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Ljusternik–Schnirelmann Theorem for the Generalized Laplacian

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$$\begin{cases} -\operatorname{div}\left(\dfrac{m(|\nabla u|)}{|\nabla u|}\nabla u\right)=\lambda g(u) & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega \end{cases}$$

has an infinite number of eigenfunctions on the level set $\int_\Omega M(|\nabla u|)=r$, where $M(t)=\int_0^{|t|} m(s)\,ds$ and $g\colon \mathbb{R}\to\mathbb{R}$ is odd satisfying some growth condition. Moreover, we show that the sequence of associated eigenvalues tends to infinity. We emphasize that no $\triangle_2$-condition is needed for M or for its conjugate, so the associated functionals are not continuously differentiable, in general. © 2000 Academic Press

1. INTRODUCTION

In this paper we shall study the problem

$$\begin{cases} -\triangle_m(u)=\lambda g(u) & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N$ is an open and bounded subset, $\triangle_m(u)=\nabla\cdot((m(|\nabla u|)/|\nabla u|)\,\nabla u)$ is *the generalized Laplacian*, $m\colon [0,\infty[\ \to [0,\infty[$ is an increasing and continuous function with $m(t)=0$ if and only if $t=0$ and $\lim_{t\to\infty} m(t)=\infty$. The function $g\colon\mathbb{R}\to\mathbb{R}$ is odd and satisfies some growth condition to be specified later.

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If $m(t)=t$ and $g(t)=t$, then the problem (1.1) is the Dirichlet eigenvalue problem for the Laplacian:

$$\begin{cases} -\triangle u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

The classical Courant minimax principle guarantees the existence of an infinite sequence of eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ of (1.2) with $\lim_{n\to\infty}\lambda_n=\infty$ (see [19, 21]).

If $m(t)=t^{p-1}$ and $g(t)=|t|^{p-2}\,t$ with $1<p<\infty$, then the problem (1.1) is reduced to the eigenvalue problem for the p-Laplacian:

$$\begin{cases} -\nabla\cdot(|\nabla u|^{p-2}\,\nabla u)=\lambda\,|u|^{p-2}\,u & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega. \end{cases}$$

Applying the Ljusternik–Schnirelmann theory for the functionals

$$F(u)=\int_\Omega |\nabla u|^p\,dx$$

and

$$G(u)=\int_\Omega |u|^p\,dx$$

in the Sobolev space $W_0^{1,\,p}(\Omega)$ we get again the existence of eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty}\lambda_n=\infty$. It is essential that the space $W_0^{1,\,p}(\Omega)$ is reflexive and separable and the corresponding functionals F and G are Fréchet-differentiable (see [9, 10, 21]). For results concerning the first eigenvalue of the p-Laplacian we refer to [2, 17].

In the general case the suitable function space for studying the problem (1.1) is the Orlicz–Sobolev space $W_0^1L_M(\Omega)$, where $M(t)=\int_0^t m(s)\,ds$. If M and $\bar{M}$ (the conjugate of M) satisfy the $\triangle_2$-condition, then the space $W_0^1L_M(\Omega)$ is separable and reflexive and hence has the usual structure (see [5]) and the functionals

$$F(u)=\int_\Omega M(|\nabla u|)\,dx$$

and

$$G(u)=\int_\Omega\int_0^u g(t)\,dt\,dx$$

are Fréchet-differentiable. Consequently, we may apply again the Ljusternik–Schnirelmann theory to get an infinite sequence of eigenvalues tending to infinity (see [8–10, 21]).

If we do not impose the Δ_2-condition on M, then the problem (1.1) becomes more complicated since the space $W_0^1 L_M(\Omega)$ is not reflexive or separable, in general, nor are the functionals F and Fréchet-differentiable. In [18] it is proved that for any $r>0$ the minimization problem

$$\inf\left\{\int_\Omega M(|\nabla u|)\,dx \;\middle|\; u\in W_0^1 L_M(\Omega),\ \int_\Omega M(u)\,dx=r\right\}$$

has a solution u_r satifying (1.1) in $W^{-1}L_{\bar{M}}(\Omega)$ for some $\lambda>0$ with $g(u)=m(u)$ without any extra condition on M.

In this paper we shall prove the existence of an infinite sequence of eigenvalues for (1.1) tending to infinity without any extra condition on M. Hence the Ljusternik–Schnirelmann theory is not available due to nonreflexivity of the space $W_0^1 L_M(\Omega)$ and lack of differentiability of F and G. Our method is based on Galerkin approximation and pseudomonotonicity of the operator $-\triangle_m$ using a modified usual structure in a general complementary system. Similar approach for smooth functionals in reflexive Banach spaces can be found in [4, 9, 10].

2. PREREQUISITIES

We begin with some preliminaries on Orlicz–Sobolev spaces. Let Ω be a bounded open subset in $\mathbb{R}^N$ and let $M\colon \mathbb{R}\to\mathbb{R}$ be an *N-function*, i.e., even, convex and continuous with $M(t)>0$ for $t>0$, $M(t)/t\to 0$ as $t\to 0$ and $M(t)/t\to+\infty$ as $t\to+\infty$. M is an N-function if and only if it can be represented in the form

$$M(t)=\int_0^{|t|} m(s)\,ds \tag{2.1}$$

where $m\colon [0,\infty[\ \to [0,\infty[$ is increasing, right continuous, $m(t)=0$ if and only if $t=0$ and $m(t)\to+\infty$ as $t\to+\infty$. We extend m to $\mathbb{R}$ by $m(t)=-m(-t)$ for $t<0$ (odd continuation). The *Orlicz class* $\mathscr{L}_M(\Omega)$ is defined as the set of real-valued measurable functions u on Ω such that

$$\int_\Omega M(u)\,dx<\infty.$$

The *Orlicz space* $L_M(\Omega)$ is the linear hull of $\mathscr{L}_M(\Omega)$. Then $L_M(\Omega)$ is a Banach space with respect to the *Luxemburg norm*

$$\|u\|_{(M)}=\inf\left\{k>0 \;\middle|\; \int_\Omega M\left(\frac{u}{k}\right)dx\leqslant 1\right\}.$$

It is easily seen that

$$\|u\|_{(M,\,r)} = \inf\left\{k>0 \;\middle|\; \int_\Omega M\left(\frac{u}{k}\right) dx \leqslant r\right\}$$

defines an equivalent norm for all $r>0$. One has $L_M(\Omega)=\mathscr{L}_M(\Omega)$ if and only if M satisfies the $\triangle_2$-*condition*: there exist $\alpha>0$ and $t_0>0$ such that

$$M(2t) \leqslant \alpha\, M(t)$$

for all $t\geqslant t_0$. Moreover, $L_M(\Omega)$ is separable if and only if M satisfies the $\triangle_2$-condition. The closure in $L_M(\Omega)$ of all bounded measurable functions is denoted by $E_M(\Omega)$. Then $E_M(\Omega)\subset\mathscr{L}_M(\Omega)$ and $E_M(\Omega)=\mathscr{L}_M(\Omega)$ if and only if M satisfies the $\triangle_2$-condition. The space E_M is $\sigma(L_M, L_{\bar{M}})$ dense in $L_M(\Omega)$. The conjugate N-function $\bar{M}$ is defined by

$$\bar{M}(t)=\sup\{ts-M(s)\mid s\in\mathbb{R}\}.$$

The function $\bar{M}$ is also an N-function and $\bar{\bar{M}}=M$. The space $L_{\bar{M}}(\Omega)$ is the dual space of $E_M(\Omega)$. The space $L_M(\Omega)$ is reflexive if and only if M and $\bar{M}$ satisfy the $\triangle_2$-condition. Note that the norm $\|\cdot\|_{(M,\,r)}$ is $\sigma(L_M, E_{\bar{M}})$ lower semicontinuous for all $r>0$. It is well-known that $L_M(\Omega)\,L_{\bar{M}}(\Omega)\subset L^1(\Omega)$. We recall also Young's inequality:

$$M(x)+\bar{M}(y)\geqslant xy \qquad \text{for all} \quad x,\, y\in\mathbb{R} \tag{2.2}$$

with equality if and only if $x=\bar{m}(y)$ or $y=m(x)$. Define

$$\operatorname{dom}(m)=\{u\in L_M(\Omega)\mid m(u)\in L_{\bar{M}}(\Omega)\}.$$

It can be shown that $E_M(\Omega)\subset\operatorname{dom}(m)\subset\mathscr{L}_M(\Omega)$ and $\operatorname{dom}(m)=L_M(\Omega)$ if and only if M satisfies the $\triangle_2$-condition. Moreover, the mapping $u\to m(u)$ is continuous from $E_M(\Omega)\to L_{\bar{M}}(\Omega)$ if and only if $\bar{M}$ satisfies the $\triangle_2$-condition (see [3, 12, 15]).

Remark 2.1. Typical examples of N-functions satisfying the $\triangle_2$-condition are $(1+|t|)\log(1+|t|)-|t|$ and $|t|^p$ for $p>1$. On the other hand, functions $e^{|t|}-|t|-1$ and $e^{|t|^p}-1$ for $p>1$ are N-functions not satisfying the $\triangle_2$-condition.

The first order Orlicz–Sobolev space of functions in $L_M(\Omega)$ with first distributional derivatives in $L_M(\Omega)$ is denoted by $W^1L_M(\Omega)$. The space $W^1E_M(\Omega)$ is defined analogously. These spaces are identified, as usual, to subspaces of the product $\prod L_M(\Omega)$. The spaces $W_0^1L_M(\Omega)$ and $W_0^1E_M(\Omega)$ are defined as the $\sigma(\prod L_M, \prod E_{\bar{M}})$ closure of $\mathscr{D}(\Omega)$ in $W^1L_M(\Omega)$ and as the norm closure of $\mathscr{D}(\Omega)$ in $W^1E_M(\Omega)$, respectively. The following spaces of distributions will also be used:

$$W^{-1}L_{\bar{M}}(\Omega)=\left\{f\in\mathscr{D}'(\Omega)\;\middle|\; f=f_0-\sum_{i=1}^{N} D_i f_i \text{ with } f_i\in L_{\bar{M}}(\Omega)\right\}$$

$$W^{-1}E_{\bar{M}}(\Omega)=\left\{f\in\mathscr{D}'(\Omega)\;\middle|\; f=f_0-\sum_{i=1}^{N} D_i f_i \text{ with } f_i\in E_{\bar{M}}(\Omega)\right\}.$$

They are endowed with their usual quotient norms. It is shown in [11] that if Ω has the segment property, then

$$\begin{pmatrix} Y & Z \\ Y_0 & Z_0 \end{pmatrix}=\begin{pmatrix} W_0^1L_M(\Omega) & W^{-1}L_{\bar{M}}(\Omega) \\ W_0^1E_M(\Omega) & W^{-1}E_{\bar{M}}(\Omega) \end{pmatrix}$$

constitutes a complementary system, i.e., Y and Z are real Banach spaces in duality with respect to a continuous pairing $\langle\cdot,\cdot\rangle$ and Y_0 and Z_0 are closed subspaces of Y and Z respectively such that, by means of $\langle\cdot,\cdot\rangle$, the dual of Y_0 can be identified to Z and that of Z_0 to Y. The pairing between $u\in Y$ and $f=f_0-\sum_{i=1}^{N} D_i f_i\in Z$ is given by

$$\langle u, f\rangle=\int_\Omega\left(u f_0+\sum_{i=1}^{N}(D_i u)\, f_i\right)dx.$$

Standard references on Orlicz and Orlicz–Sobolev spaces include [1, 6, 11–16]. The reader interested in the topological degree theory for monotone-like mappings in Orlicz–Sobolev spaces is referred to [20].

We end this section by recalling the definition of the topological index *genus*. Let X be a real Banach space. Denote the class of all closed symmetric subsets K of X with $0\notin K$ by sym_X. For each non-empty set $K\in sym_X$ we define gen K to be the smallest natural number $n\geqslant 1$ for which there exists an odd and continuous mapping $f\colon K\to\mathbb{R}^n\backslash\{0\}$. If no such number exists, then gen $K=\infty$. We also define gen $\varnothing=0$. The basic properties of genus can be found in [21]. We will list here some of them which will be utilized in Chapter 4:

(i) gen $K\leqslant$ dim X for all $K\in sym_X$

(ii) gen$\{u\in X \mid \|u\|=1\}=$ dim X

(iii) if $K\in sym_X$ is compact and $P\colon K\to X$ is odd and continuous with $0\notin P(K)$, then $P(K)\in sym_X$ and gen $K\leqslant$ gen $P(K)$.

3. A USUAL STRUCTURE IN A COMPLEMENTARY SYSTEM

Let $(Y, Y_0; Z, Z_0)$ be a complementary system with Y_0 and Z_0 separable. The space Y with $\sigma(Y, Z_0)$ topology, being a Hausdorff topological vector space, is regular. Since Y_0 is separable, the norm topology in Y_0 is Lindelöf implying $\sigma(Y, Z_0)$ topology in Y_0, being coarser, is also Lindelöf. Hence the space Y_0 with the topology $\sigma(Y_0, Z_0)$ is a regular Lindelöf space implying it is paracompact and normal (see [7]).

Now we are ready to prove the existence of a usual structure in a complementary system, with Y_0 and Z_0 separable. We shall utilize the ideas in [5], in which a similar result is proved for the reflexive case.

THEOREM 3.1. *Assume $(Y, Y_0; Z, Z_0)$ is a complementary system, Y_0 and Z_0 are separable, the norm $\|\cdot\|_Z$ is dual to $\|\cdot\|_{Y_0}$, the norm $\|\cdot\|_Y$ is dual to $\|\cdot\|_{Z_0}$ and $V \subset Y_0$ is a norm-dense linear subspace. Then there exists a sequence of mappings $P_n: Y_0 \to Y_0$, $n = 1, 2, \ldots$, satisfying*

(i) *P_n is odd and norm-continuous for all $n = 1, 2, \ldots$*

(ii) *$P_n(Y_0)$ is contained in a finite-dimensional subspace of V for all $n = 1, 2, \ldots$*

(iii) *if $\{u_n\} \subset Y_0$ and $u_n \to u \in Y$ for $\sigma(Y, Z_0)$, then $P_n(u_n) \to u$ for $\sigma(Y, Z_0)$*

(iv) *if $\{u_n\} \subset Y_0$ and $u_n \to u \in Y$ strongly, then $\|P_n(u_n)\|_Y \to \|u\|_Y$.*

Proof. Let $\{z_i\}_{i=1} \subset Z_0$ be a norm-dense countable set in Z_0 and denote

$$B_n = \{x \in Y \mid \|x\|_Y \leqslant n\}$$

$$Y_n = \left\{x \in Y \,\middle|\, |\langle x, z_1\rangle| + |\langle x, z_2\rangle| \cdots + |\langle x, z_n\rangle| < \frac{1}{n}\right\}$$

for $n = 1, 2, \ldots$. Note that the set B_n is $\sigma(Y, Z_0)$ closed, being $\sigma(Y, Z_0)$ compact due to Alaoglu's theorem, and the set Y_n is $\sigma(Y, Z_0)$ open. By regularity of $\sigma(Y, Z_0)$, for every $x \in Y$ there exists a $\sigma(Y, Z_0)$ open subset $\mathscr{W}_x \subset Y_n$ with $0 \in \mathscr{W}_x$ and $\mathscr{W}_x = -\mathscr{W}_x$ satisfying

$$(x + 2\mathscr{W}_x) \cap \{y \in Y \mid \|y\|_Y \leqslant \|x\|_Y - 1/n\} = \varnothing. \tag{3.1}$$

Since the norm $\|\cdot\|_Y$ is dual to $\|\cdot\|_{Z_0}$, the set $\{y \in Y_0 \mid \|y\|_Y \leqslant \|x\|_Y\}$ is $\sigma(Y, Z_0)$ dense in $\{y \in Y \mid \|y\|_Y \leqslant \|x\|_Y\}$ (see [11, Lemma 1.11]). Hence, on account of norm-density of V in Y_0, the set $\{v \in V \mid \|v\|_Y \leqslant \|x\|_Y\}$ is also $\sigma(Y, Z_0)$ dense in $\{y \in Y \mid \|y\|_Y \leqslant \|x\|_Y\}$. Since the set $x + \mathscr{W}_x$ is $\sigma(Y, Z_0)$ open, the intersection

$$\{v \in V \mid \|v\|_Y \leqslant \|x\|_Y\} \cap (x + \mathscr{W}_x)$$

is non-empty. Consequently, for every $x \in Y$ we may choose y_x from this intersection, i.e., $y_x \in V$ satisfies $y_x \in x + \mathscr{W}_x$ and $\|y_x\|_Y \leqslant \|x\|_Y$. Clearly

$$B_n \subset \bigcup_{x \in B_n} (y_x + \mathscr{W}_x).$$

Since the set B_n is $\sigma(Y, Z_0)$ compact, there exists a finite subcover

$$B_n \subset \bigcup_{i=1}^{k} (y_{x_i} + \mathscr{W}_{x_i}).$$

Denote

$$K_n = B_n \cap Y_0$$

$$W_i = (y_{x_i} + \mathscr{W}_{x_i}) \cap K_n$$

Then K_n is $\sigma(Y_0, Z_0)$ closed in Y_0, W_i is $\sigma(Y_0, Z_0)$ open in K_n and $K_n = \bigcup_{i=1}^{k} W_i$. Due to paracompactness of $\sigma(Y_0, Z_0)$, there exists a partition of unity $\{g_i\}_{i=1}^{k}$ such that

$$g_i \colon K_n \to [0, 1] \qquad \text{is } \sigma(Y_0, Z_0) \text{ continuous}$$

$$\sum_{i=1}^{k} g_i(x) = 1 \qquad \text{for all} \quad x \in K_n$$

$$g_i(x) = 0 \qquad \text{if} \quad x \in K_n \setminus W_i.$$

Since K_n is $\sigma(Y_0, Z_0)$ closed and $\sigma(Y_0, Z_0)$ is normal, we may extend each g_i by Tietze's extension theorem to a $\sigma(Y_0, Z_0)$ continuous map $\tilde{g}_i \colon Y_0 \to [0, 1]$. Define a mapping $Q_n \colon Y_0 \to Y_0$ by

$$Q_n(x) = \sum_{i=1}^{k} \tilde{g}_i(x)\, y_{x_i}.$$

Then Q_n is continuous from $\sigma(Y_0, Z_0)$ topology to the norm topology and hence norm-continuous. Moreover,

$$Q_n(x) - x \in Y_n \qquad \text{for all} \quad x \in K_n.$$

Indeed, suppose $x \in K_n$ and $\tilde{g}_i(x) \neq 0$. Then $x \in W_i$ implying $x \in y_{x_i} + \mathscr{W}_{x_i}$ and consequently $y_{x_i} - x \in \mathscr{W}_{x_i} \subset Y_n$. Therefore

$$Q_n(x) - x = \sum_{i=1}^{k} \tilde{g}_i(x)(y_{x_i} - x) \subset Y_n,$$

since the set Y_n is convex. Next we shall prove that

$$\|Q_n(x)\|_Y \leqslant \|x\|_Y + 1/n \qquad \text{for all} \quad x \in K_n.$$

Suppose $x \in K_n$ and $\tilde{g}_i(x) \neq 0$. Then $x \in y_{x_i} + \mathscr{W}_{x_i}$ implying $x \in x_i + 2\mathscr{W}_{x_i}$. By (3.1), $\|x\|_Y > \|x_i\|_Y - 1/n$ and consequently

$$\begin{aligned} \|Q_n(x)\|_Y &\leqslant \sum_{i=1}^{k} \tilde{g}_i(x)\, \|y_{x_i}\|_Y \leqslant \sum_{i=1}^{k} \tilde{g}_i(x)\, \|x_i\|_Y \\ &\leqslant \sum_{i=1}^{k} \tilde{g}_i(x)(\|x\|_Y + 1/n) = \|x\|_Y + 1/n. \end{aligned}$$

Define a mapping $P_n\colon Y_0 \to Y_0$ by

$$P_n(x) = \tfrac{1}{2}(Q_n(x) - Q_n(-x)).$$

Then clearly (i) and (ii) hold.

Next we shall deduce part (iii). Suppose $\{u_n\} \subset Y_0$ and $u_n \to u \in Y$ for $\sigma(Y, Z_0)$. Then $M = \sup_n \|u_n\|_Y$ is finite and consequently, $\pm u_n \in K_n$ for all $n > M$ implying

$$\|P_n(u_n)\|_Y \leqslant \tfrac{1}{2}(\|Q_n(u_n)\|_Y + \|Q_n(-u_n)\|_Y) \leqslant \|u_n\|_Y + 1/n \tag{3.2}$$

for all $n > M$. Therefore the sequence $\{P_n(u_n)\}$ is bounded in Y. Let $i \in \mathbb{N}$ be arbitrary. If $n > \max\{i, M\}$, then $\pm u_n \in K_n$ and

$$P_n(u_n) - u_n = \tfrac{1}{2}(Q_n(u_n) - u_n) - \tfrac{1}{2}(Q_n(-u_n) + u_n) \in Y_n.$$

Hence

$$\begin{aligned} |\langle z_i, P_n(u_n) - u\rangle| &\leqslant |\langle z_i, P_n(u_n) - u_n\rangle| + |\langle z_i, u_n - u\rangle| \\ &\leqslant 1/n + |\langle z_i, u_n - u\rangle| \to 0, \end{aligned}$$

when $n \to \infty$. Since the set $\{z_i\}$ is norm-dense in Z_0 and $P_n(u_n)$ remains bounded in Y, we get $P_n(u_n) \to u$ for $\sigma(Y, Z_0)$.

Finally, to prove (iv), suppose $\{u_n\} \subset Y_0$, $u \in Y$ and $\|u_n - u\|_Y \to 0$, when $n \to \infty$. Due to (3.2), $\limsup \|P_n(u_n)\|_Y \leqslant \|u\|_Y$. On the other hand, by $\sigma(Y, Z_0)$ lower semicontinuity of $\|\cdot\|_Y$, we have $\|u\|_Y \leqslant \liminf \|P_n(u_n)\|_Y$. Hence the proof is complete.

4. AN EIGENVALUE PROBLEM FOR THE GENERALIZED LAPLACIAN

Let $m: [0, \infty[\to [0, \infty[$ be an increasing continuous function with $m(t) = 0$ if and only if $t = 0$, $m(t) \to \infty$ as $t \to \infty$ and $\Omega \subset \mathbb{R}^N$ an open and bounded subset with the segment property. Let $r > 0$ be given. Let

$$\begin{pmatrix} Y & Z \\ Y_0 & Z_0 \end{pmatrix} = \begin{pmatrix} W_0^1 L_M(\Omega) & W^{-1} L_{\bar{M}}(\Omega) \\ W_0^1 E_M(\Omega) & W^{-1} E_{\bar{M}}(\Omega) \end{pmatrix}$$

be the complementary system formed by Orlicz–Sobolev spaces with the norm

$$\|u\|_Y = \| |\nabla u| \|_{(M, r)} = \left\| \left(\sum_{i=1}^N (D_i u)^2 \right)^{1/2} \right\|_{(M, r)}$$

and the corresponding quotient norm in Z. It is routine to check, based on [11, Lemmas 1.2, 1.11, and 1.12] and [19, Theorem 3.3, p. 135], that the norm $\|\cdot\|_Y$ is dual to $\|\cdot\|_{Z_0}$ and the norm $\|\cdot\|_Z$ is dual to $\|\cdot\|_{Y_0}$. Let $\{v_1, v_2, ...\} \subset \mathcal{D}(\Omega)$ be a countable norm-dense linearly independent subset in Y_0 and $\{P_n\}$ be the sequence of mappings given by Theorem 3.1 with $V = sp\{v_1, v_2, ...\}$. Denote

$$V_n = sp\{v_1, v_2, ..., v_n\}.$$

We shall denote the continuous pairing between Y and Z by $\langle \cdot, \cdot \rangle$ and the one between V_n and V_n^* by $\langle \cdot, \cdot \rangle_n$. For each $n = 1, 2, ...,$ there exists $m_n \in \mathbb{N}$ such that

$$P_n(Y_0) \subset V_{m_n}.$$

Let $g: \mathbb{R} \to \mathbb{R}$ be an odd and continuous function satisfying $g(t)\, t > 0$ for all $t \neq 0$ and

$$|g(t)| \leqslant c_1 + c_2 m(c_3 t) \qquad \text{for all} \quad t \geqslant 0, \tag{4.1}$$

where $c_1, c_2,$ and c_3 are positive constants. Define even functionals $F: D_F \to \mathbb{R}$ and $G: D_G \to \mathbb{R}$ by

$$F(u) = \int_\Omega M(|\nabla u|)\, dx$$

and

$$G(u) = \int_\Omega \int_0^u g(t)\, dt\, dx$$

with $D_F = \{u \in Y \mid F(u) < \infty\}$ and $D_G = \{u \in Y \mid G(u) < \infty\}$. Clearly both functionals vanish only at zero. A straightforward calculation gives $F, G \in C^1(V_n)$ and

$$\langle v, F'(u) \rangle_n = \int_\Omega \frac{m(|\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla v \, dx \qquad \text{for all} \quad u, v \in V_n$$

$$\langle v, G'(u) \rangle_n = \int_\Omega g(u) \, v \, dx \qquad \text{for all} \quad u, v \in V_n$$

for each $n = 1, 2, \ldots$. The growth condition (4.1) and the compact embedding $W_0^1 L_M(\Omega) \to E_M(\Omega)$ ([6, 8, 11, 20]) imply $D_G = Y$ and

$$G(u_n) \to G(u)$$

$$\langle u_n, G'(u_n) \rangle_n \to \int_\Omega g(u) \, u \, dx$$

$$\langle v, G'(u_n) \rangle_n \to \int_\Omega g(u) \, v \, dx$$

whenever $u_n \in V_n$, $u_n \to u \in Y$ for $\sigma(Y, Z_0)$ and $v \in V$. Define a mapping $\triangle_m : D_{\triangle_m} \to Z$ by

$$D_{\triangle_m} = \{u \in Y \mid m(|\nabla u|) \in L_{\bar{M}}(\Omega)\}$$

and

$$\langle v, \triangle_m(u) \rangle = -\int_\Omega \frac{m(|\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla v \, dx \qquad \text{for all} \quad u \in D_{\triangle_m} \quad \text{and} \quad v \in Y.$$

It is clear that $Y_0 \subset D_{\triangle_m} \subset D_F \subset Y$. Moreover, the following monotony properties of the mapping $-\triangle_m$ are well-known (see [11, 14]):

— the mapping $-\triangle_m$ is *monotone*, i.e.

$$\langle -\triangle_m(u) - (-\triangle_m(v)), u - v \rangle > 0 \qquad \text{for all} \quad u, v \in D_{\triangle_m} \quad \text{with} \quad u \neq v$$

— the mapping $-\triangle_m$ is *pseudomonotone*, i.e. the conditions

$$\begin{cases} \{u_n\} \subset D_{\triangle_m} \\ u_n \to u \in Y \qquad \text{for } \sigma(Y, Z_0) \\ -\triangle_m(u_n) \to \chi \in Z \qquad \text{for } \sigma(Z, Y_0) \\ \limsup \langle u_n, -\triangle_m(u_n) \rangle \leqslant \langle u, \chi \rangle \end{cases}$$

imply

$$\begin{cases} u \in D_{\triangle_m} \\ \chi = -\triangle_m(u) \\ \langle u_n, -\triangle_m(u_n) \rangle \to \langle u, \chi \rangle. \end{cases}$$

Define an even and continuous mapping $k_r\colon Y_0 \backslash \{0\} \to]0, \infty[$ by

$$k_r(u) = \frac{1}{\|u\|_Y}.$$

It is easily seen that $F(k_r(u)\, u) = r$ for all $u \in Y_0 \backslash \{0\}$. Moreover, $F(u) = r$ if and only if $k_r(u) = 1$.

In the sequel we denote

$$\mathscr{M}_r = \{u \in Y_0 \mid F(u) = r\}$$

$$\mathscr{K}_i(r) = \{K \subset \mathscr{M}_r \text{ compact and symmetric} \mid \text{gen } K \geqslant i\}$$

$$\mathscr{K}_{i,n}(r) = \{K \subset \mathscr{M}_r \cap V_n \text{ compact and symmetric} \mid \text{gen } K \geqslant i\}$$

$$c_i(r) = \sup_{K \in \mathscr{K}_i(r)} \inf_{u \in K} G(u)$$

$$c_{i,n}(r) = \sup_{K \in \mathscr{K}_{i,n}(r)} \inf_{u \in K} G(u).$$

Note that $c_{i,n}(r)$ increases with n. Choosing K to be the unit sphere of V_i, we have gen $K = i$, $K \subset \mathscr{M}_r$ and $\inf_{u \in K} G(u) > 0$. Hence $c_{i,n}(r) > 0$, when $n \geqslant i$. On the other hand, if $n < i$ then $\mathscr{K}_{i,n}(r)$ is empty.

Now we are ready to attack the problem (1.1). We start with a finite-dimensional result. We shall use the assumptions and notations given above.

LEMMA 4.1. *Let $i \in \mathbb{N}$ be given. Then there exist sequences $\{u_n\}_{n=i}^{\infty} \subset Y_0$ and $\{\lambda_n\}_{n=i}^{\infty} \subset]0, \infty[$ such that*

$$u_n \in V_n$$

$$F(u_n) = r$$

$$G(u_n) = c_{i,n}(r)$$

$$F'(u_n) = \lambda_n G'(u_n) \qquad \text{in } V_n^*$$

for all $n = i, i+1, \ldots$.

Proof. By assumptions, $F, G \in C^1(V_n)$ and $F(0)=0$. Moreover, for each $u \in V_n$ with $u \neq 0$ we have $\langle u, F'(u) \rangle_n > 0$ and $F(k_r(u)\, u) = r$. The claim follows now from the finite dimensional Ljusternik–Schnirelmann theorem ([21, Theorem 44.B]).

Next we shall study the convergence of the sequences $\{\lambda_n\}$ and $\{u_n\}$, when n tends to infinity with fixed r and i.

LEMMA 4.2. *Let $i \in \mathbb{N}$ be given and the sequences $\{u_n\}_{n=i}^{\infty} \subset Y_0$ and $\{\lambda_n\}_{n=i}^{\infty} \subset]0, \infty[$ as given by Lemma 4.1. Then there exist $\bar{u} \in D_{\triangle_m}$ and $\bar{\lambda} \in]0, \infty[$ such that $\lambda_{n'} \to \bar{\lambda}$ and $u_{n'} \to \bar{u}$ for $\sigma(Y, Z_0)$ for some common subsequence. Moreover, $F(\bar{u}) = r$, $G(\bar{u}) = \lim_{n \to \infty} c_{i,n}(r)$ and*

$$-\triangle_m(\bar{u}) = \lambda g(\bar{u}) \qquad \text{in } Z.$$

Proof. Since $\int_\Omega M(|\nabla u_n|)\, dx = r$ for all $n \geqslant i$, the sequence $\{u_n\}$ is bounded in Y. Therefore we may assume that $u_n \to \bar{u} \in Y$ for $\sigma(Y, Z_0)$ for a subsequence implying $G(u_n) \to G(\bar{u})$. Assume $\lambda_n \to \infty$, when $n \to \infty$. Since the sequence $\{c_{i,n}(r)\}_{n=i}^{\infty}$ is increasing, we have $G(\bar{u}) > 0$ implying $g(\bar{u}) \not\equiv 0$. Consequently, there exist $n_0 > i$ and $\phi \in V_{n_0}$ such that

$$\int_\Omega g(\bar{u})(\bar{u} - \phi)\, dx < 0.$$

By monotony of the mapping $-\triangle_m$ we get

$$\begin{aligned} 0 &\leqslant \langle u_n - \phi, -\triangle_m(u_n) - (-\triangle_m(\phi)) \rangle \\ &= \langle u_n - \phi, F'(u_n) - F'(\phi) \rangle_n \\ &= \lambda_n \langle u_n - \phi, G'(u_n) \rangle_n - \langle u_n - \phi, F'(\phi) \rangle_n, \end{aligned} \tag{4.2}$$

when $n \geqslant n_0$. Since

$$\langle u_n - \phi, G'(u_n) \rangle_n \to \int_\Omega g(\bar{u})(\bar{u} - \phi)\, dx < 0$$

and

$$\langle u_n - \phi, F'(\phi) \rangle_n \to \int_\Omega \frac{m(|\nabla \phi|)}{|\nabla \phi|} \nabla \phi \cdot \nabla(\bar{u} - \phi)\, dx,$$

the right hand side of (4.2) tends to $-\infty$, which is a contradiction. Consequently, the sequence $\{\lambda_n\}$ is bounded and thus we may assume that

$\lambda_n \to \bar{\lambda}$ in $\mathbb{R}$ and $u_n \to \bar{u} \in Y$ for $\sigma(Y, Z_0)$ for a common subsequence. Moreover,

$$\begin{aligned}\int_\Omega m(|\nabla u_n|)\,|\nabla u_n|\,dx &= \langle u_n, F'(u_n)\rangle_n \\ &= \lambda_n \langle u_n, G'(u_n)\rangle_n \\ &\to \lambda \int_\Omega g(\bar{u})\,\bar{u}\,dx < \infty\end{aligned}$$

implying $m(|\nabla u_n|)$ remains bounded in $L_{\bar{M}}(\Omega)$. Hence we may assume

$$-\triangle_m(u_n) \to \chi \in Z \qquad \text{for } \sigma(Z, Y_0)$$

for some $\chi \in Z$. Since

$$\langle \phi, \chi\rangle = \lim \langle \phi, F'(u_n)\rangle_n = \lim \lambda_n \langle \phi, G'(u_n)\rangle_n = \lambda \int_\Omega g(\bar{u})\,\phi\,dx$$

for all $\phi \in V$ and V is norm-dense in Y_0, we have

$$\langle \phi, \chi\rangle = \lambda \int_\Omega g(\bar{u})\,\phi\,dx \tag{4.3}$$

for all $\phi \in Y_0$. By the bipolar theorem, Y_0 is $\sigma(Y, Z)$ dense in Y. Hence (4.3) holds for all $\phi \in Y$. Consequently,

$$\begin{aligned}\lim \langle u_n, -\triangle_m(u_n)\rangle &= \lim \langle u_n, F'(u_n)\rangle_n \\ &= \lim \lambda_n \langle u_n, G'(u_n)\rangle_n \\ &= \lambda \int_\Omega g(\bar{u})\,\bar{u}\,dx = \langle \bar{u}, \chi\rangle\end{aligned}$$

implying, due to pseudomonotonicity of the mapping $-\triangle_m$, that $\bar{u} \in D_{\triangle_m}$, $-\triangle_m(\bar{u}) = \chi$ and

$$\int_\Omega m(|\nabla u_n|)\,|\nabla u_n|\,dx \to \int_\Omega m(|\nabla u|)\,|\nabla u|\,dx.$$

Standard argument (see [11, 14]) gives

$$m(|\nabla u_n|)\,|\nabla u_n| \to m(|\nabla \bar{u}|)\,|\nabla \bar{u}|$$

in $L^1(\Omega)$. Consequently, we have a majorant $h \in L^1(\Omega)$ such that

$$M(|\nabla u_n|) \leqslant m(|\nabla u_n|)\, |\nabla u_n| \leqslant h \qquad \text{a.e. in } \Omega.$$

The dominated convergence theorem implies $F(\bar{u}) = \lim F(u_n) = r$.

Finally, we shall study the convergence of the sequences $\{c_{i,n}(r)\}$ and $\{c_i(r)\}$. The proofs are analogous to ones in [9, 10]. The main difference is due to the fact that $w_n \to w$ in Y_0 does not imply $P_n(w_n) \to w$ in Y_0, in general.

LEMMA 4.3. *Let $i \in \mathbb{N}$ be given. Then $c_{i,n}(r) \to c_i(r)$, when $n \to \infty$.*

Proof. Clearly $c_{i,n}(r) \leqslant c_{i,n+1}(r) \leqslant \cdots \leqslant c_i(r)$ for all $n \in \mathbb{N}$. Suppose there exists $\varepsilon > 0$ such that $c_{i,n}(r) < c_i(r) - \varepsilon$ for all n. By the definition of $c_i(r)$, there would exist a set $K_\varepsilon \in \mathscr{K}_i(r)$ such that

$$c_i(r) - \varepsilon/2 < \inf_{w \in K_\varepsilon} G(w). \tag{4.4}$$

It is easy to check that $0 \notin P_n(K_\varepsilon)$ for every n large enough. Indeed, suppose $P_{n_k}(w_{n_k}) = 0$ with $w_{n_k} \in K_\varepsilon$ and $n_k \to \infty$. By compactness of the set K_ε, $w_{n_k} \to w \in K_\varepsilon$ for a subsequence. But part (iii) of Theorem 3.1 implies $w = 0$, which contradicts $0 \notin K_\varepsilon$. Consequently, the map

$$\Psi_n \colon w \to k_r(P_n(w))\, P_n(w)$$

is odd and continuous from K_ε to $\mathscr{M}_r \cap V_{m_n}$ for n large enough. Hence $\Psi_n(K_\varepsilon) \subset \mathscr{K}_{i,m_n}(r)$ implying

$$\inf_{w \in \Psi_n(K_\varepsilon)} G(w) \leqslant c_{i,m_n}(r)$$

for every n large enough. Thus, for every n large enough there exists $w_n \in K_\varepsilon$ such that

$$G(\Psi_n(w_n)) < c_i(r) - \varepsilon. \tag{4.5}$$

Due to compactness, $w_n \to w \in K_\varepsilon$ in Y_0 for a subsequence implying $\|P_n w_n\|_Y \to \|w\|_Y$ and $k_r(P_n w_n) \to k_r(w) = 1$. Consequently,

$$\Psi_n(w_n) \to w \qquad \text{for } \sigma(Y, Z_0)$$

implying $G(\Psi_n(w_n)) \to G(w)$, which contradicts (4.4) and (4.5).

LEMMA 4.4. *$c_i(r) \to 0$, when $i \to \infty$*

Proof. Let $\varepsilon > 0$ be arbitrary. The continuity properties of the mappings $\{P_n\}_{n=1}^{\infty}$ and G imply the existence of $n_0 \in \mathbb{N}$ and $\delta > 0$ such that

$$|G(P_{n_0}(w)) - G(w)| < \varepsilon/2 \qquad \text{for all} \quad w \in \mathcal{M}_r$$

and

$$G(w) < \varepsilon/2 \qquad \text{for all} \quad \|w\|_Y \leqslant \delta.$$

Consequently, if $K \subset \mathcal{M}_r$ is compact and symmetric with $\inf_{w \in K} G(w) > \varepsilon$, then

$$\|P_{n_0}(w)\|_Y \geqslant \delta \qquad \text{for all} \quad w \in K.$$

Hence gen $K \leqslant$ gen $P_{n_0}(K) \leqslant m_{n_0}$. Thus, if $i > m_{n_0}$ and $K \in \mathcal{K}_i(r)$, then

$$\inf_{w \in K} G(w) \leqslant \varepsilon$$

implying the claim.

Now we are ready to prove our main theorem:

THEOREM 4.5. *Let $\Omega \subset \mathbb{R}^N$ be open and bounded with the segment property and $r > 0$. Then there exist sequences $\{\bar{u}_i\}_{i=1}^{\infty} \subset W_0^1 L_M(\Omega)$ and $\{\bar{\lambda}_i\}_{i=1}^{\infty} \subset \,]0, \infty[$ such that*

$$-\nabla \cdot \left(\frac{m(|\nabla \bar{u}_i|)}{|\nabla \bar{u}_i|} \nabla \bar{u}_i \right) = \bar{\lambda}_i g(\bar{u}_i) \qquad \text{in } W^{-1} L_{\bar{M}}(\Omega) \tag{4.6}$$

and

$$\int_\Omega M(|\nabla \bar{u}_i|)\, dx = r \tag{4.7}$$

$$\int_\Omega \int_0^{\bar{u}_i} g(s)\, ds\, dx = c_i(r). \tag{4.8}$$

Moreover, $\bar{\lambda}_i \to \infty$ and $\bar{u}_i \to 0$ for $\sigma(Y, Z_0)$, when $i \to \infty$.

Proof. On account of the previous lemmas, for each $i \in \mathbb{N}$ there exist $\bar{u}_i \in D_{\triangle_m}$ and $\bar{\lambda}_i > 0$ satisfying (4.6), (4.7), and (4.8). By Lemma 4.4, $G(\bar{u}_i) \to 0$, when $i \to \infty$. Since $\bar{u}_i$ remains bounded in Y, we may deduce that $\bar{u}_i \to 0$ for $\sigma(Y, Z_0)$, when $i \to \infty$. By compact embedding and (4.1), $\bar{u}_i \to 0$ in $E_M(\Omega)$ and $g(\bar{u}_i)$ remains bounded in $L_{\bar{M}}(\Omega)$. Using (4.6) we get

$$\bar{\lambda}_i = \frac{\int_\Omega m(|\nabla \bar{u}_i|)\, |\nabla \bar{u}_i|\, dx}{\int_\Omega g(\bar{u}_i)\, \bar{u}_i\, dx} \geqslant \frac{r}{\int_\Omega g(\bar{u}_i)\, \bar{u}_i\, dx} \to \infty,$$

when $i \to \infty$.

Remark 4.6. If M and $\bar{M}$ satisfy the $\triangle_2$-condition, then an analogous result for the problem

$$\begin{cases} -\nabla \cdot \left(\dfrac{m(|\nabla u|)}{|\nabla u|} \nabla u \right) = \lambda a(x)\, m(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

is proved in [8] assuming $a \in L^\infty(\Omega)$, $a \geqslant 0$ in D and $\mu\{x \in D \mid a(x) > 0\} > 0$ for some ball $D \subset \Omega$.

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