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About the geometry of almost para-quaternionic manifolds

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ABSTRACT

We provide a general criterion for the integrability of the almost para-quaternionic structure of an almost para-quaternionic manifold $(M, \mathcal{P})$ of dimension $4m \geqslant 8$ in terms of the integrability of two or three sections of the defining rank three vector bundle $\mathcal{P}$. We relate it with the integrability of the canonical almost complex structure of the twistor space and with the integrability of the canonical almost para-complex structure of the reflector space of $(M, \mathcal{P})$. We deduce that $(M, \mathcal{P})$ has plenty of locally defined, compatible, complex and para-complex structures, provided that $\mathcal{P}$ is integrable.

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1. Introduction and main results

An almost para-complex structure on a smooth manifold M is an endomorphism J of TM such that $J^2 = \mathrm{Id}$ (where "Id" denotes the identity endomorphism) and the two distributions $\mathrm{Ker}(J \pm \mathrm{Id})$ have the same rank. We say that J is integrable (or is a para-complex structure) if its Nijenhuis tensor

$$N_J(X, Y) := [X, Y] + [JX, JY] - J\big([JX, Y] + [X, JY]\big), \quad \forall X, Y \in \mathcal{X}(M)$$

is zero.

An almost para-quaternionic structure on a smooth manifold M of dimension $4m \geqslant 8$ is a rank three sub-bundle $\mathcal{P} \subset \mathrm{End}(TM)$ locally spanned by almost para-hypercomplex structures, i.e. by triples $\{J_1, J_2, J_3\}$ where J_1 is an almost complex structure, J_2 and J_3 are anti-commuting almost para-complex structures and $J_3 = J_1 J_2$. We shall often refer to such a triple as an admissible basis of $\mathcal{P}$. The bundle $\mathcal{P}$ comes with a standard Lorenzian metric $\langle \cdot, \cdot \rangle$ defined in terms of an admissible basis by

$$\left\langle \sum_{i=1}^{3} a_i J_i, \sum_{j=1}^{3} b_j J_j \right\rangle := -a_1 b_1 + a_2 b_2 + a_3 b_3.$$

A para-quaternionic connection on $(M, \mathcal{P})$ is a linear connection on M which preserves the bundle $\mathcal{P}$. We say that $\mathcal{P}$ is integrable, or is a para-quaternionic structure, if $(M, \mathcal{P})$ has a torsion-free para-quaternionic connection; equivalently, if the torsion tensor of $\mathcal{P}$ is zero.

We begin the paper by recalling briefly, in Section 2, the theory of G-structures. We then apply these considerations to almost para-quaternionic manifolds. A central role in our paper is played by the torsion tensor of an almost para-

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quaternionic manifold $(M,\mathcal{P})$. In Section 3 we give an account on the torsion tensor of $(M,\mathcal{P})$, providing more insight of some of the results developed in Section 2 of [13]. We define a canonical family of para-quaternionic connections (also called "minimal para-quaternionic connections") on $(M,\mathcal{P})$, which consists of all para-quaternionic connections whose torsion is equal to the torsion tensor of $\mathcal{P}$. These connections are parametrized by 1-forms and are similar to the so called "Oproiu connections" of almost quaternionic manifolds, which were introduced for the first time in [10] and have been used in [2] to define a canonical almost complex structure on the twistor space of an almost quaternionic manifold.

In Section 4 we prove that if an almost para-quaternionic manifold $(M,\mathcal{P})$ admits two independent, compatible, globally defined, (para)-complex structures I_1 and I_2, such that either I_1 or I_2 is a complex structure, or otherwise both I_1 and I_2 are para-complex structures and, for any $p\in M$, $\mathrm{Span}\{I_1(p),I_2(p)\}$ is a non-degenerate 2-plane of $\mathcal{P}_p$ (with its Lorenzian metric $\langle\cdot,\cdot\rangle$), then $(M,\mathcal{P})$ is a para-quaternionic manifold (see Theorem 11). (An almost (para)-complex structure on $(M,\mathcal{P})$ is compatible with $\mathcal{P}$ if it is a section of the bundle $\mathcal{P}$; two almost (para)-complex structures I_i and I_j are independent if $I_i(p)\neq\pm I_j(p)$ at any point p.) If, however, I_1 and I_2 are both para-complex and, for any $p\in M$, the 2-plane $\mathrm{Span}\{I_1(p),I_2(p)\}$ is degenerate, then we need an additional compatible para-complex structure I_3, such that $\{I_1,I_2,I_3\}$ are pairwise independent but dependent at any point, to conclude that $\mathcal{P}$ is para-quaternionic (see Theorem 11). At the end of Section 4 we construct a class of almost para-quaternionic manifolds $(M,\mathcal{P})$ which are not para-quaternionic but admit three independent, globally defined, compatible para-complex structures I_1, I_2 and I_3, such that, at any point $p\in M$ and for any $i\neq j$, the 2-plane $\mathrm{Span}\{I_i(p),I_j(p)\}$ is degenerate (see Proposition 18). Recall that for almost quaternionic manifolds the existence of two independent, globally defined, compatible, complex structures insures the integrability of the almost quaternionic structure (see Theorem 2.4 of [2]). For conformal oriented 4-manifolds, the existence of three pairwise independent, globally defined, orthogonal complex structures is needed to deduce that the conformal structure is self-dual (see [12, p. 121]).

In Section 5 we consider the twistor space Z^- and the reflector space Z^+ of $(M,\mathcal{P})$, consisting of all compatible, complex, respectively para-complex structures of tangent spaces of M, i.e.

$$Z^{\pm}=\left\{A\in\mathcal{P}:\ A^2=\pm\mathrm{Id}\right\}\subset\mathrm{End}(TM). \tag{1}$$

It is known that a para-quaternionic connection ∇ on $(M,\mathcal{P})$ induces an almost complex structure $\mathcal{J}^{\nabla,-}$ (respectively, an almost para-complex structure $\mathcal{J}^{\nabla,+}$) on the twistor space Z^- (respectively, on the reflector space Z^+) and the way $\mathcal{J}^{\nabla,\pm}$ depend on ∇ has been studied in [8]. Our main observation in this setting is that $\mathcal{J}^{\nabla,\pm}$ are independent of the choice of ∇, provided that ∇ is minimal. Using minimal para-quaternionic connections we define an almost complex structure $\mathcal{J}^-$ on Z^- and an almost para-complex structure $\mathcal{J}^+$ on Z^+, both $\mathcal{J}^-$ and $\mathcal{J}^+$ being canonical (since they depend only on the torsion tensor of $\mathcal{P}$). We use $\mathcal{J}^{\pm}$ to prove that $(M,\mathcal{P})$ is integrable if and only if it has plenty of locally defined, compatible, complex and para-complex structures (see Theorem 21). Similar considerations hold for almost quaternionic manifolds, the role of minimal connections on $(M,\mathcal{P})$ being played by the Oproiu connections of an almost quaternionic manifold (see [2]). The geometry of twistor and reflector spaces of para-quaternionic manifolds with an additional compatible metric (the so called "para-quaternionic Hermitian" and "para-quaternionic Kähler manifolds") has already been studied in the literature, see for example, [4,6,7].

2. *G*-structures

In this section we recall the definition of the torsion tensor of a G-structure [3,9]. We follow closely the treatment developed in [5], Section 2.1.

Let G be a closed subgroup of the general linear group $GL_n(V)$, where $V=\mathbb{R}^n$. A G-structure on an n-dimensional manifold M is a principal G sub-bundle P of the frame bundle of M. A linear connection on M is adapted to the G-structure if it is induced by a G-invariant connection on P. Any two adapted connections ∇ and ∇' are related by $\nabla'=\nabla+\eta$, where $\eta\in\Omega^1(M,\mathrm{ad}(P))$ is a 1-form with values in $\mathrm{ad}(P)$, the vector bundle on M associated to the adjoint representation of G on its Lie algebra. Define the linear torsion map

$$\delta:\Omega^1\big(M,\mathrm{ad}(P)\big)\to\Omega^2(M,TM),\quad (\delta\eta)(X,Y):=\eta(X,Y)-\eta(Y,X), \tag{2}$$

where $X,Y\in TM$. The image of the torsion $T^\nabla\in\Omega^2(M,TM)$ of an adapted connection ∇ into the quotient space $\frac{\Omega^2(M,TM)}{\mathrm{Im}\,\delta}$ is independent of the choice of ∇ and is called the torsion tensor of the G-structure P. It will be denoted by T^P.

Suppose now that it is given a complement $C(\mathrm{ad}(P))$ of $\delta\Omega^1(M,\mathrm{ad}(P))$ in $\Omega^2(M,TM)$, i.e. a decomposition

$$\Omega^2(M,TM)=\delta\Omega^1\big(M,\mathrm{ad}(P)\big)\oplus C\big(\mathrm{ad}(P)\big). \tag{3}$$

The decomposition (3) identifies the quotient $\frac{\Omega^2(M,TM)}{\mathrm{Im}\,\delta}$ with $C(\mathrm{ad}(P))$. An adapted connection with torsion equal to $T^P\in C(\mathrm{ad}(P))$ is called minimal. Any two minimal connections ∇ and ∇' are related by $\nabla'=\nabla+\eta$, where $\eta\in\Omega^1(M,\mathrm{ad}(P))$ belongs to the kernel of δ.

3. Almost para-quaternionic manifolds

3.1. Torsion of almost para-quaternionic manifolds

Let $(M, \mathcal{P})$ be an almost para-quaternionic manifold of dimension $n = 4m \geqslant 8$ (in this paper we will always assume that the dimension of the almost para-quaternionic manifolds is bigger or equal to eight). The almost para-quaternionic structure $\mathcal{P}$ defines a $G = GL_m(\mathbb{H}^+)Sp(1,\mathbb{R})$ structure on M, where $Sp(1,\mathbb{R})$ is the group of unit para-quaternions acting on $\mathbb{R}^n$ and $GL_m(\mathbb{H}^+)$ is the group of automorphisms which commutes with the action of $Sp(1,\mathbb{R})$ (for details see, for example, [8]). We denote by $Z(\mathcal{P})$ and $N(\mathcal{P}) = Z(\mathcal{P}) \oplus \mathcal{P}$ the centralizer, respectively the normalizer of $\mathcal{P}$ in $\mathrm{End}(TM)$. They are vector bundles on M associated to the adjoint representations of $GL_m(\mathbb{H}^+)$ and G on their Lie algebras.

The aim of this section is to show that $\delta\Omega^1(M, Z(\mathcal{P}))$ and $\delta\Omega^1(M, N(\mathcal{P}))$ have canonical complements in $\Omega^2(M, TM)$, where δ is the linear torsion map. We then relate the torsion tensor $T^{\mathcal{P}}$ of $\mathcal{P}$ with the torsion tensor T^H of any compatible almost para-hypercomplex structure H and we determine conditions on T^H which insure the integrability of $\mathcal{P}$. We shall need these considerations (especially Corollary 9) in the proof of Theorem 11. Our arguments are similar to those employed in [5] and [2]. This section is intended mostly for completeness of the text: except the different treatment, some results of this section were already proved in [13].

Notations 1. To unify notations, we define an (almost) ϵ-complex structure on M (with $\epsilon = \pm 1$) to be an (almost) complex structure when $\epsilon = -1$ and an (almost) para-complex structure when $\epsilon = 1$. In particular, the Nijenhuis tensor of an almost ϵ-complex structure J is

$$N_J(X,Y) = \epsilon[X,Y] + [JX, JY] - J\big([JX,Y] + [X, JY]\big), \quad \forall X, Y \in \mathcal{X}(M).$$

For an admissible basis $\{J_1, J_2, J_3\}$ of $\mathcal{P}$, we define $\epsilon_i \in \{-1, +1\}$ by the conditions $J_i^2 = \epsilon_i \mathrm{Id}$, for any $i \in \{1,2,3\}$; hence $\epsilon_1 = -1$ and $\epsilon_2 = \epsilon_3 = 1$.

Notations 2. An operator, expression, etc., defined in terms of an admissible basis of $\mathcal{P}$ but independent of the choice of admissible basis will be considered, without further explanation, defined on M.

In the next lemma we show that $\delta\Omega^1(M, Z(\mathcal{P}))$ has a canonical complement in $\Omega^2(M, TM)$.

Lemma 3. *Let $\{J_1, J_2, J_3\}$ be an admissible basis of $\mathcal{P}$. Define an endomorphism*

$$P : \Omega^2(M,TM) \to \Omega^2(M,TM), \quad P(T) := \frac{2}{3}\sum_{i=1}^{3} \Pi^{0,2}_{J_i}(T), \tag{4}$$

where, for any $T \in \Omega^2(M,TM)$ and $X, Y \in TM$,

$$\Pi^{0,2}_{J_i}(T)(X,Y) := \frac{1}{4}\big\{T(X,Y) + \epsilon_i T(J_iX, J_iY) - \epsilon_i J_i\big(T(J_iX,Y) + T(X, J_iY)\big)\big\}.$$

Then P is independent of the choice of $\{J_1, J_2, J_3\}$, is a projector (i.e. $P^2 = P$) and $\mathrm{Ker}(P) = \delta\Omega^1(M, Z(\mathcal{P}))$. In particular,

$$\Omega^2(M,TM) = \delta\Omega^1\big(M, Z(\mathcal{P})\big) \oplus \mathrm{Im}(P). \tag{5}$$

Proof. Note that the expressions $\sum_{i=1}^{3} \epsilon_i T(J_iX, J_iY)$, $\sum_{i=1}^{3} \epsilon_i J_i T(J_iX, Y)$ and $\sum_{i=1}^{3} \epsilon_i J_i T(X, J_iY)$ are independent of the choice of admissible basis of $\mathcal{P}$. The same holds for P, which is a well-defined endomorphism of $\Omega^2(M,TM)$.

We now prove that P is a projector. We shall use the notation J_{ij} for the composition $J_i \circ J_j$ and ϵ_{ij} for $\epsilon_i\epsilon_j$. For any i, $\Pi^{0,2}_{J_i}$ is a projector of $\Omega^2(M,TM)$ and for any $i \neq j$,

$$\Pi^{0,2}_{J_i} \circ \Pi^{0,2}_{J_j} = \frac{1}{4}\big(\Pi^{0,2}_{J_i} + \Pi^{0,2}_{J_j} - \Pi^{0,2}_{J_{ij}}\big) + \frac{1}{16}\mathcal{E}_{ij},$$

where the endomorphism $\mathcal{E}_{ij}$ of $\Omega^2(M,TM)$ has the following expression: for any $T \in \Omega^2(M,TM)$ and $X, Y \in TM$,

$$\begin{aligned}\mathcal{E}_{ij}(T)(X,Y) = {} & \epsilon_{ij} J_j\big(T(J_{ij}X, J_iY) + T(J_iX, J_{ij}Y)\big)\\ & + \epsilon_{ij} J_i\big(T(J_{ij}X, J_jY) + T(J_jX, J_{ij}Y)\big)\\ & + \epsilon_{ij} J_{ij}\big(T(J_iX, J_jY) + T(J_jX, J_iY)\big).\end{aligned}$$

Since $\mathcal{E}_{ij}$ is anti-symmetric in i and j,

$$\Pi^{0,2}_{J_i} \circ \Pi^{0,2}_{J_j} + \Pi^{0,2}_{J_j} \circ \Pi^{0,2}_{J_i} = \frac{1}{2}\big(\Pi^{0,2}_{J_i} + \Pi^{0,2}_{J_j} - \Pi^{0,2}_{J_{ij}}\big), \tag{6}$$

for any $i \neq j$. Relation (6) implies that $P^2 = P$. We now prove that $\mathrm{Ker}(P) = \delta\Omega^1(M, Z(\mathcal{P}))$. It is easy to show, using definitions, that $\delta\Omega^1(M, Z(\mathcal{P}))$ is included in the kernel of P. Conversely, in order to show that $\mathrm{Ker}(P) \subset \delta\Omega^1(M, Z(\mathcal{P}))$ we define an endomorphism π of $\Omega^2(M, TM)$ by

$$\pi(T)(X,Y) := \frac{1}{4}T(X,Y) + \frac{1}{4}\sum_{i=1}^{3}\epsilon_i J_i T(X, J_iY) - \frac{1}{12}\sum_{i=1}^{3}\epsilon_i J_i T(J_iX, Y) - \frac{1}{12}\sum_{i,j=1}^{3}\epsilon_i\epsilon_j J_j J_i T(J_iX, J_jY),$$

where $T \in \Omega^2(M, TM)$ and $X, Y \in TM$. The endomorphism π is independent of the choice of admissible basis of $\mathcal{P}$ and its image is included in $\Omega^1(M, Z(\mathcal{P}))$. Moreover, it can be checked that

$$\delta \circ \pi(T) = T - P(T), \quad \forall T \in \Omega^2(M, TM). \tag{7}$$

Relation (7) implies the converse inclusion $\mathrm{Ker}(P) \subset \delta\Omega^1(M, Z(\mathcal{P}))$. We deduce that $\mathrm{Ker}(P) = \delta\Omega^1(M, Z(\mathcal{P}))$. Since $P^2 = P$, the decomposition (5) follows. □

Corollary 4. *Let $H = \{J_1, J_2, J_3\}$ be an admissible basis of $\mathcal{P}$ and ∇ a linear connection which preserves all J_i. Then*

$$T^H := P\big(T^\nabla\big) = -\frac{1}{6}\sum_{i=1}^{3}\epsilon_i N_{J_i} \tag{8}$$

is independent of the choice of ∇ and is the torsion tensor of the almost para-hypercomplex structure H.

Proof. If J is an almost ϵ-complex structure on a manifold M and ∇ is a linear connection on M which preserves J, then

$$\Pi_J^{0,2}\big(T^\nabla\big)(X,Y) = -\frac{\epsilon}{4}N_J(X,Y), \quad \forall X, Y \in TM. \tag{9}$$

The first claim follows from (9) and the definition of P. The second claim is trivial, from our considerations of Section 2 and from Lemma 3. □

Remark 5. The linear torsion map δ is injective for $GL_m(\mathbb{H}^+)$-structures. Given an almost para-hypercomplex structure H there is a unique linear connection, called the Obata connection, which preserves H and whose torsion is equal to T^H (see [13, Proposition 2.1]).

We need the following lemma for the proof of Proposition 7.

Lemma 6. *Let P be the projector defined in Lemma 3. For any admissible basis $\{J_1, J_2, J_3\}$ of $\mathcal{P}$ and $T \in \mathrm{Im}(P) \subset \Omega^2(M, TM)$,*

$$\sum_{i=1}^{3}\epsilon_i J_i T(J_iX, Y) = -T(X,Y), \quad \forall X, Y \in TM. \tag{10}$$

In particular,

$$\sum_{i=1}^{3}\epsilon_i \operatorname{tr}(J_i T_{J_iX}) = 0, \quad \forall X \in TM. \tag{11}$$

Above, $T_Y := T(Y, \cdot)$ denotes the interior product of a tangent vector $Y \in TM$ with the TM-valued 2-form T.

Proof. Relation (10) can be checked by writing $T = P(A)$, for $A \in \Omega^2(M, TM)$, and using the definition of P. Relation (11) follows from (10) and the observation that $\operatorname{tr}(T_X) = 0$ for any $T \in \mathrm{Im}(P)$ and $X \in TM$. □

We now state the main result of this section.

Proposition 7. *Let $\{J_1, J_2, J_3\}$ be an admissible basis of $\mathcal{P}$. The subspace*

$$C\big(N(\mathcal{P})\big) := \big\{T \in \mathrm{Im}(P)\colon \operatorname{tr}(J_iT_X) = 0,\ \forall X \in TM,\ \forall i = 1,2,3\big\}$$

is a complement of $\delta\Omega^1(M, N(\mathcal{P}))$ in $\Omega^2(M, TM)$. The projection on the second factor of the decomposition

$$\Omega^2(M, TM) = \delta\Omega^1\big(M, N(\mathcal{P})\big) \oplus C\big(N(\mathcal{P})\big) \tag{12}$$

is the map

$$\Omega^2(M, TM) \ni T \to P(T) - \delta(\tau_1 \otimes J_1 + \tau_2 \otimes J_2 + \tau_3 \otimes J_3), \tag{13}$$

where P *is the projector of Lemma 3 and the* 1*-forms* τ_i *are defined by*

$$\tau_i(X) = \frac{\epsilon_i \operatorname{tr}(J_i P(T)_X)}{n-2}, \quad \forall i \in \{1,2,3\}. \tag{14}$$

Proof. We define two subspaces of $\Omega^1(M, \mathcal{P})$:

$$\Omega^1_0(M, \mathcal{P}) := \left\{\sum_{i=1}^3 \gamma_i \otimes J_i\colon \sum_{i=1}^3 \gamma_i(J_i X) = 0,\ \forall X \in TM\right\};$$

$$\mathcal{C}(M) := \left\{T^\alpha := \sum_{i=1}^3 \epsilon_i(\alpha \circ J_i) \otimes J_i,\ \forall \alpha \in \Omega^1(M)\right\}.$$

Clearly, $\Omega^1_0(M, \mathcal{P})$ and $\mathcal{C}(M)$ have trivial intersection. Consider now an arbitrary $\mathcal{P}$-valued 1-form $\sum_{i=1}^3 \alpha_i \otimes J_i$ and define $\alpha := -\frac{1}{3}\sum_{i=1}^3 \alpha_i \circ J_i$. Then

$$\begin{aligned} A := \sum_{i=1}^3 \alpha_i \otimes J_i + T^\alpha = {} & \left(\frac{2}{3}\alpha_1 - \frac{1}{3}\alpha_2 \circ J_3 + \frac{1}{3}\alpha_3 \circ J_2\right) \otimes J_1 \\ & + \left(\frac{2}{3}\alpha_2 - \frac{1}{3}\alpha_1 \circ J_3 - \frac{1}{3}\alpha_3 \circ J_1\right) \otimes J_2 \\ & + \left(\frac{2}{3}\alpha_3 + \frac{1}{3}\alpha_1 \circ J_2 + \frac{1}{3}\alpha_2 \circ J_1\right) \otimes J_3 \end{aligned}$$

belongs to $\Omega^1_0(M, \mathcal{P})$. Therefore, $\Omega^1(M, \mathcal{P})$ decomposes as

$$\Omega^1(M, \mathcal{P}) = \mathcal{C}(M) \oplus \Omega^1_0(M, \mathcal{P}). \tag{15}$$

Next, we show that $\operatorname{Im}(P)$ decomposes as

$$\operatorname{Im}(P) = \delta\Omega^1_0(M, \mathcal{P}) \oplus C\big(N(\mathcal{P})\big). \tag{16}$$

With the previous notations, it can be checked that $\delta(A) = P\delta(\sum_{i=1}^3 \alpha_i \otimes J_i)$. This implies that $\delta\Omega^1_0(M, \mathcal{P})$ is included in $\operatorname{Im}(P)$. We now show that $\delta\Omega^1_0(M, \mathcal{P})$ and $C(N(\mathcal{P}))$ have trivial intersection. For this, we need the following observation: for any $B = \sum_{i=1}^3 \beta_i \otimes J_i \in \Omega^1_0(M, \mathcal{P})$,

$$\beta_i(X) = \frac{\epsilon_i \operatorname{tr}(J_i(\delta B)_X)}{n-2}, \quad \forall X,\ \forall i \in \{1,2,3\}. \tag{17}$$

Relation (17) can be checked using definitions: we first write the coefficients β_i of B in the form

$$\beta_1 = \frac{2}{3}\gamma_1 - \frac{1}{3}\gamma_2 \circ J_3 + \frac{1}{3}\gamma_3 \circ J_2,$$

$$\beta_2 = \frac{2}{3}\gamma_2 - \frac{1}{3}\gamma_1 \circ J_3 - \frac{1}{3}\gamma_3 \circ J_1,$$

$$\beta_3 = \frac{2}{3}\gamma_3 + \frac{1}{3}\gamma_1 \circ J_2 + \frac{1}{3}\gamma_2 \circ J_1,$$

for some 1-forms γ_1, γ_2, γ_3 and then we apply the definition of the torsion map δ, we take traces, etc., and we get (17). Suppose now that $\delta(B) \in C(N(\mathcal{P}))$. Then, from (17) and the definition of $C(N(\mathcal{P}))$,

$$0 = \operatorname{tr}\big(J_i(\delta B)_X\big) = (n-2)\epsilon_i\beta_i(X), \quad \forall X,\ \forall i \in \{1,2,3\},$$

which implies that $B = 0$. We proved that $\delta\Omega^1_0(M, \mathcal{P})$ and $C(N(\mathcal{P}))$ intersect trivially. We now prove that $\delta\Omega^1_0(M, \mathcal{P})$ and $C(N(\mathcal{P}))$ generate $\operatorname{Im}(P)$. For this, let $T \in \Omega^2(M, TM)$ and write

$$P(T) = \left(P(T) - \delta\left(\sum_{i=1}^3 \tau_i \otimes J_i\right)\right) + \delta\left(\sum_{i=1}^3 \tau_i \otimes J_i\right) \tag{18}$$

with 1-forms τ_i defined in (14). Lemma 6 and the definition of τ_i imply that $\sum_{i=1}^{3}\tau_i\otimes J_i$ belongs to $\Omega^1_0(M,\mathcal{P})$. Moreover, the first term of (18) belongs to $C(N(\mathcal{P}))$: it belongs to $\mathrm{Im}(P)$ since $\sum_{i=1}^{3}\tau_i\otimes J_i\in\Omega^1_0(M,\mathcal{P})$ and $\delta\Omega^1_0(M,\mathcal{P})$ is included in $\mathrm{Im}(P)$ (from what we already proved); moreover, for any $i\in\{1,2,3\}$ and $X\in TM$,

$$\mathrm{tr}\big(J_iP(T)_X\big)-\mathrm{tr}\left(J_i\delta\left(\sum_{k=1}^{3}\tau_k\otimes J_k\right)_X\right)=\mathrm{tr}\big(J_iP(T)_X\big)-(n-2)\epsilon_i\tau_i(X)=0$$

where the first equality holds from (17), since $\sum_{i=1}^{3}\tau_i\otimes J_i\in\Omega^1_0(M,\mathcal{P})$, and the second equality is just the definition of τ_i. The decomposition (16) follows.

We can now prove the decomposition (12). Using (16) and Lemma 3, we obtain the following decomposition of $\Omega^2(M,TM)$:

$$\Omega^2(M,TM)=\delta\Omega^1\big(M,Z(\mathcal{P})\big)\oplus\delta\Omega^1_0(M,\mathcal{P})\oplus C\big(N(\mathcal{P})\big). \tag{19}$$

On the other hand, we claim that

$$\delta\Omega^1\big(M,N(\mathcal{P})\big)=\delta\Omega^1\big(M,Z(\mathcal{P})\big)\oplus\delta\Omega^1_0(M,\mathcal{P}), \tag{20}$$

or, equivalently, that $\delta\Omega^1(M,Z(\mathcal{P}))$ and $\delta\Omega^1_0(M,\mathcal{P})$ generate $\delta\Omega^1(M,N(\mathcal{P}))$ (since $\delta\Omega^1(M,Z(\mathcal{P}))$ and $\delta\Omega^1_0(M,\mathcal{P})$ have trivial intersection, from (19)). Recall that $N(\mathcal{P})=Z(\mathcal{P})\oplus\mathcal{P}$. From (15), we notice that in order to prove (20) it is enough to show that $\delta\mathcal{C}(M)$ is included in $\delta\Omega^1(M,Z(\mathcal{P}))$. Let $T^\alpha=\sum_{i=1}^{3}\epsilon_i(\alpha\circ J_i)\otimes J_i\in\mathcal{C}(M)$, where $\alpha\in\Omega^1(M)$ is an arbitrary 1-form. It can be checked that $\delta(T^\alpha)=\delta(E^\alpha)$, where E^α, defined by

$$E^\alpha(X,Y):=-\left(\alpha(Y)X+\sum_{i=1}^{3}\epsilon_i\alpha(J_iY)J_iX+\alpha(X)Y\right),\quad\forall X,Y\in TM,$$

belongs to $\Omega^1(M,Z(\mathcal{P}))$. This implies that $\delta\mathcal{C}(M)$ is included in $\delta\Omega^1(M,Z(\mathcal{P}))$ as claimed. Decomposition (20) follows and implies, together with (19), decomposition (12). Clearly, the map (13) is the projection onto the second factor of (12). □

As a consequence, we recover Proposition 2.5 of [13].

Corollary 8. *The torsion tensor $T^{\mathcal{P}}$ of an almost para-quaternionic structure $\mathcal{P}$ is related to the torsion tensor T^H of a compatible almost para-hypercomplex structure $H=\{J_1,J_2,J_3\}$ by*

$$T^{\mathcal{P}}:=T^H-\delta(\tau_1\otimes J_1+\tau_2\otimes J_2+\tau_3\otimes J_3),$$

where, for any tangent vector X,

$$\tau_i(X):=\frac{\epsilon_i\,\mathrm{tr}(J_iT^H_X)}{n-2},\quad\forall i\in\{1,2,3\}. \tag{21}$$

Proof. The torsion tensor $T^{\mathcal{P}}\in C(N(\mathcal{P}))$ is the projection of $T^H\in\mathrm{Im}(P)$ with respect to the decomposition (16). □

We will need the following corollary in the proof of Theorem 11. This Corollary is analogue to Proposition 2.3 of [2] and can be proved in a similar way. A similar result has been proved in [13].

Corollary 9. *The torsion $T^{\mathcal{P}}$ of an almost para-quaternionic manifold $(M,\mathcal{P})$ is zero if and only if the torsion T^H of any compatible almost para-hypercomplex structure $H=\{J_1,J_2,J_3\}$ is of the form*

$$T^H=\delta\left(\sum_{i=1}^{3}\alpha_i\otimes J_i+\alpha\otimes\mathrm{Id}\right) \tag{22}$$

where $\alpha,\alpha_1,\alpha_2,\alpha_3$ are 1-forms.

3.2. Minimal para-quaternionic connections

Let $(M,\mathcal{P})$ be an almost para-quaternionic manifold. A para-quaternionic connection ∇ is minimal if its torsion T^∇ is equal to the torsion tensor $T^{\mathcal{P}}\in C(N(\mathcal{P}))$ of the almost para-quaternionic structure $\mathcal{P}$. Minimal para-quaternionic connections always exist (see [13], Proposition 2.5). Moreover, they are parametrized by 1-forms, as stated in the following lemma.

Lemma 10. *Let $\{J_1, J_2, J_3\}$ be an admissible basis of $\mathcal{P}$. Any two minimal para-quaternionic connections ∇ and ∇' on $(M, \mathcal{P})$ are related by $\nabla' = \nabla + S^\alpha$, where $\alpha \in \Omega^1(M)$ and*

$$S^\alpha_X(Y) := \alpha(Y)X - \alpha(J_1Y)J_1X + \alpha(J_2Y)J_2X + \alpha(J_3Y)J_3X \\ + \alpha(X)Y - \alpha(J_1X)J_1Y + \alpha(J_2X)J_2Y + \alpha(J_3X)J_3Y,$$

for any $X, Y \in TM$.

Proof. The statement is more general: two para-quaternionic connections ∇ and ∇' have the same torsion if and only if there is a 1-form $\alpha \in \Omega^1(M)$ such that $\nabla' = \nabla + S^\alpha$. This comes from the fact that the map which associates to a covector $\alpha \in T^*M$ the tensor $S^\alpha \in T^*M \otimes \mathrm{End}(TM)$ defined as above is an isomorphism between T^*M and $(T^*M \otimes N(\mathcal{P})) \cap (S^2T^*M \otimes TM)$, where S^2T^*M is the bundle of symmetric $(2, 0)$-tensors on M. □

4. Compatible (para)-complex structures

In our conventions, a system $\{I_i\}$ of almost complex and/or almost para-complex structures on a manifold M is independent if it is pointwise independent, i.e. for any $p \in M$, the system $\{I_i(p)\}$ is independent. In particular, two almost complex or almost para-complex structures I_1, I_2 on M are independent if $I_1(p) \neq \pm I_2(p)$ at any $p \in M$.

Our main result in this section is the following criterion of integrability of almost para-quaternionic structures. Similar results are known for conformal 4-manifolds and for almost quaternionic manifolds (see [2,11] and [12]).

Theorem 11. *Let $(M, \mathcal{P})$ be an almost para-quaternionic manifold of dimension $4m \geqslant 8$. Suppose one of the following situations holds:*

1. *There are two globally defined, independent, compatible, complex or para-complex structures I_1 and I_2 such that either I_1 or I_2 is a complex structure.*
2. *There are two globally defined, independent, compatible, para-complex structures I_1 and I_2 such that at any $p \in M$, the 2-plane $\mathrm{Span}\{I_1(p), I_2(p)\}$ is non-degenerate (with respect to the standard Lorenzian metric $\langle\cdot,\cdot\rangle$ of $\mathcal{P}$).*
3. *There are three globally defined, pairwise independent, compatible, para-complex structures I_1, I_2 and I_3, such that at any $p \in M$, $I_1(p)$, $I_2(p)$ and $I_3(p)$ are linearly dependent and for any $i \neq j$, the 2-plane $\mathrm{Span}\{I_i(p), I_j(p)\}$ is degenerate.*

Then $(M, \mathcal{P})$ is para-quaternionic.

We divide the proof of Theorem 11 into several lemmas and propositions. We begin with Lemmas 12 and 13, which are mild generalizations of (3.4.1) and (3.4.4) of [1] and can be proved in a similar way.

Lemma 12. *Let J_i $(i \in \{1, 2\})$ be two anti-commuting almost ϵ_i-complex structures on a manifold M. The Nijenhuis tensors of J_1, J_2 and $J_1 \circ J_2$ are related by*

$$2N_{J_1\circ J_2}(X, Y) = N_{J_1}(J_2X, J_2Y) - \epsilon_1 N_{J_2}(X, Y) - J_1N_{J_2}(J_1X, Y) \\ - J_1N_{J_2}(X, J_1Y) - \epsilon_2 N_{J_1}(X, Y) + N_{J_2}(J_1X, J_1Y) \\ - J_2N_{J_1}(X, J_2Y) - J_2N_{J_1}(J_2X, Y),$$

for any vector fields $X, Y \in \mathcal{X}(M)$. In particular, if J_1 and J_2 are integrable, also $J_1 \circ J_2$ is.

Recall that if $A, B \in \mathrm{End}(TM)$ are two endomorphisms of TM, their Nijenhuis bracket is a new endomorphism of TM, defined, for any vector fields $X, Y \in \mathcal{X}(M)$, by

$$[A, B](X, Y) = [AX, BY] + [BX, AY] - A\big([BX, Y] + [X, BY]\big) \\ - B\big([X, AY] + [AX, Y]\big) + (AB + BA)[X, Y].$$

Note that if J is an almost ϵ-complex structure, then $[J, J] = 2N_J$.

Lemma 13. *Let ∇ be a linear connection on a manifold M, which preserves two endomorphisms $A, B \in \mathrm{End}(TM)$. Denote by T^∇ the torsion of the connection ∇. For any vector fields $X, Y \in \mathcal{X}(M)$,*

$$[A, B](X, Y) = A\big(T^\nabla(BX, Y) + T^\nabla(X, BY)\big) \\ + B\big(T^\nabla(X, AY) + T^\nabla(AX, Y)\big) \\ - T^\nabla(AX, BY) - T^\nabla(BX, AY).$$

In the next lemma we collect some simple algebraic properties of compatible almost para-complex structures on almost para-quaternionic manifolds.

Lemma 14. *Let $(M, \mathcal{P})$ be an almost para-quaternionic manifold.*

(i) *Suppose that I_1 and I_2 are two globally defined, compatible, independent, almost para-complex structures and let $p \in M$. The 2-plane* $\mathrm{Span}\{I_1(p), I_2(p)\}$ *is degenerate if and only if $|\langle I_1(p), I_2(p)\rangle| = 1$, if and only if $\mathrm{pr}_{I_1^\perp}(I_2) = I_2 - \langle I_2, I_1\rangle I_1$ or $\mathrm{pr}_{I_2^\perp}(I_1) = I_1 - \langle I_1, I_2\rangle I_2$ is null (i.e. squares to the trivial endomorphism) at the point p.*

(ii) *Suppose that I_1, I_2 and I_3 are three pairwise independent, globally defined, compatible, almost para-complex structures, such that* $\mathrm{Span}\{I_i(p), I_j(p)\}$ *is degenerate, for any $i \neq j$ and any $p \in M$. Moreover, assume that*

$$\langle I_1, I_2\rangle\langle I_2, I_3\rangle\langle I_1, I_3\rangle = 1.$$

Then the system $\{I_1, I_2, I_3\}$ is linearly dependent at any point and (eventually changing the order of $\{I_i\}$ and replacing I_i with $-I_i$ if necessary) there is, in a neighborhood of any point, an admissible basis $\{J_1, J_2, J_3\}$ of $\mathcal{P}$ such that

$$I_1 = J_2,$$
$$I_2 = J_1 - J_2 + qJ_3,$$
$$I_3 = aJ_1 + J_2 + aqJ_3,$$

where a is a smooth function, non-vanishing and different from -1 at any point, and $q \in \{-1, 1\}$.

(iii) *Suppose that I_1, I_2 and I_3 are like in* (ii)*, but*

$$\langle I_1, I_2\rangle\langle I_2, I_3\rangle\langle I_1, I_3\rangle = -1.$$

Then the system $\{I_1, I_2, I_3\}$ is linearly independent and (eventually changing the order of $\{I_i\}$ and replacing I_i with $-I_i$ if necessary) there is a global admissible basis $\{J_1, J_2, J_3\}$ of $\mathcal{P}$ such that

$$I_1 = J_2,$$
$$I_2 = J_1 + J_2 + J_3,$$
$$I_3 = J_1 + J_2 - J_3.$$

Proof. The first statement is easy. To prove (ii), suppose that $\langle I_1, I_3\rangle = 1$. Replacing I_2 with $-I_2$ if necessary, we can moreover assume that both $\langle I_1, I_2\rangle$ and $\langle I_2, I_3\rangle$ are equal to -1. Then $I_1 + I_2$ and $I_3 - I_1$ belong to $I_1^\perp$, are null and orthogonal. Therefore, they must be proportional (the restriction of $\langle\cdot,\cdot\rangle$ to $I_1^\perp$ being non-degenerate). We deduce that $\{I_1, I_2, I_3\}$ are dependent at any point. Clearly, we can find an admissible basis of $\mathcal{P}$ such that $I_1 = J_2$ and $I_2 = J_1 - J_2 + qJ_3$, where $q \in \{-1, 1\}$. Since $I_1 + I_2$ and $I_3 - I_1$ are proportional, $I_3 = I_1 + a(J_1 + qJ_3)$ for a smooth function a. Since I_1, I_2 and I_3 are pairwise independent, a is non-vanishing and different from -1 at any point. The second claim follows. The third claim is equally easily. □

Using the previous lemmas, we can now prove Theorem 11. We first assume in Proposition 15 that $(M, \mathcal{P})$ admits a pair of (integrable) complex or para-complex structures like in the first two cases of Theorem 11 and we show that $\mathcal{P}$ is para-quaternionic. The remaining case of Theorem 11 will be treated in Proposition 16.

Proposition 15. *Let $(M, \mathcal{P})$ be an almost para-quaternionic manifold of dimension $n = 4m \geqslant 8$. Suppose that $\mathcal{P}$ admits two globally defined, independent, compatible, complex or para-complex structures I_1 and I_2, such that one of the following conditions holds:*

(i) *Either I_1 or I_2 is a complex structure.*

(ii) *Both I_1 and I_2 are para-complex structures and* $\mathrm{Span}\{I_1(p), I_2(p)\}$ *is non-degenerate at any $p \in M$.*

Then $(M, \mathcal{P})$ is a para-quaternionic manifold.

Proof. Our argument is similar to the one employed in the proof of Theorem 2.4 of [2]. In a neighborhood of any point we consider two almost ϵ_i-complex structures J_i $(i \in \{1, 2\})$, with $\langle J_1, J_2\rangle = 0$, such that $I_1 = J_1$ and $I_2 = aJ_1 + bJ_2$, where a, b are smooth functions, with b non-vanishing (this is possible since $\mathrm{pr}_{I_1^\perp}(I_2)$ is non-null when I_1 is complex – the metric on $I_1^\perp$ being positive definite – and also when both I_1 and I_2 are para-complex, from the non-degeneracy of the 2-planes $\mathrm{Span}\{I_1(p), I_2(p)\}$ and Lemma 14). Since $\langle J_1, J_2\rangle = 0$, J_1 and J_2 anti-commute and the composition $J_3 := J_1 \circ J_2$ is an almost ϵ_3-complex structure, with $\epsilon_3 := -\epsilon_1\epsilon_2 \in \{-1, +1\}$. We divide the proof into three steps.

Step one: we prove that the torsion T^H of the almost para-hypercomplex structure H defined by J_1, J_2 and J_3 has the following expression: for any vector fields X, Y,

$$-12T^H(X,Y) = 3\epsilon_2 N_{J_2}(X,Y) - \epsilon_3 J_1 N_{J_2}(J_1X,Y) - \epsilon_3 J_1 N_{J_2}(X,J_1Y) + \epsilon_3 N_{J_2}(J_1X,J_1Y).$$

We prove this in the following way: since $J_1 = I_1$ is integrable, Lemma 4 implies that

$$T^H = -\frac{1}{6}(\epsilon_2 N_{J_2} + \epsilon_3 N_{J_3}). \tag{23}$$

From Lemma 12 and the integrability of J_1,

$$2N_{J_3}(X,Y) = -\epsilon_1 N_{J_2}(X,Y) - J_1 N_{J_2}(J_1X,Y) - J_1 N_{J_2}(X,J_1Y) + N_{J_2}(J_1X,J_1Y). \tag{24}$$

Combining (23) with (24) we get our first claim.

Step two: we prove that the Nijenhuis bracket $[J_1, J_2]$ has the following expression:

$$2[J_1,J_2](X,Y) = \epsilon_2 J_3 N_{J_2}(X,Y) - \epsilon_3 J_3 N_{J_2}(J_1X,J_1Y) - \epsilon_2 J_2\big(N_{J_2}(J_1X,Y) + N_{J_2}(X,J_1Y)\big).$$

To prove this claim, we apply Lemma 13 to $A := J_1$, $B := J_2$ and the Obata connection ∇ of H (which preserves J_1, J_2 and J_3). Since $T^H = T^\nabla$, from Lemma 13,

$$\begin{aligned}[J_1,J_2](X,Y) = {} & J_1\big(T^H(J_2X,Y) + T^H(X,J_2Y)\big)\\ & + J_2\big(T^H(J_1X,Y) + T^H(X,J_1Y)\big)\\ & - T^H(J_1X,J_2Y) - T^H(J_2X,J_1Y).\end{aligned}$$

We now evaluate $T^H(J_2X,Y) + T^H(X,J_2Y)$, $T^H(J_1X,Y) + T^H(X,J_1Y)$ and $T^H(J_1X,J_2Y) + T^H(J_2X,J_1Y)$ in terms of the Nijenhuis tensor N_{J_2}. It can be checked that the Nijenhuis tensor of any almost ϵ-complex structure J has the following symmetries:

$$N_J(JX,Y) = N_J(X,JY) = -JN_J(X,Y), \quad \forall X,Y \in \mathcal{X}(M). \tag{25}$$

Using the expression of T^H determined in the first step, relation (25) for $J := J_2$ and the anti-commutativity $J_1J_2 = -J_1J_2$ we obtain

$$\begin{aligned}&6\big(T^H(J_1X,Y) + T^H(X,J_1Y)\big) = \epsilon_3 J_1 N_{J_2}(J_1X,J_1Y) - \epsilon_2 J_1 N_{J_2}(X,Y) - \epsilon_2\big(N_{J_2}(J_1X,Y) + N_{J_2}(X,J_1Y)\big),\\ &6\big(T^H(J_2X,Y) + T^H(X,J_2Y)\big) = 3\epsilon_2 J_2 N_{J_2}(X,Y) - \epsilon_3 J_2 N_{J_2}(J_1X,J_1Y),\\ &6\big(T^H(J_1X,J_2Y) + T^H(J_2X,J_1Y)\big) = 2\epsilon_2 J_2\big(N_{J_2}(J_1X,Y) + N_{J_2}(X,J_1Y)\big) + \epsilon_3 J_3\big(N_{J_2}(J_1X,J_1Y) - \epsilon_1 N_{J_2}(X,Y)\big).\end{aligned}$$

Replacing these relations in the expression of $[J_1, J_2]$ above we get our second claim.

Step three: we prove that $T^H \equiv 0$, where the sign "$\equiv$" means equality, modulo terms of the form $\delta(\alpha_1 \otimes J_1 + \alpha_2 \otimes J_2 + \alpha_3 \otimes J_3 + \alpha \otimes \mathrm{Id})$, where $\alpha, \alpha_1, \alpha_2, \alpha_3$ are 1-forms. To prove this claim, we notice that, since $I_2 = aJ_1 + bJ_2$, is integrable

$$a^2 N_{J_1} + b^2 N_{J_2} + ab[J_1,J_2] \equiv 0. \tag{26}$$

The integrability of J_1 together with (26) imply that

$$b^2 N_{J_2} + ab[J_1,J_2] \equiv 0. \tag{27}$$

On the set of points $M_0 \subset M$ where $a = 0$, (27) implies that $N_{J_2} \equiv 0$ (because b is non-vanishing). We use now the expression of $[J_1, J_2]$ determined in step two to show that (27) implies that $N_{J_2} \equiv 0$ also on $M \setminus M_0$. On $M \setminus M_0$, we can divide (27) by a and, using the expression of $[J_1, J_2]$ provided by step two, we obtain

$$-\frac{2b}{a} N_{J_2}(X,Y) \equiv \epsilon_2 J_3 N_{J_2}(X,Y) - \epsilon_3 J_3 N_{J_2}(J_1X,J_1Y) - \epsilon_2 J_2\big(N_{J_2}(J_1X,Y) + N_{J_2}(X,J_1Y)\big).$$

Replacing (X,Y) with (J_2X, J_2Y) in this relation, using again (25) for $J = J_2$ and the anti-commutativity $J_1J_2 = -J_2J_1$ we get two relations:

$$-\frac{2b}{a} N_{J_2}(X,Y) \equiv \epsilon_2 J_3 N_{J_2}(X,Y) - \epsilon_3 J_3 N_{J_2}(J_1X,J_1Y) \tag{28}$$

and

$$N_{J_2}(J_1X,Y) + N_{J_2}(X,J_1Y) \equiv 0. \tag{29}$$

Relation (29) implies that

$$N_{J_2}(J_1X,J_1Y) \equiv -\epsilon_1 N_{J_2}(X,Y). \tag{30}$$

Replacing (30) in (28) and using $\epsilon_3 = -\epsilon_1\epsilon_2$ we get $N_{J_2} \equiv 0$. From (24), $N_{J_3} \equiv 0$ as well and then, from (23), $T^H \equiv 0$. Corollary 9 implies that $T^{\mathcal{P}} = 0$. This concludes our proof. □

Proposition 16. *Let $(M,\mathcal{P})$ be an almost para-quaternionic manifold of dimension $n=4m\geqslant 8$. Suppose that $\mathcal{P}$ admits three pairwise independent, compatible, para-complex structures $\{I_1,I_2,I_3\}$, such that at any $p\in M$, $\{I_1(p),I_2(p),I_3(p)\}$ are dependent and for any $i\neq j$, $\mathrm{Span}\{I_i(p),I_j(p)\}$ is degenerate. Then $(M,\mathcal{P})$ is para-quaternionic.*

Proof. Like in the proof of Proposition 15, we will determine, in a neighborhood of any point, a suitable compatible almost para-hypercomplex structure H, for which $T^H\equiv 0$. We divide the proof into two steps.

Step one: let $H:=\{J_1,J_2,J_3\}$ be any admissible basis of $\mathcal{P}$ such that $I_1=J_2$. In particular, J_2 is integrable. We claim that the torsion T^H of H and the Nijenhuis brackets $[J_1,J_2]$, $[J_1,J_3]$ and $[J_2,J_3]$ have the following expressions: for any vector fields X and Y,

$$12T^H(X,Y)=3N_{J_1}(X,Y)-N_{J_1}(J_2X,J_2Y)+J_2\big(N_{J_1}(X,J_2Y)+N_{J_1}(J_2X,Y)\big) \tag{31}$$

and

$$\begin{aligned}
2[J_1,J_2](X,Y)&=J_1\big(N_{J_1}(J_2X,Y)+N_{J_1}(X,J_2Y)\big)+J_3\big(N_{J_1}(J_2X,J_2Y)+N_{J_1}(X,Y)\big),\\
2[J_1,J_3](X,Y)&=N_{J_1}(J_2X,Y)+N_{J_1}(X,J_2Y)-J_2\big(N_{J_1}(J_2X,J_2Y)+N_{J_1}(X,Y)\big),\\
2[J_2,J_3](X,Y)&=J_1\big(N_{J_1}(J_2X,J_2Y)+N_{J_1}(X,Y)\big)+J_3\big(N_{J_1}(J_2X,Y)+N_{J_1}(X,J_2Y)\big).
\end{aligned}$$

To prove these claims, notice that

$$T^H=-\frac{1}{6}(-N_{J_1}+N_{J_3}) \tag{32}$$

since J_2 is integrable. On the other hand, applying Lemma 12 to J_1 and J_2 and using the integrability of J_2 we get

$$2N_{J_3}(X,Y)=N_{J_1}(J_2X,J_2Y)-N_{J_1}(X,Y)-J_2N_{J_1}(X,J_2Y)-J_2N_{J_1}(J_2X,Y). \tag{33}$$

Combining (32) with (33) we obtain (31). In order to evaluate the Nijenhuis brackets $[J_1,J_2]$, $[J_1,J_3]$ and $[J_2,J_3]$ we will use Lemma 13, with ∇ the Obata connection of $\{J_1,J_2,J_3\}$, so that $T^\nabla=T^H$. Using (31), the anti-commutativity of J_1, J_2, J_3 and (25) for $J=J_1$, we obtain

$$\begin{aligned}
6\big(T^H(J_1X,J_2Y)+T^H(J_2X,J_1Y)\big)&=-2J_1\big(N_{J_1}(X,J_2Y)+N_{J_1}(J_2X,Y)\big)-J_3\big(N_{J_1}(J_2X,J_2Y)-N_{J_1}(X,Y)\big),\\
6\big(T^H(J_1X,J_3Y)+T^H(J_3X,J_1Y)\big)&=-N_{J_1}(X,J_2Y)-N_{J_1}(J_2X,Y)+J_2\big(N_{J_1}(J_2X,J_2Y)+N_{J_1}(X,Y)\big),\\
6\big(T^H(J_2X,J_3Y)+T^H(J_3X,J_2Y)\big)&=-J_1\big(3N_{J_1}(J_2X,J_2Y)+N_{J_1}(X,Y)\big),\\
6\big(T^H(J_1X,Y)+T^H(X,J_1Y)\big)&=-J_1\big(3N_{J_1}(X,Y)+N_{J_1}(J_2X,J_2Y)\big),\\
6\big(T^H(J_2X,Y)+T^H(X,J_2Y)\big)&=J_2\big(N_{J_1}(J_2X,J_2Y)+N_{J_1}(X,Y)\big)+N_{J_1}(J_2X,Y)+N_{J_1}(X,J_2Y),\\
6\big(T^H(X,J_3Y)+T^H(J_3X,Y)\big)&=-2J_1\big(N_{J_1}(J_2X,Y)+N_{J_1}(X,J_2Y)\big)+J_3\big(N_{J_1}(J_2X,J_2Y)-N_{J_1}(X,Y)\big).
\end{aligned}$$

Applying Lemma 13 we get the expressions of $[J_1,J_2]$, $[J_1,J_3]$ and $[J_2,J_3]$ as stated.

Step two: using step one, we prove that $\mathcal{P}$ is integrable. From Lemma 14 we can chose, in a neighborhood of any point, an admissible basis $\{J_1,J_2,J_3\}$ of $\mathcal{P}$ such that

$$I_1=J_2,\qquad I_2=J_1-J_2+qJ_3,\qquad I_3=aJ_1+J_2+aqJ_3,$$

where a is a smooth function, non-vanishing and different from -1 at any point, and $q\in\{-1,+1\}$. The integrability of I_1 and I_3 implies that

$$\begin{aligned}
&N_{J_1}+N_{J_3}+q[J_1,J_3]\equiv 0,\\
&[J_1,J_2]+q[J_2,J_3]\equiv 0,
\end{aligned}$$

where the sign $\equiv$ has the same meaning as in the proof of Proposition 15. Using (33) and the expressions of $[J_1,J_2]$ and $[J_2,J_3]$ previously determined, we get

$$\begin{aligned}
&(\mathrm{Id}+qJ_2)\big(N_{J_1}(J_2X,J_2Y)+N_{J_1}(X,Y)+qN_{J_1}(J_2X,Y)+qN_{J_1}(X,J_2Y)\big)\equiv 0,\\
&(\mathrm{Id}-qJ_2)\big(N_{J_1}(J_2X,J_2Y)+N_{J_1}(X,Y)+qN_{J_1}(J_2X,Y)+qN_{J_1}(X,J_2Y)\big)\equiv 0.
\end{aligned}$$

Adding these relations,

$$N_{J_1}(J_2X,J_2Y)+N_{J_1}(X,Y)+q\big(N_{J_1}(J_2X,Y)+N_{J_1}(X,J_2Y)\big)\equiv 0. \tag{34}$$

Replacing in (34) the pair (X,Y) with (J_1X,J_1Y) and using (25) for $J=J_1$ we get

$$N_{J_1}(J_2X, J_2Y) + N_{J_1}(X, Y) \equiv 0,$$
$$N_{J_1}(J_2X, Y) + N_{J_1}(X, J_2Y) \equiv 0.$$

Like in the proof of Proposition 15, we deduce that $N_{J_1} \equiv 0$. From (33) it follows that $N_{J_3} \equiv 0$ as well and therefore, $T^H \equiv 0$. We conclude that $\mathcal{P}$ is para-quaternionic. □

Proposition 16 concludes the proof of Theorem 11.

Theorem 11 raises the following question: does the existence of three globally defined, independent, compatible, para-complex structures $\{I_1, I_2, I_3\}$ on an almost para-quaternionic manifold $(M, \mathcal{P})$, such that for any $p \in M$ and $i \neq j$, the 2-plane $\mathrm{Span}\{I_i(p), I_j(p)\}$ is degenerate, imply the integrability of the almost para-quaternionic structure $\mathcal{P}$? We will now show that the answer to this question is negative.

For this, it is convenient to express the integrability of I_1, I_2 and I_3 in terms of the admissible basis $\{J_1, J_2, J_3\}$ of $\mathcal{P}$ provided by Lemma 14, i.e. related to $\{I_1, I_2, I_3\}$ by

$$I_1 = J_2, \qquad I_2 = J_1 + J_2 + J_3, \qquad I_3 = J_1 + J_2 - J_3. \tag{35}$$

Lemma 17. *The integrability of the almost para-complex structures I_1, I_2 and I_3 is equivalent to the integrability of J_2 together with the integrability of the eigenbundle of J_3 which corresponds to the eigenvalue $+1$.*

Proof. From (35), the integrability of I_1, I_2 and I_3 is equivalent to the integrability of J_2 and the following two relations:

$$N_{J_1} + N_{J_3} + [J_1, J_2] = 0,$$
$$[J_1, J_3] + [J_2, J_3] = 0.$$

We now express N_{J_3}, $[J_1, J_2]$, $[J_1, J_3]$ and $[J_2, J_3]$ in terms of N_{J_1}, using the computations of the proof of Proposition 16. To simplify notations, define, for any vector fields X and Y,

$$E(X, Y) := N_{J_1}(J_2X, J_2Y) + N_{J_1}(X, Y) \tag{36}$$

and

$$F(X, Y) := N_{J_1}(J_2X, Y) + N_{J_1}(X, J_2Y). \tag{37}$$

An easy argument shows that the previous two relations reduce to

$$(\mathrm{Id} + J_3)E(X, Y) = 0 \quad \forall X, Y. \tag{38}$$

Multiplying (38) by J_2 on the left and using (25) for $J = J_1$ we obtain

$$J_2\big(N_{J_1}(J_2X, J_2Y) + N_{J_1}(X, Y)\big) = N_{J_1}(J_2J_1X, J_2Y) - N_{J_1}(J_1X, Y). \tag{39}$$

Replacing in (39) X with J_1X we obtain

$$J_2\big(N_{J_1}(J_2J_1X, J_2Y) + N_{J_1}(J_1X, Y)\big) = -N_{J_1}(J_2X, J_2Y) + N_{J_1}(X, Y). \tag{40}$$

On the other hand, multiplying (39) on the left with J_2 we obtain

$$J_2\big(N_{J_1}(J_2J_1X, J_2Y) - N_{J_1}(J_1X, Y)\big) = N_{J_1}(J_2X, J_2Y) + N_{J_1}(X, Y). \tag{41}$$

Using (40) and (41), we get that (38) is equivalent to

$$N_{J_1}(J_3X, J_3Y) = J_3N_{J_1}(X, Y), \quad \forall X, Y. \tag{42}$$

Using (42), we easily get our claim: relation (33) expresses N_{J_3} in terms of N_{J_1}; conversely, using Lemma 12 and the integrability of J_2, we can express N_{J_1} in terms of N_{J_3} as

$$2N_{J_1}(X, Y) = -N_{J_3}(X, Y) - J_2N_{J_3}(J_2X, Y) - J_2N_{J_3}(X, J_2Y) + N_{J_3}(J_2X, J_2Y). \tag{43}$$

Using (33) and (43), it can be checked that (42) is equivalent to

$$J_3N_{J_3}(X, Y) = N_{J_3}(X, Y), \quad \forall X, Y,$$

or to the integrability of the eigenbundle of J_3 which corresponds to the eigenvalue $+1$. Our claim follows. □

Lemma 17 reduces the problem of finding independent, compatible, para-complex structures I_1, I_2 and I_3 on $(M, \mathcal{P})$, with $\mathrm{Span}\{I_i(p), I_j(p)\}$ degenerate at any $p \in M$, for any $i \neq j$, to the problem of finding admissible bases $\{J_1, J_2, J_3\}$ of $\mathcal{P}$, such that both J_2 and $\mathrm{Ker}(J_3 - \mathrm{Id})$ are integrable. We now show that these conditions, in turn, reduce to solving a certain system of partial differential equations. Indeed, since J_2 is integrable, locally $M = M_1 \times M_2$ is a product manifold and TM_1, TM_2 are the eigenbundles of J_2 which correspond to the eigenvalues 1 and -1, respectively. The four distributions $\mathcal{D}^{\pm} := \mathrm{Ker}(J_3 \mp \mathrm{Id})$, TM_1 and TM_2 are pairwise transverse. Therefore,

$$\mathcal{D}^{\pm} := \{X + A^{\pm}X,\ \forall X \in TM_1\}$$

where $A^+, A^-, A^+ - A^- : TM_1 \to TM_2$ are isomorphisms. It is easy to check that $J_2 \circ J_3 = -J_3 \circ J_2$ is equivalent to $A^+ + A^- = 0$. Note also that in terms of $A := A^+$,

$$J_1(X) = J_3(X) = A(X), \quad J_1(Y) = -J_3(Y) = -A^{-1}(Y),$$

for any $X \in TM_1$ and $Y \in TM_2$. Let $(x_1, \dots, x_{2m})$ and $(y_1, \dots, y_{2m})$ be local coordinates on M_1 and M_2 respectively. In these coordinates,

$$A\left(\frac{\partial}{\partial x_i}\right) := \sum_{j=1}^{2m} f_{ij} \frac{\partial}{\partial y_j},$$

for some smooth functions $f_{ij} = f_{ij}(x_s, y_r)$ $(1 \leqslant i, j, r, s \leqslant 2m)$, with $\det(f_{ij}) \neq 0$ at any point. The system of partial differential equations mentioned above comes from the integrability of $\mathcal{D}^+ = \mathrm{Ker}(J_3 - \mathrm{Id})$: it can be checked that $\mathcal{D}^+$ is integrable if and only if

$$\frac{\partial f_{jt}}{\partial x_i} - \frac{\partial f_{it}}{\partial x_j} + \sum_{k=1}^{2m}\left(f_{ik}\frac{\partial f_{jt}}{\partial y_k} - f_{jk}\frac{\partial f_{it}}{\partial y_k}\right) = 0, \quad \forall i \neq j,\ \forall t. \tag{44}$$

Reversing this argument, any solution (f_{ij}) of (44), with non-vanishing determinant $\det(f_{ij})$, defines an almost para-quaternionic structure $\mathcal{P}_{(f_{ij})} := \mathrm{Span}\{J_1, J_2, J_3\}$, which admits three independent, compatible, para-complex structures I_1, I_2 and I_3, related to $\{J_1, J_2, J_3\}$ by (35). The next proposition determines a class of solutions of (44), for which $\mathcal{P}$ is not para-quaternionic.

Proposition 18. *The functions $f_{ij} := f_i \delta_{ij}$ $(1 \leqslant i, j \leqslant 2m)$, where*

$$f_1 := h\left(\frac{\sum_{j=2}^{2m} x_j^2}{\sum_{j=2}^{2m} y_j^2}\right), \qquad f_i := \frac{x_i(\sum_{j=2}^{2m} y_j^2)}{y_i(\sum_{j=2}^{2m} x_j^2)}, \quad 2 \leqslant i \leqslant 2m$$

and h is a smooth real function, is a solution of (44). Moreover, on any open connected subset of $\mathbb{R}^{4m}$ on which f_i $(1 \leqslant i \leqslant 2m)$ are non-vanishing and

$$h\left(\frac{\sum_{j=2}^{2m} x_j^2}{\sum_{j=2}^{2m} y_j^2}\right) + \frac{\sum_{j=2}^{2m} x_j^2}{\sum_{j=2}^{2m} y_j^2}\dot{h}\left(\frac{\sum_{j=2}^{2m} x_j^2}{\sum_{j=2}^{2m} y_j^2}\right) \neq 0, \tag{45}$$

the associated almost para-quaternionic structure $\mathcal{P}_{(f_{ij})}$ is not para-quaternionic.

Proof. It is straightforward to check that (f_{ij}) is a solution of (44). We now show that $\mathcal{P}$ is not para-quaternionic. With the previous notations, it can be checked that, for any $i, j \in \{1, \dots, 2m\}$,

$$3T^H\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \frac{1}{f_j}\frac{\partial f_j}{\partial x_i}\frac{\partial}{\partial x_j} - \frac{1}{f_i}\frac{\partial f_i}{\partial x_j}\frac{\partial}{\partial x_i}, \tag{46}$$

where T^H is the torsion of $H := \{J_1, J_2, J_3\}$. Suppose now, by absurd, that $\mathcal{P}$ is para-quaternionic. Then T^H is of the form

$$T^H = \delta(\gamma \otimes \mathrm{Id} + \gamma_1 \otimes J_1 + \gamma_2 \otimes J_2 + \gamma_3 \otimes J_3) \tag{47}$$

for some 1-forms γ, γ_1, γ_2 and γ_3. Moreover, since

$$J_1\left(\frac{\partial}{\partial x_i}\right) = J_3\left(\frac{\partial}{\partial x_i}\right) = f_i\frac{\partial}{\partial y_i}, \qquad J_1\left(\frac{\partial}{\partial y_j}\right) = -J_3\left(\frac{\partial}{\partial y_j}\right) = -\frac{1}{f_j}\frac{\partial}{\partial x_j},$$

for any $1 \leqslant i, j \leqslant 2m$, relation (47) implies:

$$\begin{aligned} T^H\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) &= (\gamma + \gamma_2)\left(\frac{\partial}{\partial x_i}\right)\frac{\partial}{\partial x_j} + f_j(\gamma_1 + \gamma_3)\left(\frac{\partial}{\partial x_i}\right)\frac{\partial}{\partial y_j} \\ &\quad - (\gamma + \gamma_2)\left(\frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_i} - f_i(\gamma_1 + \gamma_3)\left(\frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial y_i}. \end{aligned}$$

From (46) we deduce that $(\gamma_1+\gamma_3)(\frac{\partial}{\partial x_i})=0$ for any i and

$$\frac{1}{f_i}\frac{\partial f_i}{\partial x_j}=-(\gamma+\gamma_2)\left(\frac{\partial}{\partial x_j}\right),\quad \forall i\neq j.$$

Equivalently, for any $i\neq k$, the quotient $\frac{f_i}{f_k}$ depends only on $x_i, x_k, y_1,\dots,y_{2n}$. However, from the definition of the functions f_i, this is cannot hold: take $i=1$, $k\geqslant 2$ arbitrary and use (45). We obtain a contradiction. We deduce that $\mathcal{P}$ is not para-quaternionic. □

5. Twistor and reflector spaces

Let $(M,\mathcal{P})$ be an almost para-quaternionic manifold. Denote by $\pi^\epsilon: Z^\epsilon\to M$ the reflector bundle of $(M,\mathcal{P})$ when $\epsilon=1$ and the twistor bundle of $(M,\mathcal{P})$ when $\epsilon=-1$. A para-quaternionic connection ∇ on $(M,\mathcal{P})$ induces an almost ϵ-complex structure $\mathcal{J}^{\nabla,\epsilon}$ on Z^ϵ as follows: let $H_J^{\nabla,\epsilon}$ be the horizontal space at $J\in Z^\epsilon$ of the connection ∇, acting on the bundle $\pi^\epsilon: Z^\epsilon\to M$, and

$$T_J Z^\epsilon = H_J^{\nabla,\epsilon}\oplus T_J^v Z^\epsilon \tag{48}$$

the induced decomposition of $T_J Z^\epsilon$ into horizontal and vertical subspaces. On $H_J^{\nabla,\epsilon}$, identified with $T_{\pi^\epsilon(J)}M$ by means of the differential $(\pi^\epsilon)_*$, $\mathcal{J}^{\nabla,\epsilon}$ coincides with J (viewed as an endomorphism of $T_{\pi^\epsilon(J)}M$); on $T_J^v Z^\epsilon$, it is

$$\mathcal{J}_J^\epsilon(A):=J\circ A,\quad \forall A\in T_J^v Z^\epsilon$$

and is well defined, since

$$T_J^v Z^\epsilon=\{A\in\mathcal{P}_{\pi^\epsilon(J)}:\ A\circ J+J\circ A=0\}\subset \operatorname{End}(T_{\pi^\epsilon(J)}M).$$

Consider now another para-quaternionic connection ∇'. If ∇ and ∇' have the same torsion, $\mathcal{J}^{\nabla,\epsilon}=\mathcal{J}^{\nabla',\epsilon}$ (see Corollary 3.4 of [8]).

Definition 19. Let $(M,\mathcal{P})$ be an almost para-quaternionic manifold. The twistor space Z^- has a canonical almost complex structure $\mathcal{J}^-:=\mathcal{J}^{\nabla,-}$. Similarly, the reflector space Z^+ has a canonical almost para-complex structure $\mathcal{J}^+:=\mathcal{J}^{\nabla,+}$. Here ∇ is (any) minimal para-quaternionic connection on $(M,\mathcal{P})$.

We need the following lemma in the proof of Theorem 21.

Lemma 20. *Let $(M,\mathcal{P})$ be a para-quaternionic manifold and J a compatible almost ϵ-complex structure on $(M,\mathcal{P})$. Then J is integrable if and only if the image $\operatorname{Im}(\sigma^J)\subset Z^\epsilon$ of the tautological section $\sigma^J: M\to Z^\epsilon$ defined by J is $\mathcal{J}^\epsilon$-stable.*

Proof. The proof goes like in the quaternionic case (see [2], Sections 3 and 4), with the Oproiu connections of almost quaternionic manifolds replaced by the minimal para-quaternionic connections of $(M,\mathcal{P})$. □

The next theorem is our main result of this section and is the para-quaternionic analogue of Theorem 4.2 of [2]. The equivalence between the second and the third conditions bellow has already been proved in [8].

Theorem 21. *Let $(M,\mathcal{P})$ be an almost para-quaternionic manifold of dimension $n=4m\geqslant 8$. Denote by $\mathcal{J}^+$ the canonical almost para-complex structure of the reflector space Z^+ and by $\mathcal{J}^-$ the canonical almost complex structure of the twistor space Z^- of $(M,\mathcal{P})$. The following statements are equivalent*:

(i) *$(M,\mathcal{P})$ is a para-quaternionic manifold.*
(ii) *Both $\mathcal{J}^+$ and $\mathcal{J}^-$ are integrable.*
(iii) *Either $\mathcal{J}^-$ or $\mathcal{J}^+$ is integrable.*
(iv) *For any point $p\in M$ and compatible ϵ-complex structure $I_p\in\mathcal{P}_p$, there are infinitely many compatible ϵ-complex structures defined in a neighborhood of p which extend I_p.*
(v) *Any point of M has a neighborhood on which there are defined four compatible, pairwise independent, ϵ_i-complex structures I_i $(i\in\{1,2,3,4\})$.*

Proof. Let ∇ be a minimal para-quaternionic connection on $(M,\mathcal{P})$, so that $T^\nabla=T^{\mathcal{P}}$ and $\mathcal{J}^\epsilon=\mathcal{J}^{\nabla,\epsilon}$, for $\epsilon\in\{-1,1\}$.

We show the equivalence of the first three conditions. Suppose that $\mathcal{P}$ is a para-quaternionic structure. Then $T^{\mathcal{P}}=0$, the connection ∇ is torsion free and both $\mathcal{J}^+$, $\mathcal{J}^-$ are integrable (see [8], Theorem 3.8). Conversely, suppose that $\mathcal{J}^-$ or $\mathcal{J}^+$ is integrable. Then, again from Theorem 3.8 of [8],

$$\Pi_J^{0,2}(T^{\nabla}) = 0, \quad \forall J \in \mathcal{P}, \quad J^2 = \pm \mathrm{Id}. \tag{49}$$

Relation (49) implies that $T^{\mathcal{P}} = P(T^{\nabla}) = 0$ (where P is the projector of Lemma 3). It follows that $\mathcal{P}$ is para-quaternionic.

We now show that the fourth and the fifth conditions are equivalent to any of the first three conditions. From Theorem 11, the fifth condition implies the first. Suppose now that the first (hence also the second and third) condition holds. We will prove the fourth condition when I_p is para-complex (the argument when I_p is a complex structure is similar). Since $\mathcal{J}^+$ is integrable, the distributions $T^{\pm}Z^+ := \mathrm{Ker}(\mathcal{J}^+ \mp \mathrm{Id})$ are involutive. Being transversal, there are local coordinates $(x_1, \ldots, x_{n+2})$ in a neighborhood $\mathcal{U}$ of $I_p \in Z^+$ such that

$$T^+Z^+ = \bigcap_{i=1}^{2m+1} \mathrm{Ker}(dx_i),$$
$$T^-Z^+ := \bigcap_{i=2m+2}^{4m+2} \mathrm{Ker}(dx_i).$$

The para-complex structure $\mathcal{J}^+$ preserves the fibers of the reflector projection $\pi^+ : Z^+ \to M$ and the two distributions $\mathrm{Ker}(\pi^+)_* \cap T^+Z^+$ and $\mathrm{Ker}(\pi^+)_* \cap T^-Z^+$ have rank one. Suppose they are generated on $\mathcal{U}$ by two vector fields, say X^+ and X^- respectively. From our choice of $(x_1, \ldots, x_{n+2})$,

$$dx_1(X^+) = \cdots = dx_{2m+1}(X^+) = 0,$$
$$dx_{2m+2}(X^-) = \cdots = dx_{4m+2}(X^-) = 0.$$

Eventually changing the order of $(x_1, \ldots, x_{2m+1})$, we suppose that $dx_1(X^-) \neq 0$ at I_p; similarly, we can take $dx_{2m+2}(X^+) \neq 0$ at I_p. Let $\mathcal{S}$ be a codimension two submanifold of $\mathcal{U}$, which contains I_p and is defined by

$$x^1 = f(x^2, \ldots, x^{2m+1}),$$
$$x^{2m+2} = g(x^{2m+3}, \ldots, x^{4m+2}),$$

where f, g are smooth functions with all partial derivatives $\frac{\partial f}{\partial x_j}$ $(2 \leqslant j \leqslant 2m+1)$ and $\frac{\partial g}{\partial x_j}$ $(2m+3 \leqslant j \leqslant 4m+2)$ zero at I_p. Then $\mathcal{S}$ intersects the fibers of π^+ transversally in a neighborhood of I_p and the tangent bundle $T\mathcal{S}$ is preserved by $\mathcal{J}^+$. It follows that $\mathcal{S}$ is the image of a compatible almost para-complex structure I of $(M, \mathcal{P})$, viewed as a (local) section of $\pi^+ : Z^+ \to M$. From Lemma 20 the almost para-complex structure I is integrable. Clearly, I extends I_p in a neighborhood of p. We proved that the first condition implies the fourth. The fourth condition implies the fifth. Our claim follows. □

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References

[1] D.V. Alekseevsky, S. Marchiafava, Quaternionic structures on a manifold and subordinated structures, Ann. Mat. Pura Apl. IV CLXXX (1996) 205–273.
[2] D. Alekseevsky, S. Marchiafava, M. Pontecorvo, Compatible complex structures on almost quaternionic manifolds, Trans. AMS 351 (3) (1999) 997–1014.
[3] D. Bernard, Sur la geometrie differentielle des G-structures, Ann. Inst. Fourier 10 (1960) 151–270.
[4] D.E. Blair, J. Davidov, O. Muskarov, Hyperbolic twistor spaces, Rocky Mountains J. Math. 35 (5) (2005) 1437–1465.
[5] P. Gauduchon, Canonical connections for almost-hypercomplex structures, in: V. Ancona, E. Ballico, R.M. Miro-Roig, A. Silva (Eds.), Complex Analysis and Geometry, in: Pitman Research Notes in Mathematics Series, vol. 366, 1997.
[6] E. Garcia-Rio, Y. Matsushita, R. Vasquez-Lorentzo, Para-quaternionic Kähler manifolds, Rocky Mountains J. Math. 31 (2001) 237–260.
[7] S. Ivanov, S. Zamkovoy, Para-Hermitian and para-quaternionic manifolds, Diff. Geom. Appl. 23 (2005) 205–234.
[8] S. Ivanov, I. Minchev, S. Zamkovoy, Twistor and reflector spaces of almost para-quaternionic manifolds, math.DG/0511657.
[9] S. Kobayashi, Transformation Groups, Erg. der Math., vol. 70, Springer-Verlag, 1972.
[10] V. Oproiu, Integrability of almost quaternal structures, An. St. Univ. "Al. I. Cuza" Iasi 30 (1984) 75–84.
[11] M. Pontecorvo, Complex structures on four manifolds, Math. Ann. 309 (1997) 159–177.
[12] S. Salamon, Special structures on four manifolds, Riv. Mat. Univ. Parma (4) 17 (1991) 109–123.
[13] S. Zamkovoy, Geometry of paraquaternionic Kähler manifolds with torsion, math.DG/0511595.