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Quasipolyhedral Sets in Linear Semiinfinite Inequality Systems

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ABSTRACT

This paper provides an extension to linear semiinfinite systems of a well-known property of finite linear inequality systems, the so-called Weyl property, which characterizes the extreme points of the solution set as those solution points such that the gradient vectors of the active constraints form a complete set. A class of linear semiinfinite systems which satisfy this property is identified, the p-systems. It is also shown that any p-system contains an equivalent minimal subsystem. © Elsevier Science Inc., 1997

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1. INTRODUCTION

We will deal with consistent systems of linear inequalities in $\mathbb{R}^n$. They are denoted by $\sigma = \{a_t \cdot x \geqslant b_t,\ t \in T\}$, where T is any set of indexes. Many results about these linear semiinfinite systems (LSISs for short) are found in a unified treatment in [5].

The LSISs arise in several areas of applications such as moment theory [3], game theory [6, 10], pattern recognition [4], and, especially, linear semiinfinite programming [7], where the constraint systems are LSISs. Any attempt at extending the simplex method to these optimization problems requires the characterization of the extreme feasible points.

In this paper, a sufficient condition on the system σ is given in order to show that the extreme points of its solution set $F(\sigma)$ are those solution points that satisfy at least n equations $a_t \cdot x = b_t$, $t \in T' \subset T$, for some linearly independent set $\{a_t : t \in T'\}$, i.e., those solution points such that the gradient vectors of the active constraints form a complete set. This well-known property of finite linear systems is due to Weyl [11]. E. J. Anderson and A. S. Lewis [1] have given an analogous characterization of the extreme points of a particular class of LSISs with smooth coefficients. This characterization is given by the completeness of a certain set of derivatives of the gradients of the active constraints.

We will define a class of LSIS which satisfy the Weyl property: the p-systems. The class of p-polyhedral sets is also defined. The bounded p-polyhedral sets are just the polytopes. Moreover, for any p-polyhedral set given by a system σ, an equivalent minimal subsystem is constructed; hence the solution set is g-polyhedral [5]. The class of the p-polyhedral sets is in between the polyhedral sets and the g-polyhedral sets, depending on the Weyl property.

The plan of the paper is as follows: In Section 2, we state terminology and some preliminary results on LSISs. In Section 3, the p-systems are defined and the equivalence between extreme points and Weyl's condition is proven. Finally, in Section 4, we introduce the p-polyhedral sets, and a minimal equivalent subsystem is constructed for any p-system.

2. PRELIMINARIES AND NOTATION

Given a nonempty set $B \subset \mathbb{R}^n$, cl B denotes its *closure*, int B its *interior*, ri B its *relative interior*, bd B its *boundary*, rb B its *relative boundary*, and $K\{B\}$ the *convex cone* generated by B (see [9]).

A vector $y \in \mathbb{R}^{n+1}$ is written in the form $y = (a; b)$, where $a \in \mathbb{R}^n$, $b \in R$. For the system $\sigma = \{a_t \cdot x \geqslant b_t,\ t \in T\}$, $F(\sigma)$ denotes its solution set, i.e.

$$F(\sigma) = \{x \in \mathbb{R}^n : a_t \cdot x \geqslant b_t, t \in T\}.$$

A system σ is *consistent* if $F(\sigma) \neq \emptyset$.

The following cones are associated with this system:

$$N(\sigma) := K\{(a_t; b_t) : t \in T\}$$

and

$$K(\sigma) := K\{(a_t; b_t) : t \in T; (0; -1)\}.$$

A system σ is consistent if, and only if, $(0; 1) \notin \text{cl } N(\sigma)$, or equivalently, $(0; 1) \notin \text{cl } K(\sigma)$.

An inequality $a \cdot x \geqslant b$ is called a *consequence* of σ if every solution of σ satisfies this inequality. A sufficient and necessary condition for $a \cdot x \geqslant b$ to be a consequence of σ is that $(a; b) \in \text{cl } K(\sigma)$. Given two systems σ_1 and σ_2, $F(\sigma_1) \subset F(\sigma_2)$ if, and only if, $\text{cl } K(\sigma_2) \subset \text{cl } K(\sigma_1)$. The system σ_1 and σ_2 are *equivalent* if $F(\sigma_1) = F(\sigma_2)$, i.e., $\text{cl } K(\sigma_1) = \text{cl } K(\sigma_2)$.

$F(\sigma)$ is *bounded* if, and only if, $(0; -1) \in \text{int } K(\sigma)$.

A system σ is said to be *Farkas-Minkowsky* (briefly FM) if $K(\sigma)$ is closed. σ is FM if, and only if, each consequence of σ is also a consequence of a finite subsystem. Hence, an inequality $a \cdot x \geqslant b$ is a consequence of the FM system σ if, and only if,

$$a = \sum_{t \in T'} \lambda_t a_t, \qquad b \leqslant \sum_{t \in T'} \lambda_t b_t$$

for some $\lambda_t \geqslant 0$, $t \in T' \subset T$, T' being a finite set.

An inequality $a_s \cdot x \geqslant b_s$, $s \in T$, is *redundant* in σ if it is a consequence of its complementary subsystem $\sigma_s = \{a_t \cdot x \geqslant b_t,\ t \in T,\ t \neq s\}$. A LSIS which does not contain redundant inequalities is called *minimal*.

The *reduced system* of σ is defined as the system obtained from the following sequence of operations:

(1) Elimination of any trivial inequality: $0 \cdot x \geqslant b$, with $b \leqslant 0$.

(2) *Normalization*: for each $t \in T$, both members of the inequality $a_t \cdot x \geqslant b_t$ are multiplied by $\|(a; b)\|^{-1}$.

(3) Identification of all the inequalities with the same normalized vector.

Clearly, any system σ and its reduced system are equivalent.

A convex set $P \subset \mathbb{R}^n$ is said to be *polyhedral* if it admits a finite representation, i.e., $P = F(\sigma)$ for some finite system σ. Moreover, if P is bounded, it is a *polytope*. Following M. A. Goberna and M. A. López [5], we say that a convex set is *g-polyhedral* if it admits a minimal representation.

We will identify the system $\sigma = \{a_t \cdot x \geqslant b_t,\ t \in T\}$ with the subset $\{(a_t; b_t) : t \in T\}$ in $\mathbb{R}^{n+1}$. In this way, when we write "a cluster point of σ," we mean a cluster point of the latter set. For two given systems σ_1 and σ_2, $\sigma_1 \cup \sigma_2$ will denote the LSIS obtained by adjoining all the inequalities of these systems; thus $F(\sigma_1 \cup \sigma_2) = F(\sigma_1) \cap F(\sigma_2)$.

A system σ is said to satisfy the *Weyl property* if the extreme points of $F(\sigma)$ are just those solution points x^* for which the set $\{a_t : a_t \cdot x^* = b_t,\ t \in T\}$ is complete, i.e., it contains a basis of $\mathbb{R}^n$. In 1935 Weyl proved that any finite system verifies this property. However, this property does not hold for all LSISs, as the following examples show.

EXAMPLE 2.1. The solution set of the system $\sigma = \{-(\cos t)x - (\sin t)y \geqslant -1,\ t \in [0, 2\pi]\}$ is the unitary disk in $\mathbb{R}^2$. All the points in bd $F(\sigma)$ are extreme points of $F(\sigma)$, and the Weyl property fails at each of these extreme points.

EXAMPLE 2.2. Let $\sigma = \{-(2n+1)x + n(n+1)y \geqslant 1;\ n = 1, 2, \ldots\} \cup \{x \geqslant 0;\ -y \geqslant -1\}$. Then $F(\sigma)$ is the convex hull of the set $\{(x, x^2) : x = 0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\} \cup \{(0, 1)\}$. This system fails to satisfy the Weyl property at only one extreme point, the origin.

However, there are some LSISs satisfying this property.

EXAMPLE 2.3. Let $G \subset \mathbb{R}^n$ be the convex hull of the set $\{(x, x^2) : x = 0, \pm 1, \pm 2, \pm 3, \ldots\}$. Any minimal representation of G satisfies the Weyl property.

In one direction the Weyl property is immediate. Indeed, if $x^* \in F(\sigma)$ verifies that the set $\{a_t : a_t \cdot x^* = b_t,\ t \in T\}$ is complete in $\mathbb{R}^n$, then x^* is an extreme point of $F(\sigma)$.

3. p-SYSTEMS AND THE WEYL PROPERTY

In this section we will define the p-systems and show that for these LSISs the Weyl property is verified.

DEFINITION 3.1. A consistent system σ is a *p-system* if all the cluster points $(a; b) \in \mathbb{R}^{n+1}$ of its reduced system satisfy $a \cdot x > b$ for any $x \in F(\sigma)$.

The LSIS of Example 2.3 is a p-system, while those of Examples 2.1 and 2.2 are not. Notice that $F(\sigma)$ is unbounded in Example 2.1.

REMARK 3.2. Even though FM systems inherit many properties of ordinary linear systems, that is not the case with the Weyl property, as Example 2.1 shows. A condition for an FM system σ to be a p-system is that no cluster point of its reduced system belongs to the cone $N(\sigma)$. Other examples of p-systems with an unbounded solution set are

$$\sigma = \left\{ -(2n+1)x + \frac{2n+1}{n} y + n(n+1)z \geqslant -1,\ n = 1,2,3,\dots \right\} \cup \{z \geqslant 1\}$$

and

$$\sigma = \{x + n(n+1)y \geqslant 2n+1,\ n = 1,2,3,\dots\} \cup \{x \geqslant 1\}.$$

Now, let us analyze the case in which $F(\sigma)$ is bounded.

LEMMA 3.3. *Assume $F(\sigma)$ is bounded. Then σ is a p-system if, and only if, no cluster point of its reduced system lies in* bd $K(\sigma)$.

Proof. Without loss of generality, we may assume that σ is reduced. Let $(a; b)$ be a cluster point of σ.

Suppose σ is a p-system. Put

$$\delta = \inf\{a \cdot x - b : x \in F(\sigma)\}$$

and

$$b^* = \sup\{b' : (a; b') \in \operatorname{cl} K(\sigma)\}.$$

Then $\delta > 0$ and $b < b + \delta \leqslant b^* < \infty$, because σ is a consistent p-system and $F(\sigma)$ is compact. Since $(a; b^*) \in \operatorname{cl} K(\sigma)$ and $(0; -1) \in \operatorname{int} K(\sigma)$, the accessibility lemma [9] implies that $(a; b) \in \operatorname{int} K(\sigma)$.

On the other hand, suppose that $(a; b)$ is an interior point of $K(\sigma)$. Let $\varepsilon > 0$ be such that $(a; b) + \varepsilon(0; 1) \in K(\sigma)$. Hence, the inequality $a \cdot x \geqslant b + \varepsilon$ is a consequence of σ, and so $a \cdot x > b$ for any $x \in F(\sigma)$. That is, σ is a p-system. ■

LEMMA 3.4. *Let C be a convex compact subset of the interior of $K(\sigma)$. If $\tau = \sigma \setminus (\text{int } C)$, then σ and τ are equivalent systems.*

Proof. It is enough to show that bd $K(\sigma) \subset$ cl $K(\tau)$, because $K(\sigma) \subset K\{\text{bd } K(\sigma) : (0, -1)\}$. Assume int $C \neq \emptyset$, so ri $K(\sigma) =$ int $K(\sigma) \neq \emptyset$. Take $z \in$ bd $K(\sigma)$. If $z = 0$, then $z \in$ cl $K(\tau)$. Hence, without loss of generality, we may assume that $\|z\| = 1$. Let $\{z_k\}$ be a sequence in $K(\sigma)$ converging to z, with $\|z_k\| = 1$. For each k, we have $z_k = u_k + v_k$ for some $u_k \in K(\tau)$ and $v_k \in K\{\text{int } C\}$. Since $\|z_k\| = 1$, the sequences $\{u_k\}$ and $\{v_k\}$ are both bounded or both unbounded.

Notice that $0 \notin C$, because otherwise $0 \in$ int $K(\sigma)$, which gives σ a nonconsistent system. Hence, C is a compact convex set which does not contain the origin, and so cl $K\{C\} = K\{C\}$. Therefore,

$$\text{cl } K\{\text{int } C\} \subset \text{cl } K\{C\} = K\{C\} = \bigcup_{\lambda \geqslant 0} \lambda C \subset [\text{int } K(\sigma)] \cup \{0\}.$$

It both sequences are bounded, we may assume (by taking subsequences if necessary) that $u_k \to u$ and $v_k \to v$ for some $u \in$ cl $K(\tau) \subset$ cl $K(\sigma)$, $v \in$ cl $K\{\text{int } C\} \subset [\text{int } K(\sigma)] \cup \{0\}$. Hence, $v \neq 0$ implies $v \in$ int $K(\sigma)$ and, by the accessibility lemma, $z = u + v \in$ int $K(\sigma)$, in contradiction with $z \in$ bd $K(\sigma)$. It follows that $v = 0$ and then $z = u \in$ cl $K(\tau)$.

Suppose both sequences $\{u_k\}$ and $\{v_k\}$ were unbounded. It may be assumed, with no loss of generality, that $\|u_k\| \to \infty$, $\|v_k\| \to \infty$, and that u_k, v_k never vanish. From $z_k = u_k + v_k$ and $\|z_k\| = 1$, we get that $\|u_k\| \leqslant 1 + \|v_k\|$ and $\|v_k\| \leqslant 1 + \|u_k\|$; then

$$\|u_k\| \leqslant 1 + \|v_k\| \leqslant 2 + \|u_k\|.$$

Therefore

$$\lim_{k \to \infty} \frac{\|v_k\|}{\|u_k\|} = 1.$$

For each k, put $\alpha_k = \max\{\|u_k\|, \|v_k\|\}$. Then $\{(1/\alpha_k)u_k\}$ and $\{(1/\alpha_k)v_k\}$ are bounded sequences, which we assume convergent:

$$\frac{1}{\alpha_k}u_k \to u, \qquad \frac{1}{\alpha_k}v_k \to v,$$

$u \in \mathrm{cl}\, K(\tau)$, $v \in \mathrm{cl}\, K\{\mathrm{int}\, C\} \subset [\mathrm{int}\, K(\sigma)] \cup \{0\}$, $\|u\| = \|v\| = 1$. Hence $(1/\alpha_k)z_k \to u + v$. But $\|(1/\alpha_k)z_k\| = 1/\alpha_k \to 0$ gives $u + v = 0$. Since $v \neq 0$, we have $v \in \mathrm{int}\, K(\sigma)$ and, by the accessibility lemma, $0 = u + v \in \mathrm{int}\, K(\sigma)$. This contradicts the fact that σ is a consistent system.

Therefore $\mathrm{bd}\, K(\sigma) \subset \mathrm{cl}\, K(\tau)$. ■

THEOREM 3.5. *Let σ be a p-system. If $F(\sigma)$ is bounded, then there exists an equivalent finite subsystem of σ, and therefore $F(\sigma)$ is a polytope.*

Proof. It can be assumed that σ is reduced and hence that $\mathrm{cl}\, \sigma$ is compact. An open covering of $\mathrm{cl}\, \sigma$ will be constructed. For each cluster point $(c; d)$ of σ, let $E(c; d)$ be an open ball centered at $(c; d)$ such that its closure is in $\mathrm{int}\, K(\sigma)$. The existence of such $E(c; d)$ is assured by Lemma 3.3. For each isolated point $(a; b)$ of σ, let $E(a; b)$ be an open neighborhood, $E(a; b) \cap \sigma = \{(a; b)\}$.

Due to the compactness of $\mathrm{cl}\, \sigma$, there exist a finite collection of isolated points $(a_t; b_t)$, $t \in T' \subset T$, and finitely many cluster points $(c_i; d_i)$, $i = 1, 2, \ldots, m$, such that

$$\mathrm{cl}\, \sigma \subset \bigcup_{t \in T'} E(a_t; b_t) \cup \bigcup_{i=1}^{m} E(c_i; d_i).$$

The finite system

$$\tau = \sigma \setminus \mathrm{cl}\left[\bigcup_{i=1}^{m} E(c_i; d_i)\right]$$

is equivalent to σ by an iterated application of Lemma 3.4. This completes the proof. ■

We can now show that every p-system satisfies the Weyl property.

THEOREM 3.6. *Let $\sigma = \{a_t \cdot x \geqslant b_t,\ t \in T\}$ be a p-system, and let $x^* \in F(\sigma)$. Then x^* is an extreme point of $F(\sigma)$ if, and only if, the set $\{a_t : a_t \cdot x^* = b_t,\ t \in T\}$ is complete.*

Proof. Suppose x^* is an extreme point of $F(\sigma)$. Let P be a closed rectangle in $\mathbb{R}^n$ such that $x^* \in \operatorname{int} P$. Let π be a finite system of linear inequalities with $P = F(\pi)$. Then $\sigma \cup \pi$ is a consistent p-system with $F(\sigma \cup \pi)$ a bounded set. By Theorem 3.5, there exists a finite subsystem $\tau \subset \sigma$ such that $F(\tau \cup \pi) = F(\sigma \cup \pi)$. Clearly, x^* is an extreme point of $F(\tau \cup \pi)$. The Weyl property for finite systems and the choice of P imply that x^* satisfies the desired property with respect to the finite subsystem τ and therefore with respect to σ.

The opposite implication is well known. Thus, the theorem follows. ■

In the next section it will be shown that any p-system contains a minimal (countable [5]) equivalent subsystem. In this way, any p-system can be thought as discrete, and therefore the last theorem cannot be applied to smooth LSISs. Nevertheless, Theorem 3.6 can be considered as a discrete counterpart of Theorem 4 in [1], which characterizes the extreme points through the completeness of a certain set of derivatives of the gradients of the active constraints.

4. p-POLYHEDRAL SETS AND MINIMAL SYSTEMS

Next we will define the class of convex sets called p-polyhedral sets, and study the relationship between p-systems and minimal systems.

DEFINITION 4.1. A convex set $F \subset \mathbb{R}^n$ is said to be *p-polyhedral* if F is the solution set of some p-system.

The bounded p-polyhedral sets are just the polytopes. The unitary disk of Example 2.1 is neither p-polyhedral nor g-polyhedral. Example 2.2 shows a g-polyhedral set G which is not p-polyhedral. The solution sets in Example 2.3 and Remark 3.2 are g- and p-polyhedral, but not polyhedral.

In general, a g-polyhedral set is not p-polyhedral, but we will show that every p-polyhedral set is a g-polyhedral by building a minimal equivalent subsystem of a given consistent p-system.

LEMMA 4.2. *If σ is a minimal system, then $\sigma \subset \operatorname{rb} K(\sigma)$.*

Proof. Assume $\operatorname{ri} K(\sigma) \cap \sigma \neq \emptyset$, and let $(a_s; b_s) \in \operatorname{ri} K(\sigma) \cap \sigma$. Let $b = \sup\{b' : (a_s; b') \in K(\sigma)\}$. Then $(a_s; b) \in \operatorname{rb} K(\sigma)$. By the accessibility lemma, $(a_s;(b+b_s)/2) \in \text{ri } K(\sigma)$ and hence

$$a_s = \lambda_s a_s + \sum_{t\in T'} \lambda_t a_t, \qquad \frac{b+b_s}{2} < \lambda_s b_s + \sum_{t\in T'} \lambda_t b_t$$

for some finite set $T' \subset T \setminus \{s\}$, $T' \neq \emptyset$, $\lambda_s \geqslant 0$, $\lambda_t > 0$, $t \in T'$. Thus,

$$(1-\lambda_s)a_s = \sum_{t\in T'} \lambda_t a_t, \qquad (1-\lambda_s)b_s < \sum_{t\in T'} \lambda_t b_t.$$

If $\lambda_s = 1$, then $(0; 1) \in N(\sigma)$, and the system would be inconsistent. If $\lambda_s > 1$, then $(-a_s; -b_s) \in K(\sigma)$; as $(a_s;(b+b_s)/2)$ belongs to $K(\sigma)$, the system would also be inconsistent. Finally, if $\lambda_s < 1$, then $(a_s; b_s)$ is a consequence of its complementary system, which contradicts the fact that σ is a minimal system. Therefore $\sigma \subset \text{rb } K(\sigma)$. ∎

What follows is a recursive construction of a minimal subsystem of a given p-system σ in $\mathbb{R}^n$.

CONSTRUCTION 4.3. Let P_m $(m = 1, 2, 3, \dots)$ be a sequence of closed rectangles in $\mathbb{R}^n$ such that $\bigcup_m P_m = \mathbb{R}^n$, $P_m \subset \text{int } P_{m+1}$, $\text{int } P_1 \cap F(\sigma) \neq \emptyset$. Let π^m be a minimal (finite) system of inequalities for P_m, $F(\pi^m) = P_m$. The sets $\sigma \cup \pi^m$ are consistent p-systems with a bounded solution set, $F(\sigma \cup \pi^m) = F(\sigma) \cap P_m$.

By virtue of Theorem 3.5, we get an equivalent minimal finite subsystem of $\sigma \cup \pi^1$. Let it be $\omega^1 = \tau^1 \cup \rho^1$, where $\tau^1 \subset \sigma$ and $\rho^1 \subset \pi^1$.

For a given positive integer m, assume that a finite minimal system $\omega^m = \tau^m \cup \rho^m$, with $\tau^m \subset \sigma$, $\rho^m \subset \pi^m$, $F(\omega^m) = F(\sigma) \cap P_m$, has been constructed. Again, Theorem 3.5 gives an equivalent finite subsystem of the p-system $\sigma \cup \pi^{m+1}$, which it can be assumed contains τ^m. By eliminating redundant inequalities, starting with those which do not belong to τ^m, an equivalent minimal system $\omega^{m+1} = \tau^{m+1} \cup \rho^{m+1}$, with $\tau^{m+1} \subset \sigma$, $\rho^{m+1} \subset \pi^{m+1}$, is obtained.

It is clear that $F(\omega^m) \subset F(\omega^{m+1}) \subset F(\sigma)$, and hence $K(\omega^m) \supset K(\omega^{m+1}) \supset K(\sigma)$. Here $F(\omega^m)$ is bounded, which implies that $(0; -1) \in \text{int } K(\omega^m)$. Moreover, if $\tau = \bigcup_m \tau^m$, then $F(\sigma) = F(\tau)$.

CLAIM 4.4. $\tau^m \subset \tau^{m+1}$.

Proof. Let $x \in \tau^m$. From $\tau^m \subset \sigma \subset K(\sigma) \subset K(\omega^{m+1})$, x can be written as

$$x = \sum_{t \in T} \lambda_t y_t + \sum_{s \in S} \mu_s z_s + \eta(0; -1),$$

where $y_t \in \tau^{m+1}$, $z_s \in \rho^{m+1}$, $\lambda_t, \mu_s, \eta \geqslant 0$ for all $t \in T$, $s \in S$, T and S finite sets.

By Lemma 4.2, $x \in \text{rb } K(\omega^m) = \text{bd } K(\omega^m)$. Moreover, $x \in \text{bd } K(\omega^{m+1})$, because $K(\omega^{m+1}) \subset K(\omega^m)$. Since $(0; -1) \in \text{int } K(\omega^{m+1})$, from the accessibility lemma it follows that $\eta = 0$. For the same reason, $\mu_s = 0$ for each $s \in S$, because $z_s \in \rho^{m+1} \subset \text{int } K(\pi^m) \subset \text{int } K(\omega^m)$. Therefore,

$$x = \sum_{t \in T} \lambda_t y_t, \tag{1}$$

where we may assume, without loss of generality, that $y_t \in \text{bd } K(\omega^m)$ and $\lambda_t > 0$ for all $t \in T$.

From $y_t \in K(\omega^m)$,

$$y_t = \sum_{j \in J} \alpha_j^t x_j + \sum_{k \in K} \beta_k^t w_k + \eta^t(0; -1) \tag{2}$$

for some $x_j \in \tau^m$, $w_k \in \rho^m$, $\alpha_j^t, \beta_k^t, \eta^t \geqslant 0$, for $j \in J$, $k \in K$, J and K finite sets. As before, $\eta^t = 0$.

(1) and (2) give

$$x = \alpha x + \sum_{j \in J,\, x_j \neq x} \alpha_j x_j + \sum_{k \in K} \beta_k w_k$$

for some $x_j \in \tau^m$, $w_k \in \rho^m$, $\alpha, \alpha_j, \beta_k \geqslant 0$. As x is not redundant in ω^m, we have $\alpha \geqslant 1$. If $\beta_k > 0$ for some k, it follows that $-w_k \in K(\omega^m) \subset K(\omega^1)$. But $w_k \in K(\omega^1)$, so int $F(\omega^1) = \varnothing$, which contradicts the initial assumption of the construction: int $P_1 \cap F(\sigma) = \varnothing$. Thus, $\beta_k = 0$ for each $k \in K$.

Hence, $\beta_k^t = 0$ for all $k \in K$, $t \in T$, and (2) can be written as

$$y_t = \sum_{j \in J} \alpha_j^t x_j,$$

$x_j \in \tau^m$, $\alpha_j^t \geqslant 0$. By the way in which the redundant inequalities were eliminated with ω^{m+1}, y_t cannot belong to $\tau^{m+1} \setminus \tau^m$. Hence, $y_t \in$

$\tau^{m+1} \cap \tau^m$ for any $t \in T$. Because τ^m is minimal, (1) implies that $x = y_t$ for some $t \in T$, and therefore $x \in \tau^{m+1}$, as we intended to show. ∎

CLAIM 4.5. $\bigcap_m K(\omega^m) = \text{cl } K(\tau) = \text{cl } K(\sigma) \supset K(\sigma) \supset K(\tau) = \bigcup_m K(\tau^m)$.

Proof. All the relations are immediate, except the first one.

$\text{cl } K(\tau) \subset \bigcap_m K(\omega^m)$ is obvious by construction. Let $x \in \bigcap_m K(\omega^m)$, and for each positive integer m put $x = u_m + v_m$ for some $u_m \in K(\tau^m)$ and $v_m \in K(\pi^m)$. Notice that $\bigcap_m K(\pi^m) = K\{(0; -1)\}$.

Using a similar argument as in the proof of Lemma 3.4, we may draw the conclusion that both sequences $\{u_m\}$ and $\{v_m\}$ must be bounded, and therefore we might assume that $u_m \to u$, $v_m \to v$, $u \in \text{cl } K(\tau)$, $v \in K\{(0; -1)\}$, $u + v = x$. Then each $u_m + \|v\|(0; -1) \in K(\tau^m)$ and $u_m + \|v\|(0; -1) \to x$, which implies that $x \in \text{cl } K(\tau)$. Therefore $\bigcap_m K(\omega^m) = \text{cl } K(\tau)$. ∎

CLAIM 4.6. *τ is a minimal subsystem.*

Proof. Assume that all the systems are normalized and that τ is not minimal. Let $x \in \tau$ such that $x \in \text{cl } K(\tau_x)$, $\tau_x = \tau \setminus \{x\}$. Then $\text{cl } K(\tau_x) = \text{cl } K(\sigma)$. Now, a subsystem ν of τ_x is built based on the same sequence of closed rectangles $\{P_m\}$ and procedures used in the construction of τ. Hence, for each positive integer m, there is a finite minimal system $\theta^m = \nu^m \cup q^m$, with $\nu^m \subset \tau_x$, $q^m \subset \pi^m$, $F(\theta^m) = F(\tau_x) \cap P_m = F(\sigma) \cup P_m = F(\omega^m)$, and $K(\theta^m) = \text{cl } K(\theta^m) = \text{cl } K(\omega^m) = K(\omega^m)$.

Let M be the smallest positive integer such that $x \in \tau^M$. For any $m \geqslant M$, $x \in \tau^m \subset K(\omega^m) = K(\theta^m)$. Since ω^m is minimal, it follows that x belongs to a 1-dimensional face of the polyhedral cone $K(\omega^m) = K(\theta^m)$. Hence, $x \in \theta^m$, i.e., $x \in \bigcap_{m \geqslant M} \theta^m = \bigcap_{m \geqslant M} (\nu^m \cup q^m) = \nu^M$. Therefore $x \in \nu^M \subset \tau_x = \tau \setminus \{x\}$, which is a contradiction. ∎

Thus, we have proved the following theorem.

THEOREM 4.7. *Let σ be a p-system. Then there exists a sequence of finite minimal systems $\omega^m = \tau^m \cup \rho^m$, with $\tau^m \subset \sigma$, such that for each positive integer m the following conditions hold:*

(i) $F(\omega^m) \subset F(\omega^{m+1}) \subset F(\sigma)$,

(ii) $F(\omega^m) \supset K(\omega^{m+1}) \supset K(\sigma)$,

(iii) $\tau^m \subset \tau^{m+1} \subset \sigma$.

Let $\tau = \bigcup_m \tau^m \subset gs$. *Then*

(iv) σ *and* τ *are equivalent systems*;
(v) $\bigcap_m K(\omega^m) = \text{cl } K(\tau) = \text{cl } K(\sigma) \supset K(\sigma) \supset K(\tau) = \bigcup_m K(\tau^m)$;
(vi) τ *is a minimal subsystem.*

COROLLARY 4.8. *Any p-system contains an equivalent minimal subsystem. Thus, every p-polyhedral set is g-polyhedral.*

The following example and remark are important in order to fully understand the scope of the results presented in this paper.

EXAMPLE 4.9. Let $\sigma = \{-(2n+1)x + n(n+1)z \geqslant -1;\ n = 1,2,3,\dots\} \cup \{y + z \geqslant 0;\ -y + z \geqslant 0,\ x \geqslant 0\}$. The only cluster point of the reduced system of σ is (0, 0, 1; 0), which together with the origin shows that σ is not a p-system. Notice that this system is minimal (recall Example 2.2) and satisfies the Weyl property. However, the origin is a cluster point of the set of extreme points.

REMARK 4.10. The obvious conclusion of the latter example is the existence of a minimal LSIS satisfying the Weyl property without being a p-system, which poses the question whether it is possible to find weaker conditions than the one given in the present work. Looking again at Example 4.9, it is found that $F(\sigma) = F(\sigma_1) \cap F(\sigma_2)$, where

$$\sigma_1 = \{-(2n+1)x + n(n+1)z \geqslant -1;\ n = 1,2,3,\dots\}$$
$$\cup\{y + z \geqslant 0;\ -y + z \geqslant 0\}$$

and

$$\sigma_2 = \{y + z \geqslant 0;\ -y + z \geqslant 0;\ x \geqslant 0\}.$$

σ_1 is not a p-system, because of the limit inequality $z \geqslant 0$; and $F(\sigma_1)$ has a "bad" extreme point, which is the origin. However, the origin is also an extreme point of $F(\sigma_2)$, and it is a "good" one. The system $\sigma = \sigma_1 \cup \sigma_2$ inherits the good quality of the origin from σ_2.

It is clear now that alternative weaker conditions on the system σ would involve conditions on each extreme. For example: for each extreme point $x \in F(\sigma)$ there is a p-system $\sigma[x] \subset \sigma$ such that x is an extreme point of $F(\sigma[x])$. However, this kind of statement may be regarded as trivial.

On the other hand, a pointed cone in $\mathbb{R}^n$ may be not a g-polyhedral set, but its unique extreme point may satisfy the Weyl property.

In brief, the p-polyhedral sets form a particular class of convex sets arising from LSISs for which all the extreme points are isolated and the Weyl property remains true.

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