



## Cusps in $K_L \rightarrow 3\pi$ decays

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### Abstract

The pion mass difference generates a pronounced cusp in  $K \rightarrow 3\pi$  decays, the strength of which is related to the  $\pi\pi$   $S$ -wave scattering lengths. We apply an effective field theory framework developed earlier to evaluate the amplitudes for  $K_L \rightarrow 3\pi$  decays in a systematic manner, where the strictures imposed by analyticity and unitarity are respected automatically. The amplitudes for the decay  $\eta \rightarrow 3\pi$  are also given.

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**1.** The investigation of the so-called cusp effect in  $K^+ \rightarrow \pi^+\pi^0\pi^0$  decays has become a fully competitive method for the extraction of the  $S$ -wave  $\pi\pi$  scattering lengths from experimental data. Following refined versions of the original proposal by Cabibbo [1–3], the combination  $a_0 - a_2$  has been determined from very high statistics data [4,5] to an accuracy mainly limited by remaining shortcomings in the theoretical description of the decay amplitudes. Missing ingredients are in particular (real and virtual) photon corrections. Here, an important step has recently been performed by Isidori [6], who has evaluated radiative corrections in multi-body meson decays, in particular, for the fully charged channel  $K^+ \rightarrow \pi^+\pi^+\pi^-$ , in the soft photon approximation. Once these corrections are available in all channels,  $K \rightarrow 3\pi$  decays, combined with the information gained from  $K_{e4}$  decays [7,8] and the ponium lifetime [9], have the potential to test the very precise theoretical prediction of the scattering lengths [10,11] experimentally. For recent phenomenological determinations of the scattering lengths, we refer the reader to Refs. [12–14].

As the strong impact of the unitarity cusp near the  $\pi^+\pi^-$  threshold is a universal feature of the  $\pi^0\pi^0$  scattering amplitude [15], it is present also in other decays, like  $K_L \rightarrow 3\pi^0$ ,  $\eta \rightarrow 3\pi^0$ , etc. The strength of the cusp in  $K_L \rightarrow 3\pi^0$  is reduced by about an order of magnitude compared to  $K^+ \rightarrow \pi^+\pi^0\pi^0$ , hence the experimental situation in order to gain information on  $\pi\pi$  scattering lengths is far less favourable [5]. However, the motivation to study this channel all the same is twofold: firstly, experimental efforts to at least see the cusp are under way [5]; secondly, the  $K_L \rightarrow 3\pi^0$  system provides an excellent object for exploratory studies of the most important electromagnetic effects in the cusp region, before immersing oneself into the even more relevant, but simultaneously more difficult case of  $K^+$  decays.

The  $K_L \rightarrow 3\pi$  decays have been studied with regard to the cusp phenomenon before. Ref. [2] uses unitarity, analyticity and cluster decomposition properties of the  $S$ -matrix to investigate the cusp structure. In analogy to the corresponding  $K^+$  decays discussed in the same reference, an expansion in powers of the  $\pi\pi$  scattering lengths  $a$  is used as the essential ordering principle, and the calculation is performed up to  $O(a^2)$ . In Ref. [16], in addition to analyticity and unitarity, chiral perturbation theory is

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used for the evaluation of the real parts of the  $K \rightarrow 3\pi$  decay amplitudes at one loop. For an evaluation of radiative corrections to such decays in the framework of chiral perturbation theory, see Refs. [17,18]. In the present work, however, we rely on the non-relativistic effective field theory framework developed in Ref. [3]. It is based on an effective Lagrangian, and as such satisfies all unitarity and analyticity constraints automatically. The coupling constants involved can be directly matched to  $\pi\pi$  scattering lengths, and the expansion in powers thereof as advocated in Ref. [2] emerges naturally in a generalised power counting scheme.

Our presentation closely follows that of Ref. [3], allowing for a relatively concise description of the procedure. We construct the most general non-relativistic Lagrangian required for the process in question, and match the couplings to the  $\pi\pi$  threshold parameters. Thenceforth the calculation of the decay amplitude up to two-loop order is straightforward. Our representation of tree, one-loop, and two-loop contributions correctly reproduces the analytic structure with various branch points and cusps in the Mandelstam plane throughout the physical region (and slightly beyond). The pertinent calculation of the radiative corrections within the same framework will follow in due course [19].

**2.** We consider the neutral and charged decay modes  $K_L(P_K) \rightarrow \pi^0(p_1)\pi^0(p_2)\pi^0(p_3)$  and  $K_L(P_K) \rightarrow \pi^+(p_1)\pi^-(p_2)\pi^0(p_3)$ . The kinematical variables are defined as usual:  $s_i = (P_K - p_i)^2$  with  $p_i^2 = M_i^2$ ,  $i = 1, 2, 3$ , where  $M_{\pi^+} \doteq M_\pi$  and  $M_{\pi^0}$  denote the masses of the charged and neutral pions, respectively, and  $\Delta_\pi = M_{\pi^+}^2 - M_{\pi^0}^2 \neq 0$ . In the centre-of-mass frame  $P_K = (M_K, \mathbf{0})$ , with  $M_K$  the neutral kaon mass,

$$p_i^0 = \frac{M_K^2 + M_i^2 - s_i}{2M_K}, \quad \mathbf{p}_i^2 = \frac{\lambda(M_K^2, M_i^2, s_i)}{4M_K^2}, \quad (1)$$

where  $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$  is the triangle function. Below we also use the velocities  $v_{jk}$  and kinetic energies  $T_i$ ,

$$v_{jk}^2(s_i) = \frac{\lambda(s_i, M_j^2, M_k^2)}{s_i^2}, \quad T_i = p_i^0 - M_i. \quad (2)$$

**3.** We invoke the non-relativistic framework set up in Ref. [3] for the evaluation of the pertinent decay amplitudes. In that framework, the perturbative expansion is performed in terms of two formal parameters  $\epsilon$  and  $a$ . One counts the pion and kaon masses as  $O(1)$ , the pion momenta as  $O(\epsilon)$  and the pion mass difference  $\Delta_\pi$  as  $O(\epsilon^2)$ . In addition, each four-pion vertex is counted as a quantity of order  $a$ . As these vertices are proportional to the  $\pi\pi$  scattering lengths which are small, one expects the expansion in  $a$  to converge rapidly. We refer for a further discussion of the method to the original article [3]. Here, we simply note that it is sufficient to provide the Lagrangian used—the amplitudes then follow from a straightforward application of the rules provided in Ref. [3].

**4.** The complete Lagrangian of the effective theory is  $\mathcal{L}_K + \mathcal{L}_{\pi\pi}$ , where  $\mathcal{L}_K$  contains  $K_L \rightarrow 3\pi$  vertices, and  $\mathcal{L}_{\pi\pi}$  describes elastic  $\pi\pi$  scattering. In the following, we provide the Lagrangians necessary to calculate the amplitudes for  $K_L \rightarrow 3\pi$  at order  $\epsilon^4$ ,  $a\epsilon^5$ ,  $a^2\epsilon^2$ .

We start with the  $\pi\pi$  interaction and consider the following five physical channels in  $\pi^a\pi^b \rightarrow \pi^c\pi^d$ : (ab; cd) = (1) (00; 00), (2) (+0; +0), (3) (+−; 00), (4) (+−; +−), (5) (++; ++). [We omit the channel  $\pi^-\pi^0 \rightarrow \pi^-\pi^0$ , because this amplitude is identical to  $\pi^+\pi^0 \rightarrow \pi^+\pi^0$  by charge invariance.] The Lagrangian takes the form

$$\mathcal{L}_{\pi\pi} = 2 \sum_{\pm} \Phi_{\pm}^{\dagger} W_{\pm} (i\partial_t - W_{\pm}) \Phi_{\pm} + 2\Phi_0^{\dagger} W_0 (i\partial_t - W_0) \Phi_0 + \sum_{i=1}^5 \mathcal{L}_i, \quad (3)$$

where  $\Phi_i$  is the non-relativistic pion field operator,  $W_{\pm} = \sqrt{M_{\pi}^2 - \Delta}$ ,  $W_0 = \sqrt{M_{\pi^0}^2 - \Delta}$ , with  $\Delta$  the Laplacian. Introducing further the notations

$$\begin{aligned} (\Phi_n)_{\mu} &= (\mathcal{P}_n)_{\mu} \Phi_n, & (\Phi_n)_{\mu\nu} &= (\mathcal{P}_n)_{\mu} (\mathcal{P}_n)_{\nu} \Phi_n, & (\mathcal{P}_n)_{\mu} &= (W_n, -i\nabla), \\ (\Phi_n^{\dagger})_{\mu} &= (\mathcal{P}_n^{\dagger})_{\mu} \Phi_n^{\dagger}, & (\Phi_n^{\dagger})_{\mu\nu} &= (\mathcal{P}_n^{\dagger})_{\mu} (\mathcal{P}_n^{\dagger})_{\nu} \Phi_n^{\dagger}, & (\mathcal{P}_n^{\dagger})_{\mu} &= (W_n, i\nabla), \end{aligned} \quad (4)$$

for  $n = a, b, c, d$ , one may write

$$\begin{aligned} \mathcal{L}_i &= x_i C_i (\Phi_c^{\dagger} \Phi_d^{\dagger} \Phi_a \Phi_b + \text{h.c.}) + x_i D_i \{ (\Phi_c^{\dagger})_{\mu} (\Phi_d^{\dagger})^{\mu} \Phi_a \Phi_b + \Phi_c^{\dagger} \Phi_d^{\dagger} (\Phi_a)_{\mu} (\Phi_b)^{\mu} - h_i \Phi_c^{\dagger} \Phi_d^{\dagger} \Phi_a \Phi_b + \text{h.c.} \} \\ &+ \frac{u_i E_i}{2} \{ (\Phi_c^{\dagger} (\Phi_d^{\dagger})^{\mu} - (\Phi_c^{\dagger})^{\mu} \Phi_d^{\dagger}) ((\Phi_a)_{\mu} \Phi_b - \Phi_a (\Phi_b)_{\mu}) + \text{h.c.} \} \\ &+ x_i F_i \{ (\Phi_c^{\dagger})_{\mu\nu} (\Phi_d^{\dagger})^{\mu\nu} \Phi_a \Phi_b + \Phi_c^{\dagger} \Phi_d^{\dagger} (\Phi_a)_{\mu\nu} (\Phi_b)^{\mu\nu} + 2(\Phi_c^{\dagger})_{\mu} (\Phi_d^{\dagger})^{\mu} (\Phi_a)_{\nu} (\Phi_b)^{\nu} + h_i^2 \Phi_c^{\dagger} \Phi_d^{\dagger} \Phi_a \Phi_b \\ &- 2h_i ((\Phi_c^{\dagger})_{\mu} (\Phi_d^{\dagger})^{\mu} \Phi_a \Phi_b + \Phi_c^{\dagger} \Phi_d^{\dagger} (\Phi_a)_{\mu} (\Phi_b)^{\mu}) + \text{h.c.} \} + \dots, \end{aligned} \quad (5)$$

with  $h_i = s_i^t - \frac{1}{2}(M_a^2 + M_b^2 + M_c^2 + M_d^2)$ , where  $s_i^t$  denotes the physical threshold in the  $i$ th channel. Explicitly,  $h_1 = 2M_{\pi^0}^2$ ,  $h_2 = 2M_{\pi^+}M_{\pi^0}$ ,  $h_3 = 3M_{\pi^+}^2 - M_{\pi^0}^2$ ,  $h_4 = h_5 = 2M_{\pi^+}^2$ . The ellipsis stands for terms of order  $\epsilon^6$  in the  $S$ -wave and for terms of order  $\epsilon^4$  in the  $P$ - and  $D$ -waves. The low-energy constants  $C_i, D_i, E_i, F_i$  are matched to the physical threshold amplitudes below. To simplify the resulting expressions, we have furthermore introduced the combinatorial factors  $x_1 = x_5 = 1/4$ ,  $x_2 = x_3 = x_4 = 1$ ,  $u_1 = u_3 = u_5 = 0$ ,  $u_2 = u_4 = 1$ . Finally, we note that we omit local 6-pion couplings. Their contribution to the  $K_L \rightarrow 3\pi$  amplitude is purely imaginary in the non-relativistic framework, and of order  $\epsilon^4$ .

**5.** The couplings  $C_i, D_i, E_i, F_i$  can be expressed in terms of the threshold parameters of the underlying relativistic theory. In the isospin symmetry limit, the expansion of the relativistic  $\pi\pi$  scattering amplitude reads

$$\text{Re } \bar{T}_i(s, t) = \bar{A}_i \left\{ 1 + \frac{\bar{r}_i}{4M_{\pi}^2}(s - 4M_{\pi}^2) + \frac{\bar{f}_i}{16M_{\pi}^4}(s - 4M_{\pi}^2)^2 \right\} + \frac{3}{4}\bar{A}_i^P(t - u) + \dots \quad (6)$$

The ellipsis stands for higher orders in  $\epsilon$ , e.g.  $D$ -wave contributions. The bar indicates the isospin symmetric limit, at  $M_{\pi} = 139.57$  MeV. In terms of the standard scattering lengths  $a_0, a_2$  and  $a_1$ , one has

$$\begin{aligned} 3\bar{A}_1 &= N(a_0 + 2a_2), & 2\bar{A}_2 &= Na_2, & 3\bar{A}_3 &= N(a_2 - a_0), & 6\bar{A}_4 &= N(2a_0 + a_2), & \bar{A}_5 &= Na_2, \\ 2\bar{A}_2^P &= Na_1, & 2\bar{A}_4^P &= Na_1, & \bar{A}_1^P &= \bar{A}_3^P = \bar{A}_5^P = 0, & N &= 32\pi, \end{aligned} \quad (7)$$

with  $a_0 = 0.220 \pm 0.005$ ,  $a_2 = -0.0444 \pm 0.0010$ ,  $a_0 - a_2 = 0.265 \pm 0.004$ ,  $a_1 = (0.379 \pm 0.005) \times 10^{-1} M_{\pi}^{-2}$  [11]. The products  $\bar{A}_i \bar{r}_i$  and  $\bar{A}_i \bar{f}_i$  denote effective ranges and shape parameters, respectively.

Still in the isospin symmetry limit, the couplings  $C_i$  are related to these threshold parameters according to

$$2\bar{C}_i = \bar{A}_i, \quad 8M_{\pi}^2 \bar{D}_i = \bar{A}_i \bar{r}_i, \quad 32M_{\pi}^4 \bar{F}_i = \bar{A}_i \bar{f}_i, \quad 4\bar{E}_i = 3\bar{A}_i^P, \quad (8)$$

where we have dropped higher-order terms in the threshold parameters. Taking isospin breaking into account, one finds at leading order in chiral perturbation theory [20]

$$2C_{1,2,5} = \bar{A}_{1,2,5}(1 - \eta), \quad 2C_3 = \bar{A}_3(1 + \eta/3), \quad 2C_4 = \bar{A}_4(1 + \eta), \quad (9)$$

where  $\eta = \Delta_{\pi}/M_{\pi}^2 = 6.5 \times 10^{-2}$ . Isospin breaking in the remaining couplings  $D_i, E_i, F_i$  is expected to have a negligible effect on the analysis, and we propose to use for these couplings the relations (8) also in the real world, where isospin is broken.

**6.** It remains to display the  $K_L \rightarrow 3\pi$  Lagrangian,

$$\begin{aligned} \mathcal{L}_K &= 2K^{\dagger} W_K (i\partial_t - W_K) K + L_0 (K^{\dagger} \Phi_0 \Phi_+ \Phi_- + \text{h.c.}) \\ &+ L_1 (K^{\dagger} (W_0 - M_{\pi^0}) \Phi_0 \Phi_+ \Phi_- + \text{h.c.}) + L_2 (K^{\dagger} (W_0 - M_{\pi^0})^2 \Phi_0 \Phi_+ \Phi_- + \text{h.c.}) \\ &+ L_3 (K^{\dagger} \Phi_0 (W_{\pm}^2 \Phi_+ \Phi_- + \Phi_+ W_{\pm}^2 \Phi_- - 2W_{\pm} \Phi_+ W_{\pm} \Phi_-) + \text{h.c.}) \\ &+ \frac{1}{6} K_0 (K^{\dagger} \Phi_0^3 + \text{h.c.}) + \frac{1}{2} K_1 (K^{\dagger} \Phi_0^2 (W_0 - M_{\pi^0})^2 \Phi_0 + \text{h.c.}) + \dots, \end{aligned} \quad (10)$$

where  $K$  denotes the non-relativistic field for the  $K_L$  meson,  $W_K = \sqrt{M_K^2 - \Delta}$ , and the ellipsis stands for the higher-order terms in  $\epsilon$ . The couplings  $L_i, K_i$  are assumed to be real. Their contribution to the decay matrix elements at tree-level is provided below.

The tree-level expressions for the amplitudes, generated by  $\mathcal{L}_K$ , are modified by final state interactions of the pions, generated by loops evaluated with  $\mathcal{L}_{\pi\pi}$ . We use the notation

$$\begin{aligned} \mathcal{M}_{000} &= \mathcal{M}_N^{\text{tree}} + \mathcal{M}_N^{1\text{-loop}} + \mathcal{M}_N^{2\text{-loops}} + \dots \quad [K_L \rightarrow \pi^0 \pi^0 \pi^0], \\ \mathcal{M}_{+-0} &= \mathcal{M}_C^{\text{tree}} + \mathcal{M}_C^{1\text{-loop}} + \mathcal{M}_C^{2\text{-loops}} + \dots \quad [K_L \rightarrow \pi^+ \pi^- \pi^0] \end{aligned} \quad (11)$$

for the decay amplitudes and the Condon–Shortley phase convention for the pions. Our amplitudes are normalised such that the decay rates are given by

$$d\Gamma = \frac{1}{2M_K} (2\pi)^4 \delta^{(4)}(P_f - P_i) |\mathcal{M}|^2 \prod_{i=1}^3 \frac{d^3 \mathbf{p}_i}{2(2\pi)^3 p_i^0}. \quad (12)$$

In the case of  $K_L \rightarrow 3\pi^0$ , the right-hand side must be divided by  $3! = 6$ .

7. The *tree* amplitudes are

$$\begin{aligned}\mathcal{M}_0^{\text{tree}} &= K_0 + K_1(X_1^2 + X_2^2 + X_3^2), \\ \mathcal{M}_{\pm}^{\text{tree}} &= L_0 + L_1 X_3 + L_2 X_3^2 + L_3(X_1 - X_2)^2,\end{aligned}\quad (13)$$

where  $X_i = p_i^0 - M_{\pi^0}$ . This representation is equivalent to

$$\begin{aligned}\mathcal{M}_0^{\text{tree}} &= U_0 + U_1\left(u^2 + \frac{v^2}{3}\right), \\ \mathcal{M}_{\pm}^{\text{tree}} &= V_0 + V_1(s_3 - s_c) + V_2(s_3 - s_c)^2 + V_3(s_2 - s_1)^2,\end{aligned}\quad (14)$$

where

$$\begin{aligned}u &= s_3 - s_n, & v &= s_2 - s_1, \\ s_n &= \frac{M_K^2 + 3M_{\pi^0}^2}{3}, & s_c &= \frac{M_K^2 + M_{\pi^0}^2 + 2M_{\pi}^2}{3}.\end{aligned}\quad (15)$$

The relations between the coefficients  $U_i$ ,  $V_i$  and  $L_i$ ,  $K_i$  are displayed in [Appendix A](#).

8. The *one-loop* contributions are proportional to the basic integral

$$J_{ab}(P^2) = \int \frac{d^D l}{i(2\pi)^D} \frac{1}{2w_a(\mathbf{l})2w_b(\mathbf{P}-\mathbf{l})} \frac{1}{(w_a(\mathbf{l})-l_0)(w_b(\mathbf{P}-\mathbf{l})-P_0+l_0)},\quad (16)$$

with  $w_{\pm}(\mathbf{p}) = \sqrt{M_{\pi}^2 + \mathbf{p}^2}$ ,  $w_0(\mathbf{p}) = \sqrt{M_{\pi^0}^2 + \mathbf{p}^2}$  and  $P^2 = P_0^2 - \mathbf{P}^2$ . In the limit  $D \rightarrow 4$ ,

$$J_{ab}(P^2) = \frac{i}{16\pi} v_{ab}(P^2),\quad (17)$$

which is a quantity of order  $\epsilon$ . In order to make the formulae more transparent, we modify the notation for the couplings  $C_i$ ,  $D_i$ ,  $E_i$ ,  $F_i$ ,

$$(C_1, C_2, C_3, C_4, C_5) = (C_{00}, C_{+0}, C_x, C_{+-}, C_{++}),\quad (18)$$

and analogously for the  $D_i$ ,  $E_i$ ,  $F_i$ . In the following, we use  $J_{-0} = J_{+0}$  throughout, and denote the couplings for  $\pi^- \pi^0 \rightarrow \pi^- \pi^0$  with index  $+$  as well,  $C_{-0} = C_{+0}$ , etc. We then find

$$\begin{aligned}\mathcal{M}_0^{1\text{-loop}} &= \{B_0^{(1)}(s_1)J_{00}(s_1) + (s_1 \leftrightarrow s_2) + (s_1 \leftrightarrow s_3)\} + \{B_0^{(2)}(s_1)J_{+-}(s_1) + (s_1 \leftrightarrow s_2) + (s_1 \leftrightarrow s_3)\}, \\ \mathcal{M}_{\pm}^{1\text{-loop}} &= B_{\pm}^{(1)}(s_3)J_{00}(s_3) + B_{\pm}^{(2)}(s_3)J_{+-}(s_3) + \{B_{\pm}^{(3)}(s_1, s_2, s_3)J_{+0}(s_1) + (s_1 \leftrightarrow s_2)\},\end{aligned}\quad (19)$$

with

$$\begin{aligned}B_0^{(1)}(s_1) &= (C_{00} + D_{00}Y_{1n} + F_{00}Y_{1n}^2) \left\{ K_0 + K_1 \left[ X_1^2 + 2Z_1^2 + \frac{\mathbf{Q}_1^2}{6s_1} Y_{1n} \right] \right\}, \\ B_0^{(2)}(s_1) &= 2(C_x + D_x Y_{1c} + F_x Y_{1c}^2) \left\{ L_0 + L_1 X_1 + L_2 X_1^2 + L_3 \frac{\mathbf{Q}_1^2}{3s_1} Y_{1c} \right\}, \\ B_{\pm}^{(1)}(s_3) &= (C_x + D_x Y_{3c} + F_x Y_{3c}^2) \left\{ K_0 + K_1 \left[ X_3^2 + 2Z_3^2 + \frac{\mathbf{Q}_3^2}{6s_3} Y_{3n} \right] \right\}, \\ B_{\pm}^{(2)}(s_3) &= 2(C_{+-} + D_{+-} Y_{3c} + F_{+-} Y_{3c}^2) \left\{ L_0 + L_1 X_3 + L_2 X_3^2 + L_3 \frac{\mathbf{Q}_3^2}{3s_3} Y_{3c} \right\}, \\ B_{\pm}^{(3)}(s_1, s_2, s_3) &= 2(C_{+0} + D_{+0} Y_{1nc} + F_{+0} Y_{1nc}^2) \left\{ L_0 + L_1 Z_1^- + L_2 \left[ (Z_1^-)^2 + \frac{\mathbf{Q}_1^2 q_{23}^2(s_1)}{3s_1} \right] + L_3 \left[ (Z_1^+ - X_1)^2 + \frac{\mathbf{Q}_1^2 q_{23}^2(s_1)}{3s_1} \right] \right\} \\ &\quad - \frac{1}{3} E_{+0} \frac{q_{23}^2(s_1)}{s_1 M_K} (\Delta_{\pi} (M_{\pi}^2 - M_K^2) + s_1(s_3 - s_2)) \{L_1 + 2L_2 Z_1^- + 2L_3 [X_1 - Z_1^+]\} + \mathcal{O}(\Delta_{\pi}^2).\end{aligned}\quad (20)$$

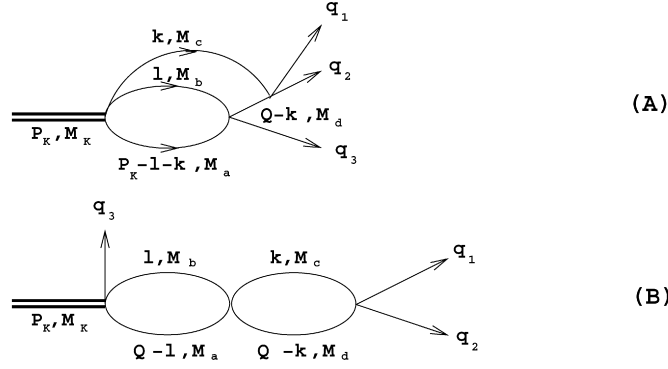


Fig. 1. Two topologically distinct non-relativistic two-loop graphs describing the final-state  $\pi\pi$  rescattering in the decay  $K \rightarrow 3\pi$ , with  $Q^\mu = (q_1 + q_2)^\mu$ .

We have used the abbreviations

$$\begin{aligned}
 Q_1^0 &= p_2^0 + p_3^0 \quad (\text{cycl.}), & \mathbf{Q}_i^2 &= \frac{\lambda(M_K^2, M_i^2, s_i)}{4M_K^2}, \\
 q_{lm}^2(s_k) &= \frac{\lambda(s_k, M_l^2, M_m^2)}{4s_k} \quad (k \neq l \neq m \neq k), \\
 Y_{in} &= s_i - 4M_{\pi^0}^2, & Y_{ic} &= s_i - 4M_\pi^2, & Y_{inc} &= s_i - (M_{\pi^0} + M_\pi)^2, \\
 Z_i &= \frac{Q_i^0}{2} - M_{\pi^0}, & Z_i^\pm &= \frac{Q_i^0}{2} \left(1 \pm \frac{\Delta\pi}{s_i}\right) - M_{\pi^0}.
 \end{aligned} \tag{21}$$

9. There are two topologically distinct two-loop graphs that describe pion–pion rescattering in the final state, see Fig. 1. At the order of accuracy we are working, it is sufficient to consider the case of non-derivative couplings. In this case, the contributions of both diagrams depend only on the variable  $s$ , where

$$Q^\mu = (q_1 + q_2)^\mu, \quad Q^2 = s. \tag{22}$$

The diagram in Fig. 1(B), apart from a factor containing coupling constants, is given by a product of two one-loop diagrams which were already calculated in Eq. (17). The non-trivial contribution from Fig. 1(A) is proportional to

$$\begin{aligned}
 \mathcal{M}(s) &= \int \frac{d^D l}{i(2\pi)^D} \frac{d^D k}{i(2\pi)^D} \frac{1}{2w_a(\mathbf{l} + \mathbf{k})} \frac{1}{w_a(\mathbf{l} + \mathbf{k}) - M_K + l^0 + k^0} \frac{1}{2w_b(\mathbf{l})} \frac{1}{w_b(\mathbf{l}) - l^0} \\
 &\times \frac{1}{2w_c(\mathbf{k})} \frac{1}{w_c(\mathbf{k}) - k^0} \frac{1}{2w_d(\mathbf{Q} - \mathbf{k})} \frac{1}{w_d(\mathbf{Q} - \mathbf{k}) - Q^0 + k^0}.
 \end{aligned} \tag{23}$$

A short discussion of this integral is given in Ref. [3]. There, it is shown that one may write

$$\mathcal{M}(s) = F(M_a, M_b, M_c, M_d; s) + \dots, \tag{24}$$

where  $F$  is ultraviolet finite and contains the full non-analytic behaviour of the two-loop diagram in the low-energy domain, whereas the ellipsis denotes terms that amount to a redefinition of the tree-level couplings in  $\mathcal{L}_K$  and which are therefore dropped. A one-dimensional integral representation for  $F$  is provided in Ref. [3]. The relevant integrals can be performed analytically—the result is displayed in Appendix B.

Below, we use the notation  $F_i(\dots; s)$  for the integral  $F(\dots; s)$ , evaluated at  $\mathbf{Q}^2 = \lambda(M_K^2, M_{\pi^i}^2, s)/4M_K^2$ , with  $i = \pm, 0$ .

Evaluating the diagrams displayed in Figs. 2 and 3, we find for the amplitudes at order  $a^2\epsilon^2$

$$\begin{aligned}
 \mathcal{M}_0^{2\text{-loops}} &= \{\mathcal{M}_0^A(s_1) + \mathcal{M}_0^B(s_1) + (s_1 \leftrightarrow s_2) + (s_1 \leftrightarrow s_3)\}, \\
 \mathcal{M}_\pm^{2\text{-loops}} &= \mathcal{M}_\pm^A(s_1, s_2, s_3) + \mathcal{M}_\pm^B(s_1, s_2, s_3),
 \end{aligned} \tag{25}$$

where

$$\begin{aligned}
 \mathcal{M}_0^A(s_1) &= 2C_{00}^2 K_0 F_0(M_{\pi^0}, M_{\pi^0}, M_{\pi^0}, M_{\pi^0}; s_1) + 8C_{+0} C_x L_0 F_0(M_{\pi^0}, M_\pi, M_\pi, M_\pi; s_1) \\
 &\quad + 4C_{00} C_x L_0 F_0(M_\pi, M_\pi, M_{\pi^0}, M_{\pi^0}; s_1), \\
 \mathcal{M}_0^B(s_1) &= C_{00}^2 K_0 J_{00}^2(s_1) + 4C_x C_{+-} L_0 J_{+-}^2(s_1) + (2C_x^2 K_0 + 2C_{00} C_x L_0) J_{+-}(s_1) J_{00}(s_1),
 \end{aligned} \tag{26}$$

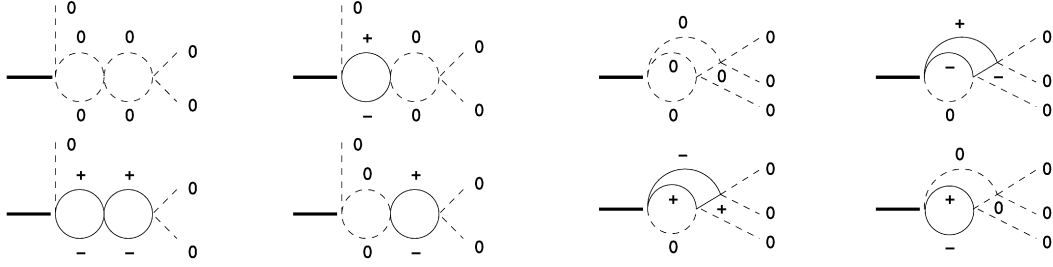


Fig. 2. Two-loop graphs contributing to the decay  $K_L \rightarrow \pi^0 \pi^0 \pi^0$  in the non-relativistic effective theory. The graphs obtained by a permutation of identical particles in the final state are not shown.

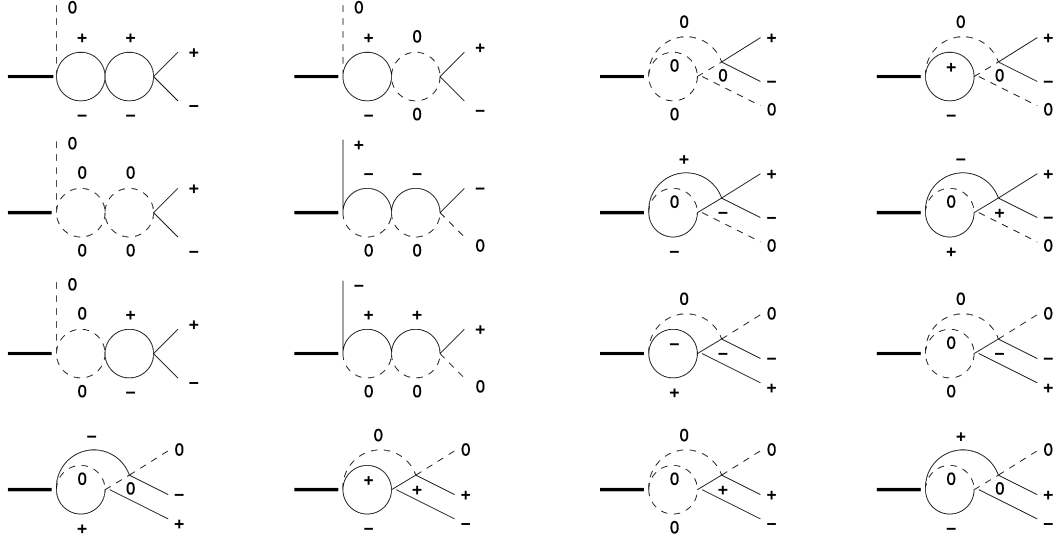


Fig. 3. Two-loop graphs contributing to the decay  $K_L \rightarrow \pi^+ \pi^- \pi^0$  in the non-relativistic effective theory.

and

$$\begin{aligned}
\mathcal{M}_{\pm}^A(s_1, s_2, s_3) &= 2C_{00}C_x K_0 F_0(M_{\pi^0}, M_{\pi^0}, M_{\pi^0}, M_{\pi^0}; s_3) + 4C_x^2 L_0 F_0(M_{\pi}, M_{\pi}, M_{\pi^0}, M_{\pi^0}; s_3) \\
&\quad + 8C_{+0}C_{-}L_0 F_0(M_{\pi^0}, M_{\pi}, M_{\pi}, M_{\pi}; s_3) + \{4C_{+0}C_{-}L_0 F_+(M_{\pi}, M_{\pi}, M_{\pi^0}, M_{\pi}; s_1) \\
&\quad + 2C_{+0}C_x K_0 F_+(M_{\pi^0}, M_{\pi^0}, M_{\pi^0}, M_{\pi}; s_1) + 4C_{+0}^2 L_0 F_+(M_{\pi}, M_{\pi^0}, M_{\pi}, M_{\pi^0}; s_1) + (s_1 \leftrightarrow s_2)\}, \\
\mathcal{M}_{\pm}^B(s_1, s_2, s_3) &= 4C_{+-}^2 L_0 J_{+-}^2(s_3) + C_{00}C_x K_0 J_{00}^2(s_3) + (2C_x C_{+-} K_0 + 2C_x^2 L_0) J_{+-}(s_3) J_{00}(s_3) \\
&\quad + \{4C_{+0}^2 L_0 J_{+0}^2(s_1) + (s_1 \leftrightarrow s_2)\}.
\end{aligned} \tag{27}$$

**10.** The decay amplitudes depend on the six *real*  $K_L \rightarrow 3\pi$  coupling constants  $L_i$ ,  $K_i$  and on the threshold parameters for  $\pi\pi$  scattering. Combining the tree- and one-loop result Eqs. (13), (19) with the two-loop contributions (25), we obtain the neutral and charged decay amplitudes up to and including terms of order  $\epsilon^4$ ,  $a\epsilon^5$  and  $a^2\epsilon^2$ , expressed in terms of the one- and two-loop integrals  $J$  and  $F$  displayed in Eqs. (17) and (B.1)–(B.2), respectively. [We have dropped some of the contributions at order  $\epsilon\Delta_\pi^2$ . In particular, *D*-waves generate contributions of this type. We expect them to be completely negligible.] This representation is valid in the whole decay region, and is the main result of this article.

The decay amplitude  $K_L \rightarrow \pi^0 \pi^0 \pi^0$  obeys what we refer to as the *threshold theorem*: the coefficient of the leading non-analytic piece, which is proportional to  $v_{+-}(s_3)$ , is given by a product of two factors, the decay amplitude  $K_L \rightarrow \pi^0 \pi^+ \pi^-$  and the scattering amplitude  $\pi^+ \pi^- \rightarrow \pi^0 \pi^0$ , both evaluated at threshold [1]. Of course, aside from the determination of the leading term in  $v_{+-}$ , our approach also allows a systematic evaluation of higher-order contributions  $v_{+-}^3, v_{+-}^5, \dots$

**11.** We now compare the content of this Letter with the work of Cabibbo and Isidori [2] (CI), who use an alternative method to construct the  $K \rightarrow 3\pi$  decay amplitudes. Conceptual aspects of the two methods were already discussed in Ref. [3] for the case of the charged kaon decays  $K^+ \rightarrow 3\pi$ . In particular, it was pointed out that the amplitudes agree at order  $a$ , whereas they differ at order  $a^2$  away from threshold, because the method used by CI does not reproduce the correct analytic properties of the amplitudes at two-loop order. [On the other hand, the two amplitudes lead to very similar results for the scattering lengths when

fitted to  $K^+ \rightarrow 3\pi$  data [5].] Analogous comments apply in the case of  $K_L \rightarrow 3\pi$  considered here. Comparing the expressions in detail, we note that the final result (4.61)–(4.67) in CI does contain some (but not all) of the terms evaluated above. In this sense, the expansion of the decay amplitudes presented here is more systematic and complete. As to the terms retained in CI, we note that, aside from obvious typos, we do agree in  $K_L \rightarrow 3\pi^0$  at order  $a$  in the physical region, and at order  $a^2$  at the thresholds  $s_i = 4M_\pi^2$ . In the charged channel  $K_L \rightarrow \pi^+\pi^-\pi^0$ , a graph is omitted in CI. It contributes at order  $a\epsilon$  and generates a cusp at the edge of physical phase space.

**12.** We add a remark concerning  $\eta \rightarrow 3\pi^0$  and  $\eta \rightarrow \pi^+\pi^-\pi^0$  decays. These processes can be analysed in a completely analogous fashion. Indeed, the  $\pi\pi$  scattering amplitudes remain the same, whereas the polynomial Lagrangian for  $\eta \rightarrow 3\pi$  can be obtained from the  $K \rightarrow 3\pi$  one by replacing field operators and particle masses in the Lagrangian (10),  $(K, M_K) \rightarrow (\eta, M_\eta)$ . The tree amplitudes analogous to Eq. (13) become

$$\begin{aligned}\mathcal{M}_0^{\eta\text{tree}} &= K_0^\eta + K_1^\eta(X_1^2 + X_2^2 + X_3^2), \\ \mathcal{M}_\pm^{\eta\text{tree}} &= L_0^\eta + L_1^\eta X_3 + L_2^\eta X_3^2 + L_3^\eta(X_1 - X_2)^2,\end{aligned}\quad (28)$$

with  $X_i = p_i^0 - M_{\pi^0}$ , and with obvious notation otherwise. The relation to an alternative expansion in the conventional  $\eta \rightarrow 3\pi$  Dalitz plot variables is provided in Appendix A. Furthermore, the one- and two-loop results in Eqs. (19), (25) can simply be taken over, with the replacements  $(K_i, L_i, M_K) \rightarrow (K_i^\eta, L_i^\eta, M_\eta)$  everywhere. Because  $\Gamma_{K_L \rightarrow \pi^+\pi^-\pi^0} / \Gamma_{K_L \rightarrow 3\pi^0} \sim \Gamma_{\eta \rightarrow \pi^+\pi^-\pi^0} / \Gamma_{\eta \rightarrow 3\pi^0}$ , we expect that the strength of the cusp effect in the neutral channel  $\eta \rightarrow 3\pi^0$  is of the same order as the one in  $K_L \rightarrow 3\pi^0$ , i.e., much less visible than in the charged channel  $K^+ \rightarrow \pi^+\pi^0\pi^0$ .

**13.** In summary, we have investigated  $K_L \rightarrow 3\pi$  decays within a non-relativistic effective Lagrangian framework. The amplitudes are calculated in a systematic double expansion in the pion momenta (counted as quantities of order  $\epsilon$ ), and in the threshold parameters of elastic  $\pi\pi$  scattering (generically denoted by  $a$ ). We provide an explicit representation of the amplitudes at order  $\epsilon^4$ ,  $a\epsilon^5$ ,  $a^2\epsilon^2$ . The representation is valid in the physical decay region, and contains the six (real)  $K_L \rightarrow 3\pi$  coupling constants  $L_i$ ,  $K_i$  and the threshold parameters  $a$ . The very same amplitude can be used, with trivial modifications described above, for a cusp analysis in  $\eta \rightarrow 3\pi$ .

Our amplitudes differ from the ones of Cabibbo and Isidori [2] when compared in detail—in particular, we do retain all terms at the above mentioned order in the low-energy expansion. For this reason, we believe that it is important to check whether our expressions for the amplitudes lead to scattering lengths that are in agreement with the ones generated with the amplitudes presented in Ref. [2].

It remains to investigate radiative corrections, which can be evaluated in the field-theoretical framework used here in a standard manner. The effects generated by the  $\pi^+\pi^-$  bound state at the  $\pi^+\pi^-$  threshold can also be investigated within the same approach [21–29], see also Ref. [30]. We plan to include these effects in forthcoming publications [19]. For the evaluation of radiative corrections in  $K^+ \rightarrow \pi^+\pi^+\pi^-$  in the framework of scalar QED, we refer the reader to the recent interesting article by Isidori [6].

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## Appendix A

The coefficients  $U_i, V_i$  are given by

$$\begin{aligned}U_0 &= K_0 + \frac{3K_1}{4M_K^2}((M_K - M_{\pi^0})^2 - s_n)^2, & U_1 &= \frac{3K_1}{8M_K^2}, \\ V_0 &= L_0 + \frac{L_1}{2M_K}((M_K - M_{\pi^0})^2 - s_c) + \frac{L_2}{4M_K^2}((M_K - M_{\pi^0})^2 - s_c)^2, \\ V_1 &= \frac{L_2}{2M_K^2}(s_c - (M_K - M_{\pi^0})^2) - \frac{L_1}{2M_K}, & V_2 &= \frac{L_2}{4M_K^2}, & V_3 &= \frac{L_3}{4M_K^2}.\end{aligned}\quad (A.1)$$

The inverse relations read

$$\begin{aligned} K_0 &= U_0 - 2U_1((M_K - M_{\pi^0})^2 - s_n)^2, & K_1 &= \frac{8}{3}M_K^2 U_1, \\ L_0 &= V_0 + V_1((M_K - M_{\pi^0})^2 - s_c) + V_2((M_K - M_{\pi^0})^2 - s_c)^2, \\ L_1 &= 4M_K V_2(s_c - (M_K - M_{\pi^0})^2) - 2M_K V_1, & L_2 &= 4M_K^2 V_2, & L_3 &= 4M_K^2 V_3. \end{aligned} \quad (\text{A.2})$$

For the decays  $\eta \rightarrow 3\pi^0$  and  $\eta \rightarrow \pi^+\pi^-\pi^0$ , the tree amplitudes (28) may be written in the alternative expansion

$$\mathcal{M}_0^{\eta \text{ tree}} = u_0 + u_1 z, \quad \mathcal{M}_{\pm}^{\eta \text{ tree}} = v_0 + v_1 y + v_2 y^2 + v_3 x^2, \quad (\text{A.3})$$

with the conventional  $\eta \rightarrow 3\pi$  Dalitz plot variables

$$x = \frac{\sqrt{3}(p_1^0 - p_2^0)}{Q_\eta}, \quad y = \frac{3(p_3^0 - M_{\pi^0})}{Q_\eta} - 1, \quad z = \frac{2}{3} \sum_{i=1}^3 \left( \frac{3p_i^0 - M_\eta}{Q_{\eta 0}} \right)^2, \quad (\text{A.4})$$

where  $Q_\eta = M_\eta - 2M_\pi - M_{\pi^0}$ ,  $Q_{\eta 0} = M_\eta - 3M_{\pi^0}$ . The coefficients of the two representations are related by

$$\begin{aligned} u_0 &= K_0^\eta + \frac{Q_{\eta 0}^2}{3} K_1^\eta, & u_1 &= \frac{Q_{\eta 0}^2}{6} K_1^\eta, & v_0 &= L_0^\eta + \frac{Q_\eta}{3} L_1^\eta + \frac{Q_\eta^2}{9} L_2^\eta, \\ v_1 &= \frac{Q_\eta}{3} \left( L_1^\eta + \frac{2}{3} Q_\eta L_2^\eta \right), & v_2 &= \frac{Q_\eta^2}{9} L_2^\eta, & v_3 &= \frac{Q_\eta^2}{3} L_3^\eta, \end{aligned} \quad (\text{A.5})$$

or reversely by

$$\begin{aligned} K_0^\eta &= u_0 - 2u_1, & K_1^\eta &= \frac{6}{Q_{\eta 0}^2} u_1, & L_0^\eta &= v_0 - v_1 + v_2, \\ L_1^\eta &= \frac{3(v_1 - 2v_2)}{Q_\eta}, & L_2^\eta &= \frac{9v_2}{Q_\eta^2}, & L_3^\eta &= \frac{3v_3}{Q_\eta^2}. \end{aligned} \quad (\text{A.6})$$

## Appendix B

The analytic expression for the two-loop function  $F$  reads

$$F(M_a, M_b, M_c, M_d, s) = \mathcal{N}(2A f_1 + B f_0) + O(\epsilon^4), \quad (\text{B.1})$$

with

$$\begin{aligned} \mathcal{N} &= \frac{1}{256\pi^3 \sqrt{s}} \left( 1 - \frac{2(M_a^2 + M_b^2)}{s_0} + \frac{(M_a^2 - M_b^2)^2}{s_0^2} \right)^{1/2} \frac{1}{\sqrt{\Delta^2 - \frac{(1+\delta)^2}{4} \mathbf{Q}^2}}, \\ f_0 &= 4(v_1 + v_2 - \bar{v}_2 + h), \\ f_1 &= \frac{4}{3}(y_1(v_1 - 1) + y_2(v_2 - 1) - \bar{y}_2(\bar{v}_2 - 1) + h), \\ h &= \frac{1}{2} \ln \left( \frac{1 + \mathbf{Q}^2/s}{1 + \bar{\mathbf{Q}}^2/\bar{s}} \right), \quad \bar{\mathbf{Q}}^2 = \mathbf{Q}^2(\bar{s}), \\ v_i &= \sqrt{-y_i} \arctan \frac{1}{\sqrt{-y_i}}, \quad i = 1, 2; \quad \bar{v}_2 = \sqrt{-\bar{y}_2} \arctan \frac{1}{\sqrt{-\bar{y}_2}}, \\ y_{1,2} &= \frac{-B \mp \sqrt{B^2 - 4AC}}{2A}, \quad \bar{y}_2 = y_2(\bar{s}), \\ A &= -\frac{\mathbf{Q}^2}{s}(M_c^2 + \Delta^2), \quad B = q_0^2 - \Delta^2 + \frac{\mathbf{Q}^2}{s} M_c^2, \quad C = -q_0^2, \\ s_0 &= M_K^2 + M_c^2 - 2M_K \left( M_c^2 + \frac{\mathbf{Q}^2(1+\delta)^2}{4} \right)^{1/2}, \\ q_0^2 &= \frac{\lambda(s, M_c^2, M_d^2)}{4s}, \quad \bar{s} = (M_c + M_d)^2, \\ \Delta^2 &= \frac{\lambda(M_K^2, M_c^2, (M_a + M_b)^2)}{4M_K^2}, \quad \delta = \frac{M_c^2 - M_d^2}{s}. \end{aligned} \quad (\text{B.2})$$



The arctan is understood to be evaluated according to

$$\arctan x = \frac{1}{2i} \ln \frac{1+ix}{1-ix}, \quad (\text{B.3})$$

and  $s$  is given a small positive imaginary part in all arguments,  $s \rightarrow s + i\epsilon$ .

The analytic formula (B.1) is exact at  $O(\epsilon^2)$ , and thus at the order considered in Ref. [3]. It differs by a few percent from the integral representation given in Ref. [3].

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