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Reflected backward stochastic differential equations with two barriers and Dynkin games under Knightian uncertainty ☆

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Abstract

This paper is concerned with a class of reflected backward stochastic differential equations (RBSDEs in short) with two barriers. The first purpose of the paper is to establish existence and uniqueness results of adapted solutions for such RBSDEs. Most of existing results on adapted solutions for RBSDEs with two barriers are heavily based on either the Mokobodski condition or other restrictive regularity conditions. In this paper, the two barriers are modeled by stochastic differential equations with coefficients satisfying the local Lipschitz condition and the linear growth condition, which enables us to weaken the regularity conditions on the boundary processes. Existence is proved by a penalization scheme together with a comparison theorem under the Lipschitz condition on the coefficients of RBSDEs. As an application, it is proved that the initial value of an RBSDE with two barriers coincides with the value function of a certain Dynkin game under Knightian uncertainty.

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Keywords: Backward stochastic differential equations; Reflecting barrier; Dynkin game; Knightian uncertainty

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1. Introduction

El Karoui et al. [3] introduced the notion of reflected backward stochastic differential equations (RBSDEs in short) with one lower barrier. An adapted solution of such an equation is forced to stay above a random barrier which is described by a continuous real-valued process. They proved an existence and uniqueness result of the solution both by the Snell envelope theory and by a penalization argument. Later, RBSDEs have been extensively studied and some existence and uniqueness results have been established by many authors. To mention just a few, Matoussi [17], Ma and Zhang [15], Lepeltier and Xu [13], Xu [21], and Li and Tang [14] discussed one-dimensional RBSDEs driven by a Brownian motion; Hamadène and Ouknine [6], Hamadène and Hassani [8], Essaky [4] studied one-dimensional RBSDEs with jumps, that is, RBSDEs are driven by a Brownian motion and an independent Poisson point process.

In [2], Cvitanić and Karatzas considered RBSDEs with two barriers and generalized the work of [3]. The solution of such an RBSDE has to stay between two prescribed continuous processes called lower and upper barriers. They proved the existence and uniqueness of the solution if, on one hand, the coefficient of such an RBSDE is Lipschitz continuous and, on the other hand, either the barriers are regular or the so-called Mokobodski condition is satisfied. The Mokobodski condition, roughly speaking, means that there exists a difference of two non-negative supermartingales between the lower barrier and the upper barrier. The regularity implies that the barriers can be uniformly approximated by Itô processes. Some of further effort to RBSDEs with two barriers can be found in [1,5,7,8,12,20] and the references therein. In these references, either the Mokobodski condition (or general Skorohod condition) or some restrictive regularity conditions are imposed. However, it should be pointed out that neither the Mokobodski condition nor regularity conditions can be satisfied easily in many practical situations. Hence it is an important issue to prove the existence and uniqueness result for an RBSDE with two barriers by weakening these conditions.

The aim of this paper is to study the solvability and applications of reflected backward stochastic differential equations with two barriers, where the two barriers are modeled by stochastic differential equations. It is found that, under this framework, the restrictive regularity conditions in [2] can be largely weakened. In order to guarantee the existence and uniqueness of strong solutions to the two barriers, we impose the local Lipschitz condition and the linear growth condition on the coefficients of equations. Under the Lipschitz condition, we prove an existence and uniqueness result of the adapted solution for such an RBSDE. The adopted technique is based on the penalization scheme together with a comparison theorem on RBSDEs with either one lower barrier or one upper barrier.

This paper is organized as follows. Section 2 contains certain notation needed in this paper and the definition of adapted solutions for RBSDEs with two barriers. Some necessary assumptions on the random barriers and RBSDEs are also proposed. In Section 3, four lemmas that associate with the existence of a unique solution for RBSDEs are given. An existence and uniqueness result derived by Hamadène and Hassani [8] for a penalized RBSDE with one upper barrier is also given in this section. Section 4 devotes to proving the existence of a unique solution to Eq. (2.1). We first deal with the case where the coefficient of Eq. (2.1) does not depend on (Y, Z). Then a contraction approach is applied to prove the required result. In the last section, we consider a Dynkin game with Knightian uncertainty. It is proved that the initial value of the solution for an RBSDE with two barriers coincides with the value function of such a Dynkin game.

2. Preliminaries

Let $T > 0$ be a fixed constant. Throughout this paper, we assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space on which a d-dimensional Brownian motion $B = (B_t)_{t\geqslant 0}$ is defined. Let $F = \{\mathcal{F}_t\}_{t\geqslant 0}$ be the natural filtration generated by B, augmented by the $\mathbb{P}$-null sets of $\mathcal{F}$, hence $\{\mathcal{F}_t\}_{0\leqslant t\leqslant T}$ satisfies the usual conditions. We denote by $\mathcal{P}$ the σ-algebra of progressively measurable sets on $[0, T] \times \Omega$. The following spaces will be used in this paper.

- $\mathbf{L}^2$: the set of $\mathcal{F}_T$-measurable variables $\xi : \Omega \to R$ with $\mathbb{E}|\xi|^2 < \infty$;
- $\mathbf{H}_d^2$: the set of $\mathcal{P}$-measurable processes $\varphi : [0, T] \times \Omega \to R^d$ with $\mathbb{E}\int_0^T |\varphi(t)|^2\,dt < \infty$;
- $\mathbf{S}^2$: the set of $\mathcal{P}$-measurable processes $\psi : [0, T] \times \Omega \to R$ with $\mathbb{E}(\sup_{0\leqslant t\leqslant T} |\psi(t)|^2) < \infty$;
- $\mathbf{A}^2$: the set of $\mathcal{P}$-measurable increasing processes $K : [0, T] \times \Omega \to R_+ := [0, \infty)$ with $K_0 = 0$, $\mathbb{E}|K_T|^2 < \infty$.

We will discuss a backward stochastic differential equation with two barriers as follows:

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, Z_s, \omega)\,ds - \int_t^T Z_s\,dB_s + K_T^+ - K_t^+ - \big(K_T^- - K_t^-\big), \\ L_t \leqslant Y_t \leqslant U_t, \quad \int_0^T (Y_t - L_t)\,dK_t^+ = \int_0^T (U_t - Y_t)\,dK_t^- = 0, \quad 0 \leqslant t \leqslant T, \end{cases} \tag{2.1}$$

where $f : [0, T] \times R \times R^d \times \Omega \to R$, generally called the coefficient, is $\mathcal{P} \times \mathcal{B}(R^{1+d})$-measurable. $K_\cdot^+$ and $K_\cdot^-$ are integrable increasing processes. The state process $Y_\cdot$ is forced to stay within the region decided by lower and upper barriers $L_\cdot$ and $U_\cdot$ with a minimal way in the sense of $\int_0^T (Y_t - L_t)\,dK_t^+ = \int_0^T (U_t - Y_t)\,dK_t^- = 0$. In order to weaken the restrictive regularity condition in [2], we assume that the barriers are described by the solutions of the following stochastic differential equations:

$$L_t = L_0 + \int_0^t b_1(s, L_s)\,ds + \int_0^t \sigma_1(s, L_s)\,dB_s, \quad L_0 \in R,\ 0 \leqslant t \leqslant T, \tag{2.2}$$

$$U_t = U_0 + \int_0^t b_2(s, U_s)\,ds + \int_0^t \sigma_2(s, U_s)\,dB_s, \quad U_0 \in R,\ 0 \leqslant t \leqslant T. \tag{2.3}$$

For (2.2) and (2.3), we assume that

(H0) both b_i and σ_i, $i = 1, 2$, satisfy the following conditions:

The local Lipschitz condition. For each $k = 1, 2, \ldots$, there is $c_k > 0$ such that

$$\big|b_i(t, x) - b_i(t, y)\big| \vee \big|\sigma_i(t, x) - \sigma_i(t, y)\big| \leqslant c_k|x - y|$$

for all $t \geqslant 0$ and those $x, y \in R$ with $|x| \vee |y| \leqslant k$, where $x \vee y = \max(x, y)$.

The linear growth condition. For all $t \geqslant 0$ and $x \in R$, there is $c > 0$ such that

$$\big|b_i(t, x)\big| \vee \big|\sigma_i(t, x)\big| \leqslant c\big(1 + |x|\big).$$

It is well known (see Mao [16]) that under above hypotheses, Eq. (2.2) (resp. Eq. (2.3)) has a unique strong solution. Moreover, for every $p > 0$,

$$\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T} |L_t|^p\Big] < \infty, \qquad \mathbb{E}\Big[\sup_{0\leqslant t\leqslant T} |U_t|^p\Big] < \infty. \tag{2.4}$$

The following assumption is also necessary throughout this paper, namely

$$L_t \leqslant U_t, \quad \forall t \in [0, T], \text{ a.s.} \tag{2.5}$$

Remark 2.1. A sufficient condition on (2.5) (see Ikeda and Watanabe [10]) is

$$L_0 \leqslant U_0, \qquad b_1(t,x) \leqslant b_2(t,x), \qquad \sigma_1(t,x) = \sigma_2(t,x), \quad \forall x \in R. \tag{2.6}$$

Remark 2.2. As pointed out in [2], if $L_t < U_t$, $0 \leqslant t \leqslant T$ holds almost surely, then the Mokobodski condition holds. In our framework, the condition that $L_t \leqslant U_t$ holds almost surely is weaker than theirs. On the other hand, since b_i and σ_i are local Lipschitz continuous, our conditions are different from those regularity conditions on the boundary processes in [2], furthermore, it is seen that the regularity conditions in [2] will be weakened.

Definition 2.1. A quadruple $(Y, Z, K^+, K^-) = (Y_t, Z_t, K_t^+, K_t^-)_{0\leqslant t\leqslant T}$ of processes with values in $R \times R^d \times R_+ \times R_+$ is called a solution of Eq. (2.1), if and only if (Y, Z, K^+, K^-) belongs to $\mathbf{S}^2 \times \mathbf{H}_d^2 \times \mathbf{A}^2 \times \mathbf{A}^2$ and satisfies (2.1).

We need the following assumptions:

(H1) The coefficient f satisfies the Lipschitz condition, that is, for any $t \geqslant 0$, $Y_1, Y_2 \in R$, $Z_1, Z_2 \in R^d$, there is a $C > 0$ such that

$$\big|f(t, Y_1, Z_1, \omega) - f(t, Y_2, Z_2, \omega)\big| \leqslant C\big[|Y_1 - Y_2| + |Z_1 - Z_2|\big], \quad \mathbb{P}\text{-a.s.}$$

(H2) The terminal value ξ belongs to $\mathbf{L}^2$, and $f(t, 0, 0, \omega)$ is $\mathscr{P}$-measurable and satisfies $\mathbb{E}\int_0^T |f(t, 0, 0, \omega)|^2\, dt < \infty$. For simplicity, $\mathbf{L}^2(0, T)$ denotes the set of all $\mathscr{P}$-measurable processes $\phi(t, \omega) : [0, T] \to R$ with $\mathbb{E}\int_0^T |\phi(t, \omega)|^2\, dt < \infty$.

3. Some technical results

In this section, we will consider a sequence of penalized equations for Eq. (2.1) with a special coefficient, and derive some estimates and representations of the solutions. Let us introduce the following BSDEs without reflections:

$$Y_t^{n,k} = \xi + \int_t^T g(s, \omega)\, ds - \int_t^T Z_s^{n,k}\, dB_s + K_T^{n,k,+} - K_t^{n,k,+} - \big(K_T^{n,k,-} - K_t^{n,k,-}\big), \tag{3.1}$$

where $K_t^{n,k,+} = \int_0^t n(Y_s^{n,k} - L_s)^-\, ds$ and $K_t^{n,k,-} = \int_0^t k(U_s - Y_s^{n,k})^-\, ds$, $n, k \in \mathbb{N}$; $g : [0, T] \times \Omega \to R$ is $\mathscr{P}$-measurable and belongs to $\mathbf{L}^2(0, T)$.

Lemma 3.1. *Assume that b_i, σ_i, $i=1,2$, satisfy* (H0). *If $g(\cdot,\cdot)$ belongs to $\mathbf{L}^2(0,T)$ and ξ satisfies* (H2), *then Eq.* (3.1) *has a unique solution $(Y^{n,k}, Z^{n,k}) \in \mathbf{S}^2 \times \mathbf{H}_d^2$ for arbitrary $n,k \in \mathbb{N}$, moreover,*

$$\mathbb{E}\big|K_T^{n,k,+}\big|^2 \leqslant C_0, \tag{3.2}$$

where C_0 is a positive constant which only depends on T, $\mathbb{E}|\xi|^2$, $\mathbb{E}\int_0^T |g(t,\omega)|^2\,dt$ and $\mathbb{E}(\sup_{0\leqslant t\leqslant T}|L_t|^2 + \sup_{0\leqslant t\leqslant T}|U_t|^2)$.

Proof. The first result directly follows from the standard theory of BSDEs (see Pardoux and Peng [18] or Yin and Mao [22]). It is only noted that $g(t,\omega)+n(L_t)^+ - k(U_t)^-$ belongs to $\mathbf{L}^2(0,T)$ according to (2.4) and the assumption on g. It remains to prove the inequality (3.2). Obviously,

$$\begin{aligned} L_t &= L_T - \int_t^T b_1(s,L_s)\,ds - \int_t^T \sigma_1(s,L_s)\,dB_s \\ &= L_T + \int_t^T g(s,\omega)\,ds - \int_t^T g(s,\omega)\,ds - \int_t^T b_1(s,L_s)\,ds - \int_t^T \sigma_1(s,L_s)\,dB_s \\ &:= L_T + \int_t^T g(s,\omega)\,ds - \int_t^T Z_s^*\,dB_s + \big(K_T^+\big)^* - \big(K_t^+\big)^* - \big[\big(K_T^-\big)^* - \big(K_t^-\big)^*\big], \end{aligned} \tag{3.3}$$

where $Z_t^* = \sigma_1(t,L_t)$ and $(K_t^{\pm})^* = \int_0^t g^{\mp}(s,\omega)\,ds + \int_0^t b_1^{\mp}(s,L_s)\,ds$. Here g^+ denotes the positive part of g, while g^- denotes the negative part. It is clear that $(K_t^+)^*$ (resp. $(K_t^-)^*$) is a continuous increasing process. Consider the following BSDEs:

$$\tilde{Y}_t^n = L_T + \int_t^T g(s,\omega)\,ds - \int_t^T \tilde{Z}_s^n\,dB_s - \big[\big(K_T^-\big)^* - \big(K_t^-\big)^*\big] + \int_t^T n\big(\tilde{Y}_s^n - L_s\big)^-\,ds. \tag{3.4}$$

Note that we can add the zero term $\int_t^T n(L_s - L_s)^-\,ds$ to the right side of (3.3). Since $d(K_t^+)^* \geqslant 0$, it then follows from the comparison theorem of BSDEs (see Peng [19]) that $L_t \geqslant \tilde{Y}_t^n$ a.s. and thus $U_t \geqslant \tilde{Y}_t^n$ a.s. By this, we can add the zero term $-k\int_t^T (U_s - \tilde{Y}_s^n)^-\,ds$ to the right side of (3.4), and apply the comparison theorem again to get $\tilde{Y}_t^n \leqslant Y_t^{n,k}$ since $d(K_t^-)^* \geqslant 0$ and $L_T \leqslant \xi$, which implies that $K_T^{n,k,+} \leqslant \tilde{K}_T^{n,+} := \int_0^T n(\tilde{Y}_t^n - L_t)^-\,dt$, thus $\mathbb{E}(K_T^{n,k,+})^2 \leqslant \mathbb{E}(\tilde{K}_T^{n,+})^2$. We will show that $\mathbb{E}(\tilde{K}_T^{n,+})^2$ has an upper bound. If we set $\hat{Y}_t^n = \tilde{Y}_t^n - (K_t^-)^*$, then (3.4) can be rewritten as

$$\hat{Y}_t^n = L_T - \big(K_T^-\big)^* + \int_t^T g(s,\omega)\,ds - \int_t^T \tilde{Z}_s^n\,dB_s + \int_t^T n\big(\hat{Y}_s^n + \big(K_s^-\big)^* - L_s\big)^-\,ds, \tag{3.5}$$

which is just the sequence of penalized BSDEs for an RBSDE with the terminal value $L_T - (K_T^-)^*$, the coefficient g and the lower barrier $L - (K^-)^*$. So by the standard method (see El Karoui [3]), we have $\mathbb{E}(\tilde{K}_T^{n,+})^2 \leqslant C_0$. This completes the proof. □

The above proof relies on the representation of stochastic differential equation for the lower process L_t. If (2.2) does not hold, Lemma 3.1 can also be derived under conditions of (2.3) and $\mathbb{E}\sup_{0\leqslant t\leqslant T}|L_t|^2<\infty$. Indeed, corresponding BSDEs similar to (3.3)–(3.5) can be constructed thanks to (2.3), and the required result follows from a similar argument and the fact of $L_t\leqslant U_t$ a.s.

Lemma 3.2. *Under the same assumptions of Lemma* 3.1*, the unique solution* $(Y^{n,k},Z^{n,k})$ *of Eq.* (3.1) *has the property of*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|Y_t^{n,k}\right|^2+\int_0^T\left|Z_t^{n,k}\right|^2dt+\left(K_T^{n,k+}\right)^2+\left(K_T^{n,k,-}\right)^2\right]\leqslant C_1, \tag{3.6}$$

where C_1 is a positive constant which only depends on T, $\mathbb{E}|\xi|^2$, $\mathbb{E}\int_0^T|g(t,\omega)|^2dt$ and $\mathbb{E}(\sup_{0\leqslant t\leqslant T}|L_t|^2+\sup_{0\leqslant t\leqslant T}|U_t|^2)$.

Proof. First, we will prove that

$$\sup_{t\in[0,T]}\mathbb{E}\left|Y_t^{n,k}\right|^2+\mathbb{E}\left[\int_0^T\left|Z_t^{n,k}\right|^2dt+\left(K_T^{n,k,+}\right)^2+\left(K_T^{n,k-}\right)^2\right]<\infty. \tag{3.7}$$

For this, we introduce a sequence of BSDEs for arbitrary $n,k\in\mathbb{N}$ in the form of

$$\begin{aligned} y_m^{(n,k)}(t)&=\xi+\int_t^T g(s,\omega)\,ds-\int_t^T z_m^{(n,k)}(s)\,dB_s+\int_t^T n\left(y_{m-1}^{(n,k)}(s)-L_s\right)^-ds\\ &\quad-\int_t^T k\left(U_s-y_{m-1}^{(n,k)}(s)\right)^-ds,\quad \left(y_0^{(n,k)}(t),z_0^{(n,k)}(t)\right)=(0,0),\quad m\in\mathbb{N}. \end{aligned} \tag{3.8}$$

For the case of $m=1$, it is easy to derive that

$$\left|y_1^{(n,k)}(t)\right|^2\leqslant K_0(k,n,T)\mathbb{E}\left(\left[|\xi|^2+\int_0^T\left|g(t,\omega)\right|^2dt+\sup_{0\leqslant t\leqslant T}|L_t|^2+\sup_{0\leqslant t\leqslant T}|U_t|^2\right]\Bigm|\mathcal{F}_t\right)$$

from the C_r-inequality and Jensen's inequality. Here and below, we use $K_i(k,n,T)$, $i=0,1,\dots,$ to denote some positive constants only depending on k, n and T. By Doob's martingale inequality, we have

$$\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}\left|y_1^{(n,k)}(t)\right|^2\right]\leqslant C_0(k,n,T), \tag{3.9}$$

where $C_i(k,n,T)>0$, $i=0,1,\dots,$ denote some constants only depending on k, n, T, $\mathbb{E}|\xi|^2$, $\mathbb{E}\int_0^T|g(t,\omega)|\,dt$ and $\mathbb{E}(\sup_{0\leqslant t\leqslant T}|L_t|^2+\sup_{0\leqslant t\leqslant T}|U_t|^2)$. Furthermore, by utilizing Jensen's inequality again, we can iteratively get

$$\begin{aligned} \mathbb{E}\left|y_m^{(n,k)}(t)\right|^2\leqslant K_1(k,n,T)\mathbb{E}\Biggl[&|\xi|^2+\int_0^T\left(\left|g(t,\omega)\right|^2+\left|y_{m-1}^{(n,k)}(t)\right|^2\right)dt\\ &+\sup_{0\leqslant t\leqslant T}\left(|L_t|^2+|U_t|^2\right)\Biggr] \end{aligned}$$

$$\leqslant C_1(k,n,T)\left[1+\mathbb{E}\int_t^T\left|y_{m-1}^{(n,k)}(s)\right|^2ds\right]$$
$$\leqslant \big(C_0(k,n,T)\vee C_1(k,n,T)\big)e^{[C_0(k,n,T)\vee C_1(k,n,T)]T}:=C_2(k,n,T). \tag{3.10}$$

Note that $\lim_{m\to\infty}\mathbb{E}\sup_{0\leqslant t\leqslant T}|y_m^{(n,k)}(t)-Y_t^{n,k}|^2=0$ since $\{Y_t^{n,k}\}$ is the limit of $\{y_m^{(n,k)}(t)\}$, $m=1,2,\dots$ in $\mathbf{S}^2$. So we can take a subsequence of $\{y_m^{(n,k)}(t)\}$ denoted by $\{y_{m_j}^{(n,k)}(t)\}$, and apply Fatou's lemma and (3.10) to obtain

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}\left|Y_t^{n,k}\right|^2\leqslant\sup_{0\leqslant t\leqslant T}\varliminf_{j\to\infty}\mathbb{E}\left|y_{m_j}^{(n,k)}(t)\right|^2\leqslant C_2(k,n,T), \tag{3.11}$$

which, together with Hölder's inequality and the C_r-inequality, yields that

$$\mathbb{E}\left[\left(K_T^{n,k,+}\right)^2+\left(K_T^{n,k,-}\right)^2\right]\leqslant C_3(k,n,T). \tag{3.12}$$

Applying Itô's formula to $|Y_t^{n,k}|^2$ and using (3.11) and (2.4), it is not hard to get

$$\mathbb{E}\int_0^T\left|Z_t^{n,k}\right|^2dt\leqslant C_4(k,n,T). \tag{3.13}$$

Note that

$$Y_t^{n,k}=\mathbb{E}\left[\xi+\int_t^T g(s,\omega)\,ds+K_T^{n,k,+}-K_t^{n,k,+}-\left(K_T^{n,k,-}-K_t^{n,k,-}\right)\Big|\mathcal{F}_t\right],$$

then by Jensen's inequality, Doob's inequality and (3.12), it is easy to get

$$\mathbb{E}\sup_{0\leqslant t\leqslant T}\left|Y_t^{n,k}\right|^2\leqslant C_5(k,n,T). \tag{3.14}$$

By applying Itô's formula to $|Y_t^{n,k}|^2$, we have

$$\begin{aligned}\left|Y_t^{n,k}\right|^2&=|\xi|^2+\int_t^T 2Y_s^{n,k}g(s,\omega)\,ds-\int_t^T 2Y_s^{n,k}Z_s^{n,k}\,dB_s\\&\quad+\int_t^T 2Y_s^{n,k}\left(dK_s^{n,k,+}-dK_s^{n,k,-}\right)-\int_t^T\left|Z_s^{n,k}\right|^2ds.\end{aligned} \tag{3.15}$$

Note that $\int_0^t Y_s^{n,k}Z_s^{n,k}\,dB_s$ is a uniformly integrable martingale by the Burkholder–Davis–Gundy inequality, (3.13) and (3.14), which is based on the fact of

$$\begin{aligned}\mathbb{E}\left(\int_0^T\left|Y_t^{n,k}\right|^2\left|Z_t^{n,k}\right|^2dt\right)^{\frac12}&\leqslant\mathbb{E}\left(\sup_{0\leqslant t\leqslant T}\left|Y_t^{n,k}\right|\left(\int_0^T\left|Z_t^{n,k}\right|^2dt\right)^{\frac12}\right)\\&\leqslant\frac12\mathbb{E}\sup_{0\leqslant t\leqslant T}\left|Y_t^{n,k}\right|^2+\frac12\mathbb{E}\int_0^T\left|Z_t^{n,k}\right|^2dt<\infty.\end{aligned}$$

By the definitions of $K_t^{n,k,+}$ and $K_t^{n,k,-}$, it is easy to know that

$$\int_t^T 2Y_s^{n,k}\big(dK_s^{n,k,+} - dK_s^{n,k,-}\big) \leqslant \int_t^T 2L_s\,dK_s^{n,k,+} - \int_t^T 2U_s\,dK_s^{n,k,-}.$$

Therefore, applying Young's inequality yields that

$$\begin{aligned}
&\mathbb{E}\big|Y_t^{n,k}\big|^2 + \mathbb{E}\int_t^T \big|Z_s^{n,k}\big|^2\,ds \\
&\quad\leqslant \mathbb{E}\Bigg[|\xi|^2 + 2\int_0^T \big|g(s,\omega)\big|^2\,ds + \int_t^T \big|Y_s^{n,k}\big|^2\,ds \\
&\qquad + \epsilon^{-1}\Big(\sup_{0\leqslant t\leqslant T}|L_t|^2 + \sup_{0\leqslant t\leqslant T}|U_t|^2\Big) + \epsilon\big(\big|K_T^{n,k,+}\big|^2 + \big|K_T^{n,k,-}\big|^2\big)\Bigg],
\end{aligned} \tag{3.16}$$

where ϵ is a sufficiently small positive constant determined later and may vary from line to line for conciseness. By the Gronwall lemma, we have

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}\big|Y_t^{n,k}\big|^2 \leqslant C_0(\epsilon,T) + \epsilon\big(\mathbb{E}\big|K_T^{n,k,+}\big|^2 + \mathbb{E}\big|K_T^{n,k,-}\big|^2\big), \tag{3.17}$$

which, together with (3.16), implies that

$$\mathbb{E}\int_0^T \big|Z_t^{n,k}\big|^2\,dt \leqslant C_1(\epsilon,T) + \epsilon\big(\mathbb{E}\big|K_T^{n,k,+}\big|^2 + \mathbb{E}\big|K_T^{n,k,-}\big|^2\big), \tag{3.18}$$

where $C_i(\epsilon,T) > 0$, $i = 0, 1, \ldots$ are similar to $C_i(k,n,T)$ except independence on n and k. We now use the Burkholder–Davis–Gundy inequality to get

$$\begin{aligned}
\mathbb{E}\sup_{0\leqslant t\leqslant T}\left|\int_t^T 2Y_s^{n,k}Z_s^{n,k}\,dB_s\right| &\leqslant 8\mathbb{E}\left(\int_0^T \big|Y_t^{n,k}\big|\big|Z_t^{n,k}\big|^2\,dt\right)^{\frac{1}{2}} \\
&\leqslant \frac{1}{2}\mathbb{E}\sup_{0\leqslant t\leqslant T}\big|Y_t^{n,k}\big|^2 + 128\mathbb{E}\int_0^T \big|Z_t^{n,k}\big|^2\,dt.
\end{aligned} \tag{3.19}$$

This, together with (3.17) and (3.18) for Eq. (3.15), gives

$$\mathbb{E}\sup_{0\leqslant t\leqslant T}\big|Y_t^{n,k}\big|^2 \leqslant C_2(\epsilon,T) + \epsilon\big(\mathbb{E}\big|K_T^{n,k,+}\big|^2 + \mathbb{E}\big|K_T^{n,k,-}\big|^2\big). \tag{3.20}$$

On the other hand, by the chain rule, we have

$$\mathbb{E}\big(K_T^{n,k,-}\big)^2 = 2\mathbb{E}\int_0^T K_s^{n,k,-}\big[dY_s^{n,k} + g(s,\omega)\,ds + dK_s^{n,k,+}\big]$$

$$\leqslant 2\mathbb{E}\big[K_T^{n,k,-}\big|Y_T^{n,k}\big|\big] - 2\mathbb{E}\int_0^T Y_s^{n,k}\,dK_s^{n,k,-}$$

$$+ 2\mathbb{E}\int_0^T K_s^{n,k,-}\big|g(s,\omega)\big|\,ds + 2\mathbb{E}\int_0^T K_s^{n,k,-}\,dK_s^{n,k,+}$$

$$\leqslant 2\mathbb{E}\big[K_T^{n,k,-}\big|Y_T^{n,k}\big|\big] + 2\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T}\big|Y_t^{n,k}\big|K_T^{n,k,-}\Big]$$

$$+ 2\mathbb{E}\Bigg[K_T^{n,k,-}\int_0^T\big|g(s,\omega)\big|\,ds\Bigg] + 2\mathbb{E}\big[K_T^{n,k,-}K_T^{n,k,+}\big]$$

$$\leqslant \frac{1}{2}\mathbb{E}\big(K_T^{n,k,-}\big)^2 + C_1(T)\Big(1 + \mathbb{E}\sup_{0\leqslant t\leqslant T}\big|Y_t^{n,k}\big|^2 + \mathbb{E}\big(K_T^{n,k,+}\big)^2\Big),$$

which, together with (3.2), gives that

$$\mathbb{E}\big(K_T^{n,k,-}\big)^2 \leqslant C_2(T)\Big(1 + \mathbb{E}\sup_{0\leqslant t\leqslant T}\big|Y_t^{n,k}\big|^2\Big). \tag{3.21}$$

Thus the required conclusion follows from (3.20), (3.21) and (3.2) by taking a sufficiently small ϵ. □

Let us point out, for each fixed $n \in \mathbb{N}$, that (3.1) is the sequence of penalized BSDEs related to a reflected BSDE with one upper barrier U, a coefficient $g(\cdot,\omega) + n(y-L)^-$ and a terminal value ξ. Note that a triple (Y,Z,K) of processes is a solution to a reflected BSDE with one upper barrier U, a coefficient f and a terminal value ξ if and only if $(-Y,-Z,K)$ is a solution for the reflected BSDE with one lower barrier associated with $(-f(t,Y,Z,\omega),-\xi,-U)$. Using this idea, Hamadène and Hassani [8] have established the following theorem in the same way as El Karoui [3].

Lemma 3.3. *(See [3].) Under the assumptions of Lemma* 3.1*, there exists a unique triple* $(Y^n, Z^n, K^{n,-})$ *of processes with values in* $R \times R^d \times R_+$*, which satisfies*

$$\begin{cases} Y^n \in \mathbf{S}^2, \quad Z^n \in \mathbf{H}_d^2, \quad K^{n,-} \in \mathbf{A}^2, \\ Y_t^n = \xi + \displaystyle\int_t^T g(s,\omega)\,ds + \int_t^T n\big(Y_s^n - L_s\big)^-\,ds - \int_t^T Z_s^n\,dB_s - \big(K_T^{n,-} - K_t^{n,-}\big), \\ Y_t^n \leqslant U_t, \quad \forall t\in[0,T], \quad \textit{and} \quad \displaystyle\int_0^T \big(U_t - Y_t^n\big)\,dK_t^{n,-} = 0. \end{cases} \tag{3.22}$$

Remark 3.1. Actually, the solution $(Y^n, Z^n, K^{n,-})$ in Lemma 3.3 is the limit of $(Y^{n,k}, Z^{n,k}, K^{n,k,-})$ of (3.1) in $\mathbf{S}^2 \times \mathbf{H}_d^2 \times \mathbf{A}^2$ as $k \to \infty$ for any fixed $n \in \mathbb{N}$, which can be proved by means of discussing a reflected BSDE with one upper barrier. Hence by Lemma 3.2, it is obvious that

$$\sup_n\left[\mathbb{E}\sup_{0\leqslant t\leqslant T}\left|Y_t^n\right|^2+\mathbb{E}\int_0^T\left|Z_t^n\right|^2dt+\mathbb{E}\left[\left(K_T^{n,+}\right)^2+\left(K_T^{n,-}\right)^2\right]\right]\leqslant C_1, \tag{3.23}$$

where $K_t^{n,+}:=\int_0^t n(Y_s^n-L_s)^-\,ds$.

The following comparison theorem plays an important role in later analysis.

Lemma 3.4. *Assume that* $(Y^n, Z^n, K^{n,-})$ *is the unique solution of* (3.22) *for each* $n\in\mathbb{N}$, *then we have,* $\mathbb{P}$*-a.s., that*

(i) $Y_t^n\leqslant Y_t^{n+1}$, $\forall t\in[0,T]$,
(ii) $K_t^{n,-}-K_s^{n,-}\leqslant K_t^{n+1,-}-K_s^{n+1,-}$, $\forall 0\leqslant s\leqslant t\leqslant T$.

Proof. By Meyer–Itô's formula to $[(Y^n-Y^{n+1})^+]^2$, we have

$$\begin{aligned}
&\mathbb{E}\left(Y_t^n-Y_t^{n+1}\right)^{+2}+\mathbb{E}\int_t^T 1_{\{Y_s^n>Y_s^{n+1}\}}\left|Z_s^n-Z_s^{n+1}\right|^2ds\\
&\quad=2\mathbb{E}\int_t^T 1_{\{Y_s^n>Y_s^{n+1}\}}\left(Y_s^n-Y_s^{n+1}\right)^+\left[n\left(Y_s^n-L_s\right)^--(n+1)\left(Y_s^{n+1}-L_s\right)^-\right]ds\\
&\qquad-2\mathbb{E}\int_t^T 1_{\{Y_s^n>Y_s^{n+1}\}}\left(Y_s^n-Y_s^{n+1}\right)^+\left(dK_s^{n,-}-dK_s^{n+1,-}\right)\\
&\quad\leqslant 2\mathbb{E}\int_t^T 1_{\{Y_s^n>Y_s^{n+1}\}}\left(Y_s^n-Y_s^{n+1}\right)^+\left[n\left(Y_s^n-L_s\right)^--(n+1)\left(Y_s^{n+1}-L_s\right)^-\right]ds\\
&\quad\leqslant 2\mathbb{E}\int_t^T 1_{\{Y_s^n>Y_s^{n+1}\}}\left(Y_s^n-Y_s^{n+1}\right)^+\left[n\left(Y_s^n-L_s\right)^--n\left(Y_s^{n+1}-L_s\right)^-\right]ds\leqslant 0,
\end{aligned}$$

where we have used the fact of $\mathbb{E}\int_t^T 1_{\{Y_s^n>Y_s^{n+1}\}}(Y_s^n-Y_s^{n+1})^+(dK_s^{n,-}-dK_s^{n+1,-})\geqslant 0$. In fact, we can compute

$$\begin{aligned}
&\mathbb{E}\int_t^T 1_{\{Y_s^n>Y_s^{n+1}\}}\left(Y_s^n-Y_s^{n+1}\right)^+\left(dK_s^{n,-}-dK_s^{n+1,-}\right)\\
&\quad=\mathbb{E}\int_t^T\left(Y_s^n-Y_s^{n+1}\right)\left(dK_s^{n,-}-dK_s^{n+1,-}\right)\\
&\quad=\mathbb{E}\int_t^T\left(Y_s^n-U_s+U_s-Y_s^{n+1}\right)\left(dK_s^{n,-}-dK_s^{n+1,-}\right)
\end{aligned}$$

$$= \mathbb{E}\int_t^T \big(U_s - Y_s^n\big)\,dK_s^{n+1,-} + \mathbb{E}\int_t^T \big(U_s - Y_s^{n+1}\big)\,dK_s^{n,-} \geqslant 0.$$

Therefore we immediately deduce that $Y_t^n \leqslant Y_t^{n+1}$ a.s. for any $t \in [0, T]$.

Let us prove the second inequality. As mentioned in Remark 3.1, $(Y^n, Z^n, K^{n,-})$ is the limit of $(Y^{n,k}, Z^{n,k}, K^{n,k,-})$ in $\mathbf{S}^2 \times \mathbf{H}_d^2 \times \mathbf{A}^2$ for any fixed $n \in \mathbb{N}$. Similarly, by Meyer–Itô's formula, it is not difficult to deduce that

$$\mathbb{E}\big(Y_t^{n,k} - Y_t^{n+1,k}\big)^{+2} + \mathbb{E}\int_t^T 1_{\{Y_s^n > Y_s^{n+1}\}}\big|Z_s^{n,k} - Z_s^{n+1,k}\big|^2\,ds$$

$$\leqslant -2\mathbb{E}\int_t^T 1_{\{Y_s^n > Y_s^{n+1}\}}\big(Y_s^{n,k} - Y_s^{n+1,k}\big)^+\big(dK_s^{n,k,-} - dK_s^{n+1,k,-}\big) \leqslant 0,$$

which implies that $Y_t^{n,k} \leqslant Y_t^{n+1,k,-}$ a.s. for any $t \in [0, T]$ and $n, k \in \mathbb{N}$. Therefore,

$$K_t^{n,-} - K_s^{n,-} = \lim_{k\to\infty}\int_s^t k\big(U_r - Y_r^{n,k}\big)^-\,dr \leqslant \lim_{k\to\infty}\int_s^t k\big(U_r - Y_r^{n+1,k}\big)^-\,dr$$
$$= K_t^{n+1,-} - K_s^{n+1,-},$$

since $K_t^{n,k,-}$ converges to $K_t^{n,-}$ in $\mathbf{A}^2$ and we can take a subsequence of $K_t^{n,k,-}$ if necessary. The proof is complete. □

Remark 3.2. Lemma 3.4 implies that for any $n \in \mathbb{N}$, $Y^n \leqslant Y^{n+1} \leqslant U$ a.s., since Y^n and U are all continuous, then by the monotone convergence theorem, there exists a $\mathcal{P}$-measurable process $Y = (Y_t)_{0\leqslant t\leqslant T}$ such that $\mathbb{P}$-a.s., for any $t \in [0, T]$, $Y_t^n \nearrow Y_t$ and $Y_t \leqslant U_t$ a.s. Obviously, the sequence $(Y^n)_{n\in\mathbb{N}}$ converges in $\mathbf{H}_1^2$ to Y by the Lebesgue dominated convergence theorem and Remark 3.1. Y also satisfies

$$\mathbb{E}\sup_{0\leqslant t\leqslant T}|Y_t|^2 \leqslant 2\mathbb{E}\sup_{0\leqslant t\leqslant T}\big|Y_t^1\big|^2 + 2\mathbb{E}\sup_{0\leqslant t\leqslant T}|U_t|^2 < \infty.$$

Furthermore, we have the following result.

Lemma 3.5. $\mathbb{E}\sup_{0\leqslant t\leqslant T}|Y_t^n - Y_t|^2 \to 0$ *as* $n \to \infty$, *namely* Y *is the limit of* Y^n *in* $\mathbf{S}^2$.

Proof. First we will prove that Y is a left continuous process, which can be proved by adopting some ideas in Hamadène and Hassani [8]. But we needn't consider the so-called local solution to a reflected BSDE with two barriers. Let $\hat{Y}^{n,k} = -Y^{n,k}$, $\hat{Z}^{n,k} = Z^{n,k}$, then BSDE (3.1) can be rewritten as

$$\hat{Y}_t^{n,k} = -\xi - \int_t^T g(s,\omega)\,ds - \int_t^T \hat{Z}_s^{n,k}\,dB_s$$
$$+ \int_t^T k\big(U_s + \hat{Y}_s^{n,k}\big)^-\,ds - \int_t^T n\big(\hat{Y}_s^{n,k} + L_s\big)^+\,ds. \tag{3.24}$$

Hence by Lemma 3.3, there exists a unique triple $(\hat{Y}^k, \hat{Z}^k, \hat{K}^{k,-}) \in \mathbf{S}^2 \times \mathbf{H}_d^2 \times \mathbf{A}^2$, which satisfies

$$\begin{cases} \hat{Y}_t^k = -\xi - \int_t^T g(s,\omega)\,ds + \int_t^T k\big(U_s + \hat{Y}_s^k\big)^- \,ds - \int_t^T \hat{Z}_s^k\,dB_s + \hat{K}_T^{k,-} - \hat{K}_t^{k,-}, \\ \hat{Y}_t^k \leqslant -L_t, \quad \forall t \in [0,T], \quad \int_t^T \big(L_s + \hat{Y}_s^k\big)^+ \,d\hat{K}_s^{k,-} = 0. \end{cases} \tag{3.25}$$

Furthermore, it follows that $\hat{Y}_t^{n,k} \searrow \hat{Y}_t^k$ for any $t \in [0,T]$ as $n \to \infty$ by applying comparison theorem to (3.24) with fixed $k \in \mathbb{N}$. By Lemma 3.4 and the monotone convergence theorem, there exists a $\mathscr{P}$-measurable process $\hat{Y}$ such that $\hat{Y}_t \leqslant -L_t$ and $\hat{Y}_t^k \nearrow \hat{Y}_t$ for any $t \in [0,T]$, $\mathbb{P}$-a.s. So we have $\lim_{k\to\infty}\lim_{n\to\infty}\hat{Y}_t^{n,k} = \hat{Y}_t$. On the other hand, since we have proved that $\lim_{n\to\infty}\lim_{k\to\infty} Y_t^{n,k} = Y_t$, we can take a subsequence $Y_t^{n,n}$ (resp. $\hat{Y}_t^{n,n}$) of $Y_t^{n,k}$ (resp. $\hat{Y}_t^{n,k}$) and note that $Y_t^{n,n} = -\hat{Y}_t^{n,n}$, thus $Y_t = -\hat{Y}_t$ a.s. This implies that Y is also the limit of a decreasing continuous sequence of processes $(-\hat{Y}_t^k)_{k\in\mathbb{N}}$. Since $Y_t^n = Y_{t-}^n = \liminf_{s\nearrow t} Y_s^n \leqslant \liminf_{s\nearrow t} Y_s$, it follows from Remark 3.2 that $Y_t \leqslant \liminf_{s\nearrow t} Y_s$ a.s. On the contrary, note that $-\hat{Y}_t^n = -\hat{Y}_{t-}^n = \limsup_{s\nearrow t}(-\hat{Y}_s^n) \geqslant \limsup_{s\nearrow t} Y_s$, then we obtain $Y_t \geqslant \limsup_{s\nearrow t} Y_s$ a.s. Therefore Y is left continuous and $\sup_{0\leqslant t\leqslant T}(Y_t^n - Y_t)^2 \searrow 0$ as $n \to \infty$ from the weak version of Dini's theorem. Then the required result follows directly from the dominated convergence theorem. □

4. Existence of a solution

This section is devoted to proving the existence of a unique solution to RBSDE (2.1). As a preparation, we first turn our attention to dealing with Eq. (2.1) with a special coefficient $g(t,\omega)$ which belongs to $\mathbf{L}^2(0,T)$. We will show that such a reflected BSDE admits a unique solution.

Theorem 4.1. *Assume that b_i, σ_i, $i = 1,2$, satisfy* (H0). *If $g(\cdot,\omega)$ belongs to $\mathbf{L}^2(0,T)$ and* (2.5) *holds, then there exists a quadruple of processes $(Y, Z, K^+, K^-) \in \mathbf{S}^2 \times \mathbf{H}_d^2 \times \mathbf{A}^2 \times \mathbf{A}^2$, where Y is given in Lemma* 3.5*, to solve the following reflected BSDE with two barriers L and U:*

$$\begin{cases} Y_t = \xi + \int_t^T g(s,\omega)\,ds - \int_t^T Z_s\,dB_s + K_T^+ - K_t^+ - \big(K_T^- - K_t^-\big), \\ L_t \leqslant Y_t \leqslant U_t, \quad \forall t \in [0,T], \quad \textit{and} \quad \int_0^T (Y_t - L_t)\,dK_t^+ = \int_0^T (U_t - Y_t)\,dK_t^- = 0. \end{cases}$$

Proof. The proof will be divided into five steps.

Step 1. We will prove that $L_t \leqslant Y_t \leqslant U_t$, $\forall t \in [0,T]$, a.s.

Indeed, we have proved that $Y_t \leqslant U_t$, $\forall t \in [0,T]$, a.s. in Lemma 3.4. It remains to show that $Y_t \geqslant L_t$, $\forall t \in [0,T]$, a.s. Note that $Y^n \leqslant Y^{n+1} \leqslant Y$ and $\mathbb{E}[\int_0^T n(Y_t^n - L_t)^- \,dt]^2 \leqslant C_1$ from Remark 3.1, then for any $0 \leqslant t \leqslant t + \Delta t \leqslant T$, we have

$$0 \leqslant \int_t^{t+\Delta t} \mathbb{E}(Y_s - L_s)^- \, ds \leqslant \mathbb{E}\int_0^T (Y_t - L_t)^- \, dt \leqslant \mathbb{E}\int_0^T \big(Y_t^n - L_t\big)^- \, dt \leqslant \frac{\sqrt{C_1}}{n} \to 0,$$

as $n \to \infty$, which implies that $Y_t \geqslant L_t$, $\forall t \in [0, T]$ almost surely since $\mathbb{E}(Y_t - L_t)^- = 0, 0 \leqslant t < T$ and Y_t and L_t are continuous.

Step 2. $(Z^n)_{n\in\mathbb{N}}$ of (3.22) is a Cauchy sequence in $\mathbf{H}_d^2$.

By Itô's formula to $(Y_t^n - Y_t^m)^2$, we have

$$\begin{aligned}
&\mathbb{E}\left[\big(Y_t^n - Y_t^m\big)^2 + \int_t^T \big(Z_s^n - Z_s^m\big)^2 \, ds\right] \\
&= \mathbb{E}\left[2\int_t^T \big(Y_s^n - Y_s^m\big)\big(n\big(Y_s^n - L_s\big)^- - m\big(Y_s^m - L_s\big)^-\big) \, ds \right. \\
&\quad \left. -2\int_t^T \big(Y_s^n - Y_s^m\big)\big(dK_s^{n,-} - dK_s^{m,-}\big)\right] \\
&\leqslant \mathbb{E}\left[2\int_t^T \big(Y_s^n - L_s\big)^- m\big(Y_s^m - L_s\big)^- \, ds + 2\int_t^T \big(Y_s^m - L_s\big)^- n\big(Y_s^n - L_s\big)^- \, ds\right] \\
&\leqslant 2\Big(\mathbb{E}\sup_{0\leqslant t\leqslant T}\big[\big(Y_t^n - L_t\big)^-\big]^2\Big)^{\frac{1}{2}}\big(\mathbb{E}\big[K_T^{m,+}\big]^2\big)^{\frac{1}{2}} \\
&\quad + 2\Big(\mathbb{E}\sup_{0\leqslant t\leqslant T}\big[\big(Y_t^m - L_t\big)^-\big]^2\Big)^{\frac{1}{2}}\big(\mathbb{E}\big[K_T^{n,+}\big]^2\big)^{\frac{1}{2}},
\end{aligned}$$

since $\int_t^T (Y_s^n - Y_s^m)(dK_s^n - dK_s^m) \leqslant 0$, $\forall t \in [0, T]$ and $n, m \in \mathbb{N}$ a.s. By Step 1 and Dini's theorem, we have $\sup_{0\leqslant t\leqslant T}[(Y_t^n - L_t)^-]^2 \searrow 0$, which implies that $\mathbb{E}\sup_{0\leqslant t\leqslant T}[(Y_t^n - L_t)^-]^2 \to 0$ as $n \to \infty$ by the monotone convergence theorem. Hence we have

$$\mathbb{E}\int_0^T \big|Z_t^n - Z_t^m\big|^2 \, dt \to 0, \quad \text{as } n, m \to \infty. \tag{4.1}$$

This means that $(Z^n)_{n\in\mathbb{N}}$ converges to a $\mathscr{P}$-measurable process Z in $\mathbf{H}_d^2$.

Step 3. $\lim_{n\to\infty} K_t^{n,+} = K_t^+$ and $\lim_{n\to\infty} Y_t^{n,-} = K_t^-$ for any $t \in [0, T]$ a.s., where K_t^+ and K_t^- are integrable increasing processes.

By Lemma 3.4, we know that $K_t^{n,-} \leqslant K_t^{n+1,-}$, $\forall t \in [0, T]$ a.s. Let $K_t^- = \liminf_{n\to\infty} K_t^{n,-}$, $\forall t \in [0, T]$. It is obvious that K^- is an increasing process. Using Fatou's lemma and Remark 3.1, we have $\mathbb{E}(K_t^-)^2 \leqslant \liminf_{n\to\infty} \mathbb{E}(K_t^{n,-})^2 \leqslant \sup_n \mathbb{E}(K_T^{n,-})^2 \leqslant C_1$, which gives that $K_t^- < \infty$ a.s. for any $t \in [0, T]$ and $K_t^{n,-} \nearrow K_t^-$ as $n \to \infty$ a.s. We also have $\mathbb{E}(K_t^{n,-} - K_t^-)^2 \to 0$ as $n \to \infty$ by applying the dominated convergence theorem. On the other hand, if we set

$$K_t^+ = -Y_t + Y_0 - \int_0^t g(s,\omega)\,ds + \int_0^t Z_s\,dB_s + K_t^-,$$

then it is easy to get that $\mathbb{E}(K_t^{n,+} - K_t^+)^2 \to 0$ for any $t \in [0,T]$ as $n \to \infty$ from Lemma 3.5, Step 1 and the fact of $\mathbb{E}(K_t^{n,-} - K_t^-)^2 \to 0$. So we can take a subsequence of $(K^{n,+})_{n\in\mathbb{N}}$ still denoted by $(K^{n,+})_{n\in\mathbb{N}}$ and have $\lim_{n\to\infty} K_t^{n,+} = K_t^+$ for any $t \in [0,T]$ a.s. Similarly, K^+ is an integrable increasing process. It is easily seen that

$$Y_t = \xi + \int_t^T g(s,\omega)\,ds - \int_t^T Z_s\,dB_s + K_T^+ - K_t^+ - \left(K_T^- - K_t^-\right). \tag{4.2}$$

Step 4. $\int_t^T (Y_t - L_t)\,dK_t^+ = 0$ and $\int_0^T (U_t - Y_t)\,dK_t^- = 0$.

Since $K_t^{n,-} \nearrow K_t^-$ and $Y_t^n \nearrow Y_t$, then $0 \leqslant \int_0^T (U_t - Y_t)\,dK_t^{n,-} \leqslant \int_0^T (U_t - Y_t^n)\,dK_t^{n,-} = 0$. By this, we have

$$\begin{aligned} 0 \leqslant \int_0^T (U_t - Y_t)\,dK_t^- &= \int_0^T (U_t - Y_t)\left(dK_t^- - dK_t^{n,-}\right) \\ &\leqslant \sup_{0\leqslant t\leqslant T} (U_t - Y_t)\left(K_T^- - K_T^{n,-}\right) \to 0, \quad n \to \infty, \end{aligned}$$

and the required conclusion follows.

It suffices to prove the reflecting condition for the lower barrier. In the spirit of a similar treatment in Peng and Xu [20], we consider the following BSDEs:

$$\tilde{Y}_t^n = \xi + \int_t^T g(s,\omega)\,ds + \int_t^T n\left(\tilde{Y}_s^n - L_s\right)^- ds - \int_t^T \tilde{Z}_s^n\,dB_s - \left(K_T^- - K_t^-\right). \tag{4.3}$$

By Meyer–Itô's formula to $[(\tilde{Y}_t^n - Y_t^n)^+]^2$, it is easy to get $\tilde{Y}_t^n \leqslant Y_t^n$ for any $t \in [0,T]$ a.s. Observe that Eq. (4.3) can be written as the following

$$\bar{Y}_t^n = \xi - K_T^- + \int_t^T g(s,\omega)\,ds + \int_t^T n\left(\bar{Y}_s^n - \left(L_s - K_s^-\right)\right)^- ds - \int_t^T \bar{Z}_s^n\,dB_s, \tag{4.4}$$

where $\bar{Y}_t^n := \tilde{Y}_t^n - K_t^-$, $\bar{Z}_t^n := \tilde{Z}_t^n$. It is easily seen that Eq. (4.4) is just the sequence of penalized equations of an RBSDE with the coefficient g, the lower barrier $L - K^-$ and the terminal value $\xi - K_T^-$. Hence by the standard result of RBSDEs with one lower barrier, there exists a triple of processes $(\bar{Y}, \bar{Z}, \bar{K}^+) \in \mathbf{S}^2 \times \mathbf{H}_d^2 \times \mathbf{A}^2$, which satisfies

$$\begin{cases} \bar{Y}_t = \xi - K_T^- + \displaystyle\int_t^T g(s,\omega)\,ds - \int_t^T \bar{Z}_s\,dB_s + \bar{K}_T^+ - \bar{K}_t^+, \\ L_t - K_t^- \leqslant \bar{Y}_t, \quad \forall t \in [0,T], \quad \text{and} \quad \displaystyle\int_0^T \left(\bar{Y}_t - \left(L_t - K_t^-\right)\right) d\bar{K}_t^+ = 0. \end{cases} \tag{4.5}$$

Indeed,

$$\bar{K}_t^+ = \lim_{n\to\infty} \int_0^t n\big(\bar{Y}_s^n - \big(L_s - K_s^-\big)\big)^- \, ds = \lim_{n\to\infty} \int_0^t n\big(\tilde{Y}_s^n - L_s\big)^- \, ds$$

in probability. So there exists a subsequence such that above convergence is almost surely. Moreover, for any $s \leqslant t$, we have, $\mathbb{P}$-a.s.,

$$\int_s^t n\big(\tilde{Y}_r^n - L_r\big)^- \, dr \geqslant \int_s^t n\big(Y_r^n - L_r\big)^- \, dr,$$

thus we have that $\bar{K}_t^+ - \bar{K}_s^+ \geqslant K_t^+ - K_s^+$ almost surely and then $d\bar{K}_t^+ \geqslant dK_t^+$. Note that Eq. (4.2) can be written as

$$Y_t - K_t^- = \xi - K_T^- + \int_t^T g(s,\omega)\, ds - \int_t^T Z_s \, dB_s + K_T^+ - K_t^+. \tag{4.6}$$

Applying Meyer–Itô's formula to $[(Y_t - K_t^- - \bar{Y}_t)^+]^2$, it is easy to obtain that $Y_t - K_t^- \leqslant \bar{Y}_t$, since $d\bar{K}_t^+ \geqslant dK_t^+$. Therefore,

$$0 \leqslant \int_0^T (Y_t - L_t)\, dK_t^+ \leqslant \int_0^T \big(\bar{Y}_t + K_t^- - L_t\big)\, dK_t^+ \leqslant \int_0^T \big(\bar{Y}_t - \big(L_t - K_t^-\big)\big)\, d\bar{K}_t^+ = 0,$$

and the proof is complete. □

We now state the existence and uniqueness result for Eq. (2.1).

Theorem 4.2. *Under the hypotheses of* (H0), (H1), (H2) *and* (2.5)*, there exists a quadruple of processes* (Y, Z, K^+, K^-) *to solve Eq.* (2.1). *The solution is unique in the following sense: if* $(Y', Z', K^{+'}, K^{-'})$ *is another solution of Eq.* (2.1), *then* $Y_t = Y_t'$, $Z_t = Z_t'$ *and* $K_t^+ - K_t^- = K_t^{+'} - K_t^{-'}$, $\forall t \in [0,T]$ *a.s.*

Proof. We will show the existence of the solution to Eq. (2.1) by applying the fixed point theorem. Let $\mathcal{D} := \mathbf{S}^2 \times \mathbf{H}_d^2$ endowed with the following norm:

$$\big\|(Y,Z)\big\|_\beta = \left(\mathbb{E}\int_0^T e^{\beta t}\big(|Y_t|^2 + |Z_t|^2\big)\, dt\right)^{\frac{1}{2}}$$

for appropriate $\beta > 0$ to be determined later. Let Φ be a map from $\mathcal{D}$ to $\mathcal{D}$. By Theorem 4.1, for any given $(\check{Y}, \check{Z}) \in \mathcal{D}$, there is a quadruple of processes $(\hat{Y}, \hat{Z}, \hat{K}^+, \hat{K}^-)$ to solve the reflected BSDE associated with $(f(t, \check{Y}_t, \check{Z}_t, \omega), \xi, L_t, U_t)$, hence $\Phi(\check{Y}, \check{Z}) = (\hat{Y}, \hat{Z})$. Let $(\check{Y}', \check{Z}')$ be another element in $\mathcal{D}$ and $\Phi(\check{Y}', \check{Z}') = (\hat{Y}', \hat{Z}')$, where $(\hat{Y}', \hat{Z}', \hat{K}^{+'}, \hat{K}^{-'})$ is a solution of the corresponding reflected BSDE. Define

$$\bar{Y}_t = \hat{Y}_t - \hat{Y}_t', \qquad \bar{Z}_t = \hat{Z}_t - \hat{Z}_t', \qquad \bar{K}_t^{\pm} = \hat{K}_t^{\pm} - \hat{K}_t^{\pm'}, \qquad \tilde{Y}_t = \check{Y}_t - \check{Y}_t',$$
$$\tilde{Z}_t = \check{Z}_t - \check{Z}_t'.$$

By Itô's formula, we have

$$\begin{aligned}
&e^{\beta t}\mathbb{E}|\bar{Y}_t|^2+\mathbb{E}\int_t^T e^{\beta s}\big[\beta|\bar{Y}_s|^2+|\bar{Z}_s|^2\big]\,ds\\
&\quad=2\mathbb{E}\int_t^T e^{\beta s}\bar{Y}_s\big[f(s,\check{Y}_s,\check{Z}_s)-f\big(s,\check{Y}'_s,\check{Z}'_s\big)\big]\,ds+2\mathbb{E}\int_t^T e^{\beta s}\bar{Y}_s\big(d\bar{K}_s^+-d\bar{K}_s^-\big)\\
&\quad\leqslant 2C\mathbb{E}\int_t^T e^{\beta s}|\bar{Y}_s|\big[|\tilde{Y}_s|+|\tilde{Z}_s|\big]\,ds\\
&\quad\leqslant 4C^2\mathbb{E}\int_t^T e^{\beta s}|\bar{Y}_s|^2\,ds+\frac{1}{2}\mathbb{E}\int_t^T e^{\beta s}\big[|\tilde{Y}_s|^2+|\tilde{Z}_s|^2\big]\,ds,
\end{aligned}$$

where we have used the fact of $\bar{Y}(d\bar{K}^+-d\bar{K}^-)\leqslant 0$, whose proof can be found in Cvitanić and Karatzas [2]. Now choose $\beta=1+4C^2$, then we immediately obtain

$$\mathbb{E}\int_0^T e^{\beta t}\big[|\bar{Y}_t|^2+|\bar{Z}_t|^2\big]\,dt\leqslant\frac{1}{2}\mathbb{E}\int_0^T e^{\beta t}\big[|\tilde{Y}_t|^2+|\tilde{Z}_t|^2\big]\,dt,$$

and the contraction property gives the required result.

Suppose that $(Y',Z',K^{+'},K^{-'})$ is another solution to Eq. (2.1). Applying Itô's formula to $(Y_t-Y'_t)^2$, we have

$$\begin{aligned}
\big|Y_t-Y'_t\big|^2=&\int_t^T 2\big(Y_s-Y'_s\big)\big(d(s,Y_s,Z_s,\omega)-f\big(s,Y'_s,Z'_s,\omega\big)\big)\,ds\\
&+\int_t^T 2\big(Y_s-Y'_s\big)\big(dK_s^+-dK_s^{+'}\big)-\int_t^T 2\big(Y_s-Y'_s\big)\big(dK_s^--dK_s^{-'}\big)\\
&-\int_t^T 2\big(Y_s-Y'_s\big)\big(Z_s-Z'_s\big)\,dB_s-\int_t^T\big|Z_s-Z'_s\big|^2\,ds.
\end{aligned}\tag{4.7}$$

Similarly, the second term and the third term in the right-hand side of (4.7) are all less than or equal to 0. It then follows that

$$\mathbb{E}\big|Y_t-Y'_t\big|^2+\frac{1}{2}\mathbb{E}\int_t^T\big|Z_s-Z'_s\big|^2\,ds\leqslant\big(2C+2C^2\big)\mathbb{E}\int_t^T\big|Y_s-Y'_s\big|^2\,ds,$$

therefore

$$\mathbb{E}\big|Y_t-Y'_t\big|^2=0\quad\text{and}\quad\mathbb{E}\int_0^T\big|Z_t-Z'_t\big|^2\,dt=0.$$

Based on these, we further use the Burkholder–Davis–Gundy inequality for (4.7) and obtain $\mathbb{E}\sup_{0\leqslant t\leqslant T}|Y_t - Y_t'|^2 = 0$, hence $Y = Y'$, $Z = Z'$ a.s. Note that

$$Y_0 - Y_t = \int_0^t f(s, Y_s, Z_s, \omega)\,ds - \int_0^t Z_s\,dB_s + K_t^+ - K_t^-,$$

$$Y_0' - Y_t' = \int_0^t f\big(s, Y_s', Z_s', \omega\big)\,ds - \int_0^t Z_s'\,dB_s + K_t^{+'} - K_t^{-'},$$

by the Burkholder–Davis–Gundy inequality, it is not hard to obtain

$$\mathbb{E}\sup_{0\leqslant t\leqslant T}\big|K_t^+ - K_t^- - \big(K_t^{+'} - K_t^{-'}\big)\big|^2 = 0.$$

The proof of Theorem 4.2 is complete. □

Remark 4.1. It should be mentioned that Theorem 4.2 still holds true if U_t satisfies (2.3) and L_t is continuous and satisfies $\mathbb{E}(\sup_{0\leqslant t\leqslant T}|L_t|^2) < \infty$. Indeed, we have pointed out that Lemma 3.1 can be similarly derived. Lemma 3.2 is independent of the representation of stochastic differential equation for L_t. Lemma 3.3 and Lemma 3.4 will consider the following sequence of RBSDEs with one lower barrier L_t,

$$\begin{cases} Y_t^k = \xi + \displaystyle\int_t^T g(s,\omega)\,ds - \int_t^T Z_s^k\,dB_s + \big(K_T^{k,+} - K_t^{k,+}\big) - \int_t^T k\big(U_s - Y_s^k\big)^-\,ds, \\ L_t \leqslant Y_t^k, \quad \forall t \in [0,T], \quad \text{and} \quad \displaystyle\int_0^T \big(Y_t^k - L_t\big)\,dK_t^{k,+} = 0. \end{cases}$$

Lemma 3.5 can be similarly proved by discussing the limit of (3.24) as $k \to \infty$. Hence in the setup of this work the regularity on the boundary processes has been weakened.

5. Dynkin games under Knightian uncertainty

Our purpose in this section is to solve a Dynkin game under Knightian uncertainty by applying the technique of RBSDEs with two barriers. Dynkin games are special stochastic games and have been studied by many authors, we here only mention Cvitanić and Karatzas [2], Hamadène and Lepeltier [5], Hamadène and Hassani [8], Hamadène and Hdhiri [9] among others. Just like these authors, we shall establish the relation between an RBSDE with two barriers and a Dynkin game under Knightian uncertainty.

In a financial market, we suppose that there are three assets whose prices are modeled by the following stochastic differential equations:

$$P_t = P_0 + \int_0^t b_0(r, P_r)\,dr + \int_0^t \sigma_0(r, P_r)\,dB_r, \quad S_0 > 0,\ 0 \leqslant t \leqslant T, \tag{5.1}$$

$$L_t = L_0 + \int_0^t b_1(r, L_r)\,dr + \int_0^t \sigma_1(r, L_r)\,dB_r, \quad L_0 > 0,\ 0 \leqslant t \leqslant T, \tag{5.2}$$

$$U_t = U_0 + \int_0^t b_2(r, U_r)\,dr + \int_0^t \sigma_2(r, U_r)\,dB_r, \quad U_0 > 0,\ 0 \leqslant t \leqslant T. \tag{5.3}$$

Assume that all the coefficients in above equations satisfy the local Lipschitz condition and the linear growth condition. We also assume that P_t, L_t and U_t are all positive for any $t \in [0, T]$ almost surely. A trivial example is the geometric Brownian motion.

We now consider a stochastic game, in which there are two players. Player 1 chooses the stopping time σ while player 2 chooses the stopping time τ. The game stops when one player decides to stop before the maturity time $T > 0$. Let $R_0(\tau, \sigma)$ represent the amount paid by player 1 to player 2 at $t = 0$. The random payoff $R_0(\tau, \sigma)$ is given by

$$\int_0^{\tau\wedge\sigma} g(u, P_u)\,du + \begin{cases} U_\sigma, & \text{if player 1 stops the game first,} \\ L_\tau, & \text{if player 2 stops the game first} \\ & \text{or both stop the game simultaneously,} \\ \xi, & \text{if neither player stops the game before } T, \end{cases}$$

namely,

$$R_0(\tau, \sigma) = \int_0^{\tau\wedge\sigma} g(u, P_u)\,du + L_\tau 1_{\{\tau \leqslant \sigma < T\}} + U_\sigma 1_{\{\sigma < \tau\}} + \xi 1_{\{\tau = \sigma = T\}}. \tag{5.4}$$

Player 1 wants to minimize, but player 2 wants to maximize the expectation $\mathbb{E} R_0(\tau, \sigma)$.

The uncertainty of this financial market comes from Brownian motion $\{B_t\}_{0\leqslant t\leqslant T}$ under the probability measure $\mathbb{P}$. However, Knightian uncertainty assumes that the financial market might evolve under the probability measure $\mathbb{P}$ or another probability measure $\mathbb{P}^\vartheta$. The parameter $\vartheta = \{\vartheta_t\}_{t\leqslant T}$ is a $\mathscr{P}$-measurable process such that $|\vartheta| \leqslant k$, where k is called the degree of Knightian uncertainty. Denote the family of those previous ϑ by Θ. We assume that $\mathbb{P}^\vartheta$ is absolutely continuous with respect to $\mathbb{P}$ and its density function is given by

$$\frac{d\mathbb{P}^\vartheta}{d\mathbb{P}} = \exp\left(-\int_0^T \vartheta_t\,dB_t - \frac{1}{2}|\vartheta_t|^2\,dt\right). \tag{5.5}$$

Thus by the Girsanov theorem, $B_t^\vartheta = B_t + \int_0^t \vartheta_s\,ds$ is a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P}^\vartheta)$. This means that the uncertainty of the financial market comes from $\{B_t^\vartheta\}_{0\leqslant t\leqslant T}$ under the probability measure $\mathbb{P}^\vartheta$. For more details on Knightian uncertainty, one can refer to Kiohiko and Hiroyuki [11]. Similar to Hamadène and Hdhiri [9], we are interested in the following value functions:

$$\underline{V} := \inf_{\vartheta\in\Theta} \sup_\tau \inf_\sigma \mathbb{E}^\vartheta\big[R_0(\tau, \sigma)\big], \tag{5.6}$$

$$\overline{V} := \sup_{\vartheta\in\Theta} \sup_\tau \inf_\sigma \mathbb{E}^\vartheta\big[R_0(\tau, \sigma)\big], \tag{5.7}$$

where $\mathbb{E}^\vartheta$ is the expectation operator with respect to $\mathbb{P}^\vartheta$. Assume $\theta \in R^d$ and $|\theta| \leqslant k$. For any $(t, p, \theta, z) \in [0, T] \times R_+ \times R^d \times R^d$ with $|\theta| \leqslant k$, we define a Hamiltonian function

$$H(t, p, \theta, z) = g(t, p) - (\theta, z).$$

Let

$$\theta^1(z) = \left(\frac{k}{\sqrt{d}} 1_{\{z^1>0\}} - \frac{k}{\sqrt{d}} 1_{\{z^1\leqslant 0\}}, \ldots, \frac{k}{\sqrt{d}} 1_{\{z^d>0\}} - \frac{k}{\sqrt{d}} 1_{\{z^d\leqslant 0\}}\right)',$$
$$\theta^2(z) = -\theta^1(z),$$

where $z = (z^1, \ldots, z^d)'$, then θ^1 and θ^2 satisfy

$$H\big(t, p, \theta^1(z), z\big) = \inf_{\theta:\, |\theta|\leqslant k} H(t, p, \theta, z), \qquad H\big(t, p, \theta^2(z), z\big) = \sup_{\theta:\, |\theta|\leqslant k} H(t, p, \theta, z). \tag{5.8}$$

Obviously, $H(t, p, \theta, z)$ satisfy the Lipschitz condition with respect to z for any given (t, p, θ), while $H(t, p, \theta^1(z), z)$ and $H(t, p, \theta^2(z), z)$ satisfy the Lipschitz condition w.r.t. z for any given (t, p). We have the following result.

Theorem 5.1. *Suppose that $g(t, P_t)$ belongs to $\mathbf{L}^2(0, T)$. Then the two barriers reflected BSDE associated with $(H(t, p, \theta, z), L, U, \xi)$ (resp. $H(t, p, \theta^1(z), z)$, $H(t, p, \theta^2(z), z)$) have a unique solution (Y, Z, K^+, K^-) (resp. $(\underline{Y}, \underline{Z}, \underline{K}^+, \underline{K}^-)$, $(\overline{Y}, \overline{Z}, \overline{K}^+, \overline{K}^-)$). If we define $\underline{\tau} = \inf\{t \in [0, T],\ \underline{Y}_t = L_t\} \wedge T$, $\overline{\tau} = \inf\{t \in [0, T],\ \overline{Y}_t = L_t\} \wedge T$, $\underline{\sigma} = \inf\{t \in [0, T],\ \underline{Y}_t = U_t\} \wedge T$, $\overline{\sigma} = \inf\{t \in [0, T],\ \overline{Y}_t = U_t\} \wedge T$, then $\underline{Y}_0 = \underline{V}$, $\overline{Y}_0 = \overline{V}$ and $\underline{V} \leqslant Y_0 \leqslant \overline{V}$.*

Proof. The existence of a unique solution follows from Theorem 4.2 since (H1) and (H2) hold under the assumptions of g as well as b_0 and σ_0. Let $\hat{\tau} = \inf\{t \in [0, T], Y_t = L_t\} \wedge T$ and $\hat{\sigma} = \inf\{t \in [0, T],\ Y_t = U_t\} \wedge T$. Note that

$$Y_t = \xi + \int_t^T g(s, P_s)\, ds - \int_t^T Z_s\, dB_s^\vartheta + K_T^+ - K_t^+ - \big(K_T^- - K_t^-\big),$$

hence

$$Y_0 = \mathbb{E}^\vartheta Y_0 = \mathbb{E}^\vartheta\left[Y_{\hat{\tau}\wedge\hat{\sigma}} + \int_0^{\hat{\tau}\wedge\hat{\sigma}} g(s, P_s)\, ds - \int_0^{\hat{\tau}\wedge\hat{\sigma}} Z_s\, dB_s^\vartheta + K^+_{\hat{\tau}\wedge\hat{\sigma}} - K^-_{\hat{\tau}\wedge\hat{\sigma}}\right].$$

Since $L_t \leqslant Y_t \leqslant U_t$ and $\int_0^T (Y_t - L_t)\, dK_t^+ = 0$, $\int_0^T (U_t - Y_t)\, dK_t^- = 0$, then $K^+_{\hat{\tau}\wedge\hat{\sigma}} = K^-_{\hat{\tau}\wedge\hat{\sigma}} = 0$. On the other hand, by the Burkholder–Davis–Gundy inequality, we have

$$\mathbb{E}^\vartheta\left[\sup_{0\leqslant t\leqslant T}\left|\int_0^t Z_s\, dB_s^\vartheta\right|\right] \leqslant 4\mathbb{E}^\vartheta\left[\int_0^T |Z_t|^2\, dt\right]^{\frac{1}{2}} \leqslant 4\sqrt{\mathbb{E}\int_0^T |Z_t|^2\, dt}\sqrt{\mathbb{E}\left(\frac{d\mathbb{P}^\vartheta}{d\mathbb{P}}\right)^2}.$$

Since $|\vartheta| \leqslant k$, by Novikov's condition and the B-D-G inequality, it is not hard to prove $\mathbb{E}[(d\mathbb{P}^\vartheta/d\mathbb{P})^2] < \infty$. Therefore $\{\int_0^t Z_s\, dB_s^\vartheta\}$ is a martingale under the probability measure $\mathbb{P}^\vartheta$. By these, we obtain

$$\begin{aligned} Y_0 &= \mathbb{E}^\vartheta\left[\int_0^{\hat{\tau}\wedge\hat{\sigma}} g(s, P_s)\, ds + Y_{\hat{\tau}\wedge\hat{\sigma}}\right] \\ &= \mathbb{E}^\vartheta\left[\int_0^{\hat{\tau}\wedge\hat{\sigma}} g(s, P_s)\, ds + L_{\hat{\tau}} 1_{\{\hat{\tau}\leqslant\hat{\sigma}<T\}} + U_{\hat{\sigma}} 1_{\{\hat{\sigma}<\hat{\tau}\}} + \xi 1_{\{\hat{\tau}=\hat{\sigma}=T\}}\right] = \mathbb{E}^\vartheta R_0(\hat{\tau}, \hat{\sigma}). \end{aligned}$$

Similar to Theorem 4.1 in Cvitanić and Karatzas [2], we can prove

$$Y_0 = \mathbb{E}^{\vartheta} R_0(\hat{\tau}, \hat{\sigma}) = \inf_{\sigma} \sup_{\tau} \mathbb{E}^{\vartheta} R_0(\tau, \sigma) = \sup_{\tau} \inf_{\sigma} \mathbb{E}^{\vartheta} R_0(\tau, \sigma).$$

In the same way, we similarly prove that

$$\underline{Y}_0 = \mathbb{E}^{\vartheta^1} R_0(\underline{\tau}, \underline{\sigma}) = \inf_{\sigma} \sup_{\tau} \mathbb{E}^{\vartheta^1} R_0(\tau, \sigma) = \sup_{\tau} \inf_{\sigma} \mathbb{E}^{\vartheta^1} R_0(\tau, \sigma),$$
$$\overline{Y}_0 = \mathbb{E}^{\vartheta^2} R_0(\bar{\tau}, \bar{\sigma}) = \inf_{\sigma} \sup_{\tau} \mathbb{E}^{\vartheta^2} R_0(\tau, \sigma) = \sup_{\tau} \inf_{\sigma} \mathbb{E}^{\vartheta^2} R_0(\tau, \sigma).$$

By the comparison theorem on RBSDEs with two barriers (see Hamadène and Hassani [8, Theorem 1.3]), we have $\underline{Y}_0 \leqslant Y_0 \leqslant \overline{Y}_0$, which implies that

$$\underline{Y}_0 = \underline{V}, \qquad \overline{Y}_0 = \overline{V}.$$

The proof is complete. □

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