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Optimal algorithms for the online time series search problem☆

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ABSTRACT

In the problem of online time series search introduced by El-Yaniv et al. (2001) [1], a player observes prices one by one over time and shall select exactly one of the prices on its arrival without the knowledge of future prices, aiming to maximize the selected price. In this paper, we extend the problem by introducing profit function. Considering two cases where the search duration is either known or unknown beforehand, we propose two optimal deterministic algorithms respectively. The models and results in this paper generalize those of El-Yaniv et al. (2001) [1].

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1. Introduction

The problem of online time series search was introduced by El-Yaniv et al. [1], where a player observes a series of n prices sequentially in order to select the highest price in the series. On the observation of each price presented, the player has to decide immediately whether to accept the price or not without the knowledge of future prices. The profit depends on the price selected by the player. El-Yaniv et al. [1] proved that if the prices are bounded within interval $[m, M]$ $(0 < m < M)$, the optimal algorithm is to accept the first price no less than $\sqrt{Mm}$ and the competitive ratio is $\sqrt{M/m}$. Damaschke, Ha and Tsigas [2] studied another case where the upper and lower bounds of prices vary as time goes on. The approach is also adopted by Lorenz, Panagiotou and Steger [3] for the k-search problem to search for the k highest (or lowest) prices in one series. All the above work assumes that the profit to accept a price is exactly the price itself, ignoring when the price shows up in the series. In many real scenarios, however, this is not the case. For example, the player may need to pay a sampling cost at each time period to observe a price, and then the accumulated sampling cost increases as time goes on. Hence, the profit to accept a price at some period can be regarded as a function of the price, such as equaling the accepted price minus the accumulated sampling cost. In this paper, we will extend the basic model in [1] by introducing profit function, and give more general results on competitiveness.

1.1. Related work

The problem of time series search has received considerable attention in mathematical economics and operations research since 1960s. It is quite related to the optimal stopping problem (see [4]) and the secretary problem (see [5,6]), both of which have many extensions such as secretary problem with discounts (see [7]) and with inspection costs (see [8]). Most of the previous work follows Bayesian approach, and algorithms are developed under assumption that prices are generated by some (e.g. uniform) distribution which is known beforehand (see [6]). Since the distribution of prices may

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not be known to the player in many situations, some research attempts to relax the assumption. Rosenfield and Shapiro [9] studied the case where the price distribution is a random variable.

Another related line is one-way trading problem. El-Yaniv et al. [1] pointed out that any algorithm for the problem can be viewed as a randomized search algorithm and they presented one optimal algorithm. Fujiwara et al. [11] studied the problem with average-case competitive analysis, adopting different optimization measures and deriving optimal strategies for those measures respectively. Kakade et al. [12] introduced several online models in modern financial markets considering two important aspects Volume Weighted Average Price trading and limit order books. They gave an extensive study for the models and related them to previous algorithms in stock trading.

1.2. Competitive ratio

Sleator and Tarjan [10] proposed to evaluate the performance of online algorithms by competitive analysis. For an arbitrary given price sequence σ, the profit of an online algorithm *ALG* is compared with that of an offline player's algorithm *OPT*, which knows all the prices in advance. Let $ALG(\sigma)$ and $OPT(\sigma)$ denote the profits of *ALG* and *OPT* in σ respectively. The competitive ratio of *ALG* is then defined as

$$\alpha = \sup_{\sigma} \frac{OPT(\sigma)}{ALG(\sigma)}$$

We also say that *ALG* is α-competitive. Given that there are not any online algorithms with competitive ratio less than α, *ALG* is called an optimal online algorithm.

The rest of the paper is organized as follows. Section 2 models the time series search problem and gives some assumptions as well. In Section 3 we investigate the case with known duration, and in Section 4 we discuss the case with unknown duration. Finally, Section 5 concludes the paper.

2. Problem statement and assumptions

The problem: A player is searching for one price of some asset in a price quotation sequence aiming to maximize the profit. There is only one chance for the player to select a price. At each time period $i = 1, 2, \dots, n$ where the time horizon n is a natural number, a price quotation p_i is received. The player has to decide immediately whether to accept the price. Once it is accepted at period i, the player cannot accept another price in later periods and the profit for p_i is denoted by $f_i(p_i)$, i.e., the *profit function* at period i. Otherwise p_i expires and p_{i+1} arrives in the next period. Note that the price series may end at some period $l \leq n$, i.e., the last price is p_l, and we call l the *duration* of the series.

For the online time series search problem, there are three basic assumptions below.

(1) The prices vary within interval $[m, M]$, where $0 < m < M$. The values of n, m, M and the functions $f_i(p), i = 1, 2, \dots, n$ are known beforehand to the player.

(2) The profit function $f_i(p)$ $(i = 1, 2, \dots, n)$ is continuous and increasing in p.

(3) For an arbitrary $p \in [m, M], f_1(p) \geq f_2(p) \geq \cdots \geq f_n(p) > 0$.

In assumption (1), if $n = 1$, the case is trivial since both online and offline players will accept the unique price with the same profit. So, we will focus on the case that $n \geq 2$ in the following. The second and third assumptions tell that at each period larger price results in larger profit and for a specific price p, the profit is larger in an earlier period than in a later one, respectively. In the following, we will divide the problem into two cases according to the knowledge of duration.

Variant 1: Known duration. The duration of the price quotation sequence is equal to n which is known to the player at the first beginning. The player can have at least a profit of $f_n(p_n)$ by accepting the last price p_n.

Variant 2: Unknown duration. The player has the only information that the duration of the price quotation sequence is at most n beforehand. At the beginning of each period the player is told whether the sequence ends at the period or not.

Note that in both variants, it is sufficient to analyze the case where $f_{i+1}(M) > f_i(m)$ for $i = 1, 2, \dots, n-1$, otherwise if $f_{i+1}(M) \leq f_i(m)$ holds at some period i, the player will accept a price and the game ends on or before period i since he gains a profit $f_j(p_j) \geq f_i(m)$ to accept p_j $(1 \leq j \leq i)$ at period j more than that to accept p_k $(i+1 \leq k \leq n)$ at period k with $f_k(p_k) \leq f_{i+1}(p_k) \leq f_{i+1}(M) \leq f_i(m)$. In the rest of the work, we will focus on the case such that $f_{i+1}(M) > f_i(m)$ for $1 \leq i \leq n-1$ in a price series.

3. Online time series search problem with known duration

In this section, we will discuss the case with known duration n, and present an optimal deterministic algorithm.

3.1. The online algorithm

The main idea of the online algorithm is to decide whether to accept or reject price p_i at period i $(1 \leq i \leq n)$. If p_i is less than some predetermined value p_i^*, then the price is rejected and goes to the next period, otherwise the price is accepted and game ends. The calculation of p_i^* is basically according to the worst case given that the online algorithm accepts p_i at period i. Before describing the algorithm formally, we give some preliminary definitions. Let

$$\alpha = \min\left\{\left\{\max\left\{\frac{f_{i+1}(M)}{f_i(m)}, \sqrt{\frac{f_2(M)}{f_i(m)}}\right\}, i = 1, 2, \ldots, n-1\right\}, \sqrt{\frac{f_2(M)}{f_n(m)}}\right\} \quad (1)$$

Note that $\alpha \geq 1$ since $f_{i+1}(M) > f_i(m)$ and $f_2(M) \geq f_n(m)$. By the definition of α, there exists a natural number r such that either $\alpha = \frac{f_{r+1}(M)}{f_r(m)}$ for $r \leq n-1$ or $\alpha = \sqrt{\frac{f_2(M)}{f_r(m)}}$ for $r \leq n$. Ties are broken by selecting the smallest r. If $\alpha = \frac{f_{r+1}(M)}{f_r(m)}$, let p_i^* $(1 \leq i \leq r)$ either be the solution of equation $\alpha f_i(p_i^*) = f_{i+1}(M)$ or $p_i^* = m$ in the case that there is no solution for the equation. Ties are broken by selecting the p_i^* with the smallest value. Otherwise if $\alpha = \sqrt{\frac{f_2(M)}{f_r(m)}}$, then let $i^* = \max\{i | f_{i+1}(M) \geq \sqrt{f_2(M) f_r(m)}\}$. Let $p_i^* = m$ for $\min\{i^*, r-1\} < i \leq r$, and for $1 \leq i \leq \min\{i^*, r-1\}$, p_i^* either be the solution of equation $\alpha f_i(p_i^*) = f_{i+1}(M)$ or $p_i^* = m$ in the case that there is no solution for the equation. Note that for $i = 1, f_2(M) > f_1(m)$ and then $\max\{\frac{f_2(M)}{f_1(m)}, \sqrt{\frac{f_2(M)}{f_1(m)}}\} = \frac{f_2(M)}{f_1(m)}$. Together with formula (1), $\alpha \leq \frac{f_2(M)}{f_1(m)}$ and thus $f_1(m) \leq \frac{f_2(M)}{\alpha} \leq f_2(M) \leq f_1(M)$, implying that p_1^* is only defined by the solution of equation $\alpha f_1(p_1^*) = f_2(M)$. Moreover, $p_r^* = m$ by the above discussion.

Algorithm *AKD* **(Algorithm with Known Duration):**

Step 1. Let $i = 1$.

Step 2. At period i, if $p_i \geq p_i^*$ then accept p_i with profit $f_i(p_i)$, otherwise if $p_i < p_i^*$, go to Step 3.

Step 3. If $i < n$ then $i = i + 1$ and go to Step 2, otherwise if $i = n$ then accept p_n and the game ends.

Note that *AKD* will accept a price on or before period r since $p_r^* = m$.

3.2. Competitive analysis

Lemma 1. *If AKD accepts* p_i *at some period* i $(1 \leq i \leq r)$*, then* $m < p_j^* \leq M$ *for* $1 \leq j \leq i-1$.

Proof. For the first inequality, if otherwise $p_j^* = m$, *AKD* will accept a previous price on or before period j since $p_j \geq m$. Moreover, $p_j^* \leq M$ due to $f_j(p_j^*) = \frac{f_{j+1}(M)}{\alpha} \leq f_{j+1}(M) \leq f_j(M)$ and assumption (2). The lemma follows. □

Lemma 1 implies that p_j^* shall be the solution of equation $\alpha f_j(p_j^*) = f_{j+1}(M)$ for $1 \leq j \leq i-1$.

Theorem 1. *AKD has competitive ratio of* α *for the online time series search problem with known duration.*

Proof. Let ε denote an arbitrarily small positive real number. We discuss two cases according to different values of α.

Case 1. $\alpha = \frac{f_{r+1}(M)}{f_r(m)}$. According to formula (1), $\frac{f_{r+1}(M)}{f_r(m)} \geq \sqrt{\frac{f_2(M)}{f_r(m)}}$ which implies $f_{r+1}(M) \geq \frac{f_2(M) f_r(m)}{f_{r+1}(M)} = \frac{f_2(M)}{\alpha} = f_1(p_1^*)$. As *AKD* accepts a price on or before period r, assume without loss of generality that it accepts p_i at period i $(1 \leq i \leq r)$. By Lemma 1, $m < p_j^* - \varepsilon < M$ for $j = 1, 2, \ldots, i-1$. So, the worst price sequence to *AKD* is $\sigma_1 = (p_1^* - \varepsilon, \ldots, p_{i-1}^* - \varepsilon, p_i^*, M, \ldots)$. The profit of *AKD* in σ_1 is $AKD(\sigma_1) = f_i(p_i^*) = \frac{f_{i+1}(M)}{\alpha}$. For *OPT*, combining $f_j(p_j^*) = \frac{f_{j+1}(M)}{\alpha}$ $(1 \leq j \leq i-1)$ and assumption (3), $f_1(p_1^*) \geq \cdots \geq f_i(p_i^*)$ As $\varepsilon \to 0$, *OPT*'s profit is as follows.

$$\begin{aligned} OPT(\sigma_1) &\approx \max\{f_1(p_1^*), \ldots, f_i(p_i^*), f_{i+1}(M)\} \\ &= \max\{f_1(p_1^*), f_{i+1}(M)\} \\ &= f_{i+1}(M) \end{aligned}$$

The last equation holds since $f_{i+1}(M) \geq f_{r+1}(M) \geq f_1(p_1^*)$. So, $\frac{OPT(\sigma_1)}{AKD(\sigma_1)} = \alpha$ in this case.

Case 2. $\alpha = \sqrt{\frac{f_2(M)}{f_r(m)}}$. According to formula (1), if $r \leq n-1$, then $\sqrt{\frac{f_2(M)}{f_r(m)}} \geq \frac{f_{r+1}(M)}{f_r(m)}$. By the definition of p_i^* in the case of $\alpha = \sqrt{\frac{f_2(M)}{f_r(m)}}$, $p^*_{\min\{i^*, r-1\}+1} = m$. So, *AKD* will accept a price on or before period $\min\{i^*, r-1\}+1$. Assume without loss of generality that it accepts p_i at some period i $(1 \leq i \leq \min\{i^*, r-1\}+1)$. We divide the case into two sub-cases according to different values of i.

Case 2.1. $1 \leq i \leq \min\{i^*, r-1\}$. In this sub-case, noting that since $i \leq i^*, f_{i+1}(M) \geq \sqrt{f_2(M) f_r(m)} = \frac{f_2(M)}{\alpha} = f_1(p_1^*)$, the worst price sequence and the following discussion are the same as those in Case 1.

Case 2.2. $i = \min\{i^*, r-1\}+1$. $i = i^*+1$ if $i^* \leq r-2$ and $i = r$ if $i^* > r-2$. We already know that $m < p_j^* - \varepsilon < M$ for $j = 1, 2, \ldots, i-1$. So, the worst price sequence to *AKD* is

$$\sigma_2 = \begin{cases} (p_1^* - \varepsilon, p_2^* - \varepsilon, \ldots, p_{i-1}^* - \varepsilon, m, M, \ldots) & : \; i = i^*+1 \\ (p_1^* - \varepsilon, p_2^* - \varepsilon, \ldots, p_{i-1}^* - \varepsilon, m, M, \ldots) & : \; i = r \leq n-1 \\ (p_1^* - \varepsilon, p_2^* - \varepsilon, \ldots, p_{i-1}^* - \varepsilon, m) & : \; i = r = n \end{cases}$$

We further discuss three sub-cases according to the three worst price sequences.

Case 2.2.1. $i = i^* + 1$ and $\sigma_2 = (p_1^* - \varepsilon, p_2^* - \varepsilon, \ldots, p_{i-1}^* - \varepsilon, m, M, \ldots)$. The profit of *AKD* is $AKD(\sigma_2) = f_i(m)$. For *OPT*, as $\varepsilon \to 0$,

$$\begin{aligned} OPT(\sigma_2) &\approx \max\{f_1(p_1^*), \ldots, f_{i-1}(p_{i-1}^*), f_i(m), f_{i+1}(M)\} \\ &= \max\{f_1(p_1^*), f_{i+1}(M)\} \\ &= f_1(p_1^*) \end{aligned}$$

where the second equation holds since $f_1(p_1^*) \geq \cdots \geq f_{i-1}(p_{i-1}^*) \geq f_i(m)$, and the third equation holds since $f_{i+1}(M) < \sqrt{f_2(M) f_r(m)} = f_1(p_1^*)$ due to $i > i^*$ and the definition of i^*. Hence, $\frac{OPT(\sigma_2)}{AKD(\sigma_2)} = \frac{f_1(p_1^*)}{f_i(m)} \leq \frac{f_1(p_1^*)}{f_r(m)} = \frac{f_2(M)/\alpha}{f_r(m)} = \alpha$ in this sub-case.

Case 2.2.2. $i = r \leq n - 1$ and $\sigma_2 = (p_1^* - \varepsilon, p_2^* - \varepsilon, \ldots, p_{i-1}^* - \varepsilon, m, M, \ldots)$. Then $AKD(\sigma_2) = f_i(m) = f_r(m)$. For *OPT*, $OPT(\sigma_2) \approx \max\{f_1(p_1^*), f_{r+1}(M)\}$ with similar reasoning as in Case 2.2.1. In the previous sub-case, we already have $\frac{f_1(p_1^*)}{f_r(m)} = \alpha$. Moreover, $\frac{f_{r+1}(M)}{f_r(m)} \leq \sqrt{\frac{f_2(M)}{f_r(m)}} = \alpha$ where the inequality holds by the condition of Case 2 and $r \leq n - 1$. So, $\frac{OPT(\sigma_2)}{AKD(\sigma_2)} \leq \alpha$ in this sub-case.

Case 2.2.3. $i = r = n$ and $\sigma_2 = (p_1^* - \varepsilon, p_2^* - \varepsilon, \ldots, p_{i-1}^* - \varepsilon, m)$. In this sub-case, $AKD(\sigma_2) = f_i(m) = f_r(m)$. For *OPT*, combining $f_1(p_1^*) \geq \cdots \geq f_{i-1}(p_{i-1}^*) \geq f_i(m)$ with $\varepsilon \to 0$, $OPT(\sigma_2) \approx f_1(p_1^*)$. Hence, $\frac{OPT(\sigma_2)}{AKD(\sigma_2)} = \alpha$. The theorem follows. □

In the following, we will show that no deterministic algorithms behave better than *AKD* in competitiveness for Variant 1 of the problem.

Theorem 2. *For the online time series search problem with known duration, no deterministic algorithm has competitive ratio less than α.*

Proof. Let *ALG* be any deterministic algorithm. We will construct a price sequence $\hat{\sigma} = (\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n)$ such that *ALG* cannot achieve a competitive ratio less than α.

The sequence $\hat{\sigma}$ is constructed as follows. First, similar to the discussion on the existence of p_1^* to equation $\alpha f_1(p_1^*) = f_2(M)$, we can define price $\hat{p}_1 \in [m, M]$ given by equation $\alpha f_1(\hat{p}_1) = f_2(M)$. At period 1, we present $\hat{p}_1$ to *ALG*, if *ALG* accepts the price, we further present the rest $n - 1$ prices $\hat{p}_2 = \cdots = \hat{p}_n = M$, otherwise we present $\hat{p}_2 = m$ and go to the next period. Similarly, at period 2, if *ALG* accepts $\hat{p}_2$, then we further present the rest $n - 2$ prices $\hat{p}_3 = \cdots = \hat{p}_n = M$, otherwise we present $\hat{p}_3 = m$ and go to the next period. This is repeated until either at some period i $(2 \leq i \leq n - 1)$, *ALG* accepts $\hat{p}_i$ or *ALG* accepts $\hat{p}_n = m$ at period n. In the first case, we further present the rest $n - i$ prices $\hat{p}_{i+1} = \cdots = \hat{p}_n = M$.

Assume that *ALG* accepts $\hat{p}_i$ at period i. We discuss three cases depending on the value of i.

Case 1. $i = 1$. In this case $OPT(\hat{\sigma}) \geq f_2(M)$ and $ALG(\hat{\sigma}) = f_1(\hat{p}_1)$ implying $\frac{OPT(\hat{\sigma})}{ALG(\hat{\sigma})} \geq \frac{f_2(M)}{f_1(\hat{p}_1)} = \alpha$.

Case 2. $2 \leq i \leq n - 1$. In this case we further divide the case into the following two sub-cases.

Case 2.1. $\frac{f_{i+1}(M)}{f_i(m)} \geq \sqrt{\frac{f_2(M)}{f_i(m)}}$. By the definition of α, $\frac{f_{i+1}(M)}{f_i(m)} \geq \alpha$. *OPT* will gain a profit satisfying $OPT(\hat{\sigma}) \geq f_{i+1}(M)$ while *ALG*'s profit $ALG(\hat{\sigma}) = f_i(m)$. Hence, $\frac{OPT(\hat{\sigma})}{ALG(\hat{\sigma})} \geq \frac{f_{i+1}(M)}{f_i(m)} \geq \alpha$.

Case 2.2. $\frac{f_{i+1}(M)}{f_i(m)} < \sqrt{\frac{f_2(M)}{f_i(m)}}$. In this case, $\sqrt{\frac{f_2(M)}{f_i(m)}} \geq \alpha$. For *OPT*, $OPT(\hat{\sigma}) \geq f_1(\hat{p}_1)$ while *ALG*'s profit $ALG(\hat{\sigma}) = f_i(m)$. Hence, $\frac{OPT(\hat{\sigma})}{ALG(\hat{\sigma})} \geq \frac{f_1(\hat{p}_1)}{f_i(m)} \geq \frac{f_2(M)}{\alpha f_i(m)} \geq \frac{\alpha^2}{\alpha} = \alpha$.

Case 3. $i = n$. By the definition of α, $\sqrt{\frac{f_2(M)}{f_n(m)}} \geq \alpha$. *OPT*'s profit satisfies $OPT(\hat{\sigma}) \geq f_1(\hat{p}_1)$ while *ALG*'s profit $ALG(\hat{\sigma}) = f_n(m)$. Hence, $\frac{OPT(\hat{\sigma})}{ALG(\hat{\sigma})} \geq \frac{f_1(\hat{p}_1)}{f_n(m)} \geq \frac{f_2(M)}{\alpha f_n(m)} \geq \frac{\alpha^2}{\alpha} = \alpha$.

According to the above discussion, *ALG* cannot have a competitive ratio less than α. The theorem follows. □

4. Online time series search problem with unknown duration

4.1. The online algorithm

Remember that in the description of algorithm *AKD* in Section 3.1, we calculate the unique value of α, which is related to the given duration. In this model with unknown duration, the online algorithm will similarly calculate α_l for each supposed duration l $(2 \leq l \leq n)$, and find the largest α_l. According to the largest α_l together with the corresponding l, the online algorithm will then calculate each p_i^* as *AKD* does. Before describing the algorithm, we give some preliminary definitions. For every natural number l $(2 \leq l \leq n)$, let

$$\alpha_l = \min\left\{\left\{\max\left\{\frac{f_{i+1}(M)}{f_i(m)}, \sqrt{\frac{f_2(M)}{f_i(m)}}\right\}, i = 1, 2, \ldots, l-1\right\}, \sqrt{\frac{f_2(M)}{f_l(m)}}\right\} \tag{2}$$

Note that $\alpha_l \geq 1$ since $f_{i+1}(M) > f_i(m)$ and $f_2(M) \geq f_l(m)$. Let $\bar{L} = \max\{L | L = \arg\max_{2 \leq l \leq n} \alpha_l\}$. Obviously, $\alpha_{\bar{L}} \geq \alpha_l$ for every l $(2 \leq l \leq n)$. By the definition of $\alpha_{\bar{L}}$, there exists a natural number s such that either $\alpha_{\bar{L}} = \frac{f_{s+1}(M)}{f_s(m)}$ for $s \leq \bar{L} - 1$ or

$\alpha_{\bar{L}} = \sqrt{\frac{f_2(M)}{f_s(m)}}$ for $s \le \bar{L}$. Ties are broken by selecting the smallest s. If $\alpha_{\bar{L}} = \frac{f_{s+1}(M)}{f_s(m)}$, let $\bar{p}_i^*$ $(1 \le i \le s)$ either be the solution of equation $\alpha_{\bar{L}} f_i(\bar{p}_i^*) = f_{i+1}(M)$ or $\bar{p}_i^* = m$ in the case that there is no solution for the equation. Ties are broken by selecting the $\bar{p}_i^*$ with the smallest value. Otherwise if $\alpha_{\bar{L}} = \sqrt{\frac{f_2(M)}{f_s(m)}}$, then let $\bar{i}^* = \max\{i | f_{i+1}(M) \ge \sqrt{f_2(M) f_s(m)}\}$. Let $\bar{p}_i^* = m$ for $\min\{\bar{i}^*, s-1\} < i \le s$, and for $1 \le i \le \min\{\bar{i}^*, s-1\}$, $\bar{p}_i^*$ either be the solution of equation $\alpha_{\bar{L}} f_i(\bar{p}_i^*) = f_{i+1}(M)$ or $\bar{p}_i^* = m$ in the case that there is no solution for the equation. For $i = 1$, $f_2(M) > f_1(m)$ and then $\max\{\frac{f_2(M)}{f_1(m)}, \sqrt{\frac{f_2(M)}{f_1(m)}}\} = \frac{f_2(M)}{f_1(m)}$. Combining formula (2) and the definition of $\bar{L}$, $\alpha_{\bar{L}} \le \frac{f_2(M)}{f_1(m)}$ and thus $f_1(m) \le \frac{f_2(M)}{\alpha_{\bar{L}}} \le f_2(M) \le f_1(M)$, implying that $\bar{p}_1^*$ is defined by the solution of equation $\alpha_{\bar{L}} f_1(\bar{p}_1^*) = f_2(M)$. Moreover, according to the above discussion, $\bar{p}_s^* = m$.

Algorithm *AUD* **(Algorithm with Unknown Duration):**

Step 1. Let $i = 1$.

Step 2. At period i, if $p_i \ge \bar{p}_i^*$ or the duration is exactly i then accept p_i and the game ends, otherwise if $p_i < \bar{p}_i^*$, go to Step 3.

Step 3. $i = i + 1$ and go to Step 2.

Note that *AUD* will accept a price on or before period s since $\bar{p}_s^* = m$.

4.2. Competitive analysis

Let $\underline{L} = \min\{L | L = \arg\max_{2 \le l \le n} \alpha_l\}$. Obviously, $\alpha_{\bar{L}} = \alpha_{\underline{L}}$. In the following we will give several lemmas.

Lemma 2. *If AUD accepts p_i at some period i $(1 \le i \le s)$, then $m < \bar{p}_j^* \le M$ for $1 \le j \le i - 1$.*

The proof of Lemma 2 is the same as that of Lemma 1. Lemma 2 implies that $\bar{p}_j^*$ shall be the solution of equation $\alpha_{\bar{L}} f_j(\bar{p}_j^*) = f_{j+1}(M)$ for $1 \le j \le i - 1$.

Lemma 3. *For each natural number $l < \underline{L}$, $\alpha_l = \sqrt{\frac{f_2(M)}{f_l(m)}} < \alpha_{\underline{L}}$.*

Proof. By the definition of α_l, there exists a natural number $i < l$ such that either $\alpha_l = \max\{\frac{f_{i+1}(M)}{f_i(m)}, \sqrt{\frac{f_2(M)}{f_i(m)}}\}$ or $\alpha_l = \sqrt{\frac{f_2(M)}{f_l(m)}}$. If $\alpha_l = \max\{\frac{f_{i+1}(M)}{f_i(m)}, \sqrt{\frac{f_2(M)}{f_i(m)}}\}$ together with $i < l < \underline{L}$, $\alpha_{\underline{L}} \le \max\{\frac{f_{i+1}(M)}{f_i(m)}, \sqrt{\frac{f_2(M)}{f_i(m)}}\} = \alpha_l$, contradicting to the definition of $\underline{L}$. So, $\alpha_l = \sqrt{\frac{f_2(M)}{f_l(m)}}$, and by the definition of $\underline{L}$, $\alpha_l < \alpha_{\underline{L}}$. The lemma follows. □

Lemma 4. *For each natural number $i < s$, $\sqrt{\frac{f_2(M)}{f_i(m)}} < \alpha_{\underline{L}}$.*

Proof. By the definition of $\alpha_{\underline{L}}$, there exists a natural number t such that either $\alpha_{\underline{L}} = \frac{f_{t+1}(M)}{f_t(m)}$ for $t \le \underline{L} - 1$ or $\alpha_{\underline{L}} = \sqrt{\frac{f_2(M)}{f_t(m)}}$ for $t \le \underline{L}$. Ties are broken by selecting the smallest t. Combining the definitions of t and s and equation $\alpha_{\bar{L}} = \alpha_{\underline{L}}$, we have $t = s$ and then $i < s = t \le \underline{L}$. By Lemma 3, $\sqrt{\frac{f_2(M)}{f_i(m)}} = \alpha_i < \alpha_{\underline{L}}$ follows. □

Theorem 3. *AUD has competitive ratio of $\alpha_{\bar{L}}$ for the online time series search problem with unknown duration.*

Proof. Let ε denote an arbitrarily small positive real number. Assume that *AUD* accepts p_i at some period i. We will discuss two cases according to different conditions for *AUD* to accept p_i.

Case 1. *AUD* accepts p_i due to $p_i \ge \bar{p}_i^*$. We will discuss two sub-cases according to different values of $\alpha_{\bar{L}}$.

Case 1.1. $\alpha_{\bar{L}} = \frac{f_{s+1}(M)}{f_s(m)}$. The discussion of this case is the same as that of Case 1 in Theorem 1, replacing p_k^* by $\bar{p}_k^*$ $(k = i, j)$.

Case 1.2. $\alpha_{\bar{L}} = \sqrt{\frac{f_2(M)}{f_s(m)}}$. By the definition of $\bar{p}_i^*$ in the case, $\bar{p}^*_{\min\{\bar{i}^*, s-1\}+1} = m$. So, *AUD* will accept a price on or before period $\min\{\bar{i}^*, s-1\} + 1$. Assume without loss of generality that it accepts p_i at period i $(1 \le i \le \min\{\bar{i}^*, s-1\} + 1)$. We further discuss two sub-cases according to different values of i.

Case 1.2.1. $1 \le i \le \min\{\bar{i}^*, s-1\}$. The discussion of this case is the same as that of Case 2.1 in Theorem 1, replacing p_k^* by $\bar{p}_k^*$ $(k = i, j)$, and replacing i^* and r by $\bar{i}^*$ and s respectively.

Case 1.2.2. $i = \min\{\bar{i}^*, s-1\} + 1$. There are four sub-cases in this case. For the cases where $i = \bar{i}^* + 1$ and where $i = s = n$, the discussions are the same as those of Case 2.2.1 and Case 2.2.3 respectively in Theorem 1. For the case where $i = s \le n - 1$ and $s \ne \bar{L}$, the discussion is the same as that of Case 2.2.2 in Theorem 1. So we will focus on the fourth case where $i = s = \bar{L} \le n - 1$. By case condition, $\alpha_{\bar{L}} = \sqrt{\frac{f_2(M)}{f_{\bar{L}}(m)}}$. By Lemma 2, $m < \bar{p}_j^* - \varepsilon < M$ for $j = 1, 2, \ldots, i - 1$. The worst price sequence to *AUD* is $\sigma_1 = (\bar{p}_1^* - \varepsilon, \bar{p}_2^* - \varepsilon, \ldots, \bar{p}_{\bar{L}-1}^* - \varepsilon, m, M, \ldots)$. The profit of *AUD* is $AUD(\sigma_1) = f_{\bar{L}}(m)$. For *OPT*, combining $f_j(\bar{p}_j^*) = \frac{f_{j+1}(M)}{\alpha_{\bar{L}}}$ $(1 \le j \le \bar{L})$ and assumption (3), $f_1(\bar{p}_1^*) \ge \cdots \ge f_{\bar{L}-1}(\bar{p}_{\bar{L}-1}^*) \ge f_{\bar{L}}(m)$ implying $OPT(\sigma_1) \approx \max\{f_1(\bar{p}_1^*), f_{\bar{L}+1}(M)\}$. We claim by

$s = \bar{L} \leq n-1$ that $\frac{f_{\bar{L}+1}(M)}{f_{\bar{L}}(m)} < \sqrt{\frac{f_2(M)}{f_{\bar{L}}(m)}} = \alpha_{\bar{L}}$ since otherwise by the definitions of $\alpha_{\bar{L}}$ and $\alpha_{\bar{L}+1}$, $\alpha_{\bar{L}+1} \geq \alpha_{\bar{L}}$ which contradicts to the definition of $\bar{L}$. Moreover, $\frac{f_1(\bar{p}_1^*)}{f_{\bar{L}}(m)} = \frac{f_2(M)}{\alpha_{\bar{L}} f_{\bar{L}}(m)} = \frac{\alpha_{\bar{L}}^2}{\alpha_{\bar{L}}} = \alpha_{\bar{L}}$. Hence, $\frac{OPT(\sigma_1)}{AUD(\sigma_1)} \approx \frac{\max\{f_1(\bar{p}_1^*), f_{\bar{L}+1}(M)\}}{f_{\bar{L}}(m)} \leq \alpha_{\bar{L}}$.

Case 2. *AUD* accepts p_i due to the duration is met in period i ($i < s$). By Lemma 2, $m < \bar{p}_j^* - \varepsilon < M$ for $j = 1, 2, \ldots, i-1$. The worst price sequence to *AUD* is $\sigma_2 = (\bar{p}_1^* - \varepsilon, \bar{p}_2^* - \varepsilon, \ldots, \bar{p}_{i-1}^* - \varepsilon, m)$. In this case, $AUD(\sigma_2) = f_i(m)$. For *OPT*, as $\varepsilon \to 0$, $OPT(\sigma_2) \approx f_1(\bar{p}_1^*)$ due to $f_1(\bar{p}_1^*) \geq \cdots \geq f_{i-1}(\bar{p}_{i-1}^*) \geq f_i(m)$. So, $\frac{OPT(\sigma_2)}{AUD(\sigma_2)} \approx \frac{f_1(\bar{p}_1^*)}{f_i(m)} = \frac{f_2(M)}{\alpha_{\bar{L}} f_i(m)}$. Combining $i < s$ and Lemma 4, $\frac{f_2(M)}{f_i(m)} < \alpha_{\underline{L}}^2$. Hence, $\frac{OPT(\sigma_2)}{AUD(\sigma_2)} < \frac{\alpha_{\underline{L}}^2}{\alpha_{\bar{L}}} = \frac{\alpha_{\bar{L}}^2}{\alpha_{\bar{L}}} = \alpha_{\bar{L}}$. The theorem follows. □

In the following we will prove that no deterministic algorithm can do better than *AUD* in competitiveness for variant 2 of the problem.

Theorem 4. *For the online time series search problem with unknown duration, no deterministic algorithm has a competitive ratio less than* $\alpha_{\bar{L}}$.

Proof. We construct a price sequence $\hat{\sigma} = (\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_{\bar{L}})$, that is, $\bar{L} = n$, and it is sufficient to prove that an arbitrary online algorithm *ALG* cannot have a profit larger than $1/\alpha_{\bar{L}}$ times of *OPT*'s in $\hat{\sigma}$. The rest reasoning is the same as that in the proof of Theorem 2. □

Remark. Note that in formulas (1) and (2), if the profit function satisfies $f_i(p_i) = p_i$ for $1 \leq i \leq n$, then $\alpha = \alpha_l = \sqrt{\frac{M}{m}}$ and thus algorithms *AKD* and *AUD* have optimal competitive ratio the same as that in El-Yaniv et al. [1].

5. Conclusion

In this paper, we extended the original online time series search problem by introducing profit function. We investigate two cases where the player knows the duration of price series and where he has no knowledge of the duration beforehand. We propose two algorithms *AKD* and *AUD*, and prove that they are optimal in the two cases respectively. The problem with profit function is a generalization of that in El-Yaniv et al. [1]. For the problem with different profit functions, it is an interesting work to design randomized algorithms to break the lower bounds of competitive ratio.

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