\(\mathcal{Q}\)-Ramsey classes of graphs

Mieczysław Borowiecki, Anna Fiedorowicz *

Faculty of Mathematics, Computer Science and Econometrics, University of Zielona Góra, ul. prof. Z. Szafrana 4a, 65-516 Zielona Góra, Poland

\section{1. Introduction}

All graphs which we consider are finite, undirected and without loops or multiple edges. Let \(G\) be a graph and let \(V(G)\) and \(E(G)\) be its vertex set and edge set, respectively. For a set \(U \subseteq V(G)\) (or \(F \subseteq E(G)\)) the subgraph of \(G\) induced by \(U\) (by \(F\), respectively) is denoted by \(G[U]\) (by \(G[F]\), respectively). The subgraph of \(G\) isomorphic to a complete graph is called a \textit{clique} of \(G\), or a \textit{k-clique}, if its order equals \(k\). The \textit{clique number} of a graph \(G\), denoted by \(\omega(G)\), is defined as the order of the largest clique in \(G\).

A \textit{k-colouring} of a graph \(G\) is a mapping \(f\) from the set of vertices of \(G\) to the set \(\{1, \ldots, k\}\) of colours. A \textit{k-colouring} is called \textit{proper}, if every two adjacent vertices have distinct colours.

The classical concept of a proper colouring has been generalised or extended in many different ways. One of such variations, namely an acyclic colouring, was presented by Grünbaum in 1973 in the paper [13]. A proper \(k\)-colouring \(f\) of a graph \(G\) is called \textit{acyclic}, if for any two distinct colours \(i\) and \(j\), the set of the edges \(xy\) such that \(f(x) = i\) and \(f(y) = j\) induces an acyclic graph.

In 1998, Yuster defined another type of graph colouring called a \textit{linear colouring} [21]. In this colouring we require that adjacent vertices have distinct colours and for any two distinct colours \(i\) and \(j\), the subgraph induced by the edges \(xy\) such that the vertex \(x\) is coloured with \(i\) and the vertex \(y\) is coloured with \(j\) induces a linear forest, \textit{i.e.} a forest in which each component is a path. A similar concept of a \textit{star colouring} of graphs was presented by Albertson et al. in [1]. For other references on acyclic, star and linear colourings, see [2–6,10,11,20].

In [8], the concept of a \(\mathcal{Q}\)-colouring of graphs was introduced, as a generalisation of all the above-mentioned. Before we formally define a \(\mathcal{Q}\)-colouring, we need to present necessary definitions and notations. We follow the notation of Borowiecki et al. [7].

\* Corresponding author.

\textit{E-mail address:} A.Fiedorowicz@wmie.uz.zgora.pl (A. Fiedorowicz).

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doi:10.1016/j.disc.2012.02.007
Let \( I \) denote the set of all finite graphs. Any nonempty subset of \( I \), which is closed with respect to isomorphism is called a graph property. The set \( I \) is called a trivial graph property. We say that a graph property \( \mathcal{P} \) is hereditary, if from the fact that \( G \in \mathcal{P} \) and \( H \subseteq G \) it follows that \( H \in \mathcal{P} \).

By \( \mathbb{L} \) we denote the set of all hereditary graph properties. Below we list some hereditary graph properties, which we use later. Let \( k \) be a positive integer.

\[
\begin{align*}
\varnothing &= \{ G \in I : E(G) = \emptyset \}, \\
\varnothing_k &= \{ G \in I : \text{the order of a component of } G \text{ is at most } k+1 \}, \\
\delta_k &= \{ G \in I : \Delta(G) \leq k \}, \\
\mathcal{D}_k &= \{ G \in I : G \text{ is } k\text{-degenerate} \}, \\
\varnothing^k &= \{ G \in I : G \text{ is } k\text{-colourable} \}. 
\end{align*}
\]

For instance, \( \mathcal{D}_1 \) is the property of being a forest and \( \varnothing^2 \) is the class of all bipartite graphs.

Any nontrivial hereditary graph property \( \mathcal{P} \) can be characterised by the set of its minimal forbidden subgraphs [7], defined as follows:

\[
\mathcal{F}(\mathcal{P}) = \{ G \in I : G \notin \mathcal{P}, \text{ but if } H \text{ is a proper subgraph of } G, \text{ then } H \in \mathcal{P} \}.
\]

Therefore, for a set \( \mathcal{F} \subseteq I \), we can define:

\[
\text{Forb}(\mathcal{F}) = \{ G \in I : \text{ for every } F \in \mathcal{F}, \ F \nsubseteq G \}.
\]

If \( \mathcal{F} = \{ F \} \), then we use the notation \( \text{Forb}(F) \).

In [7] it was proved that a nontrivial hereditary graph property \( \mathcal{P} \) can be also defined by the set

\[
\mathcal{M}(\mathcal{P}) = \{ G \in I : G \in \mathcal{P} \text{ and for any edge } e \in E(\overline{G}), \ G+e \notin \mathcal{P} \}
\]

of its maximal graphs.

Before we proceed, let us illustrate these definitions by a simple example. Consider for instance the property \( \mathcal{D}_1 \) of being a forest. Clearly, the set of its minimal forbidden subgraphs \( \mathcal{F}(\mathcal{D}_1) = \{ C_n, \ n \geq 3 \} \). It is also easy to observe that \( \mathcal{M}(\mathcal{D}_1) = \{ G \in I : G \text{ is a tree} \} \).

Consider an arbitrary set \( \mathcal{G} \subseteq I \). We say that the property \( \mathcal{P} \) is generated by the set \( \mathcal{G} \), called generator of \( \mathcal{P} \), if

\[
\mathcal{P} = \{ G \in I : \text{ there exists a graph } H \in \mathcal{G} \text{ such that } G \subseteq H \}.
\]

It is easy to see that \( \mathcal{P} \) is a hereditary graph property. In [7] it was proved that the set \( \mathcal{M}(\mathcal{P}) \) generates the hereditary property \( \mathcal{P} \). Therefore in the proofs we may restrict our attention only to the set of maximal graphs of \( \mathcal{P} \).

For definitions and notations not presented here, we refer the reader to [7,19].

### 2. Definition and basic properties of a \((k, \mathcal{Q})\)-Ramsey class

Let \( \mathcal{P}_1, \ldots, \mathcal{P}_k \in \mathbb{L} \) and \( G \in I \). A mapping \( f : V(G) \rightarrow \{1, \ldots, k\} \) is called a \((\mathcal{P}_1, \ldots, \mathcal{P}_k)\)-colouring of \( G \), if for each \( i \in \{1, \ldots, k\} \)

\[
G[\{x \in V(G) : f(x) = i\}] \in \mathcal{P}_i.
\]

Let \( \mathcal{Q} \in \mathbb{L} \) and assume \( \mathcal{Q} \subseteq \varnothing^2 \). We define a \((\mathcal{P}_1, \ldots, \mathcal{P}_k) ; \mathcal{Q})\)-colouring of a graph \( G \) as a \((\mathcal{P}_1, \ldots, \mathcal{P}_k)\)-colouring of \( G \) such that for any two distinct colours \( i \) and \( j \),

\[
G[\{xy \in E(G) : f(x) = i \text{ and } f(y) = j\}] \in \mathcal{Q}.
\]

If \( \mathcal{P}_1 = \cdots = \mathcal{P}_k = \mathcal{P} \), then a \((\mathcal{P}_1, \ldots, \mathcal{P}_k) ; \mathcal{Q})\)-colouring of \( G \) is called a \((\mathcal{P}^k ; \mathcal{Q})\)-colouring of \( G \). Furthermore, an \((\mathcal{L}^k ; \mathcal{Q})\)-colouring is called for short a \((k; \mathcal{Q})\)-colouring. By \( \mathcal{P}_1 \circ \mathcal{Q} \cdots \circ \mathcal{Q} \mathcal{P}_k \) we denote the set of all graphs having a \((\mathcal{P}_1, \ldots, \mathcal{P}_k) ; \mathcal{Q})\)-colouring. It is easy to observe that if \( \mathcal{P}_1, \ldots, \mathcal{P}_k \in \mathbb{L} \), then \( \mathcal{P}_1 \circ \mathcal{Q} \cdots \circ \mathcal{Q} \mathcal{P}_k \in \mathbb{L} \).

If \( \mathcal{Q} \) is a proper subset of \( \varnothing^2 \), then there is a significant difference between an \((\mathcal{P}^k ; \mathcal{Q})\)-colouring and a proper \(k\)-colouring. To see this, observe first that since \( \mathcal{Q} \) is a hereditary graph property and \( \mathcal{Q} \subseteq \varnothing^2 \), there exist integers \( t_1, t_2 \geq 1 \) such that the complete bipartite graph \( K_{t_1, t_2} \notin \mathcal{Q} \). Let \( t = \min(t_1, t_2) \). We prove that for any \( k \geq 2 \) there is a bipartite \( t\)-degenerate graph \( G \) which does not admit any \((\mathcal{P}^k ; \mathcal{Q})\)-colouring. Such a graph can be constructed as follows. Assume \( t_1 \leq t_2 \). We start with the graph \( K_{n_1} \), with \( n_1 = (t_1 - 1)k + 1 \). Denote its vertex set by \( A \). For each \( t_1 \)-subset \( S \) of \( A \), add \( n_2 = (t_2 - 1)(k - 1) + 1 \) new independent vertices and join each of them to all vertices of \( S \). Denote the set of all vertices added in this way by \( B \).

Clearly, \( G \) is bipartite, with bipartition \((A, B)\), and \( t\)-degenerate. Assume to the contrary that \( G \) has an \((\mathcal{P}^k ; \mathcal{Q})\)-colouring. Then there is a set \( U \subseteq A \) containing \( t_1 \) vertices which are all coloured with the same colour, say colour 1. Consider the set \( U' \subseteq B \) of \( n_2 \) vertices which are adjacent to all vertices of \( U \). Clearly, a vertex \( y \in U' \) cannot have colour 1. But from the fact that \( K_{t_1, t_2} \notin \mathcal{Q} \) it follows that we cannot use a colour \( c \in \{2, \ldots, k\} \) more than \( t_2 - 1 \) times on the vertices of \( U' \). But \( n_2 = (t_2 - 1)(k - 1) + 1 \), a contradiction follows.

We use the concept of a \((k; \mathcal{Q})\)-colouring of a graph to define a \((k, \mathcal{Q})\)-Ramsey class. This notion was introduced in [8], as an extension of the well-known concept of a vertex Ramsey class of graphs. We prove that some important graph classes,
such as $k$-degenerate graphs, $k$-trees or hom-properties, are $(k, \emptyset)$-Ramsey classes of graphs. We also present sufficient conditions for a graph property to be a $(k, \emptyset)$-Ramsey class. For a reference concerning Ramsey classes of graphs, see [18].

Let $k \geq 2$ be an integer and $\emptyset \in \mathbb{L}$ such that $\emptyset \subseteq \Theta^2$. A graph property $\mathcal{P}$ is called a $(k, \emptyset)$-Ramsey class, if for every $G \in \mathcal{P}$ there exists a graph $H \in \mathcal{P}$ such that in any $(k, \emptyset)$-colouring of $H$ there is a colour $\ell \in \{1, \ldots, k\}$ such that $G \subseteq H[V_\ell]$, where $V_\ell$ denotes the set of vertices of $H$ coloured with $\ell$.

It is easy to observe that if a graph property $\mathcal{P}$ is a $(k, \emptyset)$-Ramsey class, for some $k \geq 2$, then $\mathcal{P}$ is also an $(l, \emptyset)$-Ramsey class for each $l \in \{2, \ldots, k\}$. The opposite is not true; see for instance Theorem 2. Another useful observation is that if a property $\mathcal{P}$ is a $(k, \emptyset)$-Ramsey class and $\emptyset' \subseteq \Theta$, then $\mathcal{P}$ is also a $(k, \emptyset')$-Ramsey class. It is worth noticing that if $\mathcal{P}$ is a vertex Ramsey class in the usual sense (in our notation it means $\mathcal{P}$ is a $(2, \emptyset^2)$-Ramsey class), then $\mathcal{P}$ is also a $(2, \emptyset)$-Ramsey class for any hereditary graph property $\emptyset \subseteq \Theta^2$.

The following lemma gives a sufficient condition for a graph property not to be a $(k, \emptyset)$-Ramsey class.

**Lemma 1.** Let $\emptyset \subseteq \Theta^2$ be a hereditary graph property. If a graph property $\mathcal{P}$ satisfies $\mathcal{P} \subseteq \text{Forb}(F_1) \circ \cdots \circ \mathcal{Q} \text{Forb}(F_k)$, where $F_1 \in \mathcal{P}$ and $F_i \subseteq F_k$, for $i \in \{1, \ldots, k\}$, then $\mathcal{P}$ is not a $(k, \emptyset)$-Ramsey class.

**Proof.** Assume to the contrary that $\mathcal{P}$ is a $(k, \emptyset)$-Ramsey class. Therefore there exists a graph $H \in \mathcal{P}$ such that in any $(k, \emptyset)$-colouring of $H$ we obtain a monochromatic copy of $F_k$ if $\emptyset \subseteq \emptyset' \subseteq \emptyset$. But $\mathcal{P} \subseteq \text{Forb}(F_1) \circ \cdots \circ \mathcal{Q} \text{Forb}(F_k)$; hence $H$ has a $(\mathcal{Q} \text{Forb}(F_1), \ldots, \mathcal{Q} \text{Forb}(F_k))$-colouring. A contradiction follows from the fact that $F_i \subseteq F_k$, for $i \in \{1, \ldots, k\}$. □

### 3. Degenerate graphs, $k$-trees and chordal graphs

In this section, we consider $k$-degenerate graphs, $k$-trees and partial $k$-trees, and also chordal graphs of clique number at most $k$. We prove that all these properties are $(k, \emptyset)$-Ramsey classes for the property $\emptyset = \text{Forb}(K_{2,t}) \cap \Theta^2$, where $t \geq 2$.

First we present generators of $k$-degenerate graphs and partial $k$-trees.

For completeness we shall recall necessary definitions. Let $k$ be a positive integer. A $k$-tree is a graph defined inductively as follows: a complete graph of order $k$ is a $k$-tree. If $G$ is a $k$-tree, and $K$ is a $k$-clique of $G$, then a graph obtained from $G$ by adding a new vertex and joining it by new edges to all vertices of $K$ is a $k$-tree. Any subgraph of a $k$-tree is called a partial $k$-tree. The *tree-width* of a graph $G$, denoted by $tw(G)$, is zero if $G$ is edgeless; otherwise it is the smallest integer $k$ such that $G$ is a partial $k$-tree.

Let

$$T_k = \{G \in \mathcal{L} : G \text{ is a } k\text{-tree}\}$$

$$T_{W_k} = \{G \in \mathcal{L} : tw(G) \leq k\}$$

Let $G$ be a graph. We define a function $\lambda$, called *level function*, which assigns to each vertex $x \in V(G)$ a nonnegative integer. The value assigned by $\lambda$ to $x$ is called its *level*. The level of a set $S \subseteq V(G)$ is defined as $\max\{\lambda(x) : x \in S\}$. The level of a subgraph $G' \subseteq G$ equals the level of the set $V(G')$ in $G$.

In order to construct a generator of the property $\mathcal{D}_{kh}$, we define a family of $k$-degenerate graphs equipped with a level function. Assume $k \geq 1$. The graph $D(k, 0, 0)$ is the complete graph on $k$ vertices, each with level 0. Let $l, r$ be positive integers. The graph $D(k, l, r)$ and its level function are obtained from $D = D(k, l - 1, r)$ (or from $D = D(k, 0, 0)$, if $l = 1$) in the following way: for any $k$-subset $S \subseteq V(D)$, which has level $l - 1$, add $r$ independent vertices. Then join each of them to all vertices of $S$, and assign to the new vertices level $l$. It is easy to observe that for any integers $k, l$ and $r$, which satisfy $k \geq 1$ and $l, r \geq 0$, the graph $D(k, l, r)$ is $k$-degenerate. The set $\{D \in \mathcal{L} : D = D(k, l, r), l, r \geq 0\}$ is a generator of the property $\mathcal{D}_{kh}$, as stated by the following lemma.

**Lemma 2.** For any graph $G \in \mathcal{D}_{kh}$, $k \geq 1$, there exist integers $l, r \geq 0$ such that $G \subseteq D(k, l, r)$.

**Proof.** Let $k$ be a positive integer and $G$ be a maximal $k$-degenerate graph. We proceed by induction on the order $n$ of $G$. If $n$ is at most $k$, then $G \subseteq D(k, 0, 0)$. Assume that for $1 < i < n$ the lemma is true. Let us consider the maximal $k$-degenerate graph $G$ of order $n$, $n > k$. Let $x$ be the vertex of degree $k$ in $G$ (such a vertex exists, because $G$ is maximal $k$-degenerate). Let $G' = G - x$. From the inductive hypothesis, there is a graph $D = D(k, l, r)$ such that $G' \subseteq D$. It is quite easy to observe that from the construction of $D(k, l, r)$ we choose all $k$-subsets $S \subseteq V(D(k, l - 1, r))$, which have level $l - 1$. It turns out that if we additionally require that the subset $S$ induces a clique in $D(k, l - 1, r)$, then we will obtain a $k$-tree. Below we define this construction formally. We follow Ding et al. [9].

Let $k, l, r$ be positive integers. The graph $T(k, 0, 0)$ is the complete graph on $k$ vertices, each with level 0. Let $T(k, l, r)$ be the graph obtained from $T = T(k, l - 1, r)$ (or from $T = T(k, 0, 0)$, if $l = 1$) in the following way: for each $k$-clique $K$ of $T$ which has level $l - 1$, add $r$ independent vertices and join each of them to all vertices of $K$, and assign to the new vertices level $l$.

**Observation 1** ([9]). For any integers $k, l$ and $r$, which satisfy $k \geq 1$ and $l, r \geq 0$, the graph $T(k, l, r)$ is a $k$-tree. Furthermore, for any $k$-tree $G$ there are integers $l$ and $r$ such that $G \subseteq T(k, l, r)$. One can also observe that the same holds if the graph $G$ is a partial $k$-tree.
It follows that for any $k \geq 1$, the set \( \{ T \in \mathcal{I} : T = T(k, l, r), l, r \geq 0 \} \) is a generator of the property \( \mathcal{T} \mathcal{W}_k \). We use this fact later.

Now we partially answer the question, for which properties \( \mathcal{Q} \subseteq \Theta^2 \), the property \( \mathcal{D}_k \) of being a \( k \)-degenerate graph is a \( (k, \mathcal{Q}) \)-Ramsey class. We use the graphs \( D(k, l, r) \) and \( T(k, l, r) \) defined above. We start with an easy observation about 1-degenerate graphs, i.e. forests.

**Proposition 1.** If \( \mathcal{Q} = \text{Forb}(K_{1,t}) \cap \Theta^2 \), where \( t \geq 2 \), then for any \( k \geq 2 \), the property \( \mathcal{D}_1 \) is a \( (k, \mathcal{Q}) \)-Ramsey class.

**Proof.** Let \( G \in \mathcal{D}_1 \). Take any tree \( T \) such that \( G \subseteq T \). Consider the tree \( T' = T(1, l, r) \), where \( l \) equals the diameter of \( T \) and \( r = (k - 1)(t - 1) + \Delta(T) \). It is easy to see that in any \((k; \mathcal{Q})\)-colouring of \( T' \) we have a monochromatic copy of \( T \), and hence of \( G \).

Next we consider the property \( \mathcal{D}_2 \) of \( k \)-degenerate graphs for \( k \geq 2 \). We prove that if a complete bipartite graph \( K_{2,t} \), is a forbidden subgraph for the property \( \mathcal{Q} \), then \( \mathcal{D}_2 \) is a \( (k, \mathcal{Q}) \)-Ramsey class.

**Theorem 1.** If \( \mathcal{Q} = \text{Forb}(K_{2,t}) \cap \Theta^2 \), where \( t \geq 2 \), then for any \( k \geq 2 \), the property \( \mathcal{D}_2 \) is a \( (k, \mathcal{Q}) \)-Ramsey class.

**Proof.** The proof consists of two steps. At the beginning we construct a \( k \)-degenerate graph \( G_k^* \) with the property that in any \((k; \mathcal{Q})\)-colouring of \( G_k^* \) there is a monochromatic \( k \)-clique. Next we use this graph and Lemma 2 to show, for a given graph \( G \in \mathcal{D}_k \), how to construct \( H \in \mathcal{D}_k \) such that in any \((k; \mathcal{Q})\)-colouring of \( H \) we have a monochromatic subgraph isomorphic to \( G \).

The graph \( G_k^* \) is constructed as follows. Let \( s = (t - 1)(k - 1) + 1 \). Take a graph \( K_{2,s} \) and \( s - 2k + 2 \) copies of \( T = T(k, k - 2, s) \). Denote by \( T_1, \ldots, T_{k+1} \) the copies of \( T \). Let \( S_1, \ldots, S_{k+1} \) be the \( k \)-cliques of \( K_{2,s} \). The vertices of the clique of level 0 in the copy \( T_i \) are identified with the vertices of the clique \( S_i \). The levels of the vertices in the obtained graph are the same, as in \( T_1, \ldots, T_{k+1} \).

Observe that in \( G_k^* \) there are \( k + 1 \) vertices of level 0. Hence if \( f \) is a \((k; \mathcal{Q})\)-colouring of \( G_k^* \), then at least two vertices of level 0, say \( v_1, v_2 \), are coloured with the same colour, say colour 1. From the construction of \( G_k^* \) it follows that \( v_1 \) and \( v_2 \) have \((k - 1)s > s \) common neighbours of level 1. We have a \((k; \mathcal{Q})\)-colouring, hence there is a vertex \( v_3 \), adjacent both to \( v_1 \) and \( v_2 \), such that \( f(v_3) = 1 \). Therefore, \( v_1, v_2, v_3 \) belong to a monochromatic 3-clique. Consider \( s \) vertices of level 2 which are adjacent to \( v_1, v_2 \) and \( v_3 \). Similarly as above, at least one of these neighbours, say \( v_4 \), is also coloured with 1. Clearly, we have a monochromatic 4-clique. By the similar analysis of vertices of the next levels, up to level \( k - 2 \), we get that in \( G_k^* \), there is a \( k \)-clique \( S \) such that all vertices of \( S \) have colour 1.

Let \( G \in \mathcal{D}_k \). By Lemma 2, there is a \( k \)-degenerate graph \( D = D(k, l, r), l, r \geq 1 \), such that \( G \subseteq D \). Hence, we can consider the graph \( D \) instead of \( G \). We construct for this graph \( D \) a \( k \)-degenerate graph \( H \) with the property that in any \((k; \mathcal{Q})\)-colouring of \( H \) we have a monochromatic subgraph isomorphic to \( D \).

Let \( s' = r + (k - 1)(t - 1) \). We claim that if \( f' \) is a \((k; \mathcal{Q})\)-colouring of \( D' = D(k, l, s') \) such that all vertices of level 0 are coloured with the same colour, then there is a monochromatic subgraph isomorphic to \( D \). To see that, let \( U \) be a \( k \)-subset of level \( l \leq l \) in \( D' \) and assume that all vertices of \( U \) are coloured with the same colour, say colour 1. Consider the set \( U' \) containing the vertices of level \( l + 1 \) adjacent to all vertices of \( U \). From the construction of \( D' \) we have \(|U'| = s' \) and \( |D'[U \cup U']| \geq K_{2,s'} \). But \( K_{2,s'} \notin \mathcal{Q} \), hence for each colour \( c \in \{2, \ldots, k\} \) at most \( l - 1 \) vertices of \( U' \) can be coloured with \( c \). The fact \( s' = r + (k - 1)(t - 1) \) implies that at least \( r \) vertices of \( U' \) have colour 1.

Now we are ready to construct the graph \( H \). Let \( p \) be the number of \( k \)-cliques in \( G_k^* \). Take the graph \( G_k^* \) and \( p \) copies of \( D' \). Next, identify the vertices of each \( k \)-clique of \( G_k^* \) with the vertices of the \( k \)-clique of level 0 in a corresponding copy of \( D' \).

Clearly, \( H \in \mathcal{D}_k \). From the above it follows that in any \((k; \mathcal{Q})\)-colouring of \( H \) we have a monochromatic subgraph isomorphic to \( D \), and hence to \( G \).

Next we consider properties \( \mathcal{T}_k \) and \( \mathcal{T} \mathcal{W}_k \). We prove that if \( \mathcal{Q} = \text{Forb}(K_{2,t}) \cap \Theta^2 \), \( t \geq 2 \), then for any \( k \geq 2 \) both \( \mathcal{T}_k \) and \( \mathcal{T} \mathcal{W}_k \) are \((k, \mathcal{Q})\)-Ramsey classes.

**Theorem 2.** If \( \mathcal{Q} = \text{Forb}(K_{2,t}) \cap \Theta^2 \), where \( t \geq 2 \), then for any \( k \geq 2 \), \( \mathcal{T}_k \) is a \((k, \mathcal{Q})\)-Ramsey class. Moreover, \( \mathcal{T}_k \) is not a \((k + 1, \mathcal{Q})\)-Ramsey class.

**Proof.** For any \( G \in \mathcal{T}_k \) there is a graph \( T(k, l, r) \) such that \( G \subseteq T(k, l, r) \); see Observation 1. Hence it is enough to show how to obtain a required \( k \)-tree \( H \) for \( T(k, l, r) \). We use a construction analogous to the one presented in the proof of Theorem 1. The graph \( H \) is constructed as follows: we start with the graph \( G_k^* \) (defined as in the proof of Theorem 1), next we add as many copies of the graph \( T' = T(k, l, r + (k - 1)(t - 1)) \), as there are \( k \)-cliques in \( G_k^* \). The vertices of each \( k \)-clique of \( G_k^* \) are identified with the vertices of the \( k \)-clique of level 0 in a corresponding copy of \( T' \). Observe that \( H \in \mathcal{D}_k \). An analysis similar to that in the proof of Theorem 1 yields to the conclusion that in any \((k; \mathcal{Q})\)-colouring of \( H \) we have a monochromatic subgraph isomorphic to \( T(k, l, r) \), and hence to \( G \).

It remains to prove that \( \mathcal{T}_k \) is not a \((k + 1, \mathcal{Q})\)-Ramsey class. Observe first that an arbitrary \( k \)-tree \( G \) has an \((\Theta^{k+1}; \mathcal{D}_1)\)-colouring. (More precisely, every proper \((k + 1)\)-colouring of a \( k \)-tree of order \( n \geq k + 1 \) is also an \((\Theta^{k+1}; \mathcal{D}_1)\)-colouring, which was proved in [10].) Obviously, every \((\Theta^{k+1}; \mathcal{D}_1)\)-colouring of a graph \( G \) is also an \((\Theta^{k+1}; \mathcal{Q})\)-colouring, since \( \mathcal{D}_1 \subseteq \mathcal{Q} \). Therefore, from the fact that \( \Theta = \text{Forb}(K_{2,t}) \) and according to Lemma 1, \( \mathcal{T}_k \) is not a \((k + 1, \mathcal{Q})\)-Ramsey class.
In [8] it was proved that for any \( k \geq 2 \), the property \( \mathcal{T} W_k \) is a \((2, \mathcal{D}_1)\)-Ramsey class. We use Theorem 2 to generalise this result.

**Corollary 1.** If \( \mathcal{Q} = \text{Forb}(K_{2,t}) \cap \Theta^2 \), where \( t \geq 2 \), then for any \( k \geq 2 \), \( \mathcal{T} W_k \) is a \((k, \mathcal{Q})\)-Ramsey class. Moreover, \( \mathcal{T} W_k \) is not a \((k + 1, \mathcal{Q})\)-Ramsey class.

**Proof.** The proof follows from the fact that for any \( G \in \mathcal{T} W_k \) there is a \( k \)-tree \( T \) such that \( G \subseteq T \) and from Theorem 2. \( \square \)

Another corollary of Theorem 2 concerns chordal graphs. Let us recall that a graph \( G \) is called chordal, if it does not contain an induced cycle of length greater than 3. For \( k \geq 1 \), let

\[ \mathcal{C} \mathcal{H}_k = \{ G \in \mathcal{I} : G \text{ is chordal and } \omega(G) \leq k + 1 \}. \]

**Corollary 2.** If \( \mathcal{Q} = \text{Forb}(K_{2,t}) \cap \Theta^2 \), where \( t \geq 2 \), then for any \( k \geq 2 \), \( \mathcal{C} \mathcal{H}_k \) is a \((k, \mathcal{Q})\)-Ramsey class. Moreover, \( \mathcal{C} \mathcal{H}_k \) is not a \((k + 1, \mathcal{Q})\)-Ramsey class.

In the proof of Corollary 2 we use the following lemma, due to Kloks.

**Lemma 3 ([15]).** If \( G \in \mathcal{C} \mathcal{H}_k \) and \( |V(G)| \geq k + 1 \), then there is a \( k \)-tree \( T \) which is a triangulation of \( G \).

In [12], it was proved that any chordal graph \( G \) can be acyclically coloured with \( l \) colours, where \( l = \omega(G) \). In our terminology, it means that any chordal graph \( G \) admits an \((\Theta^2; \mathcal{D}_1)\)-colouring. If \( \mathcal{D}_1 \subseteq \mathcal{Q} \subseteq \Theta^2 \), then any \((\Theta^2; \mathcal{D}_1)\)-colouring of a graph \( G \) is also an \((\Theta^2; \mathcal{Q})\)-colouring of \( G \). This fact implies the following.

**Proposition 2.** If \( \mathcal{Q} \) is a hereditary graph property such that \( \mathcal{D}_1 \subseteq \mathcal{Q} \subseteq \Theta^2 \) and \( G \) is a chordal graph with \( \omega(G) = l \), then \( G \) has an \((\Theta^2; \mathcal{Q})\)-colouring.

**Proof of Corollary 2.** Let \( \mathcal{Q} = \text{Forb}(K_{2,t}) \cap \Theta^2 \), where \( t \geq 2 \). Let \( k \geq 2 \) be an integer. From Lemma 3 it clearly follows that for each \( G \in \mathcal{C} \mathcal{H}_k \) there is a \( k \)-tree \( T \) such that \( G \) is a spanning subgraph of \( T \). But every \( k \)-tree is chordal and its clique number is at most \( k + 1 \). Hence, \( T \in \mathcal{C} \mathcal{H}_k \). By Theorem 2, the property \( \mathcal{C} \mathcal{H}_k \) is a \((k, \mathcal{Q})\)-Ramsey class. Proposition 2 and Lemma 1 yield that \( \mathcal{C} \mathcal{H}_k \) is not a \((k + 1, \mathcal{Q})\)-Ramsey class, because \( \mathcal{D}_1 \subseteq \mathcal{Q} \). \( \square \)

### 4. Hom-properties and perfect graphs

In this section we give, in Lemma 4, sufficient conditions for a graph property to be a \((k, \mathcal{Q})\)-Ramsey class. We use this lemma to prove that if a hereditary graph property \( \mathcal{Q} \) is a proper subset of \( \Theta^3 \), then both hom-properties generated by a connected graph and perfect graphs with bounded clique number are \((k, \mathcal{Q})\)-Ramsey classes, for any integer \( k \geq 2 \).

We start with an easy observation about the property \( \Theta^k, k \geq 1 \). In [8], the authors proved that if \( \mathcal{Q} = \text{Forb}(F) \cap \Theta^2 \), where \( F \) is a connected bipartite graph, then \( \Theta^k \) is a \((2, \mathcal{Q})\)-Ramsey class for any \( k \geq 2 \). Using the same method it can be shown that \( \Theta^k \) is a \((t, \mathcal{Q})\)-Ramsey class for any \( k \), \( t \geq 2 \).

Now we present a generalisation of the above result. We start by defining the property \( \Theta^k \) in terms of a *homomorphism* of graphs. Let \( G, H \in \mathcal{I} \). Recall that a graph \( G \) is *homomorphic* to \( H \), denoted \( G \rightarrow H \), if there is a mapping \( h : V(G) \rightarrow V(H) \), called *homomorphism*, which preserves the edges, i.e. if \( e = uv \in E(G) \), then \( h(e) = h(u)h(v) \in E(H) \).

For a graph \( H \in \mathcal{I} \), we define

\[ \rightarrow H = \{ G \in \mathcal{I} : G \rightarrow H \}. \]

The property \( \rightarrow H \) is called the *hom-property generated by \( H \)*. Observe that the property \( \rightarrow H \) is hereditary. For a more comprehensive view on hom-properties, we refer the reader to [16].

It is obvious that \( \Theta^k \rightarrow \mathcal{K}_k \). We generalise the result concerning the property \( \Theta^k \), presented in [8], to all hom-properties generated by a connected graph.

**Theorem 3.** Let \( H \in \mathcal{I} \) be a connected graph. If \( \mathcal{Q} = \text{Forb}(F) \cap \Theta^2 \), where \( F \) is a bipartite graph with at least one edge, then for any \( k \geq 2 \), the property \( \rightarrow H \) is a \((k, \mathcal{Q})\)-Ramsey class.

In the proof of Theorem 3 we use a lemma which gives sufficient conditions for a graph property to be a \((k, \mathcal{Q})\)-Ramsey class. We use the notion of a *disjunction* of graphs, introduced by Harary et al. in [14]. The *disjunction* \( G_1 \vee G_2 \) of graphs \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) is a graph \( G = (V, E) \) such that \( V = V_1 \times V_2 \) and \( u \in (u_1, u_2) \) is adjacent to \( v = (v_1, v_2) \) whenever \( u_1v_1 \in E_1 \) or \( u_2v_2 \in E_2 \).

**Lemma 4.** Let \( \mathcal{Q} = \text{Forb}(F) \cap \Theta^2 \), where \( F \) is a bipartite graph with at least one edge. If the graph property \( \mathcal{P} \) satisfies the following two conditions:

1. If \( G \in \mathcal{P} \), then there is a connected graph \( G' \in \mathcal{P} \) such that \( G \subseteq G' \),
2. If \( G \in \mathcal{P} \), then for any \( l \geq 1 \), the graph \( G \vee \overline{K}_l \in \mathcal{P} \),

then \( \mathcal{P} \) is a \((k, \mathcal{Q})\)-Ramsey class for any \( k \geq 2 \).
Proof. Let $G \in \mathcal{P}$. The property $\mathcal{P}$ satisfies the first condition; hence there exists a connected graph $G' \in \mathcal{P}$ such that $G \subseteq G'$. Therefore, it is enough to prove that for any connected graph $G'$ there exists a graph $H \in \mathcal{P}$ such that in any $(k; \mathcal{Q})$-colouring of $H$ there is a monochromatic copy of $G'$.

Let $(A, B)$ be any bipartition of the set of vertices of $F$ into independent sets and take $t = \max(|A|, |B|, 2)$. Property $\mathcal{Q}$ is a hereditary graph property; hence $K_{t,t} \not\in \mathcal{Q}$. Let $G = G' \cup \overline{K}_t$, where $s = (t - 1)k + 1$. The second condition yields $H \in \mathcal{P}$. We prove that in any $(k; \mathcal{Q})$-colouring of $H$, there is a monochromatic copy of $G'$. Assume $V(G') = \{u_1, \ldots, u_n\}$ and $V(\overline{K}_t) = \{u_1, \ldots, u_t\}$. For a vertex $v_i \in V(G')$, $W_i$ denotes the subset $(\{v_i, u_1\}, \ldots, \{v_i, u_t\})$ of $V(H)$. Observe that if $v_i v_j \in E(G')$, then $H[W_i \cup W_j] \cong K_{s,s}$.

Consider an arbitrary vertex $v_i$ of $G'$. Since $|W_i| = s$, in the set $W_i$ at least $t$ vertices are coloured with the same colour, say colour 1. Let $v_j$ be a neighbour of $v_i$ in $G'$. From the fact that $H[W_i \cup W_j] \cong K_{s,s}$ and $K_{t,t} \not\in \mathcal{Q}$, it clearly follows that in $W_i$ there are at most $t - 1$ vertices in each colour $c \in \{2, \ldots, k\}$. Hence there are at least $t$ vertices coloured with colour 1. The graph $G'$ is connected, and so is $H$, thus we can apply this reasoning to other neighbours of the vertices $v_i$ and $v_j$ and, repeatedly, to the whole graph $G'$. From the above we have that for each $v_i \in V(G')$, the set $W_i$ contains at least $t$ vertices with colour 1. But $H = G' \cup \overline{K}_t$, therefore, we obtain a monochromatic subgraph isomorphic to $G'$. □

Proof of Theorem 3. If $H \cong K_1$, then $\rightarrow H = \emptyset$. Since $\emptyset$ is a $(k, \mathcal{Q})$-Ramsey class for $k \geq 2$, we can assume $K_2 ^* \subseteq H$. It is obvious that if $G \rightarrow H$, then $(G \cup \overline{K}_t) \rightarrow H$ for any $l \geq 1$. Thus the property $\rightarrow H$ satisfies the second condition of Lemma 4.

Observe that if $G \rightarrow H$, then we can always find an integer $s \geq 1$ such that $G \subseteq H \cup \overline{K}_s$ (for instance, we can take $s = |V(G)|$). From the fact that $H$ is connected it follows $H \cup \overline{K}_s$ is connected too, and hence the property $\rightarrow H$ satisfies also the first condition of Lemma 4. Therefore, $\rightarrow H$ is a $(k, \mathcal{Q})$-Ramsey class for any $k \geq 2$. □

We also use Lemma 4 to prove that if $\mathcal{Q} = \text{Forb}(F) \cap \Theta^2$, where $F$ is a bipartite graph with at least one edge, then the property

$$\text{Perf}_{k} = \{G \in I : G \text{ is perfect and } \omega(G) \leq k + 1\}$$

is a $(t, \mathcal{Q})$-Ramsey class for any $t, k \geq 2$.

In the proof of the above-mentioned result, the following observation will be useful.

Lemma 5. For any integers $l \geq 1$ and $k \geq 2$, if $G \in \text{Perf}_k$, then $G \cup \overline{K}_l \in \text{Perf}_k$.

Lemma 5 is a corollary of the Substitution lemma due to Lovász, which we present below for the sake of completeness.

Let $G_1$ and $G_2$ be disjoint graphs, $x \in V(G_1)$. The substitution of $G_2$ for $x$ in $G_1$ creates a new graph by removing $x$ and its incident edges from $G_1$, and adding an edge between each vertex of $G_2$ and each vertex in $N_{G_1}(x)$.

Theorem 4 ([17]). The graph obtained from a perfect graph $G_1$ by the substitution of a perfect graph $G_2$ for a vertex of $G_1$ is a perfect graph.

It is easy to see that $G \cup \overline{K}_l$ can be obtained from $G$ by substitution of $\overline{K}_l$ for each vertex of $G$. Hence, if $G$ is perfect, then so is $G \cup \overline{K}_l$. Moreover, $\omega(G) = \omega(G \cup \overline{K}_l)$.

Theorem 5. If $\mathcal{Q} = \text{Forb}(F) \cap \Theta^2$, where $F$ is a bipartite graph with at least one edge, then the property $\text{Perf}_k$ is a $(t, \mathcal{Q})$-Ramsey class for any $t, k \geq 2$.

Proof. Let $k \geq 2$. The fact that $\text{Perf}_k$ satisfies the second condition of Lemma 4 follows from Lemma 5. It remains to prove that also the first condition is fulfilled. Assume the contrary. Let $G \in \text{Perf}_k$ be a graph which is not connected and such that in $\text{Perf}_k$ there is no connected graph containing $G$. Additionally, assume that $G$ is a graph with this property and with the smallest possible number of components. Consider vertices $x$ and $y$ which are in different components of $G$. Let $G'$ be the graph obtained from $G$ by adding the edge $xy$. We claim that $G'$ belongs to $\text{Perf}_k$. Indeed, if $U \subseteq V(G)$ and $\omega(G[U]) \geq 2$, then $\omega(G'[U]) = \omega(G[U])$ and $\chi(G'[U]) = \chi(G[U])$. If $\omega(G[U]) = 1$ and either $x$ or $y$ does not belong to $U$, then $\omega(G'[U]) = 1 = \chi(G'[U])$. Otherwise, if $x, y \in U$, then $\omega(G'[U]) = 2 = \chi(G'[U])$. Therefore, $G' \in \text{Perf}_k$, contrary to the choice of $G$. □

Acknowledgements

The authors would like to thank anonymous referees for many helpful comments and constructive suggestions. The second author’s research was partially supported by grant no. N N201 370136 of the Ministry of Science and Higher Education of Poland.

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