Benchmarks in nonlocal elasticity defined by Eringen’s integral model

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**Abstract**

In this paper we present low-residual approximate solutions for nonlocal 1D and 2D elasticity problems defined according to Eringen’s integral model. The benchmarks in the 1D cases are defined by prescribing the stress field while the unknown fields are the strains or the displacements. For the 2D cases we define problems with equilibrated tractions and evaluate the approximate displacement field. Meanwhile a Fourier series as well as a set of Chebyshev polynomials are used as the basis functions for the main unknown fields. We increase the number of the approximation functions to decrease the norm of the residuals and repeat the procedure until reasonable accuracy is obtained for the final solution. Since the procedure is very time consuming, in some benchmark problems we present the calculated coefficients and in some other we give some point-wise values for further use. The results presented in this paper are particularly useful for the validation and convergence studies when numerical methods are to be used for the solution of the nonlocal elasticity problems.

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1. Introduction

The classical theory of elasticity is widely used to solve a large number of engineering problems. However, in some problems the theory is not capable of modeling the material behavior accurately. Well-known examples/situations are: the modeling of Micro/Nano structures when the size effect becomes prominent, studies of elastic waves when dispersion effect is taken to account, and stress analyses at crack tips when the singularity of the solution is of concern. According to Eringen (1987), lack of an internal characteristic length scale implies such limitations in this theory and causes failure of the modeling in physical problems in which the influence of microstructural effects is significant. Several remedies have been proposed since the late 60s to circumvent the failure of the classical continuum theories in such situations. In the studies by Kröner (1967), Kunin (1984) and Krumhansl (1968) elastic materials with long range cohesive forces, elastic media with microstructures, and continuum approaches derived from an atomic lattice theory were considered, respectively. Improved formulations were proposed later by Edelen and Laws (1971), Edelen et al. (1971) and Eringen and Edelen (1972). In the category of nonlocal elasticity problems with linear homogeneous and isotropic materials, extensive studies by Eringen and Kim (1974) and Eringen et al. (1977) can be traced in the literature. In the aforementioned studies the main difference between the presented nonlocal theory and the classical one lies just in the stress–strain relations. In such cases the stresses at a generic point of the domain are considered dependent on the strains at the neighborhood of the point. With the use of such constitutive relations, it was shown that the singularity effect at a crack tip disappears (see Artan and Yelkenci, 1996; Zhou et al., 1999).

In continuum boundary-value problems, a nonlocal theory was proposed by Rogula (1982) for which the existence of the fundamental solutions was also investigated. Studies on the uniqueness of the solutions using nonlocal theories can be seen in the works by Altan (1989a,b). The readers can find comprehensive surveys of nonlocal plasticity and damage models in the review papers by Bažant and Jirásek (2002) or by Jirásek and Rolshoven (2003). In the realm of numerical solutions, using nonlocal theories, the studies by Polizzotto (2001) and Pisano et al. (2009) on the use of the finite element method and the research by Schwartz et al. (2012) on the application of the boundary element method should be mentioned here.

A strong tendency towards using numerical methods is seen in the recent studies (see Pisano et al., 2009; Polizzotto, 2001; Schwartz et al., 2012). However, it is well understood that all numerical methods are prone to numerical errors and therefore the availability of solutions with low errors seems to be vital for demonstrating the validity of the results. In the current study the objective is to present such benchmark problems for further studies. To this end we present approximate solutions by the use of well-known orthogonal bases; i.e., Fourier series and Chebyshev polynomials. In order to demonstrate the accuracy of the solutions we define appropriate norms in terms of the residuals. By increasing the number of the bases we search for the solutions in which the residual norms are reasonably low.
Both one and two dimensional (1D/2D) cases are to be considered in this paper. For the 1D cases we start from determinate problems in which the stress field is known a priori and thus the equilibrium equation is satisfied in advance. For such problems we shall find the strain field. This helps us to obtain fast convergence in the solutions. The displacement field may easily be found by a simple integration algorithm. It will be shown that both of the orthogonal bases perform well in the approximation, while in some special cases one is better than another. Having presented the 1D determinate benchmark problems, we give some further results for the indeterminate ones simply by writing compatibility conditions. The final results will be given in a series of tables for further use.

For the 2D problems we directly use the approximation of the displacement field. The solution process is extremely time-consuming. We present a benchmark problem in which the tractions are predefined at the boundaries. It will be shown that the use of Chebyshev polynomials leads to faster convergence with less residual norms when compared with the use of Fourier series. For further use we shall present the coefficients of the bases in an Appendix.

The layout of the paper is as follows. In the next section an overview of the nonlocal model used in this paper is given. In Section 3, the approximations used in the solutions of the 1D and 2D problems are described. In Section 4 the principle of the virtual work used for the solution is explained. The basis functions used for the construction of the approximate strain or stress fields are elaborated on in Section 5. The discussion on the numerical solutions and the final results for the benchmark problems are given in Section 6. Finally in Section 7 we summarize the conclusions made throughout the paper.

2. Nonlocal model; an overview

We consider an elastic body occupying \( \Omega \) in a 1D/2D space. According to Eringen’s model (Eringen, 2002) the stress values at a generic point as \( x = [x, y] \) depend on the stresses at other points of the domain, here known as \( x' = [x', y'] \). The strain and stress fields should satisfy the following constitutive integral equation

\[
\sigma(x) = \int_\Omega \xi(x, x') \kappa(x') \mathrm{d}x', \quad \forall x, x' \in \Omega
\]  

(1)

In the above relation \( \sigma \) and \( \kappa \) represent the stress and strain tensors arranged in vectors while \( \mathbf{D} \) is the matrix of material constants in the classical elasticity theory. The strains are defined in terms of the displacements as in the classical elasticity theory. Also, \( \xi(x, x') \) is a positive attenuation function, playing the role of a measure for the dependence of the stresses at \( x \) to the strains at \( x' \) (the volume fraction at \( x' \) is represented by \( \mathrm{d}x' \) in (1)). The attenuation function is chosen such that it reaches its maximum at \( x = x' \) and decays to zero for large distances between \( x \) and \( x' \), i.e.,

\[
\lim_{|x - x'| \to \infty} \xi(x, x') = 0.
\]  

(2)

The attenuation function \( \xi(x, x') \) also satisfies the following condition

\[
\int_\Omega \xi(x, x') \mathrm{d}x = 1,
\]  

(3)

analogous to the Dirac delta function, e.g., when a very sharp attenuation function is to be used. In (3) \( \Omega \) is an infinite domain embedding the main computational domain \( \Omega \). It is clear that the sharpness of \( \xi(x, x') \) represents an internal characteristic length for the material. Such a characteristic length may be defined experimentally or by matching the dispersion curves of the plane waves with those of the atomic lattice dynamics and therefore \( \lambda(x, x') \) can be chosen accordingly (Eringen, 2002).

An alternative nonlocal constitutive relation defined in Altan (1989b) and Eringen (2002) incorporates the classical elasticity constitutive relation in a weighted fashion

\[
\sigma(x) = \zeta_1 \mathbf{D} \kappa(x) + \zeta_2 \int_\Omega \xi(x, x') \kappa(x') \mathrm{d}x',
\]  

(4)

with \( \zeta_1 \) and \( \zeta_2 \) being two weight factors so that

\[
\zeta_1 + \zeta_2 = 1.
\]  

(5)

In fact in Eq. (4) the material is considered as a two-phase material, i.e., one with local and another with nonlocal characteristics. Obviously choosing \( \zeta_1 = 0 \) in (4) leads to Eq. (1). It is clear that even when \( \zeta_1 \neq 0 \), the formula given in (4) does not generally yield to a constant stress field when a constant strain field is considered for \( \xi(x) \). Conversely, a constant strain field is not obtained by considering a constant stress field except for the regions far from the boundaries. This is a feature of the formula given in (4) whose validity must be confirmed by experimental observations/evidences (see Eringen, 2002). However, it may be easily shown that for the regions far from stress or strain concentrations, a constant strain field may lead to a constant stress field. This can be shown by considering an infinite domain in (4), so that the boundary layer effects become small enough, and assuming a constant strain field as \( \varepsilon \)

\[
\sigma(x) = \zeta_1 \mathbf{D} \kappa + \zeta_2 \int_\Omega \xi(x, x') \kappa(x') \mathrm{d}x',
\]  

(6)

Now in view of (3) and (5), it is concluded that

\[
\sigma = (\zeta_1 + \zeta_2 \times 1) \mathbf{D} \kappa = \mathbf{D} \kappa.
\]  

(7)

However, such a conclusion may not easily be made for the problems defined on bounded domains. Nevertheless for the problems in which the influence/supporting domain of the attenuation function \( \xi(x, x') \) is relatively very small (see Remark 1), one may expect that the constant strain and stress fields are recoverable at the regions far from the boundaries. This may be considered as a necessary, but not sufficient, condition for the validity of a solution algorithm in the problems with nonlocal stress–strain relation as in (4).

**Remark 1.** Since the attenuation function has a very small value at large distances from the source point \( x \), it may be defined on a compact support within an influence distance \( L_k \), i.e.,

\[
\xi(x, x') = 0, \quad \forall |x - x'| > L_k.
\]  

(8)

The value of the influence distance \( L_k \) depends on the selected attenuation function and the internal length (to be defined later). Considering the influence distance leads to lower computation costs especially in methods such as the finite element method (Pisano et al., 2009). We shall give some results in one of the 1D numerical examples solved in Section 6 to give an insight to the effect of considering \( L_k \).

3. Approximation

In this paper we consider benchmark problems with a priori known stress field, in the 1D cases, or prescribed traction, in the 2D cases. In the former case the stress field must satisfy the equilibrium equation and thus the main unknown is the strain field.
3.1. One-dimensional problems

In this case Eq. (4) takes the form of
\[ \sigma(x) = \zeta_1 E \varepsilon(x) + \zeta_2 \int_0^L \alpha(x, x') E \varepsilon(x') dx', \quad 0 \leq x \leq L, \]  
(9)
where \( E \) denotes the Young’s modulus of the material and \( L \) is the domain length. With a prescribed stress as \( \sigma(x) \), the main unknown may be considered as the strain function \( \varepsilon(x) \). Note that in 1D problems the displacement field may be evaluated by a direct integration from the strain values. Therefore for the 1D problem in (9), we approximate the strain field as
\[ \varepsilon(x) \approx \sum_{i=1}^M \mathbf{c}_i f_i(x). \]  
(10)
In (10) the function \( f_i(x) \) represents a set of bases used for the approximation (e.g., Fourier series terms etc.) and thus \( \mathbf{c}_i \) represents the associated unknown coefficients. By substitution of (10) in (4), the following residual is defined
\[ R_e(x) = \sigma(x) - \left[ \zeta_1 E \varepsilon(x) + \zeta_2 \int_0^L \alpha(x, x') E \varepsilon(x') dx' \right]. \]  
(11)
After selecting the basis functions \( f_i(x) \), the rest of the procedure pertains to the reduction of a suitable norm of \( R_e(x) \).

Remark 2. The reader may note that, instead of approximating the strain field as in (10), one may start from the approximation of the displacement field. In that case the derivative of the displacement function appears in (11). Our experience in the solution of the 1D problems shows that the direct approximation of the strain field leads to the best convergence while increasing the number of the basis functions (i.e., \( M \) in (10)). This, however, is in contrast with the 2D cases described below. □

3.2. Two-dimensional problems

In this case the displacement field is approximated as
\[ \mathbf{u}(x) = \mathbf{u}(x) = \sum_{i=1}^M \mathbf{c}_i f_i(x), \]  
(12)
where \( \mathbf{u} = [u, v]^T \) denotes the vector of the displacements and \( \mathbf{u} = [u, v]^T \) represents the approximated field. Here, \( f_i(x) \) is a representative of the 2D basis functions and \( \mathbf{c}_i = [c_i^1, c_i^2]^T \) is the vector of the associated unknowns. The strain values are found as
\[ \varepsilon(x) = S \mathbf{u}(x), \quad \hat{\varepsilon}(x) = \hat{S} \mathbf{u}(x), \]  
(13)
where \( S \) is the well-known operator for defining strains in the classical elasticity theory.

The approximated stress field in this case takes the form of
\[ \hat{\sigma}(x) = \zeta_1 E \hat{S} \mathbf{u}(x) + \zeta_2 \int_0^L \alpha(x, x') E \hat{S} \mathbf{u}(x') dx'. \]  
(14)

With a set of basis functions \( f_i(x) \) selected, the rest of the procedure pertains to finding an equilibrated form of \( \mathbf{u}(x) \).

Remark 3. The reader may note again that, unlike the 1D cases, we cannot easily start from the direct approximation of the strain field. The reason lies in the uniqueness of the solution. It is well understood that with a given strain field in 2D, it is not always possible to find the displacement field unless the strain field satisfies the compatibility conditions (Timoshenko and Goodier, 1969). This means that if the strains are to be approximated directly, then the residuals of the compatibility conditions must be defined and taken into account. This effect is clearly in contrast with the 1D cases for which the displacement field can be uniquely defined from the strain field (and thus clearly affect the convergence of the solution). □

4. Finding an equilibrated stress field

To obtain an equilibrated stress field, we employ the virtual work definition
\[ \delta W_{\text{int}} = \delta W_{\text{ext}}. \]  
(15)
In (15) \( \delta W_{\text{int}} \) and \( \delta W_{\text{ext}} \) respectively denote the virtual work by the internal stresses and the external tractions. For a given virtual displacements \( \delta \mathbf{u} \), the virtual strains are defined as \( \delta \varepsilon = \hat{S} \delta \mathbf{u} \) in a manner analogous to (13). We use (15) in two forms; one for the 1D problems and another for the 2D problems.

4.1. For the 1D cases

In these cases for which we assume an exact stress field, one may write the following relation for both exact and approximated stress fields, i.e., \( \sigma \) and \( \hat{\sigma} \)
\[ \int_0^L \delta \varepsilon \sigma(x) dx = \int_0^L \delta \varepsilon \hat{\sigma}(x) dx = \int_0^L \delta \varepsilon \hat{\sigma}(x) dx = \delta W_{\text{int}}, \quad \int_0^L \delta \varepsilon \hat{\sigma}(x) dx = \delta W_{\text{int}}. \]  
(16)

Since the right hand sides of the above relations are identical, the following orthogonality condition is concluded
\[ \int_0^L \delta \varepsilon (\sigma - \hat{\sigma}) dx = 0, \]  
(17)
or
\[ \int_0^L \delta e R_e dx = 0, \]  
(18)
with \( R_e \) defined in (11). If the basis functions in (10) are reused in defining \( \delta \varepsilon \) as
\[ \delta \varepsilon(x) = \sum_{j=1}^M (\delta d_j f_j(x)), \]  
(19)
then, noting that the coefficients \( \delta d_j \) in (19) are arbitrary, Eq. (18) generates \( M \) algebraic equations as
\[ \int_0^L f_j R_e dx = 0, \quad j = 1, \ldots, M, \]  
(20)
which are to be solved for the coefficients \( c_i \) when (10) is substituted in (11) and the result is used in (20). The system of equations may be written as
\[ \mathbf{A} \mathbf{c} = \mathbf{B}, \]  
(21)
in which \( \mathbf{A} \) is a \( M \times M \) matrix with the following entries
\[ A_{ij} = \int_0^L f_i(\zeta_1 f_j + \zeta_2 \int_0^L \alpha f_j(x') dx'), \]  
(22)
and \( \mathbf{B} \) is an array with the following components
\[ B_k = \int_0^L f_k \sigma dx. \]  
(23)

4.2. For the 2D cases

In these cases we use (15) as its well-known form
\[ \int_\Omega \delta \varepsilon^T \hat{\sigma} d\Omega = \int_\Gamma \delta \mathbf{u}^T \mathbf{t} d\Gamma, \]  
(24)
In (24), \( t \) is the vector of tractions defined on the boundary of the domain denoted by \( \Gamma \). The virtual strains are evaluated from the virtual displacements as

\[
\delta \varepsilon = S \delta u = S \left( \sum_{j=1}^{M} \delta c_f_j(x) \right) = \left( \sum_{j=1}^{M} \delta c_j S F_j(x) \right), \quad F_j(x) = I \times f_j(x),
\]

where \( I \) is a \( 2 \times 2 \) identity matrix. By considering \( \delta c_j = [\delta c_1, \delta c_2]^T \) as an arbitrary array, the relation in (24) leads to \( M \) pairs of algebraic equations as

\[
\int_{\Omega} S F_j^T \delta \varepsilon d\Omega = \int_{\Gamma} F_j t d\Gamma, \quad j = 1, \ldots, M.
\]

The system of equations may again be written as (21) with \( A \) being a \( 2M \times 2M \) matrix such that

\[
[A]_{2:2} = \int_{\Omega} S F_i^T \left( \chi^1 \text{DSF}_i \right) + \int_{\Omega} \chi \text{DSF}_i d\Omega d\Omega
\]

and \( B \) as

\[
[B]_{2:1} = \int_{\Gamma} F_i t d\Gamma.
\]

The final approximate solution is written by inserting the coefficient \( c = A^{-1} B \) in (12). In the case that \( A \) is ill-conditioned, we use pseudo-inverse of \( A \) to solve the system of equations.

5. The basis functions

The basis functions \( f_j(x) \) are chosen so that they satisfy the completeness condition. To this end, we employ the well-known bases such as Fourier series or series of Chebyshev polynomials. Our experience shows the convergence behavior of each set differs for the 1D or 2D cases. Prior to giving more details, it is essential to mention that we use a normalized coordinate system for the solution. For the 1D problems we define

\[
\zeta = \frac{2x}{L} - 1, \quad 0 \leq x \leq L,
\]

and for the 2D problems, since the benchmarks are defined on rectangular domains, we define

\[
\zeta = \frac{2x}{a} - 1, \quad \eta = \frac{2y}{b} - 1, \quad 0 \leq x \leq a, \quad 0 \leq y \leq b,
\]

where \( a \) and \( b \) are the edge lengths of the domain.

5.1. Fourier series

We choose the Fourier series as below:

5.1.1. Fourier series for the 1D problems

With the normalized coordinate defined in (29), the basis functions in (10) are chosen as

\[
\hat{\varepsilon} = \sum_{i=1}^{p} c_i f_i(x) = \sum_{i=1}^{P} c_i N_i(\zeta) + \sum_{n=1}^{P} c_{n,0} \sin \left( \frac{n \pi (\zeta + 1)}{2} \right), \quad P \geq 1,
\]

in which \( N_i(\zeta) \) represents the well-known Lagrangian shape functions (and of order \( P - 1 \)). By considering the first summation, the strain values at both ends of the domain are directly taken into account as two unknowns (note that a sine series vanishes at both ends). In most of the cases we use \( P = 1, 2 \), i.e., we consider a constant or linearly varying strain along a line between the two end strains while letting the Fourier terms represent the deviation from the linear variation. For few cases we may use \( P > 2 \) depending on our experience on the convergence of the solution. Note that such a polynomial terms reproduce the exact strain field when the non-local effect is ignored, i.e., when \( \zeta_2 = 0 \). We specify the order of \( P \) in the examples. Note again that by using the basis functions in (31) in (22) and (23), the orthogonality property of the series term no longer helps to obtain a diagonal coefficient matrix \( A \) and thus the matrix is a populated one.

5.1.2. Fourier series for the 2D problems

The basis functions in this case are defined here as

\[
\hat{u}(x) = \sum_{i=1}^{4} c_i f_i(x) = \sum_{i=1}^{4} a_i N_i(\zeta, \eta)
\]

\[
+ \sum_{n=1}^{m_1} \sum_{m=1}^{m_2} b_{nm} \sin \left( \frac{n \pi (\zeta + 1)}{2} \right) \sin \left( \frac{m \pi (\eta + 1)}{2} \right)
\]

\[
+ \sum_{n=1}^{m_1} \sum_{m=1}^{m_2} c_{nm} \sin \left( \frac{n \pi (\zeta + 1)}{2} \right) \cos \left( \frac{m \pi (\eta + 1)}{2} \right)
\]

\[
+ \sum_{n=1}^{m_1} \sum_{m=1}^{m_2} d_{nm} \cos \left( \frac{n \pi (\zeta + 1)}{2} \right) \sin \left( \frac{m \pi (\eta + 1)}{2} \right).
\]

In the above relation \( a_i, b_{nm}, c_{nm}, d_{nm} \) play the role of the vector coefficient \( c \) in (12) and thus the total number of the basis functions is \( M = 3 \cdot (n_1 \times m_1) + 4 \) (and the total number of the unknowns is \( 2M \)). Moreover, \( N_i(\zeta, \eta) \) are the well-known 2D bi-linear shape functions defined on a domain with the normalized coordinates \( -1 \leq \zeta \leq 1 \) and \( -1 \leq \eta \leq 1 \). In the section of numerical experiments, we further simplify the displacement field (32) for some special cases, e.g., using symmetry etc.

5.2. Chebyshev polynomials

With the normalized coordinates defined in (29) or (30), the Chebyshev polynomials (Abramowitz and Stegun, 1965) take the form of

\[
T_0(\zeta) = 1, \quad T_1(\zeta) = \zeta, \quad \ldots \quad T_{n-1}(\zeta) = \frac{2}{T_n(\zeta) - T_{n-1}(\zeta)},
\]

where \( T_n(\zeta) \) denotes a Chebyshev polynomial of the first kind and of order \( n \). The following orthogonality condition also exists

\[
\int_{-1}^{1} \frac{1}{\sqrt{1 - \zeta^2}} T_n(\zeta) T_m(\zeta) d\zeta = \left\{ \begin{array}{ll}
0 & n \neq m, \\
p & n = m = 0, \\
p/2 & n \neq m.
\end{array} \right.
\]

Such an orthogonality condition, in our studies, just helps conditioning the coefficient matrix in one of the cases below.

5.2.1. Chebyshev polynomials for the 1D problems

The basis functions in (10) are chosen as

\[
\hat{\varepsilon} = \sum_{i=1}^{M} c_i f_i(x) = \sum_{i=0}^{P} c_i T_i(\zeta).
\]

Note that, unlike (31), there is no need to consider additional polynomial terms since the basis functions are a complete set of polynomials. The rest of formulation is as given in (11), (22) and (23). The elements of \( A \) take the form of

\[
A_{ij} = E \left\{ \int_{1}^{L} \left( \int_{-1}^{1} T_i(\zeta) T_j(\zeta) d\zeta \right) dz \right\} + \rho \int_{1}^{L} \left( \int_{-1}^{1} \frac{\partial T_i(\zeta)}{\partial \zeta} T_j(\zeta) d\zeta \right) dz.
\]
and the elements of $B$ are as
\[ B_k = \frac{L}{2} \int_{-1}^{1} T_k(\xi) \sigma(\xi) d\xi. \]  

**Remark 4.** In order to use the orthogonality condition (34) one may consider the basis functions as (35) and the functions for defining the virtual strain (see (19)) as
\[ \delta \epsilon = \sum_{i=1}^{M} \delta d_{f_i} = (1 - \xi^2)^{-1/2} \sum_{i=1}^{M} \delta d_{T_i(\xi)}. \]  

By writing (18), the elements of $A$ take the form of
\[ A_{kl} = E \left\{ \frac{L}{2} \left( \int_{-1}^{1} (1 - \xi^2)^{-1/2} T_k(\xi) T_l(\xi) d\xi \right) + \frac{\zeta_2 L^2}{4} \left( \int_{-1}^{1} \int_{-1}^{1} \alpha(\xi, \zeta') (1 - \zeta^2)^{-1/2} T_k(\xi) T_l(\xi') d\xi d\zeta' \right) \right\}. \]  

Then the orthogonality condition (34) helps to simplify the first term in the right hand side of (39). For instance when $k \neq l$ we have
\[ A_{kl} = E \left\{ \frac{L}{2} \left( \int_{-1}^{1} \int_{-1}^{1} \alpha(\xi, \zeta') (1 - \zeta^2)^{-1/2} T_k(\xi) T_l(\xi') d\xi d\zeta' \right) \right\}, \]  

which does not necessarily vanish. Also the elements of $B$ take the form of
\[ B_k = \frac{L}{2} \int_{-1}^{1} (1 - \xi^2)^{-1/2} T_k(\xi) \sigma(\xi) d\xi, \]  

which gives the projection of $\sigma$ on $T_k$. As is seen the use of (34) does not necessarily decrease the computational cost but at least the full spectral decomposition of the right-hand side of the equations, as given in (41), is obtained. \(\square\)

### 5.2.2. Chebyshev polynomials for the 2D problems

The basis functions in this case are defined as
\[ u(x) = \sum_{m=1}^{M} c_m f_i(x) = \sum_{n=0}^{m_1} \sum_{m=0}^{m_2} a_{nm} T_n(\xi) T_m(\eta). \]  

Here again $a_{nm}$ plays the role of the vector coefficient $c_i$ and thus the total number of the basis functions is $M = (m_1 + 1) \times (m_2 + 1)$. By using a mapping for defining one index from two indices as
\[ k = k(n, m) \quad n = 0, \ldots, n_1, \quad m = 0, \ldots, m_1, \]  

or
\[ l = l(s, p) \quad s = 0, \ldots, n_1, \quad p = 0, \ldots, m_1, \]  

the elements of $A$ take the form of
\[
[A_{kl}]_{2 \times 2} = \frac{a b}{4} \int_{-1}^{1} \int_{-1}^{1} [ST_k(\xi)]^T D [ST_l(\xi)] d\xi d\eta + \frac{\zeta_2 a^2 b^2}{16} \int_{-1}^{1} \int_{-1}^{1} \alpha(\xi, \zeta') [ST_k(\xi)]^T D [ST_l(\xi')][ST_k(\xi')] d\xi' d\eta. \]  

in which $\xi = [\xi, \eta]^T$, $\zeta' = [\zeta, \eta']^T$ and $T_k(\xi) = I \times T_n(\xi) T_m(\eta)$, $T_l(\xi) = I \times T_s(\xi) T_p(\eta)$.

Also the elements of $B$ are as
\[
[B_{kl}]_{2 \times 1} = \frac{a}{2} \int_{-1}^{1} [T_k(\xi, \eta = 1)]^T t(\xi) d\xi + \frac{a}{2} \int_{-1}^{1} [T_l(\xi, \eta = -1)]^T t(\xi) d\xi \]  

As mentioned earlier, in the section of numerical experiments, we further simplify the displacement field (42) for some special cases, e.g., using symmetry etc.

**Remark 5.** If the orthogonality condition (34) is to be used, one may consider the basis functions as (42) and the functions for defining the virtual strains (see (23)) as
\[ \delta \epsilon = \sum_{i=1}^{M} \delta d_{i} (1 - \xi^2)^{-1/2} (1 - \eta^2)^{-1/2} [ST_i(\xi)]. \]  

However such a virtual strain field does not necessarily correspond to a virtual displacement field (see also **Remark 3**). Therefore using (48) in (16) does not have a physical meaning (i.e., virtual work). Nevertheless, the expression (48) may be considered as a set of weights, with arbitrary coefficients, for the reduction of the residuals as in (18). Using (48) gives the following relation for the elements of $A$
\[
[A_{kl}]_{2 \times 2} = \frac{a b}{4} \int_{-1}^{1} \int_{-1}^{1} (1 - \xi^2)^{-1/2} (1 - \eta^2)^{-1/2} [ST_k(\xi)]^T D [ST_l(\xi)] d\xi d\eta + \frac{\zeta_2 a^2 b^2}{16} \int_{-1}^{1} \int_{-1}^{1} \alpha(\xi, \zeta') [ST_k(\xi)]^T D [ST_l(\xi')][ST_k(\xi')] d\xi' d\eta. \]  

Again, unlike the 1D cases in (39), it is not possible to further simplify (49) since the derivatives of the Chebyshev polynomials are

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**Fig. 1.** Loading conditions for nonlocal elastic bar: (a) mid and end concentrated forces; (b) linear body force.
not necessarily orthogonal functions (note that the operator $S$ is operating on the basis functions). □

6. Numerical solutions and benchmarks

In this section we present some 1D and 2D benchmark problems.

6.1. One-dimensional problems

Before giving further details, first of all we define the following norm for the measurement of the largeness of the residuals in 1D problems

$$\eta = \left( \int_\Omega \frac{R_j^2}{\sigma^2} \, d\Omega \right)^{1/2}. \quad (50)$$

Also the following norm is defined

$$\tilde{\eta} = \left( \frac{\sum_{i=1}^{NP} (\epsilon_i - \tilde{\epsilon}_i)^2}{\sum_{i=1}^{NP} (\epsilon_i)^2} \right)^{1/2}, \quad (51)$$

for measuring the deviation of two independent sets of the results, e.g., form two successive numerical solutions as $\epsilon_1$ and $\epsilon_2$ with different numbers of basis functions. In the above relation $NP$ is the number of points selected inside the domain (for instance $NP = 1000$).

Several 1D nonlocal elastic bars have been devised most of which are determinate problems, i.e., when the stress field is known a priori through the equilibrium equation. Having obtained the solution for the determinate problems, we give some more results for the indeterminate benchmarks. The following normalized strain is defined for the presentation of the results:

$$\tilde{\epsilon} = \frac{EA}{L} \tilde{\epsilon}, \quad F = \int_0^L |b(x)| \, dx + |F|, \quad (52)$$

where $|F|$ denotes the magnitude of the force at the end of the bar in the determinate problems. For the indeterminate problems we use $F = 0$. Also in (52), $b(x)$ denotes the body force of the problem whose variation in each case is obtained by $b(x) = -A(d(\sigma(x))/dx)$ for the determinate problems (for the indeterminate problems we use the body forces defined for the determinate ones). Moreover, all examples have been solved with an attenuation function as

$$\sigma(x,a) = \frac{1}{2l} \exp\left(-\frac{|x| - a}{l}\right). \quad (53)$$

### Table 1

<table>
<thead>
<tr>
<th>Load case</th>
<th>The end forces, body forces and the stress fields $P$ in Eq. (31)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>$F_1 = 0$, $F_2 = F$, $\sigma(x) = 0$, etc.</td>
</tr>
<tr>
<td>b</td>
<td>$F_1 = F_2 = F$, $\sigma(x) = \frac{2\pi}{\lambda} x$, etc.</td>
</tr>
<tr>
<td>c</td>
<td>$q_1 = q_2 = q_3$, $\sigma(x) = \frac{2\pi}{\lambda} (1 - \frac{x}{L})$</td>
</tr>
<tr>
<td>d</td>
<td>$q_1 = 0$, $q_2 = q_3$, $\sigma(x) = \frac{2\pi}{\lambda} (1 - \frac{x}{L})$</td>
</tr>
</tbody>
</table>

### Table 2

<table>
<thead>
<tr>
<th>Load case</th>
<th>Fourier sine series</th>
<th>Chebyshev polynomials</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>$L = 60$</td>
<td></td>
</tr>
<tr>
<td>b</td>
<td>$L = 60$</td>
<td></td>
</tr>
<tr>
<td>c</td>
<td>$L = 60$</td>
<td></td>
</tr>
<tr>
<td>d</td>
<td>$L = 60$</td>
<td></td>
</tr>
</tbody>
</table>

$\eta_p = \left( \int_\Omega \frac{R_j^2}{\sigma^2} \, d\Omega \right)^{1/2}$.  \quad (50)

$\tilde{\eta} = \left( \frac{\sum_{i=1}^{NP} (\epsilon_i - \tilde{\epsilon}_i)^2}{\sum_{i=1}^{NP} (\epsilon_i)^2} \right)^{1/2}, \quad (51)$

### Table 3

<table>
<thead>
<tr>
<th>Load case</th>
<th>Chebyshev polynomials</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>1.41421353600</td>
</tr>
<tr>
<td>b</td>
<td>1.4071527515</td>
</tr>
<tr>
<td>c</td>
<td>1.43294215765</td>
</tr>
<tr>
<td>d</td>
<td>1.4156807185</td>
</tr>
</tbody>
</table>

$\eta_r = \left( \int_\Omega \frac{R_j^2}{\sigma^2} \, d\Omega \right)^{1/2}$.  \quad (50)

$\tilde{\eta} = \left( \frac{\sum_{i=1}^{NP} (\epsilon_i - \tilde{\epsilon}_i)^2}{\sum_{i=1}^{NP} (\epsilon_i)^2} \right)^{1/2}, \quad (51)$

### Table 4

<table>
<thead>
<tr>
<th>Load case</th>
<th>Fourier sine series</th>
<th>Chebyshev polynomials</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>$M = 35$</td>
<td></td>
</tr>
<tr>
<td>b</td>
<td>$M = 35$</td>
<td></td>
</tr>
<tr>
<td>c</td>
<td>$M = 35$</td>
<td></td>
</tr>
<tr>
<td>d</td>
<td>$M = 35$</td>
<td></td>
</tr>
</tbody>
</table>

$\eta_p = \left( \int_\Omega \frac{R_j^2}{\sigma^2} \, d\Omega \right)^{1/2}$.  \quad (50)

$\tilde{\eta} = \left( \frac{\sum_{i=1}^{NP} (\epsilon_i - \tilde{\epsilon}_i)^2}{\sum_{i=1}^{NP} (\epsilon_i)^2} \right)^{1/2}, \quad (51)$

### Table 5

<table>
<thead>
<tr>
<th>Load case</th>
<th>Chebyshev polynomials</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>1.41421353600</td>
</tr>
<tr>
<td>b</td>
<td>1.4071527515</td>
</tr>
<tr>
<td>c</td>
<td>1.43294215765</td>
</tr>
<tr>
<td>d</td>
<td>1.4156807185</td>
</tr>
</tbody>
</table>
with \( l \) being a characteristic length to be defined for each problem as a fraction of \( L \).

6.1.1. Determinate problems

The stress values are defined through the equilibrium condition by considering the body force. The loading conditions are shown in Fig. 1. The force characteristics, the exact stress values and the corresponding order \( P \), used in (31), for the cases considered in this paper are given in Table 1. As is seen in Table 1, for load case (b) instead of the concentrated load in Fig. 1a, which produces a discontinuous stress field, we have used a uniformly distributed load on a small segment of the bar (in the present study the segment is equal to \( L/20 \)). The number of the utilized functions for a good approximation is been shown in Table 2. As is seen in Table 2, the use of Chebyshev polynomials leads to better accuracy and convergence except for load case (b) in which a concentrated load is applied at the center of the bar. Therefore in Table 3 we report the strain values obtained by Chebyshev polynomials except for load case (b) for which the strains are obtained by Fourier series.

Fig. 2a also shows the variation of the strain obtained for load case “a” in Table 1 using \( l = L/20 \). As is seen in this figure (or in Table 3), the strain values obtained by the nonlocal elasticity theory differ from those of the classical theory (or local theory) just at points where a sharp variation of stress happens. For further use, the coefficients obtained for the normalized strain field, in this load case, are presented in Appendix A. To give an insight to the solution accuracy, the recovered stress field is also shown in Fig. 2b (a normalized stress is used in this figure as \( \sigma = \sigma A / F \)).

It is worthwhile to mention that the problem has also been solved by Pisano and Fuschi (2003) and the following strain field has been introduced

\[
\varepsilon(x) = \varepsilon_0 - \frac{\lambda}{2} \sum_{k=0}^{\infty} \left( \varepsilon^j(x = x) + \varepsilon^j(x = L - x) \right)
\]

\[
\lambda = -\varepsilon_0 / 2 C_1.
\]

The normalized strain obtained, using \( \varepsilon \) in (54) in place of \( \varepsilon \) in (52), may be seen in Fig. 2a. Small discrepancies are observed between the two solutions. The source of such discrepancies may be found by recovering the stress values (i.e., substitution of Eq. (30) in Pisanono and Fuschi (2003) into Eq. (6) of the same reference or Eq. (9) in this paper). Fig. 2b shows that, while the normalized stress field is

---

**Fig. 2.** The results obtained by various methods for load case “a” in Table 1 and for \( l = L/20 \): (a) the normalized strain distributions and, (b) the recovered normalized stress fields (\( \sigma = \sigma A / F \)).
Fig. 3. Normalized strain field in: (a) load case "b" \( F_1 = F_2 = F \), (b) load case "c" \( q_1 = q_2 = q_0 \), (c) load case "d" \( q_1 = 0, q_2 = q_0 \).

Table 4
Error norms for different values of \( L_e \) for load case (a) and \( l = L/20 \).

<table>
<thead>
<tr>
<th>( L_e )</th>
<th>2( l )</th>
<th>3( l )</th>
<th>4( l )</th>
<th>5( l )</th>
<th>6( l )</th>
<th>7( l )</th>
<th>8( l )</th>
<th>9( l )</th>
<th>10( l )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \eta_\alpha )</td>
<td>6.93E−4</td>
<td>2.90E−4</td>
<td>1.19E−4</td>
<td>4.81E−5</td>
<td>1.94E−5</td>
<td>7.96E−6</td>
<td>3.23E−6</td>
<td>1.24E−6</td>
<td>5.28E−7</td>
</tr>
<tr>
<td>( \eta_\beta )</td>
<td>6.30E−2</td>
<td>2.12E−2</td>
<td>7.28E−3</td>
<td>2.51E−3</td>
<td>8.60E−4</td>
<td>2.90E−4</td>
<td>9.80E−5</td>
<td>3.20E−5</td>
<td>1.09E−5</td>
</tr>
</tbody>
</table>
recovered by the method presented in this study, in the results obtained by Eq. (54) some discrepancies from the exact stress field are seen at the two ends. This indicates that the stress field is not fully recovered. Nevertheless, the relation presented by Pisano and Fuschi (2003) is very appealing since it not only provides a simple expression for the solution but also it is useable for a wide range

Fig. 4. Normalized strain fields for different values of $l$ in the indeterminate problems: (a) mid force and $x_1 = L/2$; (b) constant body force $q_1 = q_2 = q_0$; (c) linear body force $q_1 = 0$, $q_2 = q_0$. 

of values for the parameters \( \varepsilon_1, \varepsilon_2 \) and \( l \). To improve the solution we modify the strain relation in (54) as

\[
\varepsilon(X) = \varepsilon_0 - \varepsilon_1 \frac{l}{2} \left[ e^{h_1 |x - h_2|/l} - e^{h_2 |x + h_2|/l} \right] + \varepsilon_2 \frac{l}{2} \left[ e^{h_1 |x + h_1|/l} - e^{h_1 |x + h_1|/l} \right],
\]

(55)

and find the optimal values for \( h_1, h_2 \) and \( h_3 \) through the minimization of an error norm for the recovered stresses (for various values of \( l \)). The following value are found for \( \varepsilon_1 = \varepsilon_2 = 0.5 \)

\[
h_1 = 1.1953654412422674, \quad h_2 = 0.9428090415825793, \quad h_3 = 0.2308483213427966.
\]

The normalized strain field and the recovered normalized stresses obtained from the relation (55) are also shown in Fig. 2a and b. As is seen, not only the modified strain field agrees with the one presented in this paper but also the stress field is fully recovered. Eqs. (54) and (55) inherently indicate that the solution to the problems may be found in the form of exponential functions. The use of exponential basis functions (EBFs) in the solution of elasticity problems has been addressed by Boroomand et al. (2010) and the latest developments may be found in the recent paper by Hashemi et al. (2013). The readers may follow our forthcoming papers on the use of EBFs in the solution of 1D/2D nonlocal elasticity problems.

Focusing on the other load cases in Table 2, we present the strain distribution for load case “b” in Fig. 3a. As expected, the sharp variation of strain at the concentrated load in the classical theory is replaced by a smooth one in the nonlocal elasticity. Fig. 3b and c show the strains obtained for load cases “c” and “d”.

Similar conclusions made for Fig. 3a are valid for these load cases. The point-wise values of the strains obtained for all aforementioned cases are given in Table 3 for further use.

As the final discussion in this paper, we study the effect of \( L_\infty \), i.e., considering a compact support for the attenuation function in the constitutive equation (Eq. (9) for 1D bar) as mentioned in Remark 1. The effect of choosing different values of the influence distance \( L_\infty \) is shown in Table 4. In this table \( \eta \) represents the values of the norm between the results considering the full domain and those considering the influence distance \( L_\infty \) for the attenuation function. As is seen, the differences are quite small, meaning that the results previously obtained may still be used for \( L_\infty \gg 2l \).

### 6.1.2. Indeterminate problems

Having found the solution of a 1D determinate problem, one may easily solve the associated indeterminate problem through a superposition algorithm. For instance, if in load cases “b” to “d” the end point is restrained against movement, the problem can be solved by the superposition of that load case with load case “a” and the application of the compatibility condition at the end point. To do this, load case “a” is first solved with an unknown \( h_3 \) and find the optimal values for \( h_1, h_2, \) and \( h_3 \) through the minimization of an error norm for the recovered stresses (for various values of \( l \)). The following value are found for \( \varepsilon_1 = \varepsilon_2 = 0.5 \)

\[
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### 6.2. Two-dimensional problems

The solution of 2D nonlocal elasticity problems is very time consuming, especially when low-residual solutions are of concern. Here again, prior to giving further details, we need a suitable norm to demonstrate that the numerical solution is close enough to the exact solution. As mentioned earlier, it is not possible to consider a predefined equilibrated stress field and find the associated displacement field as in the 1D cases. The validity and accuracy of the solution may be understood by measuring the level of satisfaction of the equilibrium equations and the boundary conditions. To this end the following residuals are defined here

\[
R_\Omega = \mathbf{S}^T \sigma + b \quad \text{in} \quad \Omega.
\]

(57)
\( \mathbf{R}_f = \mathbf{n} \sigma - \mathbf{t} \) on \( \Gamma_t \). (58)

In (57), \( \mathbf{b} \) is the body force. We are interested in the following norms
\[
\| \mathbf{R}_g \| = \left( \int_\Omega \mathbf{R}_g^T \mathbf{R}_g \, d\Omega \right)^{1/2} \quad \text{and} \quad \| \mathbf{R}_f \| = \left( \int_\Gamma \mathbf{R}_f^T \mathbf{R}_f \, d\Gamma \right)^{1/2},
\]
but the difficulty is that they are not dimensionless and thus the estimation of their largeness/smallness is not an easy task. One may compare the residual norms with the norm of the prescribed values; for instance \( \| \mathbf{R}_g \| \) and \( \| \mathbf{R}_f \| \) may be compared with \( \| \mathbf{b} \| \) and \( \| \mathbf{t} \| \), respectively. However, for problems with no body force, as the one we solve here, one may use an appropriate scale of \( \| \mathbf{t} \| \) for the equilibrium residuals. On this basis we define the following norm as an error indicator
\[
\eta = \frac{\sqrt{A_\Omega} \| \mathbf{R}_g \| + \sqrt{L_\Gamma} \| \mathbf{R}_f \|}{2 \sqrt{L_\Gamma} \| \mathbf{t} \|}, \quad \| \mathbf{t} \| = \left( \int_\Gamma \mathbf{t}^T \mathbf{t} \, d\Gamma \right)^{1/2}. \quad (60)
\]
In the above relation \( A_\Omega \) is the area occupied by the 2D domain \( \Omega \) and \( L_\Gamma \) is the total length of its boundary \( \Gamma = \partial \Omega \). With the above error indicator we proceed to examine the validity of the solution in a 2D benchmark problem. We consider a problem on a square domain \( 0 \leq x \leq L, \ 0 \leq y \leq L \) (or \( a = b = L \)). The tractions are defined as
\[
\mathbf{t}_{i,x=0,y} = (-\sigma_0,0)^T, \quad \mathbf{t}_{i,x=L,y} = (\sigma_0,0)^T, \quad \mathbf{t}_{i,y=0} = \mathbf{t}_{i,y=L} = \mathbf{0}, \quad (61)
\]
with \( \sigma_0 \) being the magnitude of the tractions. The Young’s modulus and the Poisson’s ratio are considered as; \( E \) and \( \nu = 0.2 \) respectively.

The following normalized displacement, strain and stress fields are defined for the presentation of the results:
\[
\mathbf{u} = \frac{E}{\sigma_0} \mathbf{u}, \quad \mathbf{e} = \frac{E}{\sigma_0} \mathbf{e}, \quad \sigma = \frac{1}{\sigma_0} \sigma. \quad (62)
\]

We present the solution of the problem under plane stress conditions and with \( \zeta_1 = \zeta_2 = 0.5 \) while the attenuation function is
\[
\alpha(x,x') = \frac{1}{\pi^2} \exp \left( -\frac{(x-x')^2}{r^2} \right), \quad (63)
\]
with \( r = L/10 \).

We solve the problem using both Fourier series and Chebyshev polynomials. Due to the symmetry of the geometry and the loading, we can reduce the number of the unknowns in each case. For instance, in using Fourier series in (32) we consider the terms producing anti-symmetric \( u \) and symmetric \( v \). The same strategy may be employed when Chebyshev polynomials are used.

Convergence of the solutions with respect to the number of terms used is demonstrated in Fig. 5. As is seen, the solution with Chebyshev polynomials converges faster than the one solved with Fourier series. Moreover, in both cases the residual norm deceases to a minimum value and then grows. Such an effect is due to the ill-conditioned coefficient matrices, \( \mathbf{A} \) in (21), confronted during the solution process and is directly related to the precision used during the computation (here double precision). The reader may note that although we use a complete set of functions, such as those in

Fig. 6. Distribution of the normalized strains in the 2D benchmark when Chebyshev polynomials are used for \( m_1 = n_1 = 12 \): (a) distribution of \( \bar{\epsilon}_x \) (b) distribution of \( \bar{\epsilon}_y \), (c) distribution of \( \bar{\gamma}_{xy} \).
Fig. 7. Distribution of the normalized strain $\tilde{\varepsilon}_y$ in the 2D benchmark when Fourier series is used for $m_1 = n_1 = 8$.

Fig. 8. Distribution of the normalized stresses in the 2D benchmark when Chebyshev polynomials are used and $m_1 = n_1 = 12$: (a) distribution of $\sigma_x$, (b) distribution of $\sigma_y$, (c) distribution of $\tau_{xy}$. 
Chebyshev polynomials, the orthogonality condition does not play an important role especially in 2D cases (see Remark 5). It is also seen that the use of Chebyshev polynomials leads to a minimum residual which is much less than the minimum residual obtained by the use of Fourier series.

Fig. 6a to c show the distribution of the normalized strain components \( (\varepsilon_x, \varepsilon_y, \text{and} \gamma_{xy}) \) when Chebyshev polynomials are used \((m = n_1 = 12, M = 196 \text{ in } (42))\). For further use, we present the coefficients of the polynomials for the normalized displacement field (see (62)) in Appendix A.

The results obtained by Fourier series, corresponding to the minimum residual in Fig. 5, for \( \varepsilon_y \) are shown in Fig. 7 \((m = n_1 = 8, M = 196 \text{ in } (32))\). As is seen, even with 196 terms, the normalized strain is obtained with an oscillatory distribution (in comparison with \( \varepsilon_y \) in Fig. 6b). It is obvious that the results obtained by the use of Chebyshev polynomials are superior. The stresses obtained by Chebyshev polynomials are shown in Fig. 8a to c. As is seen, along with \( \sigma_{rr} \), other stress components \( \sigma_{\theta \theta} \) and \( \tau_{\theta r} \) are also induced, however, they are considerably less than \( \sigma_{rr} \).

### 7. Conclusions

Low-residual approximate solutions for nonlocal 1D and 2D elasticity problems defined with Eringen's integral model are presented in this paper. For the 1D cases, determine problems are first defined and then indeterminate problems are solved through a superposition approach. In the determine problems the stress field, resulting from the equilibrium equation, is prescribed and then the strain field is found as the main unknown. For the 2D cases, the problems are defined with the self-equilibrated tractions while the main unknown is the displacement field. Fourier series and a set of Chebyshev polynomials are used as the basis functions for the main unknown fields. In the 1D case, the use of both sets of the bases leads to fast convergence and very low residual values. In three benchmark problems out of the four 1D determine problems considered in this paper, the results obtained by Chebyshev polynomials are superior to those obtained by Fourier series. The same effect is seen in the 2D problem solved. Since the solution is extremely time-consuming the point values from the strains for the 1D polynomials and the coefficients obtained for the 2D problem have been reported in a series of tables for further use.

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### References


