Multivalued superposition operators in ideal spaces of vector functions.1

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ABSTRACT

The paper is concerned with boundedness properties of nonlinear superposition operators generated by multi-valued functions \( f: \Omega \times \mathbb{R}^m \to \mathcal{P}(\mathbb{R}^n) \) between ideal spaces (Banach lattices) of vector functions. In particular, sufficient conditions on the spaces \( X \) and \( Y \) are given under which any superposition operator from \( X \) into \( \mathcal{P}(Y) \) is locally bounded or bounded on bounded sets.

The present paper is concerned with boundedness properties of multi-valued nonlinear superposition operators in ideal spaces of vector functions. Although ideal spaces of scalar functions have been extensively studied (see, for instance, the detailed references in the books [5], [6], [7], [9]), in the vector case such spaces have been introduced [10] and studied (see e.g. [13], [14]) only quite recently. Important examples of ideal spaces arising in applications are Orlicz spaces (see e.g. [8], [11], [12]). The results presented in this paper may be considered as natural higher-dimensional and multi-valued extensions of corresponding results from [2], see also Chapter 2 of [1]. However, the structure of the spaces and operators we are going to study is much more complicated than in the single-valued scalar case, and several new features occur here. Thus, the multi-valued higher-dimensional theory presented here is by no means a straightforward generalization of the single-valued scalar theory.

1. IDEAL SPACES OF VECTOR FUNCTIONS

In this section, we shall briefly recall some facts from the theory of ideal
spaces of vector functions which will be needed in the sequel. Let $\Omega$ be an arbitrary non-empty set, $\mathcal{M}$ some $\sigma$-algebra of subsets of $\Omega$, $\mu$ a complete non-negative $\sigma$-finite and countably additive measure on $\mathcal{M}$, and $\mu_*$ some normalized measure which is equivalent to $\mu$ (i.e. has the same null sets). It is known that one may represent $\Omega$ as a disjoint union of a "continuous" part $\Omega^c$ and a "discrete" part $\Omega^d$, i.e. $\mu$ is atomic-free on $\Omega^c$ and purely atomic on $\Omega^d$. By $S(\Omega, \mathbb{R}^k)$ we denote the (complete) metric space of all (classes of) measurable functions on $\Omega$ with values in $\mathbb{R}^k$, equipped with the metric induced by convergence in measure (see e.g. [4]), and equipped with the pairing

$$(1) \quad \langle x, y \rangle = \int_{\Omega} (x(s), y(s)) d\mu(s),$$

where $(\cdot, \cdot)$ denotes the usual scalar product in $\mathbb{R}^k$. In what follows, we denote by $\mathcal{P}(V)$ (resp. $\text{Cl}(V), \text{Cp}(V), \text{CV}(V), \text{ClCV}(V), \text{CpCV}(V)$) the class of all (resp. all closed, compact, convex, closed convex, compact convex) non-empty subsets of a metric linear space $V$. All multi-valued maps (called "multis" in the sequel, for short), we are going to study take their values in one of these classes.

Let us call a multi $B : \Omega \to \text{Cl}(\mathbb{R}^k)$ Castaing–measurable if it admits a Castaing representation, i.e. a countable family of (single-valued) measurable selections $b_n$ of $B$ such that the set $\{b_n(s) : n = 1, 2, \ldots\}$ is dense in $B(s)$ for almost all $s \in \Omega$. It is well-known [3] that Castaing–measurability is equivalent to the usual measurability of multis.

A Banach space $X \subset S(\Omega, \mathbb{R}^k)$ with norm $\| \cdot \|_X$ is called ideal space if the relations $x \in X$ and $\theta \in L_\infty(\Omega, \mathbb{R})$ (the space of all real essentially bounded functions on $\Omega$), imply that also $\theta x \in X$ and $\|\theta x\|_X \leq \|\theta\|_{L_\infty} \|x\|_X$. Examples of ideal spaces are Lebesgue spaces, Orlicz spaces (which arise in the study of strongly nonlinear problems), or Lorentz and Marcinkiewicz spaces (which arise in interpolation theory for linear and nonlinear operators).

With every ideal space $X$ one may associate its vector-support $\text{supp}_X$; this is the minimal (with respect to inclusion) Castaing–measurable multi on $\Omega$ with the property that $x(s) \in \text{supp}_X(s)$ for any $x \in X$. Given an ideal space $X$, the associate space $X'$ to $X$ consists, by definition, of all functions $y \in S(\Omega, \mathbb{R}^k)$ such that $y(s) \in \text{supp}_X(s)$ and $|\langle x, y \rangle| < \infty$ (see (1)) for all $x \in X$. Equipped with the norm

$$(2) \quad \|y\|_{X'} = \sup\{\langle x, y \rangle : \|x\|_X \leq 1\},$$

$X'$ is also an ideal space. An ideal space $X$ is called perfect if $X$ coincides with its second associate space $X''$.

2. MULTI-VALUED SUPERPOSITION OPERATORS

Recall that the sets $\text{Cp}(\mathbb{R}^k)$ and $\text{CpCV}(\mathbb{R}^k)$ are complete separable metric spaces with respect to the Hausdorff metric

$$(3) \quad h(A, B) = \max\{\sup_{a \in A} \text{dist}(a, B), \sup_{b \in B} \text{dist}(b, A)\},$$

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while the sets $\text{Cl}(\mathbb{R}^k)$ and $\text{ClCv}(\mathbb{R}^k)$ are complete separable metric spaces with respect to the metric

\[
d(A, B) = \sum_{j=0}^{\infty} \frac{h(A \cap B_j(\mathbb{R}^k), B \cap B_j(\mathbb{R}^k))}{1 + h(A \cap B_j(\mathbb{R}^k), B \cap B_j(\mathbb{R}^k))}.
\]

(Here and in what follows, we denote by $B_r(X)$ the closed ball with centre at 0 and radius $r>0$ in a normed space $X$). The topology induced by the restriction of the metric (4) to the sets $C_p(\mathbb{R}^k)$ and $C_pCv(\mathbb{R}^k)$ coincides with that induced by the Hausdorff metric (3).

A multi $f: \Omega \times \mathbb{R}^m \to \mathcal{P}(\mathbb{R}^n)$ is called *superpositionally measurable* (or *sup-measurable*, for short) if, for any (single-valued) function $x \in S(\Omega, \mathbb{R}^m)$, the multi

\[
(5) \quad F_x(s) = f(s, x(s))
\]

is measurable, i.e. belongs to $S(\Omega, \mathcal{P}(\mathbb{R}^n))$. Likewise, $f$ is called *weakly superpositionally measurable* (or *weakly sup-measurable*, for short) if, for any $x \in S(\Omega, \mathbb{R}^m)$, the multi (5) admits at least a measurable selection. Obviously, every sup-measurable multi is also weakly sup-measurable.

Given a weakly sup-measurable multi $f: \Omega \times \mathbb{R}^m \to \mathcal{P}(\mathbb{R}^n)$, the superposition operator (or Nemytskij operator) $N_f$ generated by $f$ associates to each $x \in S(\Omega, \mathbb{R}^m)$, by definition, the set of all measurable selections of the multi (5), i.e.

\[
(6) \quad N_f(x) = \{ y : y \in S(\Omega, \mathbb{R}^n), y(s) \in F_x(s) \text{ for almost all } s \in \Omega \}.
\]

An important property of the operator (6) is its *local determination*; this means that $x(s) = y(s)$ almost everywhere on $D \in \mathcal{M}$ implies that also $N_f(x)(s) = N_f(y)(s)$ almost everywhere on $D$. Equivalently, this may be expressed as

\[
(7) \quad P_D N_f(x) = P_D N_f(P_D x) \quad (D \in \mathcal{M}),
\]

where $P_D$ denotes the multiplication operator by the characteristic function of $D \in \mathcal{M}$, i.e.

\[
P_D x(s) = \chi_D(s) x(s) = \begin{cases} x(s) & \text{if } s \in D, \\ 0 & \text{if } s \not\in D. \end{cases}
\]

There are two important classes of sup-measurable multis. A multi $f: \Omega \times \mathbb{R}^m \to \text{Cl}(\mathbb{R}^n)$ is called a *Carathéodory multi* if $f(s, \cdot) : \mathbb{R}^m \to \text{Cl}(\mathbb{R}^n)$ is continuous for (almost) all $s \in \Omega$, and $f(\cdot, u) : \Omega \to \text{Cl}(\mathbb{R}^n)$ is Castaing-measurable for all $u \in \mathbb{R}^m$. More generally, $f$ is called a *Shragin multi* if there exists a null set $N_0 \subseteq \Omega$ such that, for any closed Borel subset $C \subseteq \mathbb{R}^n$, the set $\{(s, u) : s \in \Omega \setminus N_0, u \in \mathbb{R}^m, f(s, u) \subseteq C\}$ belongs to $\mathcal{M} \otimes \mathcal{B}_m$, the minimal product algebra generated by the sets $M \in \mathcal{M}$ and the Borel subsets $B \subseteq \mathbb{R}^m$.

We mention that every Carathéodory multi is a Shragin multi, and every Shragin multi is sup-measurable.

In what follows, we shall be concerned basically with two boundedness properties of the superposition operator (6). These boundedness properties are
motivated, for instance, by certain problems involving cones in ordered Orlicz spaces of vector functions [15]. Given two metric spaces \( X \subseteq S(\Omega, \mathbb{R}^m) \) and \( Y \subseteq S(\Omega, \mathbb{R}^n) \) and a subset \( G \subseteq X \), the operator \( N_f \) is called \textit{bounded} between \( G \) and \( \mathcal{P}(Y) \) if, for any bounded subset \( M \) of \( G \), the image \( N_f(M) = \bigcup_{x \in M} N_f(x) \) is bounded in \( Y \). Likewise, \( N_f \) is called \textit{quasi-bounded} if one may choose, for any \( x \in M \), a function \( y_x \in N_f(x) \) such that the set \( \{ y_x : x \in M \} \) is bounded in \( Y \); loosely speaking, this means that one may majorize at least a representative system of selections of \( N_f(M) \). The notions of \textit{locally bounded} and \textit{locally quasi-bounded} superposition operators are defined similarly.

In what follows, we use the notation

\[ \| M \|_* = \sup_{m \in M} |m|, \quad |M|_* = \inf_{m \in M} |m| \]

for any bounded subset \( M \) of a normed space \( X \). Obviously, this defines two disjointly sublinear functionals on \( Cl(X) \), i.e.

\[
\begin{align*}
\| M + N \|_* &\leq \| M \|_* + \| N \|_* , \\
\| \lambda M \|_* &\leq |\lambda| \| M \|_* , \\
\| M - N \|_* &\leq \| M \|_* + \| N \|_* , \\
\| \lambda M \|_* &\leq |\lambda| \| M \|_* ,
\end{align*}
\]

whenever \( M \cap N = \emptyset \).

We give now two lemmas on the (quasi-)boundedness of the operator (6) on the whole space \( S(\Omega, \mathbb{R}^m) \). The proofs are straightforward, and so we shall not present them.

**Lemma 1.** Suppose that \( f : \Omega \times \mathbb{R}^m \to Cl(\mathbb{R}^n) \) is a sup-measurable (resp. weakly sup-measurable) multi. Then the corresponding superposition operator \( N_f : S(\Omega, \mathbb{R}^m) \to \mathcal{P}(S(\Omega, \mathbb{R}^n)) \) is bounded (resp. quasi-bounded) if and only if, for each \( r > 0 \), the scalar function

\[
\Phi^*(s) = \sup_{x \in f(r)} |N_f(x)(s)|^*
\]

(resp. the scalar function

\[
\Phi_*^*(s) = \sup_{x \in f(r)} \|N_f(x)(s)\|_*^*
\]

is finite and measurable on \( \Omega \); here \( f(r) \) denotes the set of all functions \( x \in S(\Omega, \mathbb{R}^m) \) whose graphs are contained (almost everywhere) in \( \Omega \times B_r(\mathbb{R}^m) \).

**Lemma 2.** Suppose that \( f : \Omega \times \mathbb{R}^m \to Cl(\mathbb{R}^n) \) is a Carathéodory or Shragin multi. Then the corresponding superposition operator \( N_f : S(\Omega, \mathbb{R}^m) \to \mathcal{P}(S(\Omega, \mathbb{R}^n)) \) is bounded (resp. quasi-bounded) if and only if, for each \( r > 0 \), the scalar function

\[
\varphi^*(s) = \sup_{|u| \leq r} \|f(s, u)\|^*
\]

(resp. the scalar function

\[
\varphi_*^*(s) = \sup_{|u| \leq r} \|f(s, u)\|_*^*
\]
\[ \varphi_s(s) = \sup_{|u| \leq r} \|f(s, u)\|_* \]
is finite and measurable on \( \Omega \).

3. LOCAL BOUNDEDNESS

The preceding two lemmas characterize the (quasi-)boundedness of multi-valued superposition operators in the space \( S(\Omega, \mathbb{R}^k) \) of all measurable functions. In this and the following sections, we shall be concerned with boundedness properties of superposition operators between ideal spaces. As we have distinguished sup-measurability and weak sup-measurability in the preceding section, we shall now distinguish acting conditions for \( N_f \) between two ideal spaces \( X \) and \( Y \) (i.e. \( N_f(x) \subseteq Y \) for all \( x \in X \)) from quasi-acting conditions (i.e. \( N_f(x) \cap Y \neq \emptyset \) for all \( x \in X \)). Without loss of generality, we formulate all results for the case of continuous measures (i.e. \( \Omega = \Omega^c \)) and discrete measures (i.e. \( \Omega = \Omega^d \)) separately.

**Theorem 1.** Let \( X \subset S(\Omega, \mathbb{R}^m) \) and \( Y \subset S(\Omega, \mathbb{R}^n) \) be two ideal spaces over \( \Omega = \Omega^c \), where \( Y \) is perfect. Suppose that \( f: \Omega \times \mathbb{R}^m \to \text{Cl}(\mathbb{R}^n) \) is a weakly sup-measurable multi. Assume that the interior \( G \) of the domain of definition \( \mathcal{D}(N_f) \) of the operator (6), acting between \( X \) and \( \mathcal{P}(Y) \), is non-empty. Then \( N_f \) is locally bounded on \( G \).

**Theorem 2.** Let \( X \subset S(\Omega, \mathbb{R}^m) \) and \( Y \subset S(\Omega, \mathbb{R}^n) \) be two ideal spaces over \( \Omega = \Omega^d \), where \( Y \) is perfect. Suppose that \( f: \Omega \times \mathbb{R}^m \to \text{Cl}(\mathbb{R}^n) \) is a weakly sup-measurable multi. Assume that the interior \( G \) of the domain of definition \( \mathcal{D}(N_f) \) of the operator (6), acting between \( X \) and \( \mathcal{P}(Y) \), is non-empty. For \( s \in \Omega \), let

\[ I(s) = \{ u : u \in \text{supp}_X(s), f(s, u) \subseteq \text{supp}_Y(s) \}, \]

and denote by \( \text{ri}_X I(s) \) the relative interior of \( I(s) \) with respect to \( \text{supp}_X(s) \). Then \( N_f \) is locally bounded on \( G \) if and only if, for almost all \( s \in \Omega \), the function

\[ \beta_s(u) = \|f(s, u)\|_* \]
(see (8)) is bounded on each compact subset of \( \text{ri}_X I(s) \).

**Proof of Theorem 1.** Suppose that the assertion is false. Without loss of generality, we may suppose that \( B_{\delta}(X) \subseteq G \) for some \( \delta > 0 \), and \( N_f \) is locally unbounded at 0. Then we may find sequences \( x_n \in X \) and \( y_n \in Y \) such that

\[ \|x_n\| \leq \delta 2^{-n}, y_n \in N_f(x_n), \|y_n\| > n2^n. \]

Since \( \Omega = \Omega^c \), we may choose a partition \( \{D_{n,j} : j = 1, \ldots, 2^n \} \) of \( \Omega \) with \( \mu_*(D_{n,j}) = 2^{-n} \) for \( j = 1, \ldots, 2^n \). For at least one index \( j(n) \) we have \( |P_{D_{n,j(n)}} y_n| \)
Since otherwise \( \| y_n \| \leq n2^n \) contradicting (10). Let 
\[
\Omega_{n,m} = D_{n,j(m)} \setminus \bigcup_{k=m}^{\infty} D_{k,j(k)}.
\]
Since 
\[
\mu_\phi\left( \bigcup_{k=0}^{\infty} D_{k,j(k)} \right) \leq \sum_{k=0}^{\infty} 2^{-k} = 2^{1-m} \to 0 \quad (m \to \infty)
\]
and \( Y \) is perfect, we conclude that \( \| P_{\Omega_{n,m}} y_n \| \to \| P_{D_{n,j(n)}} y_n \| \) as \( m \to \infty \). This means that with each \( n \in \mathbb{N} \) we may associate an \( n' \in \mathbb{N} \), \( n' > n \), such that \( \| P_{\Omega_{n',y}} y_n \| > n \). By induction, we thus may construct a sequence \( n_1, n_2, \ldots, n_k = n_{k-1}, \ldots \) of natural numbers such that the sets \( \Omega_k = \Omega_{n_k, n_{k+1}} \) are mutually disjoint and satisfy 
\[
\mu_\phi(\Omega_k) < 2^{-n_k}, \quad \| P_{\Omega_k} y_n \| > n_k.
\]
The function \( x_\ast = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} P_{\Omega_k} x_{n_k} \) belongs then to \( G \), since 
\[
\| x_\ast \| \leq \sum_{k=1}^{\infty} \| P_{\Omega_k} x_{n_k} \| \leq \sum_{k=1}^{\infty} \delta 2^{-k} = \delta.
\]
Now, by the local determination of the operator \( N_f \) (see (7)), we have 
\[
P_{\Omega_k} N_f(x_\ast) = P_{\Omega_k} N_f(P_{\Omega_k} x_\ast) = P_{\Omega_k} N_f(x_{n_k}).
\]
Since \( y_{n_k} \in N_f(x_{n_k}) \), we further conclude that \( P_{\Omega_k} y_{n_k} \in P_{\Omega_k} N_f(x_{n_k}) = P_{\Omega_k} N_f(x_\ast) \). By assumption, the multi \( f \) is weakly sup-measurable, and hence we may find a measurable function \( y_0 \in N_f(x_\ast) \). Let 
\[
y_\ast(s) = \begin{cases} y_{n_k}(s) & \text{if } s \in \Omega_k, \\ y_0(s) & \text{if } s \in \Omega_0,
\end{cases}
\]
where \( \Omega_0 = \Omega \setminus \bigcup_{k=1}^{\infty} \Omega_k \). Since \( P_{\Omega_k} y_\ast = P_{\Omega_k} y_{n_k} \in P_{\Omega_k} N_f(x_\ast) \), on the one hand, and \( P_{\Omega_0} y_\ast = P_{\Omega_0} y_0 \in P_{\Omega_0} N_f(x_\ast) \), on the other, we have \( y_\ast \in N_f(x_\ast) \). But 
\[
\| y_\ast \| \geq \| P_{\Omega_0} y_\ast \| = \| P_{\Omega_0} y_{n_k} \| > n_k \to \infty,
\]
a contradiction. \( \square \)

**PROOF OF THEOREM 2.** Assume first that the function (9) is bounded on each compact subset of \( r_i X I(s) \), and suppose again that \( B_\delta(X) \subseteq G \) for some \( \delta > 0 \), and \( N_f \) is locally unbounded at 0. For each \( s \in \Omega \), the inclusion 
\[
B_\delta(X) \cap \text{supp}_X(s) \subseteq r_i X I(s)
\]
holds. Without loss of generality, let \( \Omega \) be of the form \( \Omega = \Omega^2 = \{ s_1, s_2, \ldots, s_n, \ldots \} \), and let \( P_n = P_{\{s_{n+1}, s_{n+2}, \ldots \}} \) denote the multiplication operator by the characteristic function of \( \{s_{n+1}, s_{n+2}, \ldots \} \). By the boundedness of the function (9), all the numbers 
\[
c_n = \sum_{k=1}^{n} \sup \{ \| w \| : w \in f(s_k, u), \| P_{\{s_k\}} u \| \leq \delta \}
\]
are finite. Since \( N_f \) is locally unbounded at 0, by assumption, we find
sequences \( x_n \in X \) and \( y_n \in Y \) such that

\[
\|x_n\| \leq 2^{-n}, \quad y_n \in N_f(x_n), \quad \|y_n\| > c_n + n.
\]

By the perfectness of \( Y \), we have \( \|(I - P_m)y_n\| \to \|y_n\| \) as \( m \to \infty \). This means that with each \( n \in \mathbb{N} \) we may associate an \( n' \in \mathbb{N} \), \( n' > n \), such that \( \|(I - P_m)y_n\| > c_n + n \), hence

\[
\|(P_n - P_{n'})y_n\| \geq \|(I - P_m)y_n\| - \|(I - P_n)y_n\|
\]

\[
\geq c_n + n - |N_f(I - P_n)x_n|^* \\
\geq c_n + n - c_n = n.
\]

By induction, we may construct a sequence \( n_1, n_2, \ldots, n_k = n_{k-1} \ldots \) of natural numbers such that

\[
\|(P_{n_k} - P_{n_{k-1}})x_n\| \leq 2^{-n_k}, \quad \|(P_{n_k} - P_{n_{k+1}})x_n\| > n_k.
\]

The function \( x_\delta = \sum_{k=1}^{\infty} (P_{n_k} - P_{n_{k+1}})x_n \) belongs then to \( G \), since

\[
\|x_\delta\| \leq \sum_{k=1}^{\infty} \|(P_{n_k} - P_{n_{k+1}})x_n\| \leq \sum_{k=1}^{\infty} 2^{-k} = \delta.
\]

Choose any function \( y_0 \in N_f(x_\delta) \) and put

\[
y_\star = \sum_{k=1}^{\infty} (P_{n_k} - P_{n_{k+1}})y_{n_k} + (I - P_{n_k})y_0.
\]

Then \( y_\star \in N_f(x_\delta) \), but

\[
\|y_\star\| \geq \|(P_{n_k} - P_{n_{k+1}})y_\star\| = \|(P_{n_k} - P_{n_{k+1}})y_{n_k}\| > n_k \to \infty,
\]

again a contradiction. This proves the "if" part of Theorem 2. To prove the "only if" part, we have to show that the local boundedness of the operator (6) on \( G \) implies the boundedness of the function (9) on each compact subset of \( ri_X I(s) \). By the classical Heine-Borel theorem, it suffices in turn to prove that, for fixed \( s_0 \in \Omega \), the function \( \beta_{s_0} \) is locally bounded at each \( u_0 \in ri_X I(s_0) \).

For \( u_0 \in ri_X I(s_0) \) we have also

\[
U_\delta = (u_0 + B_\delta(X)) \cap supp_x(s_0) \subseteq G
\]

for \( \delta > 0 \) sufficiently small. Given \( u \in U_\delta \) and \( x \in B_\delta(X) \subseteq G \) with \( x(s_0) = 0 \), we have

\[
N_f(P_{\{s_0\}} u \cap P_{\Omega \setminus \{s_0\}} x)(s) = \begin{cases} f(s_0, u) & \text{if } s = s_0, \\
P_{\Omega \setminus \{s_0\}} N_f(P_{\Omega \setminus \{s_0\}} x)(s) & \text{if } s \neq s_0. \end{cases}
\]

We show that all elements \( z \in X \) with \( \|z - P_{\{s_0\}} u_0\| < \varepsilon = \min\{\delta, \varepsilon\} \) belong to \( G \). By the local boundedness of \( N_f \) on \( G \), this implies the local boundedness of \( f(s_0, \cdot) \) at \( u_0 \) as claimed. Thus, for these elements \( z \) we get

\[
\varrho \geq \varepsilon > \|z - P_{\{s_0\}} u_0\| = \|(z(s_0) - u_0) x_{\{s_0\}} + P_{\Omega \setminus \{s_0\}} z\|
\]

\[
\geq \max\{\|(z(s_0) - u_0) x_{\{s_0\}}\|, \|P_{\Omega \setminus \{s_0\}} z\|\},
\]

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which shows, in particular, that \( z(s_0) \in U_\delta \). The statement follows now from (12). □

4. LOCAL QUASI-BOUNDEDNESS

In this section, we shall derive parallel results for the local quasi-boundedness of the operator (6) between ideal spaces. To this end, a further notion is in order. We say that an ideal space \( Y \) has the **Fatou property** if, whenever \( B : \Omega \rightarrow \text{Cl}(\mathbb{R}^n) \) is a measurable multi such that the set \( Y_B \) of all selections \( y \in Y \) of \( B \) is non-empty, and \( D_m \in \mathcal{M} \) is a sequence of sets with \( D_m \uparrow \Omega \), we have

\[
\lim_{m \to \infty} \|P_{D_m} Y_B\|_* = \|Y_B\|_*.
\]

**THEOREM 3.** Let \( X \subset S(\Omega, \mathbb{R}^m) \) and \( Y \subset S(\Omega, \mathbb{R}^n) \) be two ideal spaces over \( \Omega = \Omega^d \), where \( Y \) is perfect and has the Fatou property. Suppose that \( f : \Omega \times \mathbb{R}^m \rightarrow \text{Cl}(\mathbb{R}^n) \) is a sup-measurable multi. Assume that the interior \( G \) of the domain of definition \( \mathcal{D}(N_f) \) of the operator (6), quasi-acting between \( X \) and \( \mathcal{P}(Y) \), is non-empty. Then \( N_f \) is locally quasi-bounded on \( G \).

**THEOREM 4.** Let \( X \subset S(\Omega, \mathbb{R}^m) \) and \( Y \subset S(\Omega, \mathbb{R}^n) \) be two ideal spaces over \( \Omega = \Omega^d \), where \( Y \) is perfect and has the Fatou property. Suppose that \( f : \Omega \times \mathbb{R}^m \rightarrow \text{Cl}(\mathbb{R}^n) \) is a sup-measurable multi. Assume that the interior \( G \) of the domain of definition \( \mathcal{D}(N_f) \) of the operator (6), quasi-acting between \( X \) and \( \mathcal{P}(Y) \), is non-empty. For \( s \in \Omega \), let

\[
J(s) = \{ u : u \in \text{supp}_X(s), f(s, u) \cap \text{supp}_Y(s) \neq \emptyset \},
\]

and denote by \( \text{ri}_X J(s) \) the relative interior of \( J(s) \) with respect to \( \text{supp}_X(s) \). Then \( N_f \) is locally quasi-bounded on \( G \) if and only if, for almost all \( s \in \Omega \), the function

\[
\gamma_s(u) = \|f(s, u)\|_*
\]

(see (8)) is bounded on each compact subset of \( \text{ri}_X J(s) \).

**PROOF OF THEOREM 3.** As in the proof of Theorem 1, we suppose that \( B_0(x) \subseteq G \) for some \( \delta > 0 \), and \( N_f \) is locally unbounded at 0. Choose sequences \( x_n \in X \) and \( y_n \in Y \) such that

\[
|x_n| \leq \delta 2^{-n}, \quad y_n \in N_f(x_n), \quad n 2^n + 1 < \|y_n\| \leq \|N_f(x_n)\|_* + 1.
\]

Let \( \{D_{n,j} : j = 1, \ldots, 2^n\} \) be a partition of \( \Omega \) with \( \mu_*(D_{n,j}) = 2^{-n} \) for \( j = 1, \ldots, 2^n \). Since the functionals (8) are disjointly subadditive, we have

\[
\|N_f(x_n)\|_* = \|N_f\left( \sum_{j=1}^{2^n} P_{D_{n,j}} x_n \right)\|_* = \|\sum_{j=1}^{2^n} P_{D_{n,j}} N_f(x_n)\|_* \leq \sum_{j=1}^{2^n} \|P_{D_{n,j}} N_f(x_n)\|_*.
\]

Consequently, for at least one index \( j(n) \) we have \( \|P_{D_{n,j(n)}} N_f(x_n)\|_* > n \). Let \( \Omega_{n,m} \) be defined as in the proof of Theorem 1. Since \( Y \) has the Fatou property, we conclude that \( \|P_{D_{n,m}} N_f(x_n)\|_* \rightarrow \|P_{D_{n,m}} N_f(x_n)\|_* \) as \( m \to \infty \). This means that
with each \( n \in \mathbb{N} \) we may associate an \( n' \in \mathbb{N}, n'>n \), such that \( \|P_{\Omega_{n,n'}}N_f(x_n)\|_\star > n \).

By induction, we construct a sequence \( n_1, n_2, \ldots, n_k = n_{k-1}' \), \( k \), of natural numbers such that the sets \( \Omega_k = \Omega_{n_k,n_{k+1}} \) are mutually disjoint and satisfy

\[
\mu_\star(\Omega_k) < 2^{-n_k}, \quad \|P_{\Omega_k}N_f(x_{n_k})\|_\star > n_k.
\]

The function \( x_* = \sum_{k=1}^\infty P_{\Omega_k}x_{n_k} \) belongs then to \( G \), since

\[
\|x_*\| \leq \sum_{k=1}^\infty \|P_{\Omega_k}x_{n_k}\| = \sum_{k=1}^\infty 2^{-k} = \delta.
\]

By the quasi-acting hypothesis on \( N_f \), we find a function \( y_* \in N_f(x_*) \cap Y \). But the relation \( P_{\Omega_k}y_* \subset P_{\Omega_k}N_f(x_*) \cap Y \) implies that

\[
\|y_*\| \geq \|P_{\Omega_k}y_*\| \geq \|P_{\Omega_k}N_f(x_*)\|_\star = \|P_{\Omega_k}N_f(x_{n_k})\|_\star > n_k \rightarrow \infty,
\]

a contradiction. \( \Box \)

The **Proof of Theorem 4** is completely analogous to that of Theorem 2, and therefore we shall drop it.

5. **BOUNDEDNESS AND QUASI-BOUNDEDNESS**

Following the terminology of [2], we shall call an ideal space \( X \) a **split space** if one can find a sequence \( \sigma(n) \) of natural numbers, depending only on \( X \), with the following property: given a sequence of functions \( x_n \in B_1(X) \) with disjoint supports, one can decompose each \( x_n \) in the form

\[
x_n = x_{n,1} + \cdots + x_{n,\sigma(n)}
\]

such that the functions \( x_{n,1}, \ldots, x_{n,\sigma(n)} \) are also mutually disjoint, and for each choice \( s = (s(1), \ldots, s(n), \ldots) \) of natural numbers \( s(n) \in \{1, \ldots, \sigma(n)\} \) the function \( x_s = \sum_{n=1}^\infty x_{n,s(n)} \) belongs also to \( B_1(X) \). As a matter of fact, almost all ideal spaces arising in applications are split spaces. In particular, every Orlicz space \( L_M \) (equipped with the usual Luxemburg norm) is a split space (see [1], [2]). Since in case \( \Omega = \Omega^c \) there exist only trivial split spaces, we shall restrict ourselves to the case \( \Omega = \Omega^c \) in what follows.

**THEOREM 5.** Let \( X \subset S(\Omega, \mathbb{R}^m) \) and \( Y \subset S(\Omega, \mathbb{R}^n) \) be two ideal spaces over \( \Omega = \Omega^c \), where \( X \) is a split space, and \( Y \) is perfect. Suppose that \( f : \Omega \times \mathbb{R}^m \rightarrow \text{Cl}(\mathbb{R}^n) \) is a weakly sup-measurable multi. Assume that the interior \( G \) of the domain of definition \( \mathcal{D}(N_f) \) of the operator (6), acting between \( X \) and \( \mathcal{P}(Y) \), is non-empty. Then \( N_f \) is bounded on each ball contained in \( G \).

**THEOREM 6.** Let \( X \subset S(\Omega, \mathbb{R}^m) \) and \( Y \subset S(\Omega, \mathbb{R}^n) \) be two ideal spaces over \( \Omega = \Omega^c \), where \( X \) is a split space, and \( Y \) is perfect and has the Fatou property. Suppose that \( f : \Omega \times \mathbb{R}^m \rightarrow \text{Cl}(\mathbb{R}^n) \) is a sup-measurable multi. Assume that the interior \( G \) of the domain of definition \( \mathcal{D}(N_f) \) of the operator (6), quasi-acting between \( X \) and \( \mathcal{P}(Y) \), is non-empty. Then \( N_f \) is quasi-bounded on each ball contained in \( G \).
PROOF OF THEOREM 5. We may suppose that $B_1(X) \subseteq G$ and $N_f$ is unbounded on $B_1(X)$. Choose sequences $z_n \in X$ and $y_n \in Y$ such that

\[(16) \quad |z_n| \leq 1, \quad y_n \in N_f(z_n), \quad |y_n| > 2^n n\sigma(n),\]

where $\sigma(n)$ is the sequence occurring in the definition of the split space $X$. As in the proof of Theorem 1, we may associate with the functions $y_n$ a sequence of mutually disjoint subsets $\Omega_n$ such that $\mu(\Omega_n) \leq 2^{-n}$ and $|P_{\Omega_n}y_n| > n\sigma(n)$. The functions $x_n = P_{\Omega_n}z_n$ belong then to $B_1(X)$ and have disjoint supports. Decompose each function $x_n$ in the form (15), and choose a corresponding partition $\{D_{n,1}, \ldots, D_{n,\sigma(n)}\}$ of $\Omega_n$ such that $x_{n,j}$ vanishes outside $D_{n,j}(j = 1, \ldots, \sigma(n))$. Since

$$P_{\Omega_n}y_n \in P_{\Omega_n}N_f(x_n) = \sum_{j=1}^{\sigma(n)} P_{D_{n,j}}N_f(x_n),$$

we have

\[(17) \quad |P_{\Omega_n}y_n| = \left| \sum_{j=1}^{\sigma(n)} P_{D_{n,j}}y_n \right| \leq \sum_{j=1}^{\sigma(n)} |P_{D_{n,j}}y_n|.
\]

Consequently, we may choose indices $s(n) \in \{1, \ldots, \sigma(n)\}$ such that $|P_{D_{n,s(n)}}y_n| > n$, by (16). Let $x_* = \sum_{n=1}^{\infty} x_{n,s(n)}$, where $x_{n,s(n)} = P_{D_{n,s(n)}}x_n$. Since $X$ is a split space, we have $x_* \in B_1(X)$, and hence $N_f(x_*) \subseteq Y$. Fix $y_0 \in N_f(x_*)$, and let

$$y_* = \begin{cases} P_{D_{n,s(n)}}y_n(s) & \text{if } s \in D_{n,s(n)}, \\ y_0(s) & \text{if } s \in D_0, \end{cases}$$

where $D_0 = \Omega \setminus \bigcup_{k=1}^{\infty} D_{k,s(k)}$. As in the proof of Theorem 3, we conclude that, on the one hand, $y_* \in N_f(x_*)$, and, on the other,

$$\|y_*\| \geq \|P_{D_{n,s(n)}}y_*\| = \|P_{D_{n,s(n)}}y_n\| > n \to \infty,$$

a contradiction. \(\square\)

PROOF OF THEOREM 6. Supposing again that $N_f$ is unbounded on $B_1(X)$, choose sequences $z_n \in X$ and $y_n \in Y$ such that

\[(18) \quad |z_n| \leq 1, \quad y_n \in N_f(z_n), \quad 2^n n\sigma(n) + 1 < |y_n| = |N_f(z_n)|_* + 1,
\]

where $|M|_*$ is defined as in (8). Define $\Omega_n, x_n, x_{n,j},$ and $D_{n,j}$ as in the proof of the preceding Theorem 5. The estimate (17) reads now

\[(19) \quad \|P_{\Omega_n}N_f(x_n)\|_* = \left| \sum_{j=1}^{\sigma(n)} P_{D_{n,j}}N_f(z_n) \right|_* \leq \sum_{j=1}^{\sigma(n)} \|P_{D_{n,j}}N_f(z_n)\|_*.
\]

by the disjoint subadditivity of the functional (8). Fixing indices $s(n) \in \{1, \ldots, \sigma(n)\}$ with $|P_{D_{n,s(n)}}N_f(z_n)|_* > n$, let $x_* = \sum_{n=1}^{\infty} x_{n,s(n)}$, where $x_{n,s(n)} = P_{D_{n,s(n)}}x_n$. Further, choosing $y_* \in N_f(x_*) \cap Y$, we get

$$\|y_*\| \geq \|P_{D_{n,s(n)}}y_*\| \geq \|P_{D_{n,s(n)}}N_f(z_n)\|_* > n,$$

the same contradiction as before. \(\square\)
Observe that the local boundedness properties of the operator (6) have been ensured by specific properties of the “target space” $Y$ (namely the perfectness property, see Theorems 1 and 2, or Fatou property, see Theorems 3 and 4), while the global boundedness properties of the operator (6) are due to specific properties of the “source space” $X$ (namely the split property, see Theorems 5 and 6). As far as continuity properties of the multivalued nonlinear superposition operator (6) are concerned, the situation is quite different. In contrast to linear operators, a nonlinear operator may be bounded without being continuous, or continuous without being bounded. A parallel study of the multivalued nonlinear superposition operator (6) from the viewpoint of continuity will be carried out in a subsequent paper.

REFERENCES