Journal of Pure and Applied Algebra 74 (1991) 73–84
North-Holland

A note on the variety of projectors

C. Musili*

Mathematics Research Centre, University of Warwick, Coventry CV4 7AL, United Kingdom, and School of Mathematics & CIS, University of Hyderabad, Central University, Hyderabad, India 500 134

Communicated by C.A. Weibel
Received 16 November 1989

Abstract

Musili, C., A note on the variety of projectors, Journal of Pure and Applied Algebra 74 (1991) 73–84.

Somewhat analogous to the case of the variety of Complexes, the variety P of projectors, i.e., idempotents (of rank d on an n-space), is shown to be a principal affine open subset of the product of the Grassmannian $\mathrm{Gr}(d, n)$ and its dual $\mathrm{Gr}(n-d, n)$. Also P is identified with the affine coset space $\mathrm{GL}(n)/H$ for a closed reductive subgroup H of the form $\mathrm{GL}(d) \times \mathrm{GL}(n-d)$; consequently, P is nonsingular and of dimension $2d(n-d)$. The coordinate ring R of P is described explicitly by generators and relations as the subring of left translation H-invariants of $k[\mathrm{GL}(n)]$ as an immediate consequence of the classical Hodge Standard Monomial Basis readily available for R just as for the homogeneous coordinate ring of $\mathrm{Gr}(d, n)$ for its Plücker embedding. The $\mathrm{GL}(n)$-module structure of R is shown to be the direct limit of the filtered family of representations of $\mathrm{GL}(n)$:

$$m\omega_d \otimes m\omega_{n-d} \otimes (-m)\,\mathrm{det.}, \quad m \in \mathbb{Z}^+,$$

where ω_d and ω_{n-d} are the fundamental weights of $\mathrm{GL}(n)$ corresponding to $\mathrm{Gr}(d, n)$ and $\mathrm{Gr}(n-d, n)$, respectively, and det. is the determinant character of $\mathrm{GL}(n)$.

Introduction

In [7], Strickland has studied (among other things) the variety of projectors P, namely, the space of all $n \times n$ matrices A in $\mathbf{gl}(n, k)$ (over a field k, algebraically closed and of arbitrary characteristic), such that $A^2 = A$ and rank $A = d$ (fixed). It is obvious that P is closed in $\mathbf{gl}(n, k)$ (see 2.1, below, for the defining equations). Modulo only these equations, she has constructed an explicit linear basis for the coordinate ring R of P using the techniques of Hodge Algebras (cf. [1]), and

* British Council Visiting Fellow for Mathematics at Warwick.

concludes that these equations do generate the prime ideal of P. The linear basis for R is given by certain double standard products of determinants, etc.

In this note, we identify P with two other familiar spaces, conclude a little more about P and describe R more concretely. This is done simply by putting some well-known facts together so as to make the inter connexions transparent.

First, we look at the adjoint action of the group $G = \mathrm{GL}(n, k)$ on its Lie Algebra $\mathbf{gl}(n, k)$ and notice that P is the orbit through the point

$$A_0 = \mathrm{diag}(1^d, 0^h), \quad d + h = n .$$

A simple calculation shows that the centraliser of A_0 is the closed reductive subgroup (in the block form), namely,

$$H = \mathrm{diag}(\mathrm{GL}(d, k), \mathrm{GL}(h, k)) .$$

Hence (cf. [6]), the geometric quotient G/H exists, is affine, nonsingular and of dimension $2d(n-d)$. It is well known that the orbit map $G/H \to P$ is an isomorphism of varieties since A_0 is semi-simple. This gives us the fact that R is the subring of left translation H-invariants in the ring of regular functions $k[G]$ on G (see Corollary 3, below).

Second, we notice that H is the intersection of two maximal parabolic subgroups Q and Q' of G, where

$$Q = \{g \in G \mid \text{left-hand bottom } h \times d \text{ block of } g \text{ is } 0\}$$

and

$$Q' = \{g \in G \mid \text{right-hand top } d \times h \text{ block of } g \text{ is } 0\} .$$

This allows us to identify G/H with a G-orbit for the diagonal action of G on the product of the Grassmannians $G/Q \times G/Q'$ and then conclude easily that the inclusion morphism $G/H \hookrightarrow G/Q \times G/Q'$ is an open immersion onto its image which is a certain principal open subset 'det $\neq 0$' (see Proposition 4, below). This result is somewhat analogous to the one that the variety of Complexes is a principal affine open subset of a union of Schubert varieties in the Flag variety (cf. [5]).

Next, we take the Plücker embedding of the product of the Grassmannians $G/Q \times G/Q'$, followed by the Segre embedding, and determine the open subset 'det $\neq 0$'. Thereby, we get R as the homogeneous localisation of a graded ring S at the homogeneous element 'det' (which is of degree 1 in S). In the process, we observe that the graded piece S_m of S is nothing but the representation space of the tensor product $\rho_m = m\omega_d \otimes m\omega_h$, where ω_d and ω_h are the dth and hth fundamental weights of G. Now R is filtered by $\{E_m\}$, where

$$E_m = S_m/\det^m \quad \text{and} \quad E_m \subseteq E_{m+1}, \qquad \text{for all } m \in \mathbb{Z}^+ .$$

But each E_m is again the same as S_m as a vector space. But, as G-modules, we have

$$S_m = m\omega_d \otimes m\omega_h, \quad \text{whereas } E_m = m\omega_d \otimes m\omega_h \otimes (\det)^{-m} .$$

Thus we get (see Corollary 7 and Theorem 8, below):

$$S = \bigoplus_m (m\omega_d \otimes m\omega_h) \quad \text{and} \quad R = \varinjlim (m\omega_d \otimes m\omega_h \otimes (\det)^{-m})$$

Consequently, we have the well-known classical standard monomial bases of Hodge, ready made for S and R, giving also generators and relations explicitly.

1. The variety of projectors P

1.1. Let k be an algebraically closed field of arbitrary characteristic. Let V be a vector space of finite dimension n (over k). Fix an integer d, $1 \le d \le n-1$. Let

$$P = \{ f \in \mathrm{End}_k V \mid f^2 = f \text{ and rank } f = d \} ,$$

i.e., the space of all projectors (or idempotents) on V of rank d. We fix a basis $\{e_1, e_2, \dots, e_n\}$ for V. With respect to this basis, we write the matrix A of an endomorphism f as

$$A = (a_{ij}), \quad \text{where } f(e_j) = \sum a_{ij} e_i .$$

We shall use f or A interchangeably. Now P is simply the set of all $n \times n$ matrices A over k such that $A^2 = A$ and rank $A = d$. We fix the following notation:

1.2. For an integer r, $1 \le r \le n-1$, we denote by $[r]$ the following set with the natural partial order (Bruhat order), namely:

$$[r] = \{ I = (i_1, i_2, \dots, i_r) \in \mathbb{N}^r \mid 1 \le i_1 < \cdots < i_r \le n \}$$

with the coordinate-wise comparison as the partial order. We then have the basis $\{e_I \mid I \in [r]\}$ for $\Lambda^r(V)$, induced by the basis fixed for V, where

$$e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_r}, \quad \text{for all } I = (i_1, \dots, i_r) \in [r] .$$

1.3. Let $\mathrm{Gr}(r, n)$ be the Grassmannian of r-dimensional subspaces of V and its Plücker embedding

$$p: \mathrm{Gr}(r, n) \hookrightarrow \mathbb{P}(\Lambda^r(V)).$$

For a point $W \in \mathrm{Gr}(r, n)$, we write the Plücker coordinates of $p(W)$ as

$$p(W) = (\ldots, p_I, \ldots).$$

The $\{p_I\}$, considered as coordinate functions on $\Lambda^r(V)$, is simply the dual basis of $\Lambda^r(V)^*$ dual to the basis $\{e_I\}$. In fact, we shall interpret the functions p_I more concretely as follows: take a basis $v_1, \ldots, v_r$ for $W \in \mathrm{Gr}(r, n)$ and write each v_i as a column vector in the basis of V. Let $L = (v_1, \ldots, v_r)$ be the $n \times r$ matrix of these columns. For $I \in [r]$, the coordinate p_I is then the maximal minor of L corresponding to the row indices I. Now we recall the following well-known theorem:

Theorem 1 (cf. [3, 4]). (Folklore) (1) *The homogeneous coordinate ring* $k[\mathrm{Gr}(r, n)]$ *for the Plücker embedding is given by the graded k-algebra generated by the coordinate functions* p_I, $I \in [r]$, *subject to the quadratic relations satisfied by the minors of a generic matrix* L.

(2) *The graded component* $k[\mathrm{Gr}(r, n)]_m$ *of degree m has a standard monomial basis*:

$$\{p_{I_1} \cdots p_{I_m} \mid I_1 \le I_2 \le \cdots \le I_m \text{ in } [r]\}$$

(*i.e*, *standard monomials of degree m in the Plücker coordinates*). □

Remark. Let Φ denote the ample generator of $\mathrm{Pic}(\mathrm{Gr}(r, n))$. Then, we have

$$k[\mathrm{Gr}(r, n)]_m = H^0(\mathrm{Gr}(r, n), \Phi^{\otimes m})$$

and

$$k[\mathrm{Gr}(r, n)] = \bigoplus_m H^0(\mathrm{Gr}(r, n), \Phi^{\otimes m}).$$

1.4. Having fixed d, $1 \le d \le n-1$, we write $h = n - d$ throughout in what follows. We will be working with $\mathrm{Gr}(d, n)$ and $\mathrm{Gr}(h, n)$ (each being dual to the other). To avoid notational confusion, we shall have the following convention: (i) the Plücker coordinates p_I with $I \in [d]$ for $\mathrm{Gr}(d, n)$ and q_J with $J \in [h]$ for $\mathrm{Gr}(h, n)$ and (ii) the line bundles Φ on $\mathrm{Gr}(d, n)$ and Ψ on $\mathrm{Gr}(h, n)$.

2. *P* as a quotient variety

2.1. Given $f \in P$, let $V_0 = \text{kernel } f$ and $V_1 = \text{image } f$ so that $\dim_k V_1 = d$ and $V = V_1 \oplus V_0$. With respect to the basis of V, obtained by choosing arbitrary bases for V_1 and V_0, the matrix of f is simply $A_0 = \mathrm{diag}(1^d, 0^h)$. In other words, all the elements of P are similar to A_0, or equivalently, P is the $\mathrm{GL}(n, k)$-conjugacy class of matrices in $\mathbf{gl}(n, k)$ containing A_0. Since A_0 is semi-simple (i.e., diagonal), it is well known that the conjugacy class through A_0 is a closed subset of $\mathbf{gl}(n, k)$. However, it is quite straightforward to write down the equations defining P (cf. [7]). In fact, $A = (a_{ij}) \in P$ if and only if the following three conditions are satisfied:

(i) $A^2 = A$: $a_{ij} = \sum a_{ik}a_{kj}$ for all i, j,

(ii) rank $A \leq d$: all the $d+1$ by $d+1$ minors of A or 0,

(iii) the characteristic polynomial, $p(x) = x^n - s_1x^{n-1} + \cdots + (-1)^n s_n$ of A, is $(x-1)^d x^h$, i.e., $s_i = \binom{d}{i}$ for $i \leq d$ (by (ii), it follows that $s_i = 0$ for $i \geq d+1$).

We equip P with the canonical reduced structure, i.e., P is an affine variety.

2.2. Since P is an orbit for $G = \mathrm{GL}(n, k)$ acting on $\mathbf{gl}(n, k)$ by inner conjugation, P is naturally identified with the space of cosets $G/H = \{gH \mid g \in G\}$, where H is the centraliser of A_0. It is trivial to see that

$$H = \mathrm{diag}(\mathrm{GL}(d, k), \mathrm{GL}(h, k)) .$$

Thus H is a closed reductive subgroup of G. Now we recall the following theorem:

Theorem 2 (cf. [6]). (1) *The geometric quotient G/H exists as a variety.*

(2) *G/H is affine* (*since H is reductive*).

(3) *G/H is nonsingular* (*being a homogeneous variety*). □

It is easy to see that the differential of the orbit map $G/H \to P$ is surjective for dimension reasons and hence we have the following:

Corollary 3. (1) *The orbit map $G/H \to P$ is an isomorphism of varieties, consequently,*

(2) *the coordinate ring R of P is the subring of H-invariants in $k[G]$ for the left translations, i.e., for $f \in k[G]$, $g \in H$, $x \in G$, $(g \cdot f)(x) = f(xg)$,*

(3) *R is a regular geometric k-algebra and*

(4) $\dim P = \dim R = 2d(n-d) = \dim G - \dim H$. □

Our aim now is to describe the ring R more closely. We proceed as follows.

3. The coordinate ring R of P

3.1. From now on we shall work with G/H. Look at the maximal parabolic subgroups Q and Q' of G, where

$$Q = \{ g \in G \mid \text{left-hand bottom } h \times d \text{ block of } g \text{ is } 0 \} ,$$

and

$$Q' = \{ g \in G \mid \text{right-hand top } d \times h \text{ block of } g \text{ is } 0 \} .$$

It is obvious that $H = Q \cap Q'$. Recall that we have a natural identification of the projective varieties:

$$G/Q \simeq \mathrm{Gr}(d, n) \quad \text{and} \quad G/Q' \simeq \mathrm{Gr}(h, n) ,$$

where the maps are simply sending an element $g = (L \mid M) \in G$ (L being the $n \times d$ matrix of the first d columns of g and M the last h columns), to the Plücker coordinates given by L and M, respectively. For the linear action of G on V, by

$$g(e_j) = \sum g_{ij} e_i , \quad g = (g_{ij}) \in G ,$$

we see that Q is the stabiliser of $e_- = e_{(1,\ldots,d)}$ in $\mathrm{Gr}(d, n)$ and Q' is the stabiliser of $e_+ = e_{(n-h+1,\ldots,n)}$ in $\mathrm{Gr}(h, n)$. Hence H is the stabiliser of $(e_-, e_+) \in \mathrm{Gr}(d, n) \times \mathrm{Gr}(h, n)$. This gives the diagonal morphism

$$G \to G/Q \times G/Q' = \mathrm{Gr}(d, n) \times \mathrm{Gr}(h, n)$$

which factors through the inclusion

$$\theta : G/H \hookrightarrow \mathrm{Gr}(d, n) \times \mathrm{Gr}(h, n) .$$

We have the following easy proposition:

Proposition 4. *The inclusion θ is an open immersion.*

Proof. It is clear that θ is a monomorphism of varieties, i.e., an isomorphism onto its image. It is therefore enough to show that Image θ is an open subset. Let $U =$ Image θ. Let $z \in U$. By definition, there is a $g = (L \mid M) \in G$ such that

$$z = (\ldots, p_I(L), \ldots; \ldots, q_J(M), \ldots) .$$

Since g is a nonsingular matrix, we have $\det(g) \neq 0$. But we have, by the Laplace expansion of the determinant, that

$$\det(g) = \sum_{I \cap J = \emptyset} \mathrm{sgn}\, \sigma_{I,J} p_I(L) q_J(M) ,$$

where the summation runs over all $I \in [d]$ and $J \in [h]$ with $I \cap J = \emptyset$ and $\sigma = \sigma_{I,J}$ is the permutation, given by $\sigma(r) = i_r$ for all $r \leq d$ and $\sigma(d+s) = j_s$ for all $s \leq h$, if

$$I = (i_1, \ldots, i_d) \quad \text{and} \quad J = (j_1, \ldots, j_h) .$$

Thus U is contained in the principal affine open subset

$$\det = \sum_{I \cap J = \emptyset} \pm p_I q_J \neq 0 ,$$

in the product variety $\mathrm{Gr}(d, n) \times \mathrm{Gr}(h, n)$, read via the Segre embedding. Conversely, suppose $z \in \det \neq 0$, say,

$$z = (\ldots, p_I, \ldots; \ldots, q_J, \ldots) .$$

This means that there are $g_1, g_2 \in G$ such that $g_1 = (L \mid *)$ and $g_2 = (* \mid M)$ with

$$z = (\ldots, p_I(L), \ldots; \ldots, q_J(M), \ldots) .$$

But then there are nonzero scalars λ and μ such that

$$p_I(L) = \lambda p_I \quad \text{and} \quad q_J(M) = \mu q_J \qquad \text{for all } I \text{ and } J .$$

Hence

$$\begin{aligned} \det(L \mid M) &= \sum_{I \cap J = \emptyset} \pm (p_I(L) q_J(M)) \\ &= \sum_{I \cap J = \emptyset} \pm (\lambda p_I \cdot \mu q_J) \\ &= \lambda\mu \Big(\sum_{I \cap J = \emptyset} \pm p_I \cdot q_J \Big) \\ &\neq 0 . \end{aligned}$$

Thus $g = (L \mid M) \in G$ and $\theta(g) = z$, as required. This completes the proof. □

3.2. We shall now determine the coordinate ring of '$\det \neq 0$'. Let S be the homogeneous coordinate ring of $\mathrm{Gr}(d, n) \times \mathrm{Gr}(h, n)$ for the Segre embedding. We have

$$\begin{aligned} S &= \bigoplus_m H^0(\mathrm{Gr}(d, n) \times \mathrm{Gr}(h, n), (\Phi \times \Psi)^{\otimes m}) \\ &= \bigoplus_m H^0(\mathrm{Gr}(d, n), \Phi^{\otimes m}) \otimes H^0(\mathrm{Gr}(h, n), \Psi^{\otimes m}) \\ &= \bigoplus_m (k[\mathrm{Gr}(d, n)]_m \otimes k[\mathrm{Gr}(h, n)]_m) . \end{aligned}$$

We identify this with a subring of the polynomial algebra $k[X]$, where

$$X = (X_{ij}), \quad 1 \leq i, j \leq n$$

is a matrix of indeterminates over k, as described in the following:

Proposition 5. *The ring S is isomorphic to the subalgebra of $k[X]$ generated by the bi-homogeneous products*

$$\{P_I(X)Q_J(X) \mid I \in [d] \text{ and } J \in [h]\},$$

where $P_I(X)$ is the $d \times d$ minor of the first d columns of X corresponding to the row indices I and similar meaning for $Q_J(X)$ with respect to the last h columns of X.

Proof. It is well known that S is generated by $P_I \otimes Q_J$ modulo the Segre relations

$$(P_I \otimes Q_J)(P_{I'} \otimes Q_{J'}) - (P_I \otimes Q_{J'})(P_{I'} \otimes Q_J)$$

for all $I, I' \in [d]$ and $J, J' \in [h]$. But these are obviously satisfied by the products $P_I Q_J$ in $k[X]$. We need to show that the only other relations satisfied by $P_I Q_J$ are the quadratic relations among the P_I's and Q_J's independently. This is shown in the next section (see Corollary 7, below). □

4. Standard monomial basis for S

4.1. For $I \in [d]$ and $J \in [h]$, for simplicity we write

$$P_I(X)Q_J(X) = (I \mid J).$$

We partially order the pairs (I, J) by defining

$$(I, J) \leq (I', J') \quad \text{if} \quad I \leq I' \text{ and } J \geq J'.$$

We *note the reversal* in the order of second factor.
A monomial of degree m in $(I \mid J)$'s, say

$$(I_1 \mid J_1) \cdots (I_m \mid J_m),$$

is called a *standard monomial* if the pairs (I_i, J_i) are totally ordered, i.e.,

$$I_1 \leq I_2 \leq \cdots \leq I_m \quad \text{and} \quad J_m \leq J_{m-1} \leq \cdots \leq J_1,$$

i.e., a standard monomial in $(I \mid J)$'s is simply a product of a standard monomial in the P_I's and another in the Q_J's. Now we have the following theorem:

Theorem 6. *The graded ring S, generated by the $(I \mid J)$'s, has a linear basis consisting of all distinct standard monomials in the $(I \mid J)$'s.*

Proof. That the standard monomials span S is immediate from the fact that a monomial of the form $P_{I_1} \cdots P_{I_m}$ can be written as a sum of standard monomials $P_{I_{1,i}} \cdots P_{I_{m,i}}$ (i.e., $I_{1,i} \leq \cdots \leq I_{m,i}$) and similarly a monomial $Q_{J_1} \cdots Q_{J_m}$ can be written as a sum of standard monomials $Q_{J_{1,j}} \cdots Q_{J_{m,j}}$ (i.e., $J_{1,j} \geq \cdots \geq J_{m,j}$) and hence any monomial in $(I \mid J)$ takes the form

$$(I_1 \mid J_1) \cdots (I_m \mid J_m) = \sum_{i,j} (\text{coefft.})(I_{1,i} \mid J_{1,j}) \cdots (I_{m,i} \mid J_{m,j}),$$

which is a sum of standard monomials, as required.

4.2. The linear independence of the standard monomials in S can be established in two ways: (i) By a trick of Hodge or (ii) by a geometric approach by means of 'Schubert varieties' in the product $\mathrm{Gr}(d, n) \times \mathrm{Gr}(h, n)$.

(i) Hodge's method (cf. [3, Last Chapter]), consists in attaching certain numerical weights to the variables X_{ij} of the matrix X in such a way that the principal diagonal term of the minor $P_I(X)$, namely,

$$d(P_I) = d(P_I(X)) = X_{i_1 1} \cdot X_{i_2 2} \cdots X_{1_d d}, \quad \text{if } I = (i_1, i_2, \ldots, i_d);$$

is singled out as a monomial of largest numerical weight among all the monomial terms of that minor. Consequently, it follows that any linear dependency of the monomials $P_{I_1} \cdots P_{I_m}$ (standard or not), goes down to a linear dependency of the genuine monomials $d(P_{I_1}) \cdots d(P_{I_m})$.

To finish the proof, the *point to be noted* is that the set of monomials (in X_{ij}) of the form $d(P_{I_1}) \cdots d(P_{I_m})$ are distinct (only) for the set of distinct standard monomials $\{P_{I_1} \cdots P_{I_m} \mid I_1 \leq \cdots \leq I_m\}$. This method works in toto for the monomials $\{(I_1 \mid J_1) \cdots (I_m \mid J_m)\}$ as well, since the minors P_I and Q_J are based on the independent sets of variables

$$\{X_{ij} \mid 1 \leq j \leq d\} \quad \text{and} \quad \{X_{ij} \mid d+1 \leq j \leq n\},$$

respectively. This completes the proof. □

4.3. The geometric method is to set up an inductive procedure by means of a family of subvarieties of $\mathrm{Gr}(d, n) \times \mathrm{Gr}(h, n)$. The family is the natural one, namely:

$$\{Y_I \times Z_J \mid I \in [d] \text{ and } J \in [h]\}$$

where Y_I is the Schubert subvariety of $\mathrm{Gr}(d, n)$ given by I in the Bruhat decomposition for which $\mathrm{Gr}(d, n) = Y_{(1,2,\ldots,d)}$; whereas the Z_J is the one in $\mathrm{Gr}(h, n)$ with $\mathrm{Gr}(h, n) = Z_{(n-h+1,\ldots,n)}$.

The most *important observation* to make here is that a coordinate function $P_I Q_J$ on $\mathrm{Gr}(d, n) \times \mathrm{Gr}(h, n)$ is nonvanishing on $Y_{I'} \times Z_{J'}$ if and only if $I' \leq I$ and $J' \geq J$. Now induction on degrees of the standard monomials and restrictions to this family of subvarieties do the job verbatim as for the case of the Grassmannians (cf. [4]). The next result is the following:

Corollary 7. (1) *The mth graded component* S_m *of* S *is isomorphic to* $k[\mathrm{Gr}(d, n)]_m \otimes k[\mathrm{Gr}(h, n)]_m$ *and hence the only relations satisfied by the products* $P_I Q_J$ *are the ones satisfied by the* P_I*'s and* Q_J*'s independently.*

(2) *The functions* $P_I Q_J$ *are semi-invariants for* H *with determinant as the character (for the linear action of* H *on the matrix* X *by multiplication on the right).*

5. Standard monomial basis for R

We have seen that

$$
\begin{aligned}
R &= \text{coordinate ring of } \det \neq 0 \text{ in } \mathrm{Gr}(d, n) \times \mathrm{Gr}(h, n) \\
&= S_{(\det)}, \text{ the homogeneous localisation of } S \text{ at } \det \\
&= \text{the } k\text{-subalgebra of the field } k(X) \text{ generated by the set} \\
&\qquad \{P_I Q_J / \det \mid I \in [d] \text{ and } J \in [h]\}\,.
\end{aligned}
$$

Thus the rational functions $P_I Q_J/\det$, which are regular functions on the group G, give a natural coordinate system on P, viewed as a quotient variety of G. Since S is a graded integral domain and $R = S_{(\det)}$ with det an element of degree 1 in S, it is obvious that the family

$$E_m = S_m/(\det)^m\,, \quad m \in \mathbb{Z}^+$$

gives a filtration on R, where we identify E_m canonically with the subset of E_{m+1} as

$$E_m = S_m \cdot \det/(\det)^{m+1} \subset S_{m+1}/(\det)^{m+1} = E_{m+1}\,.$$

Further, E_m is isomorphic to S_m as a vector space. Thus we have the following summary:

Theorem 8. (1) *The coordinate ring* R *of the variety of projectors* P*, as the subring of left translation* H*-invariants in the ring* $k[G]$*, is generated by the obvious set of invariants*

$$\{P_I Q_J/\det = (I \mid J)/\det \mid I \in [d] \text{ and } J \in [h]\} .$$

In particular, H has no polynomial invariants in $K[G]$.

(2) *The generators* $P_I Q_J/\det$ *are subject only to the well-known quadratic relations satisfied by the Plücker coordinates* P_I*'s and* Q_J*'s independently.*

(3) *R has a canonical filtration of G-modules* E_m *generated by* E_1, *where* E_m *is the representation space of the tensor product* $\rho_m = m\omega_d \otimes m\omega_d^* \otimes (\det)^{-m}$ *with* ω_d *being the dth fundamental weight and* $\omega_d^* = \omega_h$, *the hth fundamental weight of G.*

(4) *Each of the filters* E_m *of R has a linear basis consisting of all distinct standard monomials*

$$\{P_{I_1} Q_{J_1} \cdots P_{I_m} Q_{J_m}/\det^m \mid I_1 \leq \cdots \leq I_m;\, J_m \leq \cdots \leq J_1\}$$

with

$$I_i \in [d] \text{ and } J_i \in [h] , \quad 1 \leq i \leq m , \ m \in \mathbb{Z}^+ . \qquad \square$$

6. Some remarks

6.1. For the representation point of view, it seems more natural to identify P with the quotient space $\mathrm{SL}(n)/H'$ rather than $\mathrm{GL}(n)/H$, where $H' = H \cap \mathrm{SL}(n)$.

The coordinate ring R of P is the subring of H'-invariants of $k[\mathrm{SL}(n)]$ and, as an $\mathrm{SL}(n)$-module, R is the direct limit of the representation spaces E_m of the tensor product $m\omega_d \otimes m\omega_d^*$ for $\mathrm{SL}(n)$.

6.2. The standard monomial bases for the filters E_m of R, as in Theorem 8, above, are *not consistent* with the inclusions $E_m \subseteq E_{m+1}$. However, there seems to exist a '*Good module filtration*' for R, in the sense of [2], which is a refinement of this filtration. Fixing then standard monomial bases for the sections of this refined filtration, we get a standard monomial basis globally for R. In practical terms, this is not constructive in general.

6.3. There is an affine open covering for P,

$$\{U_{I,J} \mid I \in [d], J \in [h], I \cap J = \emptyset\} ,$$

such that each $U_{I,J}$ is the complement of a hypersurface in an affine $2dh$-space with explicit patching data. The corresponding cocycle in Pic P is not zero, hence P, which is already locally factorial, is *not* globally factorial. In fact, the restriction of the line bundle $\Phi \times \Psi$ on $\mathrm{Gr}(d, n) \times \mathrm{Gr}(h, n)$ to P is a nonprincipal divisor on P. As for the description of $U_{I,J}$, we take

$$U_{I,J} = P_I Q_J/\det \neq 0\,, \quad I \cap J = \emptyset\,.$$

This is simply the hypersurface $\det \neq 0$ in the product of the big cells in Gr(d, n) and Gr(h, n) corresponding to the Bruhat decompositions for which Gr(d, n) = Y_I and Gr(h, n) = Z_J (see 4.3 above, for the notation).

Acknowledgment

It gives me great pleasure to thank once again Prof. E.C. Zeeman, Prof. R.W. Carter, Dr. M. Reid and Dr. S. Donkin for the excellent opportunities they have so generously made possible for me and to Ms. Elaine Shiels for her unfailing help during my stays at Warwick.

References

[1] C. De Concini, D. Eisenbud and C. Procesi, Hodge algebras, Asterisque 91 (1982) 1–87.
[2] S. Donkin, Good Filtrations for Rational Modules for Reductive Groups, Proceedings of Symposia in Pure Mathematics (Amer. Mathematical Soc., Providence, RI, 1987) Part 1, 69–80.
[3] W.V.D. Hodge and D. Pedoe, Methods of Algebraic Geometry, Vol. II (Cambridge Univ. Press, Cambridge, 1952).
[4] C. Musili, Postulation formula for Schubert varieties, J. Indian Math. Soc. 36 (1972) 143–171.
[5] C. Musili and C.S. Seshadri, Schubert varieties and the variety of complexes, Arithmetic and Geometry, Vol. II, Progress in Mathematics 36 (Birkhauser, Basel, 1983) 329–359.
[6] R.W. Richardson, Affine coset spaces of reductive algebraic groups, Bull. London Math. Soc. 9 (1977) 38–41.
[7] E. Strickland, On the variety of projectors, J. Algebra 106 (1987) 135–147.