\big(\mathrm{Exp}^w\big)^s \big/ \big(\mathrm{Exp}^w\big)^{s+1}.$$

When evaluated at the augmentation ideal N_A of A-algebra $A \otimes S^1$ with A a cofibrant connective S-algebra, this spectral sequence converges to the reduced Topological Hochschild Homology and is the topological analogue of the fundamental spectral sequence for André–Quillen homology:

$$E_1^{s,t} = \pi_{t-s}\left[\left(\bigwedge^{s-1} \Sigma TAQ(A)\right)_{h\Sigma_{s-1}}\right] \quad \textit{for } t \geqslant s \geqslant 0.$$

Proof. The first part of the corollary is trivial as it is simply the spectral sequence associated with the homotopy inverse limit system $\mathrm{holim}_n I/I^{n+1}$ (for a reference see [2, Chapter 9], for example).

For the second part, first observe that by Proposition 2.3 the canonical map

$$\mathrm{Exp}^w(N_A) \to \mathrm{holim}_n I/I^{n+1}(N_A) = \mathrm{holim}_n Q_n \Gamma B_* \Gamma(N_A)$$

is a weak equivalence. The Proposition 2.3 applies because the connectivity of A implies that N_A is 0-*connected.* Thus, the above spectral sequence converges to $\pi_*(\mathrm{Exp}^w(N_A)) \cong \pi_*(N_A)$, and since by a theorem of McClure, Schwänzle and Vogt $THH(B) \cong B \otimes S^1$ [6], N_A is the *reduced* Topological Hochschild Homology of A.

To recognize the Topological André–Quillen Homology in this set up, we note that $Q_2(\Gamma N_A) \simeq \Sigma Q_2(\Gamma I_A)$ where I_A is the fiber of the multiplication map $A \wedge A \xrightarrow{\mu} A$. This implies that $Q_2|B_*(\Gamma N_A)|$ and $\Sigma Q_2|B_*(\Gamma I_A)|$ are weakly equivalent. To see this, we observe that for any $A - NUCA\,N$, the weak equivalences $|\Gamma B_* \Gamma(N)| \to |B_* \Gamma(N)| \to N$ induce a composite weak equivalence

$$\mathbf{L}Q_2\big|B_*\Gamma(N)\big| \cong Q_2\big|\Gamma B_*\Gamma(N)\big| \to Q_2\big|B_*\Gamma(N)\big| \to Q_2\Gamma N \cong \mathbf{L}Q_2 N.$$

The first map is a weak equivalence, since Q_2 commutes with realizations, which follows from the fact that Q_2 has a right adjoint $Z : \mathcal{M}_A \to \mathcal{N}_A$, which takes a module M to an $A - NUCA\,M$ with a 0-multiplication (for a detailed proof see [1, Section 3], and thus, Q_2 preserves colimits. A proof of existence of the total derived functor $\mathbf{L}Q_2$ using the above adjoint pair can also be found in [1]. Hence, the second map $Q_2|B_*\Gamma(N)| \to$

$Q_2 \Gamma N$ is also a weak equivalence, which proves the desired result. Thus, we have a sequence of isomorphisms:

$$\begin{aligned}\pi_*\big[I/I^2(N_A)\big] &\cong \pi_*\big[Q_2\Gamma B_*\Gamma(N_A)\big] \cong \pi_*\big[Q_2 B_*\Gamma(N_A)\big] \cong \pi_*\big[Q_2\big|B_*\Gamma(N_A)\big|\big] \\ &\cong \pi_*\big[\Sigma Q_2\big|B_*\Gamma(I_A)\big|\big] \cong \pi_*[\Sigma Q_2\Gamma I_A] \cong \pi_*\big[\Sigma TAQ(A)\big].\end{aligned}$$

Hence $\pi_*(I^s/I^{s+1}) \cong \pi_*[(\bigwedge^s I/I^2)_{h\Sigma_s}] \cong \pi_*[(\bigwedge^s \Sigma TAQ(A))_{h\Sigma_s}]$ which completes the proof. □

Corollary 2.8. *For a cofibrant connective S-algebra A, $TAQ(A) \simeq * \iff \widetilde{THH}(A) \simeq *$ (where $\widetilde{THH}$ is the reduced Topological Hochschild Homology).*

Proof. If $\widetilde{THH}(A) \simeq *$, than $N_A \simeq *$. Hence $Q_2\Gamma N_A \simeq *$. Thus, using the observation made in the above corollary, $\Sigma TAQ(A) \simeq *$, which implies $TAQ(A) \simeq *$.

Now suppose $TAQ(A) \simeq *$. Then using Propositions 2.4 and 2.5 we can claim that N_A is weakly equivalent to the homotopy inverse limit of a directed system whose first element is $TAQ(A)$ and is thus contractible and whose layers (or fibers) are the homotopy orbits of smash powers of $TAQ(A)$ and as a result are also contractible. Hence $N_A \simeq *$. □

In conclusion, we would like to consider our spectral sequence for the special case, of algebras A over Eilenberg–MacLane spectra Hk, with k a field. For the remainder both *TAQ* and *THH* will be taken over Hk. By [4, Chapter IX] we have a spectral sequence

$$HH_{p,q}\big(\pi_*(A)\big) \Longrightarrow \pi_{p+q} THH(A).$$

Rationally, we have an analogous spectral sequence *TAQ*. Thus, in characteristic 0, for the case of Hk-algebras the spectral sequence of the Corollary 2.7 can be obtained from its analogue in discrete algebra by simply using the above two spectral sequences. In particular, if A itself is an Eilenberg–MacLane spectrum, then our spectral sequence coincides with its discrete analogue.

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