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Existence of Solutions for Vector Optimization

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Abstract—In this paper, we prove the existence of a weak minimum for constrained vector optimization problem by making use of vector variational-like inequality and preinvex functions.

Keywords—Vector optimization problem, Vector variational-like inequality problem, Preinvex function, Weak minimum, KKM-Fan theorem.

1. INTRODUCTION

We denote the norm on $\mathbb{R}^m$ by $\|\cdot\|$. $(\mathbb{R}^m, \mathbb{R}^m_+)$ is an ordered Hilbert space with an ordering in $\mathbb{R}^m$ defined by the convex cone $\mathbb{R}^m_+$,

$$\forall y,\, x \in \mathbb{R}^m, \qquad y \leq x \Leftrightarrow x - y \in \mathbb{R}^m_+.$$

If $\operatorname{int} \mathbb{R}^m_+$ denotes the interior of $\mathbb{R}^m_+$, then a weak ordering in $\mathbb{R}^m$ is also defined by

$$\forall y,\, x \in \mathbb{R}^m, \qquad y \not< x \Leftrightarrow x - y \notin \operatorname{int} \mathbb{R}^m_+.$$

A subset $K \subset \mathbb{R}^n$ is said to have "η-connectedness" property if, for each $x,\, y \in K$, $\lambda \in [0,\, 1]$, there exists a vector $\eta(x,\, y) \in \mathbb{R}^n$, such that $y + \lambda\eta(x,\, y) \in K$.

Consider a vector optimization problem

$$w\text{-}\min f(x) \text{ subject to } x \in K, \tag{1.1}$$

where $K \subset \mathbb{R}^n$ is nonempty "η-connected" set, $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a vector-valued function, and w-min denotes weak minimum. We note that the problem (1.1) has a weak minimum at $x = x_0 \in K$ if and only if

$$f(x) - f(x_0) \notin -\operatorname{int} \mathbb{R}^m_+, \qquad \forall x \in K,$$

see, for example, [1,2].

Following [3], a function $f : K \longrightarrow \mathbb{R}^m$ is called $\mathbb{R}^m_+$-invex, with respect ot a function $\eta : K \times K \longrightarrow \mathbb{R}^m$, if, for each $x,\ y \in K$

$$f(x) - f(y) - \langle f'(y),\ \eta(x,\, y)\rangle \in \mathbb{R}^m_+, \tag{1.2}$$

where $f'(y)$ denotes the Fréchet derivative of f at y. If we take $\mathbb{R}$ and $\mathbb{R}_+$ in place of $\mathbb{R}^m$ and $\mathbb{R}^m_+$, respectively, then f is called invex. Invex functions were first considered by Hanson [4],

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX who showed that if, instead of the usual convexity conditions, the objective function and each of the constraints of a nonlinear program are all invex for the same $\eta(x, y)$ then the sufficiency of the Kuhn-Tucker conditions [5], and weak [6] duality still holds. Morever, Craven and Glover [7] showed that the class of real-valued invex functions is equivalent to the class of functions whose stationary points are global minima.

Following Ben-Israel and Mond [8] and Hanson and Mond [9], consider a function $f : K \longrightarrow \mathbb{R}^m$ having the property that there exists a function $\eta : K \times K \longrightarrow \mathbb{R}^m$ such that, for each $x,\ y \in K$ and $\lambda \in [0,1]$, $y + \lambda\eta(x, y) \in K$ and

$$\lambda f(x) + (1-\lambda) f(y) - f(y + \lambda\eta(x, y)) \in \mathbb{R}^m_+. \tag{1.3}$$

It is observed that if f is Fréchet differentiable and satisfies (1.3) that f also satisfies (1.2). This can be seen by rewriting (1.3) as

$$\lambda(f(x) - f(y)) - [f(y + \lambda\eta(x,y)) - f(x)] \in \mathbb{R}^m_+,$$

and then divided by $\lambda > 0$ and taking the limit as $\lambda \longrightarrow 0_+$ gives

$$f(x) - f(y) - \langle f'(y), \eta(x,y)\rangle \in \mathbb{R}^m_+.$$

In view of this observation, the functions satisfying (1.3) will be called $\mathbb{R}^m_+$-preinvex. It is also noted that the set K should have the "η-connectedness" property. Note also that if $\eta(x, y) \equiv \alpha(x, y)(x - y)$ where $0 < \alpha(x, y) < 1$, then K should be star-shaped [10]. If we take $\mathbb{R}$ and $\mathbb{R}_+$ in place of $\mathbb{R}^m$ and $\mathbb{R}^m_+$, respectively, and if f satisfies (1.3), then f is called preinvex.

In many papers, see, for example, [11–14] dealing with the existence of optimal solutions for vector optimization, some kind of compactness in the value space of the objective functions is assumed. This is often difficult to verify in applications.

In this paper, a sufficient condition is given for the existence of optimal solutions for problem (1.1) by making use of vector variational-like inequality and preinvex functions.

2. EXISTENCE OF SOLUTIONS

First we establish the equivalence relation between the vector optimization problem (1.1) and the vector variational-like inequality problem of finding $x_0 \in K$ such that

$$\langle f'(x_0), \eta(x, x_0)\rangle \notin -\operatorname{int} \mathbb{R}^m_+, \qquad \forall x \in K. \tag{2.1}$$

We need the following lemma [15].

LEMMA 2.1. *Let (X, P) be an ordered topological vector space with a closed, pointed and convex cone P with $\operatorname{int} P \neq \emptyset$. Then, $\forall x, y \in X$, we have*

(i) *$y - x \in \operatorname{int} P$ and $y \notin \operatorname{int} P$ imply $x \notin \operatorname{int} P$;*
(ii) *$y - x \in P$, and $y \notin \operatorname{int} P$ imply $x \notin \operatorname{int} P$;*
(iii) *$y - x \in -\operatorname{int} P$ and $y \notin -\operatorname{int} P$ imply $x \notin -\operatorname{int} P$; and*
(iv) *$y - x \in -P$ and $y \notin -\operatorname{int} P$ imply $x \notin -\operatorname{int} P$.*

Let $W = \mathbb{R}^m \setminus (-\operatorname{int} \mathbb{R}^m_+)$.

THEOREM 2.1. *Let the set K be satisfy the η-connectedness property, and let the function f be $\mathbb{R}^m_+$-preinvex and Fréchet differentiable. Then the vector optimization problem (1.1) and the vector variational-like inequality problem (2.1) have the same solution set.*

PROOF. Let x_0 be a weak minimum of problem (1.1). If $x \in K$ and $0 < \alpha \leq 1$, then $x_0 + \alpha\eta(x, x_0) \in K$ since K have the η-connectedness property. Hence,

$$\alpha^{-1}\left[f\left(x_0 + \alpha\eta\left(x, x_0\right)\right) - f\left(x_0\right)\right] \in W, \qquad \forall \alpha \in (0,1].$$

Since W is closed and f is Fréchet differentiable, it follows that

$$\langle f'(x_0), \eta(x, x_0)\rangle \notin -\operatorname{int} \mathbb{R}^m_+.$$

Conversely, let $x_0 \in K$ satisfy (2.1) Since f is $\mathbb{R}^m_+$-preinvex and K has the η-connectedness property, then it follows that

$$f(x) - f(x_0) - \langle f'(x_0), \eta(x, x_0)\rangle \in \mathbb{R}^m_+.$$

Hence,

$$f(x) - f(x_0) \notin -\operatorname{int} \mathbb{R}^m_+ \qquad \text{(by (iv) of Lemma 2.1).}$$

This completes the proof.

Now we are able to prove the main result of this paper, but before stating this, we quote the following theorem (KKM-Fan Theorem [16]) which play an important role in the proof of our main result.

THEOREM 2.2. *Let E be a subset of the topological vector space X. For each $x \in E$, let a closed set $F(x)$ in X be given such that $F(x)$ is compact for at least one $x \in E$. If the convex hull of every finite subset $\{x_1, x_2, \dots, x_n\}$ of E is contained in the corresponding union $\cup_{i=1}^n F(x_i)$, then $\cap_{x\in E} F(x) \neq \emptyset$.*

THEOREM 2.3. *Let K be a nonempty compact convex set in $\mathbb{R}^n$ and satisfy the η-connectedness property; let $f : K \longrightarrow \mathbb{R}^m$ be Fréchet differentiable and $\mathbb{R}^m_+$-preinvex; let the set*

$$\{y \in K : \langle f'(x), \eta(y, x)\rangle \in -\operatorname{int} \mathbb{R}^m_+\}$$

be convex for each fixed $x \in K$; let η be continuous and satisfy $\eta(x, x) = 0$ for each $x \in K$. Then the vector optimization problem (1.1) has a weak minimum x_0.

PROOF. By Theorem 2.1, it is sufficient to show that the vector variational-like inequality problem (2.1) has a solution x_0. For $y \in K$, define

$$F(y) = \{x \in K : \langle f'(x), \eta(y, x)\rangle \notin -\operatorname{int} \mathbb{R}^m_+\}.$$

Let $\{x_1, x_2, \dots, x_m\} \subset K$. Claim that the convex hull of every finite subset $\{x_1, x_2, \dots, x_m\}$ is contained in the corresponding union $\cup_{i=1}^m F(x_i)$, i.e., $\operatorname{conv}\{x_1, x_2, \dots, x_m\} \subset \cup_{i=1}^m F(x_i)$. Indeed let $\alpha_i \geq 0$, $1 \leq i \leq m$, with $\sum_{i=1}^m \alpha_i = 1$. Suppose that $x = \sum_{i=1}^m \alpha_i x_i \notin \cup_{i=1}^m F(x_i)$, then

$$\langle f'(x), \eta(x_i, x)\rangle \in -\operatorname{int} \mathbb{R}^m_+, \qquad \forall i.$$

Since, for each fixed $x \in K$, the set

$$G(x) = \{y \in K : \langle f'(x), \eta(y, x)\rangle \in -\operatorname{int} \mathbb{R}^m_+\}$$

is convex. Hence, by convex property of $G(x)$, we have

$$\left\langle f'\left(\sum_{i=1}^m \alpha_i x_i\right), \eta\left(\sum_{i=1}^m \alpha_i x_i, \sum_{i=1}^m \alpha_i x_i\right)\right\rangle \in -\operatorname{int} \mathbb{R}^m_+,$$

which is a contradiction and our claim is then verified.

Next, we claim that for each $y \in K$, $F(y)$ is closed. Indeed, let $y \in K$ and let a sequence $\{x_k\} \subset F(y)$ satisfy $\|x_k - x\| \longrightarrow 0$.

Since $f'(.)$ is continuous on K, $\{f'(x_k)\} \longrightarrow f'(x)$ uniformly on K. Then

$$\begin{aligned}\|\langle f'(x_k), \eta(y,x_k)\rangle - \langle f'(x), \eta(y,x)\rangle\| &\\ \leq \|\langle f'(x_k), \eta(y,x_k)\rangle - \langle f'(x), \eta(y,x_k)\rangle\| &\\ + \|\langle f'(x), \eta(y,x_k)\rangle - \langle f'(x), \eta(y,x)\rangle\| &\\ \leq \|f'(x_k) - f'(x)\| \, \|\eta(y,x_k)\| + \|f'(x)\| \, \|\eta(y,x_k) - \eta(y,x)\| &\\ \longrightarrow 0 &\end{aligned}$$

as $n \longrightarrow \infty$, since η is continuous on K.

Since W is closed and $\langle f'(x_k), \eta(y,x_k)\rangle \in W$, $\forall k$,

$$\langle f'(x), \eta(y,x)\rangle \in W,$$

that is

$$\langle f'(x), \eta(y,x)\rangle \notin -\operatorname{int} \mathbb{R}^m_+,$$

and our claim is verified. Also $F(y)$ is nonempty since $y \in F(y)$, for all $y \in K$. Since $F(y)$ is closed, and hence, compact for each $y \in K$. Thus, by Theorem 2.2, it follows that there exists at least one point $x_0 \in \cap_{y\in K} F(y)$, that is $x_0 \in K$ such that

$$\langle f'(x_0), \eta(y,x_0)\rangle \notin -\operatorname{int} \mathbb{R}^m_+, \qquad \forall y \in K.$$

This completes the proof.

It is remarked that the results of this paper generalize some results of Chen and Craven [17].

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